Almost toric fibrations, with applications

August 26, 2021

Plan for these two lectures:

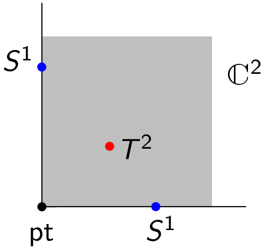
1. From toric fibrations to almost toric fibrations
2. Three applications of ATFs:
    - Exotic Lagrangian tori
    - The embedding function $E(1, a) \overset{s}{\hookrightarrow} B^4(A)$
    - Non-isotopic symplectic embeddings of cubes

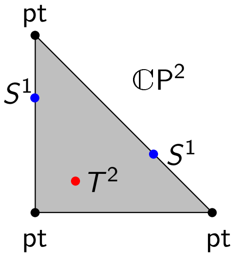
Today: **From toric fibrations to almost toric fibrations**

Recall: Toric fibration of (M, ω):

smooth surjection $\pi \colon M \to \mathbb{R}^n$ such that the fibers are Lagrangian tori T^n or subtori

Examples

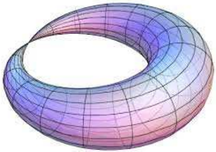




From now on: dim $= 4$ always

Almost toric fibration of (M, ω):

allow also the next best singularity (nodal/focus-focus):



Local normal form of such a singularity C:

There are Darboux coordinates $(\mathbf{x}, \mathbf{y})$ near C such that

$$\pi^{-1}(0) = C$$

and π near C is given by

$$\pi(\mathbf{x}, \mathbf{y}) = (x_1 y_1 + x_2 y_2, \, x_1 y_2 - x_2 y_1) \, .$$

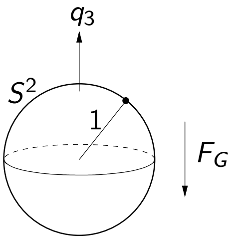
If we view $\mathbf{x}, \mathbf{y}$ as complex coordinates:

$$\mathbf{x} = x_1 + \mathrm{i}\, x_2, \quad \mathbf{y} = y_1 + \mathrm{i}\, y_2,$$

then $\pi(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{x}}\,\mathbf{y}$.

Hence C is the same singularity that appears in Lefschetz fibrations of symplectic manifolds.

More intuitive/dynamical approach: spherical pendulum:



Hamiltonian of this system:

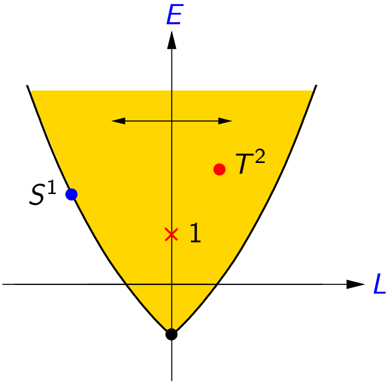
$$\begin{array}{rcl} T^*\mathbb{R}^3 \supset T^*S^2 & \to & \mathbb{R} \\ (q,p) & \overset{E}{\mapsto} & \frac{1}{2}\|p\|^2 + q_3 \end{array}$$

S^1-symmetry (by rotation around q_3-axis) gives preservation of angular momentum:

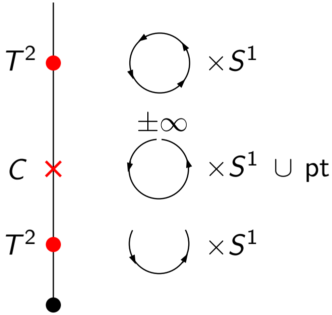
$$\begin{array}{rcl} T^*\mathbb{R}^3 \supset T^*S^2 & \to & \mathbb{R} \\ (q,p) & \overset{L}{\mapsto} & q_1\,p_2 - q_2\,p_1 \end{array}$$

Energy-momentum mapping:

$$\begin{array}{rcl} T^*\mathbb{R}^3 \supset T^*S^2 & \to & \mathbb{R}^2 \\ (q,p) & \overset{\mu}{\mapsto} & (L(q,p), E(q,p)) \end{array}$$



Fibers over the axes (planar penduli):



Hence $\mu^{-1}(0,1) =$ pinched torus C

So there is only one new fiber.

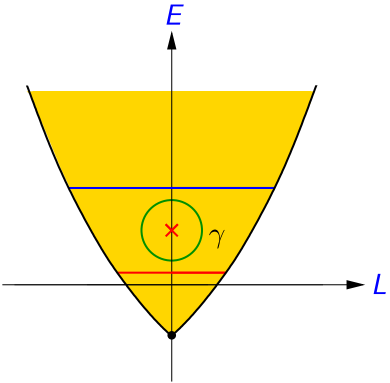
On $\text{Int}(\mu(T^*S^2)) \setminus (0,1)$ have T^2-fibration

Lemma This fibration is not trivial.

Indeed: It is trivial $\Leftrightarrow$ It is trivial over γ

$\Leftrightarrow$ $E^{-1}(1-\varepsilon) \cong E^{-1}(1+\varepsilon)$

But $E^{-1}(1-\varepsilon) \cong S^3 \neq \mathbb{R}\mathrm{P}^3 = T_1^*S^2 \cong E^{-1}(1+\varepsilon)$

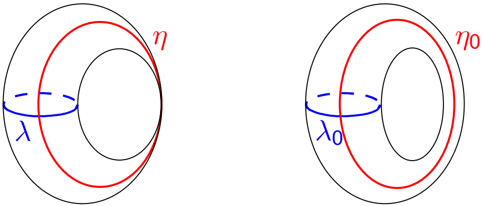


So there is non-trivial monodromy around γ. What is it?

On C choose curves η and λ.

- η: generated by X_E
- λ: generated by X_L (the vanishing cycle)

Take curves η_0, λ_0 on $T^2_{\gamma_0}$ close to η, λ

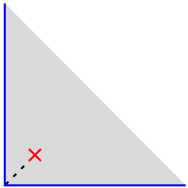


Then going around γ gives monodromy

$$\eta_0 \mapsto \eta_0 - \lambda_0, \quad \lambda_0 \mapsto \lambda_0, \quad i.e.\ M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

(Picard–Lefschetz formula; cf. e.g. Heckman)

Notation: $\pi^{-1}(\triangle) :=$



Meaning:

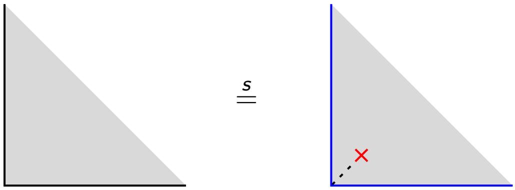
- $\times$ denotes a focus focus singularity
- - - - indicates the monodromy

Note: Over **every** boundary point there is an S^1;
the set over the boundary is a **smooth** cylinder;
the corner is an "illusion", it is smoothened by the monodromy

Recall: $\mu^{-1}(\triangle)$ is the open 4-ball

Lemma Nodal trade (Margaret Symington)

$$\pi^{-1}(\triangle) \overset{s}{=} \mu^{-1}(\triangle) = B^4$$



Sketch of proof

Step 1: diffeomorphic

From theory of Lefschetz fibrations one knows:

$\pi^{-1}(\otimes) \cong T^2 \times B^2$ with a -1-framed 2-handle attached along a simple closed curve in $T^2 \times \{\times\}$

Hence: $\pi^{-1}(\triangle) \cong S^1 \times B^3$ with a 2-handle attached ...

Since $S^1 \times B^3 \cong B^4$ with a 1-handle attached:

$\pi^{-1}(\triangle) \cong B^4$ with a 1-handle attached with a 2-handle attached

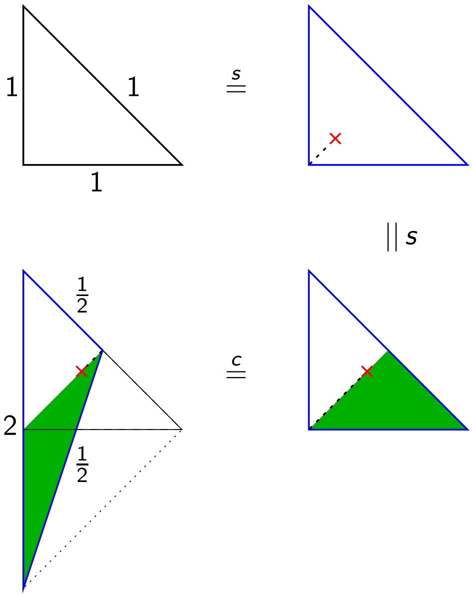
The 1-handle and 2-handle cancel.

Step 2: symplectomorphic

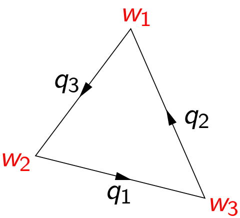
Eliashberg–Floer–McDuff:

Any symplectic form ω on B^4 that agrees with ω_0 near the boundary is diffeomorphic to ω_0, by a diffeomorphism fixing a neighbourhood of the boundary.

Nodal slide + transferring the cut (**Renato Vianna**):

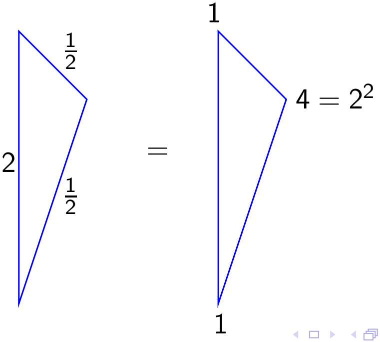


A rational triangle is (up to scaling) determined by its three integral angles:

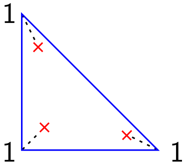


$w_1 := \|q_2 \times q_3\|, \ldots$

Example:



Now start with toric fibration of $\mathbb{C}P^2$, traded at all vertices:



and apply the above "nodal slide + transferring the cut" operation again and again.

Which triangles does one get?

"Recall" the Markov equation:

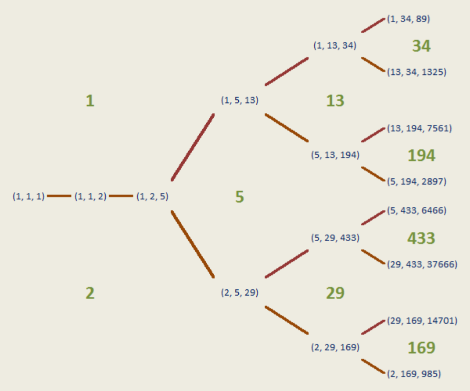
$$a^2 + b^2 + c^2 = 3abc, \qquad a, b, c \in \mathbb{N}$$

Exercise (Markov 1879) *If (a, b, c) is a solution, then also*

$$M(a, b, c) := (a, b, 3ab - c)$$

is a solution, and all solutions are obtained this way from $(1, 1, 1)$.

Get the Markov tree



Exercise (Vianna; Galkin–Usnich–Mikhalkin)

The triangles one gets from $\triangle(1,1,1)$ by

"nodal slide + transferring the cut" operations

are exactly the Markov triangles $\triangle(a^2, b^2, c^2)$, where (a, b, c) is a Markov triple.

More precisely: "nodal slide + transferring the cut" at the vertex with integral angle c transforms

$$\triangle(a^2, b^2, c^2) \rightsquigarrow \triangle(a^2, b^2, (3ab - c)^2).$$

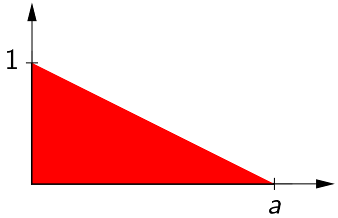
First application: Solution to $E(1,a) \overset{s}{\hookrightarrow} B^4(A)$

For $a, b > 0$:

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}$$

Can assume $b = 1,\ a \geq 1$

Moment polytope under $\mu(z_1, z_2) = \left(\pi|z_1|^2, \pi|z_2|^2\right)$:



Wish to know:

$$c(a) = \inf \left\{ A \mid E(1,a) \stackrel{s}{\hookrightarrow} B^4(A) \right\}$$

Obvious lower bound: volume constraint: $c(a) \geq \sqrt{a}$

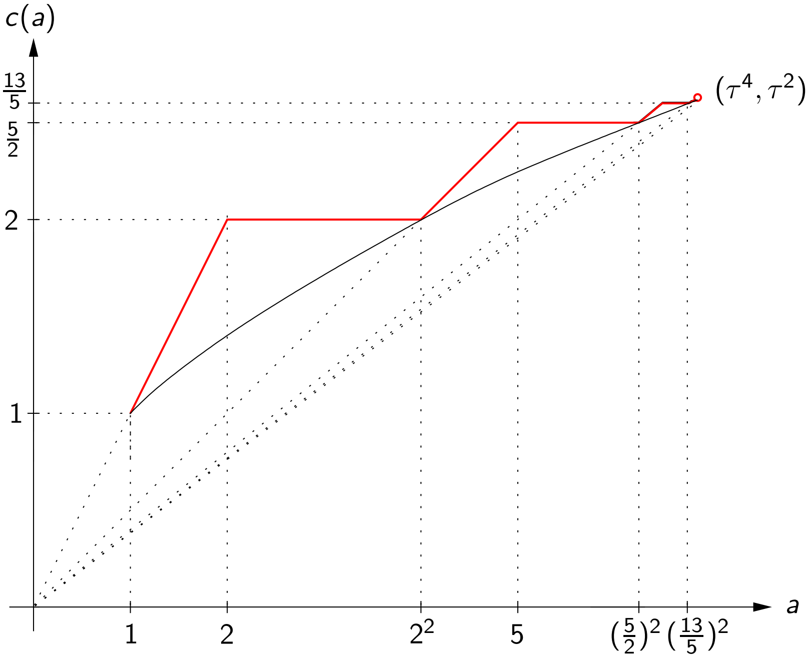
Answer (McDuff–S 2012)

1. For $a \in [1, \tau^4]$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio:
   $c(a)$ is given by the Fibonacci stairs

2. For $a \in \left[\tau^4, 8\frac{1}{36}\right]$: $c(a) = \sqrt{a}$ except for nine disjoint "steps"

3. $c(a) = \sqrt{a}$ for $a \geq 8\frac{1}{36}$

From now on: Focus on Fibonacci stairs



c(a)
13/5
5/2
2
1
(τ^4, τ^2)
a
1
2
2^2
5
(5/2)^2
(13/5)^2

Description

Fibonacci numbers

$$f_{-1} = 1, \quad f_0 = 0, \quad f_{n+1} = f_n + f_{n-1}$$

$g_n = f_{2n-1}$ odd-index Fibonacci numbers

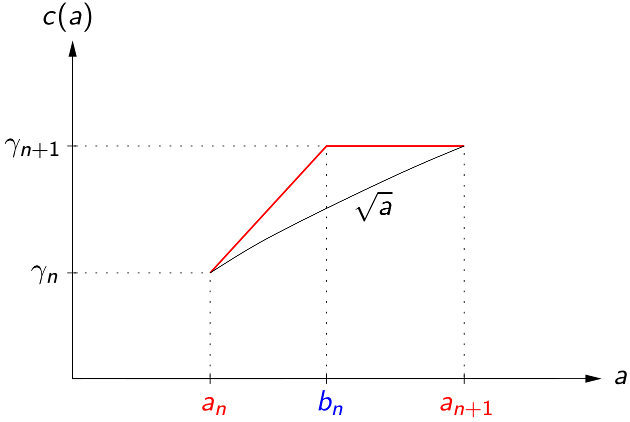
$$(g_0,\, g_1,\, g_2,\, g_3,\, g_4, \dots) = (1,\, 1,\, 2,\, 5,\, 13, \dots)$$

sequence $\gamma_n = \frac{g_{n+1}}{g_n}$

$$(\gamma_0,\, \gamma_1,\, \gamma_2,\, \gamma_3, \dots) = \left(1,\, 2,\, \frac{5}{2},\, \frac{13}{5}, \dots\right),$$

converges to τ^2

n'th step of the Fibonacci stairs



$$a_n = \gamma_n^2 = \left(\frac{g_{n+1}}{g_n}\right)^2 \quad \text{and} \quad b_n = \frac{g_{n+2}}{g_n}$$

Lemma It suffices to show that

$$c(a_n) \le \gamma_n \quad \text{and} \quad c(b_n) \ge \gamma_{n+1} \qquad (*)$$

Proof

- volume constraint, $(*)$, and monotonicity of $c(a)$ $\Rightarrow$
  $c(a_n) = \gamma_n$ and $c(b_n) = \gamma_{n+1}$, horizontal segment

- slanted segment: Take any $a \ge 1$ and $\lambda > 1$.

$$E(1, a) \overset{s}{\hookrightarrow} B^4(\mu) \iff E(\lambda, \lambda a) \overset{s}{\hookrightarrow} B^4(\lambda\mu)$$

Since $E(1, \lambda a) \subset E(\lambda, \lambda a)$, get

$$c(\lambda a) \le \lambda c(a) \quad \text{i.e.} \quad \frac{c(\lambda a)}{\lambda a} \le \frac{c(a)}{a}$$

Proof of $c(b_n) \geq \gamma_{n+1}$:

by cutting $E(1, a)$ into a collection of balls, symplectic blowing-up, existence of certain holomorphic spheres in blow-up of $\mathbb{C}P^2$

Proof of $c(a_n) \leq \gamma_n$:

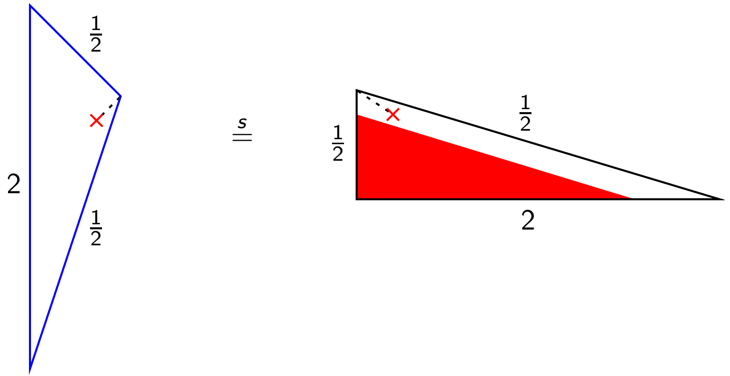
by these methods and more (more SWT and inflation)

But: these full embeddings

$$E(1, a_n) \overset{s}{\hookrightarrow} B^4(\sqrt{a_n})$$

are obtained at once and "explicitly" by the above almost toric fibrations of $\mathbb{C}P^2$!

After one operation:



Since the cuts can be chosen as short as we like, we obtain

$$E(\tfrac{1}{2}-\varepsilon, 2-\varepsilon) \subset \mathbb{C}\mathrm{P}^2(1).$$

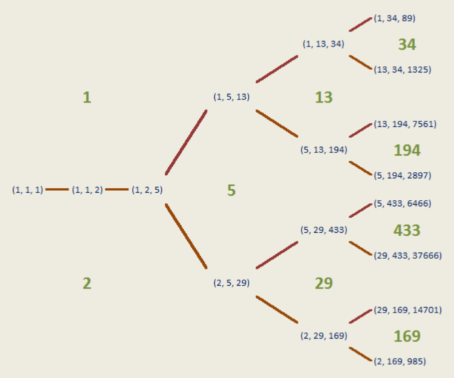
It is known (McDuff–Polterovich) that ellipsoid embeddings can always be made disjoint from $\mathbb{C}\mathrm{P}^1$, hence get

$$E(\tfrac{1}{2}-\varepsilon, 2-\varepsilon) \overset{s}{\hookrightarrow} \mathbb{C}\mathrm{P}^2(1) \setminus \mathbb{C}\mathrm{P}^1 \overset{s}{=} B^4(1)$$

hence $\qquad E(1-\varepsilon, 4-\varepsilon) \overset{s}{\hookrightarrow} B^4(2)$

In general, we have an ellipsoid triangle $\mu(E)$ in $\triangle(a^2, b^2, c^2)$ only if one of a, b, c is 1.

These Markov triangles form the upper branch in the Markov tree:



The entries form exactly the odd-index Fibonacci numbers g_n:

$$1, 1, 2, 5, 13, 34, 89, \ldots :$$

Lemma The solutions (a, b) of

$$a^2 + b^2 + 1 \;=\; 3\,a\,b$$

are exactly the (g_n, g_{n+1})

i.e., from the ATFs we exactly get the full fillings

$$E(1, a_n) \overset{s}{\hookrightarrow} B^4(\sqrt{a_n}).$$

Proof Consider the the upper branch

$$(1,1,1) \longrightarrow \dots \longrightarrow (1, a_n, b_n) \longrightarrow (1, a_{n+1}, b_{n+1}) \longrightarrow \dots$$

where $a_n < b_n$ and

$$a_{n+1} = b_n, \qquad b_{n+1} = 3b_n - a_n$$

Assume by induction that $(a_n, b_n) = (g_n, g_{n+1})$. Then

$$\begin{aligned} (a_{n+1}, b_{n+1}) &= (b_n, 3b_n - a_n) \\ &= (g_{n+1}, 3g_{n+1} - g_n) \\ &= (g_{n+1}, g_{n+2}) \end{aligned}$$

where the Fibonacci relation $3g_{n+1} - g_n = g_{n+2}$ is readily checked.

Two more applications of ATFs:

- Exotic Lagrangian tori
- Non-isotopic symplectic embeddings of cubes

First:

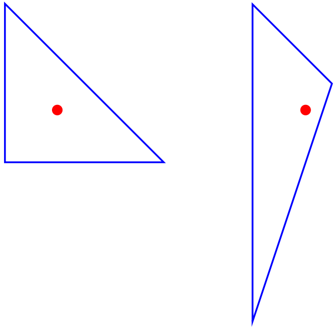
Theorem (Vianna; Galkin–Mikhalkin)

There are infinitely many different monotone Lagrangian tori in $\mathbb{C}\mathrm{P}^2$ and hence in $B^4(1)$.

"different" means: not Hamiltonian isotopic, and, in fact, not symplectomorphic

More precisely:

Let $\pi_{a,b,c}\colon \mathbb{C}\mathrm{P}^2 \to \triangle(a^2, b^2, c^2)$ be an ATF as above. Let $T_{a,b,c}$ be the torus over the center point:



Then $T_{a,b,c} \neq T_{a',b',c'}$ if $(a, b, c) \neq (a', b', c')$.

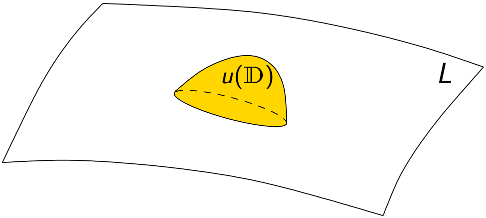
One method:

(**Eliashberg–Polterovich**;
Vianna, **Mikhalkin**, **Pascaleff–Tonkonog**)

Count the number of holomorphic discs

$$u\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$$

of Maslov index 2



Easier method: Versal deformations (**Chekanov**)

Idea: Study a symplectic invariant for tori **nearby** L

Example: **Displacement energy**

For $H\colon [0,1]\times M \to \mathbb{R}$ define the cost function

$$\|H\| = \text{int}_0^1 \left(\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x) \right) dt$$

For $A \subset M$ look at the optimal transport problem

$$e\,(A) = \inf_{H\in\mathcal{H}} \left\{ \|H\| \mid \varphi_H^1(A) \cap A = \varnothing \right\}$$

Exercise $e(D^2(a)) = e(S^1(a)) \leq a$

Hint: One always has

$$e(\psi(A)) = e(A) \quad \text{for all } \psi \in \mathrm{Symp}(M, \omega)$$

(look at $H \circ \psi^{-1}$)

For the exercise, take a ψ that maps $D^2(a)$ to (a rounding of) $[0,1] \times [0,a]$

Theorem $e(D^2(a)) = a$

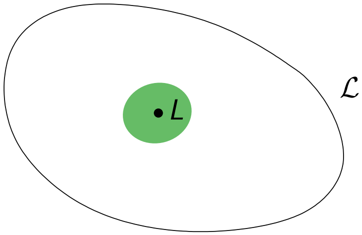
There is no easy proof.

The first and most elementary is by **Hofer 1990** by the calculus of variations for the action functional of classical mechanics.

A harder proof is by Chekanov's theorem:

$$e(L) \geq \text{minimal area of a non-constant } J\text{-holomorphic disc with boundary on } L$$

$e(\mathsf{T}_{\mathsf{Cliff}}, \mathbb{C}\mathsf{P}^2) = \infty$, but not at nearby tori:



Weinstein: locally

$$\{ \text{ Lagrangian tori near } L \} \,/\, \text{Ham} \;=\; H^1(L;\mathbb{R})$$

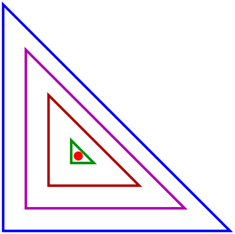
Since $e\colon \mathcal{L} \to [0,\infty]$ is Ham-invariant, obtain function germ

$$e_L\colon (H^1(L;\mathbb{R}), 0) \to [0,\infty]$$

Proposition Assume that x is not the center of $\triangle = \mu(\mathbb{C}P^2)$. Then

$$e(T(x)) = \mathrm{dist}_{\mathrm{int}}(x, \partial\triangle)$$

In other words: The level lines of $e_{T_{\mathrm{Cliff}}}$ in $\mathbb{C}P^2$ are:

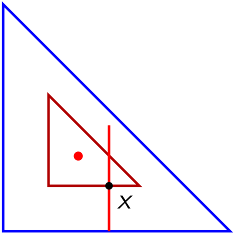


"Proof": Use symplectic reduction twice (Brendel):

$\leq$: Take a segment in $\triangle$ from x to $\partial\triangle$ of integral length

$$d := \mathrm{dist}_{\mathrm{int}}(x, \partial\triangle)$$

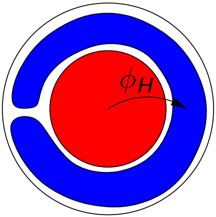
and extend this segment to a segment in $\triangle$ of length $> 2d$.



Over this segment we find a disc D^2 in M of area $> 2d$ containing the smaller factor $S^1(d)$ of $T(x)$.

Take a compactly supported Hamiltonian diffeomorphism ϕ_H of D^2 such that

$$\phi_H(S^1(d)) \cap S^1(d) = \emptyset \quad \text{and} \quad \|H\| \leq d + \varepsilon.$$



By cutting off we can extend H to $\widetilde{H}$ on M such that

$$\|\widetilde{H}\| \leq \|H\| \leq d + \varepsilon.$$

Then also

$$\phi_{\widetilde{H}}(T(x)) \cap T(x) = \emptyset$$

and so $e(T(x)) \leq d$.

$\geq$: Recall that $\mathbb{C}\mathrm{P}^2$ is a symplectic reduction of $S^5 \subset \mathbb{C}^3$.
The preimage of $T(a,b)$ is $T(a,b,1-a-b)$.

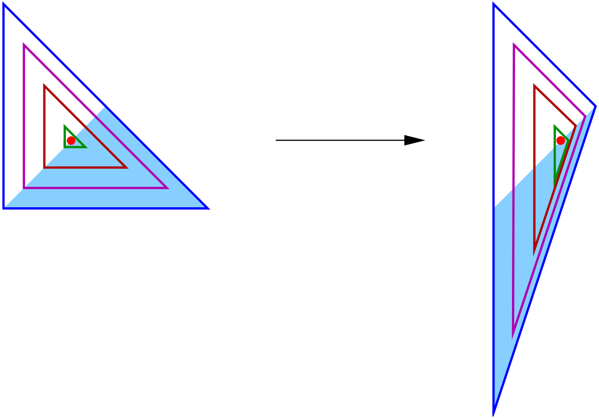
A Hamiltonian isotopy of $\mathbb{C}\mathrm{P}^2$ that disjoins $T(a,b)$ lifts to a Hamiltonian isotopy of $\mathbb{C}^3$ of the same Hofer norm that disjoins $T(a,b,1-a-b)$:

$$e\big(T(a,b);\mathbb{C}\mathrm{P}^2\big) \geq e\big(T(a,b,1-a-b;\mathbb{C}^3)\big).$$

By **Sikorav** (extending Hofer's argument, by the calculus of variations for the action functional of classical mechanics) or, again, by Chekanov's theorem:

$$\begin{aligned} e\big(T(a,b,1-a-b);\mathbb{C}^3\big) &= \min\{a,b,1-a-b\} \\ &= \mathrm{dist}_{\mathrm{int}}(T(a,b),\partial\triangle). \end{aligned}$$

Since the transferring the cut is done by a half-shear (in $SL(2, \mathbb{Z})$):



(away from thin neighbourhoods of rays!)
i.e. the level lines know the ATF up to $SL(2, \mathbb{Z})$

Hence the set of integral angles of the ATF is an invariant of the central torus. $\square$

Third application:
Non-isotopic symplectic embeddings of cubes

The problem

Take a "good" compact set $K \subset (\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$
like a ball or a cube of an ellipsoid or a polydisc

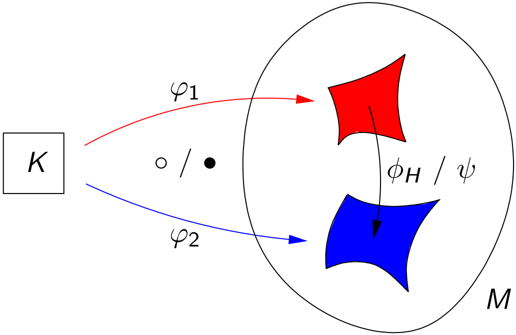
(M^{2n}, ω) connected symplectic manifold

$\mathrm{Emb}_\omega(K, M) :=$ space of symplectic embeddings $K \to M$

Question 1 Is $\mathrm{Emb}_\omega(K, M)$ non-empty ?

Question 2 Is $\mathrm{Emb}_\omega(K, M)$ connected ?
If not, what is $\pi_0(\mathrm{Emb}_\omega(K, M))$?

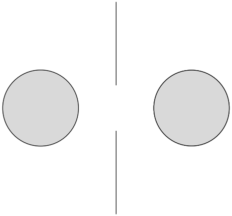
Question 3 What is the topology $(\pi_k,\ H_k)$ of $\mathrm{Emb}_\omega(K, M)$?



K
φ_1
φ_2
$\circ$ / $\bullet$
ϕ_H / ψ
M

Let's look at Question 2.

Examples



1. **Gromov**'s **camel theorem**: For the camel-space $\mathcal{C}(1)$ in $\mathbb{R}^{2n}$,

$$\mathrm{Emb}_\omega(\mathrm{B}^{2n}(a), \mathcal{C}(1))$$

   is **not** connected for $a > 1$

2. **McDuff**: $\mathrm{Emb}_\omega(\mathrm{B}^4(a), \mathring{\mathrm{B}}^4(A))$ is connected $\forall\, a < A$

ATFs can be used to say something on Question 2 for the domain a cube $C^4(a) := D^2(a) \times D^2(a)$:

Theorem 1 $M = \mathring{B}^4(3)$ or $\mathbb{C}P^2(3)$

(i) Consider the sequence

$$s_n = \frac{1}{g_n^2 + g_{n+1}^2}, \qquad n \geq 0$$

where g_n is the n'th odd-index Fibonacci number. Hence

$$(s_0, s_1, s_2, s_3, \dots) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{29}, \frac{1}{194}, \dots\right).$$

Then for $a \in (1, 1 + s_n)$ there are at least $n + 1$ non-equivalent symplectic embeddings of $C^4(a)$ into $\mathring{B}^4(3)$ and $\mathbb{C}P^2(3)$.

(ii) There are infinitely many non-equivalent symplectic embeddings of $C^4(1)$ into $\mathring{B}^4(3)$ and $\mathbb{C}P^2(3)$.

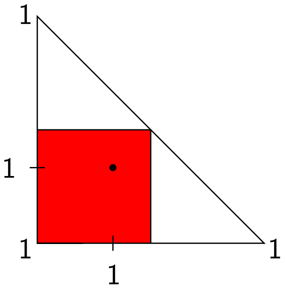
Proof

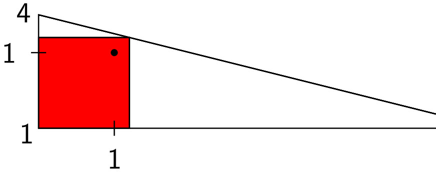
Have a symplectomorphism $\Phi\colon \mathrm{ATF}_1 \to \mathbb{C}P^2$

For $a < 1 + \frac{1}{5}$: $C^4(a) \subset \mathrm{ATF}_1$

Hence obtain $\varphi := \Phi|_{C^4(a)}\colon C^4(a) \overset{s}{\hookrightarrow} \mathbb{C}P^2$

Also have `id`: $C^4(a) \subset \mathrm{TF} = \mathbb{C}P^2$





Claim: $\mathtt{id} \not\sim \varphi$ if $a \in [1, 1+\frac{1}{5})$

Proof: If not, $\exists$ a symplectomorphism ψ of $\mathbb{C}\mathrm{P}^2$ such that

$$(\psi \circ \mathtt{id})(\mathrm{C}^4(a)) \;=\; \varphi(\mathrm{C}^4(a)).$$

Restricting to the central torus $L := \mathrm{T}(1) \times \mathrm{T}(1)$ we obtain

$$\psi(L) = \varphi(L).$$

But the Clifford torus L and $\varphi(L)$ are not symplectomorphic ! $\square$

The same question for $C^4(a) \overset{s}{\hookrightarrow} C^4(2)$ is more interesting: