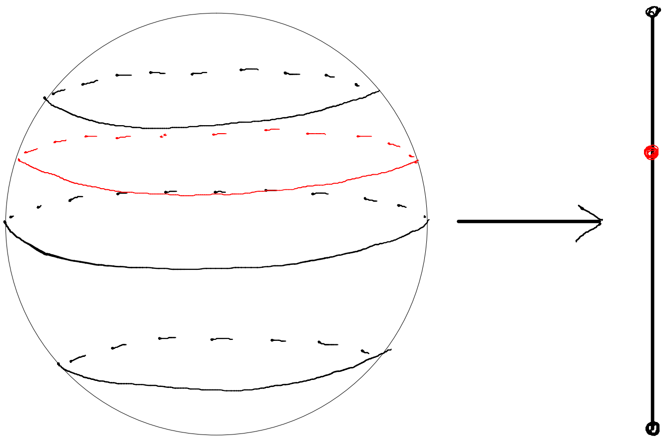
I. Toric symplectic manifolds & Delzant construction.

(GGTI Online Seminar on toric symplectic manifolds)



Joé BRENDEL – Université de Neuchâtel (23 August 2021)

Outline:

§0. Symplectic geometry & Ham. torus actions
§1. Toric manifolds
§2. Example: $\mathbb{CP}^2$
§3. Symplectic Reduction
§4. The Delzant construction.

(Exercises: green stars (*) & see last slide.)

References:

*) A. Cannas da Silva — Symplectic toric mflds.
*) E. Meinrenken — Symplectic geometry (script)
*) J. Evans — Lagrangian torus fibrations
*) V. Guillemin — Moment maps and combinatorial invariants of Hamiltonian T^n-spaces.
*) M. Audin — Torus actions on sympl. manifolds.

...

§0. Overview: Symplectic manifolds & Hamiltonian torus actions.

Def: Let (M,ω) be a symplectic manifold, i.e. 1) $\omega \in \Omega^2_{cl}(M)$
2) ω non-degen.

$$TM \xrightarrow{\cong} T^*M, \quad X \mapsto \iota(X)\omega = \omega(X,\cdot)$$

Let $H: M \to \mathbb{R}$ smooth function. Can associate a vector field $X_H \in \mathfrak{X}(M)$ to H by setting

$$\boxed{dH = \iota(X)\omega} \qquad \text{(Hamilton's equation)}$$

Def: The function H is called Hamiltonian (function).
The vector field X_H is called Hamiltonian vector field.
Its flow $\phi^t_H: M \to M$ is called Hamiltonian flow.

Rk: In general, Hamiltonians may be time-dependent.

Hamiltonian flows (*), 1) preserve the symplectic form, $(\phi_H^t)^* \omega = \omega$

2) preserve their generating Hamiltonian $H \circ \phi_H^t = H$. (in the autonomous case)

Example: $M = \mathbb{C} = \mathbb{R}^2$, $\omega = dx \wedge dy$,

$H = |z|^2 = x^2 + y^2$

Hamiltonian flow?

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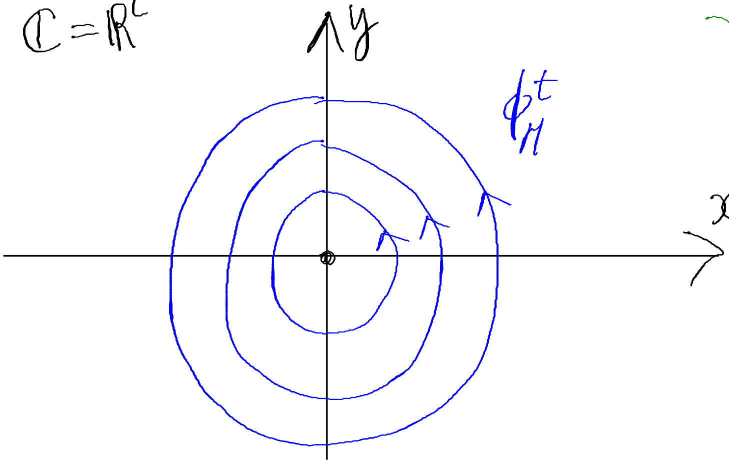
$H = |z|^2 = x^2 + y^2$

$\mathbb{C} = \mathbb{R}^2$



$(*)_2$

Note that $\phi_H^{\pi} = id$.

$\rightsquigarrow$ Defines an S^1-action.

More generally:

Def.: A smooth S^1-action is called Hamiltonian iff it is induced by a Hamiltonian flow.

Example: $M = S^2$, $\omega =$ area form from $S^2 \subseteq \mathbb{R}^3$.
$H(x,y,z) = z$

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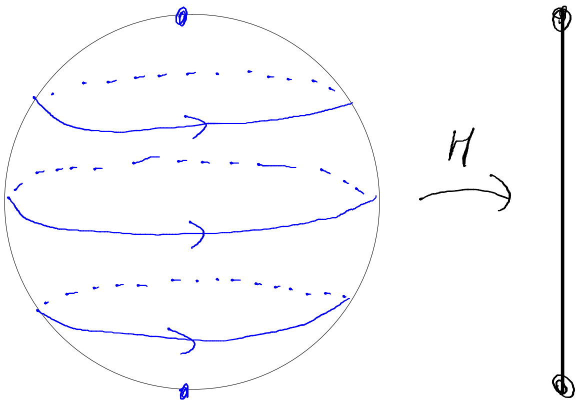


Hamiltonian flow = rotation around z-axis.

Hamiltonian torus actions:

Def: Two Hamiltonian circle actions commute if their Hamiltonians H, F satisfy:

$$\{H,F\} = \omega(X_H, X_F) = 0$$

(Poission bracket)

$(8)_3$ **Fact:** $\{H,F\} = 0 \implies \phi_H^t \circ \phi_F^s = \phi_F^s \circ \phi_H^t$.

$\implies (H,F)$ induce a $(T^2 = S^1 \times S^1)$-action.

Def: A torus action $T^k \circlearrowright M$ is called Hamiltonian if it is of the above type.

Def: The map: $\mu = (H_1, \dots, H_k) : M \to \mathbb{R}^k$ is called a moment map.

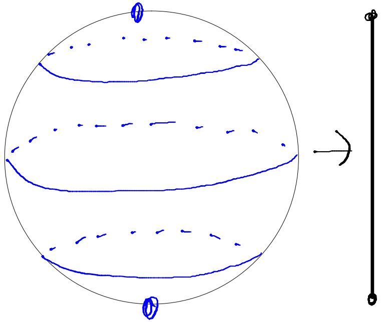
Such moment maps have remarkable properties!

Thm: (Atiyah / Guillemin—Sternberg)
The image of the moment map $\Delta := \mu(M) \subseteq \mathbb{R}^k$ is a convex rational polytope.

Def.: The polytope Δ is called moment polytope.

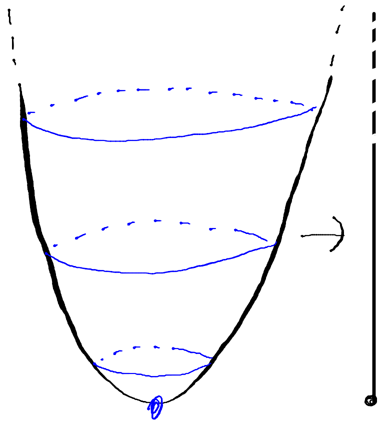
§1. Toric manifolds

Recall:



$S^2 \longrightarrow [-1,1]$

$(x,y,z) \longmapsto z$



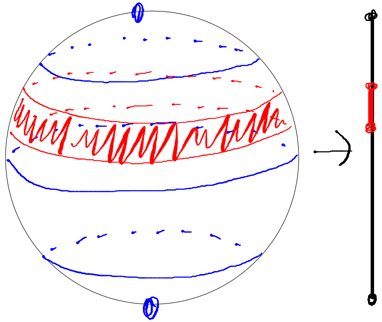
$\mathbb{R}^2 \longrightarrow \mathbb{R}_{\geq 0}$

$(x,y) \longmapsto x^2+y^2$

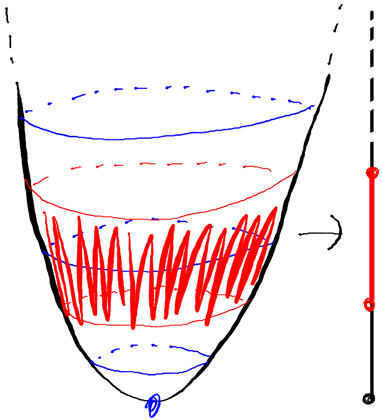
§1. Toric manifolds

Recall:

$(*)_4$



$$S^2 \longrightarrow [-1,1]$$
$$(x,y,z) \longmapsto z$$



$$\mathbb{R}^2 \longrightarrow \mathbb{R}_{\geq 0}$$
$$(x,y) \longmapsto x^2+y^2$$

Remark: The area of the red slices is proportional to the length of the red segments.

i.e. the moment map intertwines the geometry of Δ with the symplectic geometry of M.

Ambitious goal:

$$\{\text{Hamiltonian } T^k\text{-spaces}\} \xrightarrow{A./G.-S.} \{\text{Polytopes}\}$$

Reconstruct the manifold ??

Ambitious goal:

$$\{\text{Hamiltonian } T^k\text{-spaces}\} \xrightarrow{\text{A/G-S.}} \{\text{Polytopes}\}$$

true in a subset.

$\boxed{\text{Delzant's theorem: } \{\text{Toric symplectic mflds}\}/\sim \;\rightleftarrows\; \{\text{Delzant polytopes}\}/\sim \quad \text{Delzant construction.}}$

Def: (M^{2n}, ω) is called toric if there is an effective Hamiltonian T^n-action.

We get the "philosophy":

Symplectic geometry of toric manifolds $\longleftrightarrow$ Combinatorics of Delzant polytopes

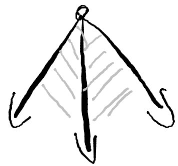
For us: 1) Embedding problems (balls, ellipsoids ...)

2) Lagrangian submanifolds, exotic tori: Chekanov, Vianna etc....

Def: A rational polytope $\Delta \subseteq \mathbb{R}^n$ is Delzant if it is: simple, smooth.

(*)

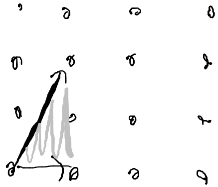
n facets at each vertex



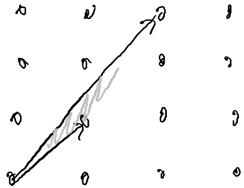
form a $\mathbb{Z}$-basis of $\mathbb{Z}^n \subseteq \mathbb{R}^n$ at each vertex.

Smoothness of polytopes:

smooth.

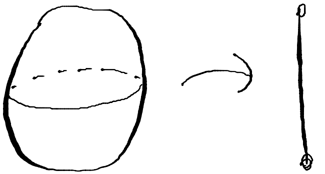


non-smooth.

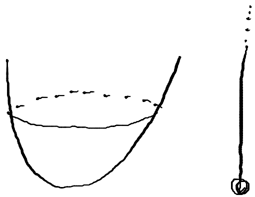


smooth.

Examples: 2-dimensional:



$S^2 \to [-1,1]$



$\mathbb{C} \to \mathbb{R}_{\geq 0}$

$S^1 \times \mathbb{R} \to \mathbb{R}$

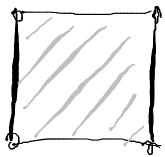
compact

non-compact

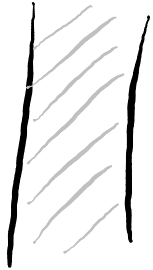
4-dimensional:

take products ...

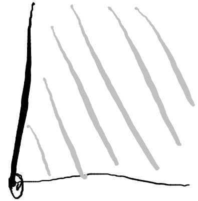
$S^2 \times S^2 \longrightarrow$



$S^2 \times T^*S^1 \longrightarrow$

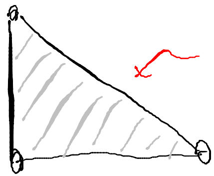


$\mathbb{C}^2 \longrightarrow$



... or not ...

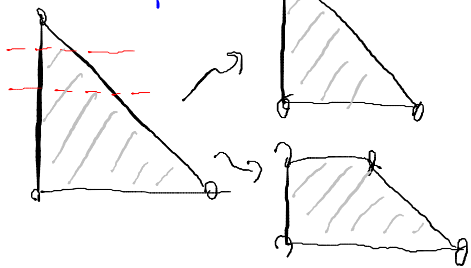
$\mathbb{CP}^2 \longrightarrow$



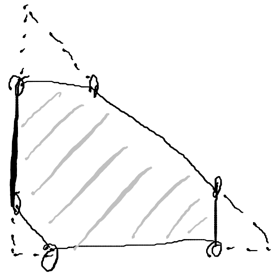
divisor at ∞.

... or take blow-ups:

$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \longrightarrow$

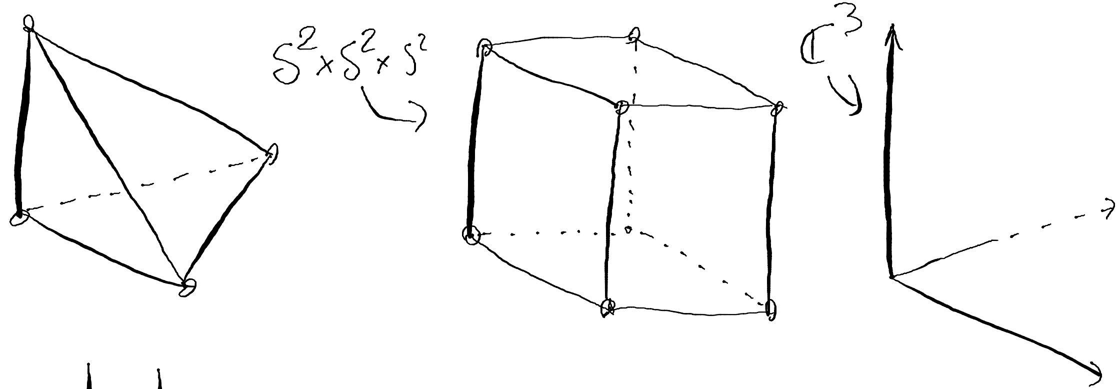


$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \longrightarrow$

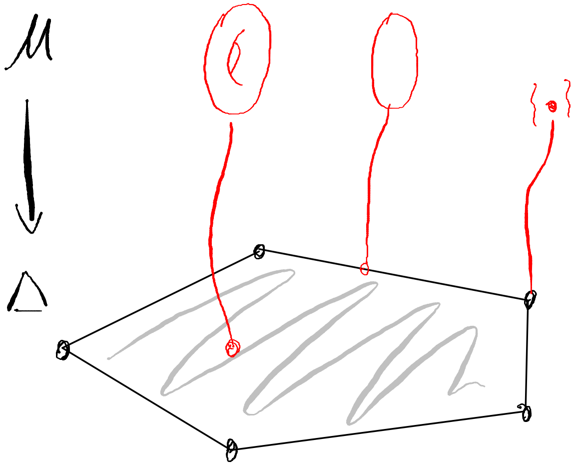


6-dimensional:

$\mathbb{CP}^3 \to$ $S^2 \times S^2 \times S^2 \to$ $\mathbb{C}^3 \to$



Fibration structure:



Recall: Hamiltonians are invariant under their flows,

$$\Longrightarrow \mu(t.x) = \mu(x)$$

In the toric case:

fibers = orbits of T^n-action.

$$\mu : \underset{\substack{\circlearrowleft \\ T^n}}{M} \to \Delta$$

is a quotient map.

§2. Example : $\mathbb{CP}^2$

$\mathbb{CP}^2 = \{[z_1 : z_2 : z_3]\}$,

Idea : Start with $\mathbb{C}^3$, and construct $\mathbb{CP}^2$

Symplectic: $(\mathbb{C}^3, \omega_0)$ $(\mathbb{CP}^2, \omega_{FS})$

Can be done by symplectic reduction.

Toric structure on $\mathbb{C}^3$:

$$T^3 \circlearrowright \mathbb{C}^3 \; : \; (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \circ (z_1, z_2, z_3) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3)$$

generated by :

$$\mu : \mathbb{C}^3 \longrightarrow \mathbb{R}^3_{\geq 0}$$
$$(z_1, z_2, z_3) \mapsto (|z_1|^2, |z_2|^2, |z_3|^2)$$

Take $S^1 \subseteq T^3$, $e^{i\theta} \mapsto (e^{i\theta}, e^{i\theta}, e^{i\theta})$ subtorus

induced by $H(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + |z_3|^2$

$= \mu_1 + \mu_2 + \mu_3$

$= \langle \mu, (1,1,1) \rangle$. induced Hamiltonian.

Consider $H^{-1}(1) = S^5 \subseteq \mathbb{C}^3$: Since ϕ_H^t preserves $H^{-1}(c)$

Hopf fibration.

$$\begin{cases} \overset{S^1}{\circlearrowright} H^{-1}(1) = S^5 \hookrightarrow \mathbb{C}^3 \\ \quad \downarrow \\ \quad \mathbb{CP}^2 \end{cases}$$

Take $S^1 \subseteq T^3$, $e^{i\theta} \mapsto (e^{i\theta}, e^{i\theta}, e^{i\theta})$ subtorus

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Consider $H^{-1}(1) = S^5 \subseteq \mathbb{C}^3$: Since ϕ_H^t preserves $H^{-1}(c)$

Symplectic reduction.

$$\begin{array}{ccc} S^1 \circlearrowright H^{-1}(1) = S^5 & \hookrightarrow & (\mathbb{C}^3, \omega_0) \\ \downarrow & & \\ (\mathbb{CP}^2, \omega_{FS}) & & \end{array}$$

$$\boxed{\text{Q: Where is the toric structure?}}$$

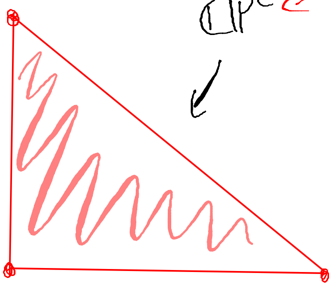
We want $T^2 \circlearrowright \mathbb{CP}^2$ Hamiltonian.

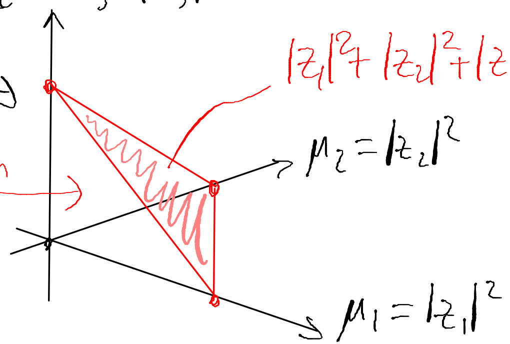
$$\begin{tikzcd}
T^3 \circlearrowright S^5 \arrow[r, hook] \arrow[d, squiggly] & (\mathbb{C}^3, \omega_0) \circlearrowleft T^3 \\
T^2 \cong T^3/S^1 \circlearrowright (\mathbb{CP}^2, \omega_{FS}) &
\end{tikzcd}$$

$$S^5 \xrightarrow{/S^1} (\mathbb{CP}^2, \omega_{FS})$$

"residual" action.

Moment maps: $\mathbb{CP}^2 \xleftarrow{\text{reduction}} S^5 \subseteq \mathbb{C}^3$





inclusion

§3. Symplectic Reduction

<u>Idea</u>: $M \circlearrowleft G$ smooth free group action $\Rightarrow$ M/G quotient manifold.

$(M,\omega) \circlearrowleft G$ Hamiltonian group action & some conditions $\Rightarrow$ $(\tilde{M}, \tilde{\omega})$ symplectic quotient manifold.

<u>Thm</u>: (Marsden–Weinstein, ... for torus actions)

Let $M \circlearrowleft T^k$ Hamiltonian torus action with moment map

$$\mu : M \longrightarrow \mathbb{R}^k.$$

Let $c \in \mathbb{R}^k$ s.th. $\mu^{-1}(c) \circlearrowleft T^k$ freely.

$\Rightarrow \exists \tilde{\omega}$ symplectic form on $\mu^{-1}(c)/T^k =: \tilde{M}$ s.th.

$$\begin{tikzcd} \mu^{-1}(c) \arrow[r, hook, "i"] \arrow[d, "/T^k"', "\pi"] & (M,\omega) \\ (\tilde{M}, \tilde{\omega}) & \end{tikzcd} \qquad \boxed{\pi^* \tilde{\omega} = i^* \omega}$$

Exercises: (See last pages for hints)

① Check that Hamiltonians flows preserve the symplectic form and the generating Hamiltonian

$$(\phi_H^t)^* \omega = \omega, \quad H \circ \phi_H^t = H$$

② Compute the Hamiltonian flows for:

i) $M = \mathbb{R}^2$, $\omega = dx \wedge dy$, $H(x,y) = x^2 + y^2$

ii) $M = S^2$, $\omega =$ area form, $H(x,y,z) = z$
(from $S^2 \hookrightarrow \mathbb{R}^3$ as round sphere)

Check that they indeed induce circle actions!

③ (On the Poisson bracket)

i) Let $F, G \in \mathcal{C}^\infty(M)$ s.th $\{F, G\} = 0$. Check that F is constant along ϕ_G^t
(G —— " —— ϕ_F^t)

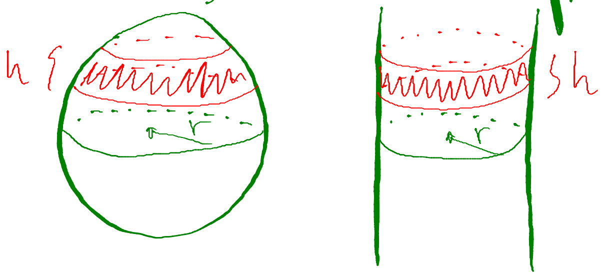
ii) Check that $X_{\{F,G\}} = [X_F, X_G]$
(here $[\cdot,\cdot]$ is the Lie bracket of vector fields)

iii) Deduce the claim from the talk:

$$\{F,G\} = 0 \underset{(\dagger)}{\Longrightarrow} \phi_F^t \circ \phi_G^s = \phi_G^s \circ \phi_F^t$$

iv) Does the converse of $(\dagger)$ hold? Give a counterexample or a proof.

(4) Prove Archimedes' principle, i.e. the red slices have equal areas.



Can you prove it without differential geometry (like Archimedes)?
I can't ....

5 (On the Delzant condition of polytopes)

i) Let $v \in \mathbb{Z}^2 \setminus \{0\}$. Is there always $w \in \mathbb{Z}^2 \setminus \{0\}$ s.th. the vertex formed by (v, w) is smooth?
If not: Counterexample. If yes, prove it and discuss uniqueness of w.

ii) Show that any two Delzant vertices are equivalent under $GL(n; \mathbb{Z})$.

iii) Let $B^{2n}_{(r)} := \{ x \in \mathbb{R}^{2n} \mid \|x\| \leq r \}$ the Euclidean ball. Compute its image under the standard moment map of $\mathbb{R}^{2n}$.
Use ii) to deduce that every vertex of a Delzant moment polytopes gives a symplectic Ball embedding. (See also Felix's talk)

There are some computational rules that may come in handy for exercises ① – ③ :

$\mathcal{L}_X$: Lie derivative, $\iota(X)$: contraction, d : exterior derivative.

on vector fields: $\mathcal{L}_X(Y) = [X,Y]$ ("Cartan's magic formula")

on differential forms: $\mathcal{L}_X\alpha = d\iota(X)\alpha + \iota(X)d\alpha$

$$\mathcal{L}_X d\alpha = d\mathcal{L}_X\alpha$$

① Differentiate and use $\frac{d}{dt}(\phi_H^t)^*(\circ) = \mathcal{L}_{X_H}(\circ)$

② Choose smart coordinates (especially on S^2...)

③ i), ii) : Differentiate

iv) Take the easiest (non-constant) functions on $\mathbb{R}^2 = \{(x,y)\}$

⑤ ii) Can be proved by a one-line sentence.