§4. The Delzant construction.

Recall : $\{\text{toric manifolds}\}/\sim \xrightarrow{1:1} \{\text{Delzant polytopes}\}/\sim$

$$(M,\omega,\mu) \longmapsto \Delta := \mu(M)$$

The Delzant construction gives an inverse map.

Idea : i) Start with $(\mathbb{C}^k, \omega_0, \mu_0)$, where

$$\mu_0(z_1,\dots,z_k) = (|z_1|^2,\dots,|z_k|^2) \in \mathbb{R}^k_{\geq 0}$$

generates standard $T^k \circlearrowright \mathbb{C}^k$ action.

ii) Construct (M,ω) as a symplectic reduction of $(\mathbb{C}^k,\omega_0)$ by a subtorus $K \leq T^k$

iii) The residual action of $T^n \cong T^k/K$ turns (M,ω) into a toric manifold.

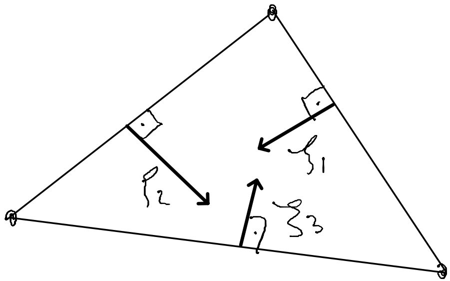
$$\begin{tikzcd}
T^k \circlearrowright Z = \mu_K^{-1}(0) \arrow[r, hook] \arrow[d, "\pi"', "/K"] & (\mathbb{C}^k, \omega_0) \circlearrowleft T^k \\
T^n \cong T^k/K \circlearrowright (M, \omega) &
\end{tikzcd}$$

$$\boxed{\pi^* \omega = \omega_0}$$

Delzant construction — the recipe:

Let $\Delta \subseteq \mathbb{R}^n$ be a Delzant polytope. Describe as:

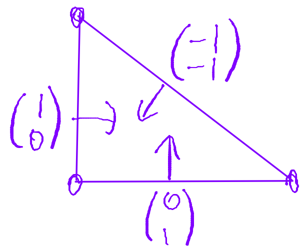
$$\Delta = \{ x \in \mathbb{R}^n \mid \underbrace{\langle x, \xi_i \rangle + \lambda_i \geq 0}_{\substack{\text{intersection of } k \text{ halfspaces} \\ (k = \#\text{ facets of } \Delta)}} \}$$



where ξ_i primitive, inward pointing normal vectors to the facets.

(Example: $\mathbb{CP}^2$ $\quad x = (x_1, x_2)$

$$x_1 + 1 \geq 0$$
$$x_2 + 1 \geq 0$$
$$-x_1 - x_2 + 1 \geq 0$$



)

Define the linear map

$$\Xi : \mathbb{R}^k \longrightarrow \mathbb{R}^n \quad \text{by} \quad \Xi = (\xi_1 | \dots | \xi_k)$$

Note that $\Xi(\mathbb{Z}^k) \subseteq \mathbb{Z}^n$ lattice embedding

$\rightsquigarrow$ $\Xi : T^k \longrightarrow T^n$ map on tori.

<u>Define</u>: $\boxed{K := \ker \Xi \subseteq T^k}$ $\rightarrow$ $(k-n)$-dim. subtorus.

(<u>Example</u>: $(\mathbb{C}P^2)$ $\quad \Xi = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} : T^3 \longrightarrow T^2$

with $K = \ker \Xi = \{(e^{i\theta}, e^{i\theta}, e^{i\theta})\}$)

Perform symplectic reduction with respect to K.

$K \hookrightarrow \underbrace{T^k \circlearrowright \mathbb{C}^k}_{\text{std. action}}$ with moment map $\underbrace{j^*\mu_0}_{=\mu_K} : \mathbb{C}^k \to \mathbb{R}^{k-n}$.

$j : \underset{\underset{T^{k-n}}{\cong}}{K} \hookrightarrow T^k$,

$$\boxed{\begin{array}{ccc} \mu_K^{-1}(c) & \hookrightarrow & (\mathbb{C}^k, \omega_0) \\ \downarrow \pi & & \\ (M, \omega) & & \end{array}}$$

Check: Δ Delzant $\Rightarrow$ K acts freely on $\mu_K^{-1}(0)$.

(Example: $(\mathbb{CP}^2)$

$$\left.\begin{array}{l} j : S^1 \to T^3 \\ \quad e^{i\theta} \mapsto (e^{i\theta}, e^{i\theta}, e^{i\theta}) \\ \Rightarrow j^*\mu_0 = |z_1|^2 + |z_2|^2 + |z_3|^2 \end{array}\right\} \to \begin{array}{ccc} S^5 & \hookrightarrow & \mathbb{C}^3 \\ \downarrow & & \\ \mathbb{CP}^2 & & \end{array}$$

)

The symplectic quotient 1) is toric,
2) has moment polytope Δ.

Recall: Short exact sequence of tori:

$$1 \longrightarrow K \xhookrightarrow{j} T^k \xrightarrow{\pi} T^n \longrightarrow 1$$

$$\begin{tikzcd}
T^k \circlearrowright \mu_K^{-1}(c) \arrow[r, hook] \arrow[d, "/K"] & (\mathbb{C}^k, \omega_0) \\
T^n = T^k/K \circlearrowright M &
\end{tikzcd}$$

This induced action is toric,
check by computing its moment map.

$$\begin{tikzcd}
M \arrow[dr, dashed, bend right, "\mu"'] & \mu_K^{-1}(c) \arrow[l, two heads] \arrow[r, hook] & \mathbb{C}^k \arrow[d, "\mu_0"] \arrow[dr, bend left, "\mu_K"] & \\
& \mathbb{R}^n \arrow[r, hook, "\pi^*"] & \mathbb{R}^k \arrow[r, "j^*"] & \mathbb{R}^n
\end{tikzcd}$$

Application of Delzant's construction:

"Understand data on (M,ω) as K-invariant/equivariant data on $(\mathbb{C}^k,\omega_0)$"

1) Understanding J-holomorphic spheres / disks in (M,ω): Cho–Oh, Vortex equations Salamon–Cieliebak, Woodward etc.

2) Understanding anti-symplectic involutions on (M,ω): (B.–Kim–Moon '19, B. '20)

3) Understanding Lagrangians / toric fibres in (M,ω):

$$
\begin{array}{ccc}
T(a_1,\dots,a_k) \subseteq Z & \hookrightarrow & (\mathbb{C}^k,\omega_0) \\
\downarrow \quad\quad \downarrow & & \\
T_x \subseteq (M,\omega) & &
\end{array}
$$

product tori

$\Uparrow$ lift

toric fibres.

References:

*) A. Cannas da Silva — Symplectic toric mflds.

*) E. Meinrenken — Symplectic geometry (script)

*) J. Evans — Lagrangian torus fibrations

*) V. Guillemin — Moment maps and combinatorial invariants of Hamiltonian T^n-spaces.

*) M. Audin — Torus actions on sympl. manifolds.

...

Thanks for your attention!