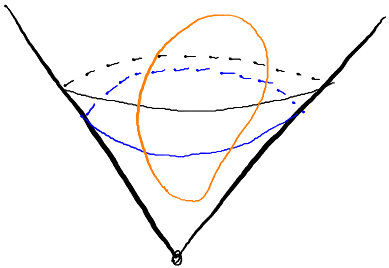
II. The Chekanov torus.

GGTI Online Seminar on Symplectic toric manifolds



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Outline: §3. Symplectic reduction
§4. Background: Lagrangian tori
§5. Construction of the Chekanov torus
§6. Displacement energy and versal deformations
§7. The Chekanov torus is exotic.

References:

*) Yu. Chekanov, Lagrangian tori in a symplectic vector space and global symplectomorphisms, '96
*) Ya. Eliashberg, L. Polterovich, The problem of Lagrangian knots in four-manifolds, '97
*) F. Schlenk, Yu. Chekanov, Notes on monotone Lagrangian twist tori, '10
*) J. Brendel, Real Lagrangian tori and versal deformations, '20

§3. Symplectic Reduction

Idea: $M \circlearrowleft G$ smooth free group action $\Rightarrow$ M/G quotient manifold.

$(M,\omega) \circlearrowleft G$ Hamiltonian group action & some conditions $\Rightarrow$ $(\tilde{M},\tilde{\omega})$ symplectic quotient manifold.

Thm: (Marsden–Weinstein, ... for torus actions)

Let $M \circlearrowleft T^k$ Hamiltonian torus action with moment map
$$\mu : M \to \mathbb{R}^k.$$
Let $c \in \mathbb{R}^k$ s.th. $\mu^{-1}(c) \circlearrowleft T^k$ freely.

$\Rightarrow$ $\exists\, \tilde{\omega}$ symplectic form on $\mu^{-1}(c)/T^k =: \tilde{M}$ s.th.

$$\begin{tikzcd} \mu^{-1}(c) \arrow[r, hook, "\iota"] \arrow[d, "\pi"', "/T^k"] & (M,\omega) \\ (\tilde{M},\tilde{\omega}) & \end{tikzcd}$$

$$\boxed{\pi^*\tilde{\omega} = \iota^*\omega}$$

idea of proof: From $\mu^{-1}(c) \overset{\iota}{\subseteq} M$, get $\iota^*\omega \in \Omega^2(\mu^{-1}(c))$

Problem: This form $\iota^*\omega$ is degenerate.

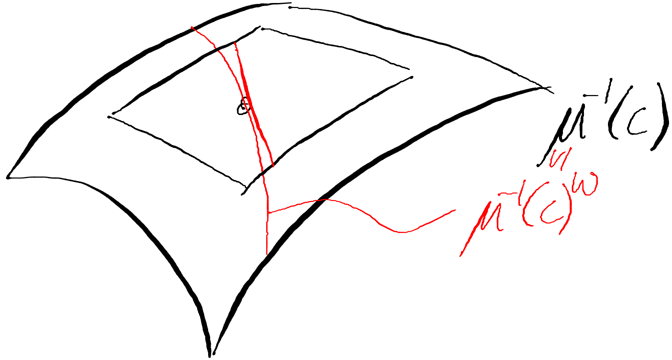
But: $\mu^{-1}(c)$ is coisotropic, i.e.
$T_x\mu^{-1}(c)^{\omega} \subseteq T_x\mu^{-1}(c)$

① Coisotropic reduction. Let (V,ω) symplectic vector space.

Ex.: prove this claim

$(\ast)$: Let $E \subseteq V$ coisotropic subspace, i.e. $E^\omega \subseteq E$. Then the vector space $W = E/E^\omega$ carries a natural symplectic form, $(W, \tilde{\omega})$.

②



Recall:

$$\begin{tikzcd} \mu^{-1}(c) \arrow[d, two heads, "/T^k"] \\ \tilde{M} \end{tikzcd}$$

Ex: prove this.

$(*)_1$ "Miracle": $T_x\mu^{-1}(c)^\omega = T_x\mathcal{O}_x$

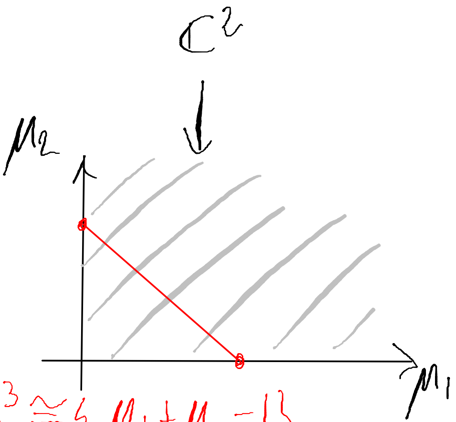
where $\mathcal{O}_x = T^k_\circ x$ orbit.

The group action divides out the "degenerate part of $\iota^*\omega$".

$\rightsquigarrow$ get symplectic form $\tilde{\omega}$ on quotient.

"Toric reduction" by examples:

(1)



$S^3 \cong \{\mu_1 + \mu_2 = 1\}$

$$\left.\begin{array}{l} S^3 \hookrightarrow (\mathbb{C}^2, \omega_0) \\ \Big\downarrow \; /S^1 = \{(e^{i\theta}, e^{i\theta})\} \\ (S^2, \omega) \end{array}\right\} (*)_2$$

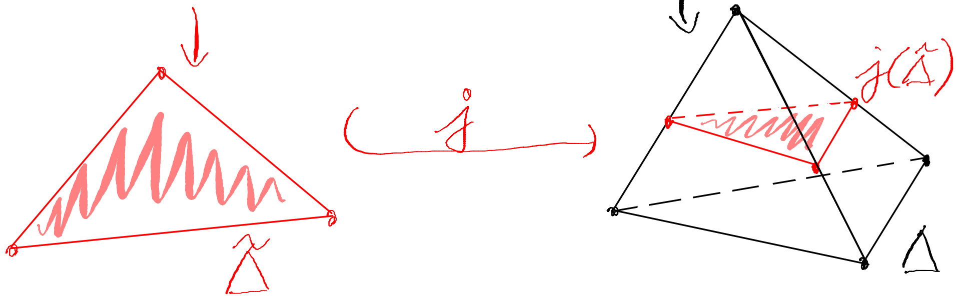
(2)

$\mathbb{R}^2 \times S^1 \cong \{\mu_1 - \mu_2 = c\}$

$$\begin{array}{l} \mathbb{R}^2 \times S^1 \hookrightarrow (\mathbb{C}^2, \omega_0) \\ \Big\downarrow \; /S^1 = \{(e^{i\theta}, e^{-i\theta})\} \\ (\mathbb{R}^2, \omega_0) \end{array}$$

General principle:

$$(\tilde{M}, \tilde{\omega}) \Longleftarrow \mu^{-1}(j(\tilde{\Delta})) \subseteq (M, \omega)$$



$\exists$ toric reduction: $M \longrightarrow \tilde{M}$ $\iff$ $\exists$ "nice" embedding of Delzant polytopes: $\tilde{\Delta} \hookrightarrow \Delta$

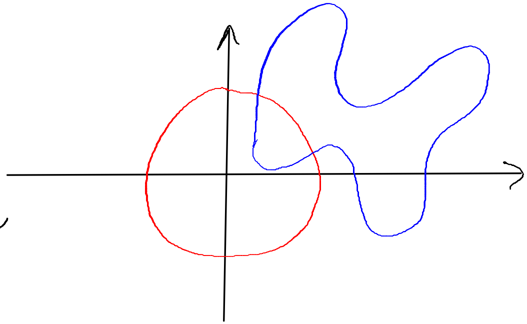
§4. Background on Lagrangian tori

Recall: $L \subseteq M$ is called Lagrangian if 1) $\dim L = \frac{1}{2}\dim M$
2) $\omega|_{TL} = 0$

Examples of Lagrangian tori:

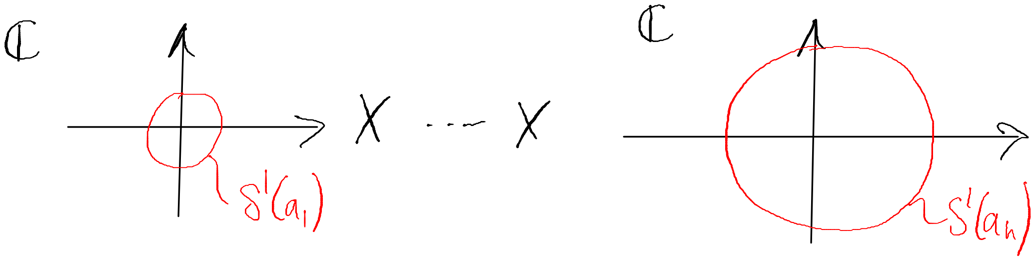
1) Circles in $\mathbb{C} = \mathbb{R}^2$

$S^1(a) :=$ circle containing area $a \geqslant 0$.



Fact: Let $\gamma \subseteq \mathbb{R}^2$ be a simple closed curve containing area$_\gamma = a$. Then $\gamma \cong S^1(a)$ Hamiltonian isotopic.

2) Product tori in $\mathbb{C}^n$:



$$T^n(a_1,\dots,a_n) := S^1(a_1)\times\dots\times S^1(a_n) \subseteq \mathbb{C}^n \text{ Lagrangian.}$$

Question: Are all Lagrangian tori $L \subseteq (\mathbb{C}^n, \omega_0)$ up to Hamiltonian isotopy of this type?

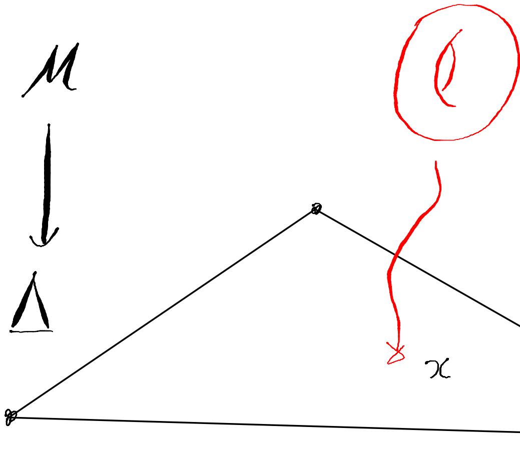
For $n=1$: Yes, as we have seen.

$n=2$: No! (Chekanov, Eliashberg–Polterovich)

$(*)_4$

3) Toric fibres

$$\begin{array}{c} M \\ \downarrow \\ \Delta \end{array}$$



$T_x = \mu^{-1}(x)$
Lagrangian torus
$=$ orbit of T^n-action.

The product tori $\subseteq \mathbb{C}^n$ are also of this type!

$\boxed{\text{Question: Are there other tori?}}$

Call them exotic.

§5. Construction of the Chekanov torus

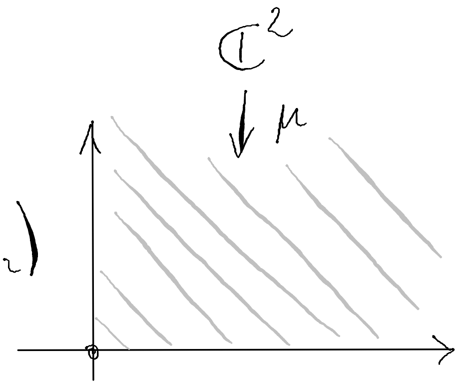
Lifting trick:

$$(\ast)_\sigma \qquad \left.\begin{tikzcd} L \arrow[d] & \subseteq & Z \arrow[d] \arrow[r, hook] & (M,\omega) \\ \tilde{L} & \subseteq & (\tilde{M},\tilde{\omega}) & \end{tikzcd}\right\} \text{symplectic reduction.}$$

Idea: View Chekanov torus as lifted from a certain reduction.

Use the toric structure:

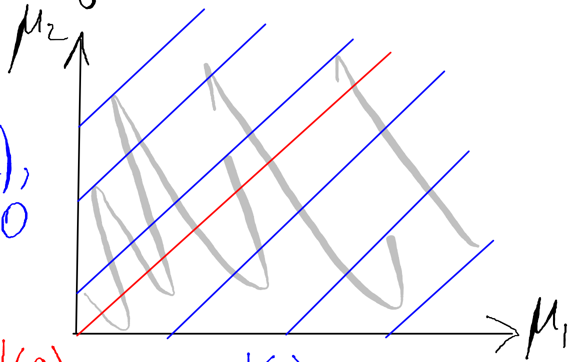
$$(e^{i\theta_1}, e^{i\theta_2}).(z_1,z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$



Perform symplectic reduction with respect to

$$S^1=\{(e^{i\theta},e^{-i\theta})\}\leq T^2$$

generated by: $H=|z_1|^2-|z_2|^2=\mu_1-\mu_2$

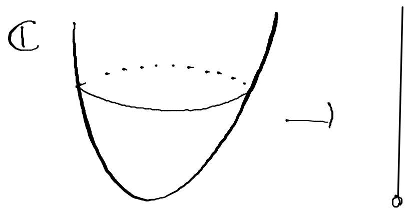


regular $\left\{ H^{-1}(c),\ c<0 \right.$

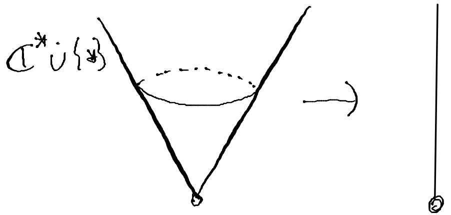
$H^{-1}(0)$

singular

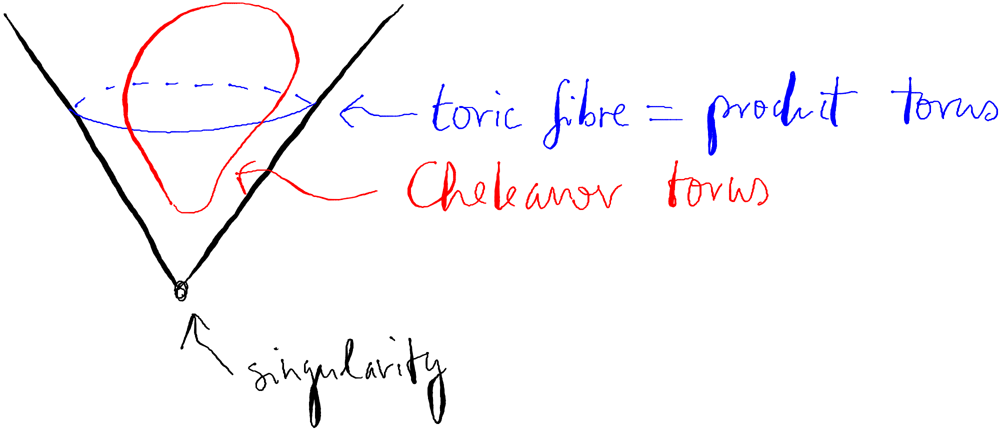
$\underbrace{H^{-1}(c),\ c>0}_{\text{regular}}$



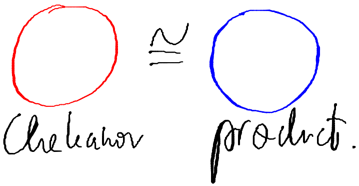
regular reduced space



singular reduced space.



(!) <u>Note</u>: In the regular case:



Remark: this construction is purely toric. Can be carried out in any toric manifold.

Question: How to distinguish $T^2_{Ch} \subseteq \mathbb{C}^2$ from product tori?

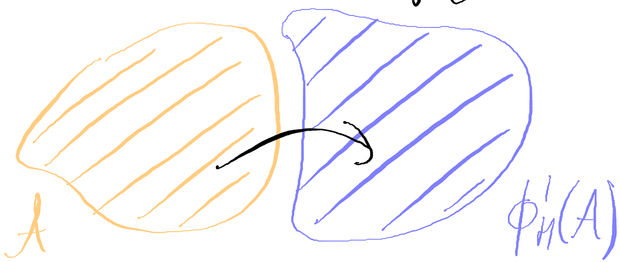
"Classical" invariants fail to distinguish T^2_{Ch} from $T(a,a)$ for some $a>0$.

§6. Displacement energy & Versal deformations.

Def: Let $A \subseteq (M,\omega)$ subset. The displacement energy of A is given by:

$$e(A) = \inf\{\|H\| \in (0,+\infty) \mid \phi_H^1(A) \cap A = \emptyset\}$$

(and $:= \infty$ for the empty set.)



Unfortunately, $e(T^2_{Ch}) = e(T(a,a))$!

Chekanov's idea: Look at "Lagrangian neighbours" of T^2_{Ch} and $T(a,a)$ and see how the invariant behaves on them! ("Versal deformations")

Let $L \subseteq (M,\omega)$ be any compact Lagrangian.

Weinstein's theorem: There is a tubular neighborhood of L that looks like the zero section in T^*L.

$\exists\, \mathcal{U}, V, \varphi$ s.th. $\varphi : T^*L \supseteq \mathcal{U} \longrightarrow V \subseteq M$, $\varphi^*\omega = d\lambda_{can}$

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & V \\ \cup\!\shortmid & & \cup\!\shortmid \\ L & \xrightarrow{\cong} & L \end{array}$$

Fact ①: Lagrangians $(\mathcal{C}^1\text{-})$ close to the zero-section $L \subseteq T^*L$ are graphs of closed one-forms.

$$\Omega^1_{cl}(L) \supseteq \hat{\mathcal{U}} \xrightarrow{\cong} \hat{\mathcal{V}} \subseteq \{\text{Lagrangians in } T^*L\}$$
$$\alpha \longmapsto \Gamma_\alpha$$

Fact ②: Two such Lagrangians are Hamiltonian isotopic iff their forms are cohomologous.

$$H^1_{dR}(L;\mathbb{R}) \supseteq \hat{\mathcal{U}}/_\sim =: \mathcal{U} \to \mathcal{V} := \hat{\mathcal{V}}/_\sim$$
$$\subseteq \left\{\begin{array}{l}\text{Lagrangians up to (small)}\\ \text{Hamiltonian isotopies}\end{array}\right\}$$

$$\boxed{\text{In conclusion: Nearby Lagrangians} \cong \mathcal{U} \subseteq H^1(L;\mathbb{R})}$$

We combine this with displacement energy to obtain:

$$
\begin{tikzcd}
H^1(L;\mathbb{R}) \supseteq \mathcal{U} \arrow[r, "\cong"] \arrow[rr, bend right, "S_L"'] & \{\text{Lagrangians}\}/_{\sim} \arrow[r, "e(\cdot)"] & \mathbb{R} \cup \{+\infty\}
\end{tikzcd}
$$

§7. The Chekanov torus is exotic.

Let $L := T(1,1)$ and $L' := T^2_{Ch}$.
Determine the versal deformations S_L and $S_{L'}$ in order to distinguish L and L'.

$\boxed{\text{We start with } S_L:}$

① identify $H^1(L;\mathbb{R}) \cong \mathbb{R}^2 \supseteq \mathcal{U} \longrightarrow \{\text{Lagrangians}\}$ explicitly.

Natural choice: $(a,b) \mapsto T(1+a, 1+b)$

② Recall that $S_L(a,b) = e(T(1+a,1+b))$ ← determine displ. energy!

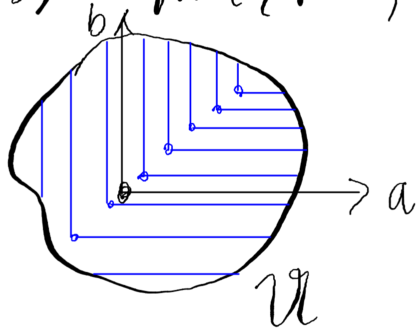
Find $e(T(x,y)) = \min\{x,y\}$.

$\Big(\; e(T(x,y)) \leq \min\{x,y\}$ "easy" by writing down an explicit displacement.

$e(T(x,y)) \geq \min\{x,y\}$ "hard", see Hofer, Sikorav... $\Big)$

$\rightsquigarrow \quad S_L(a,b) = \min\{1+a, 1+b\}$

level sets :

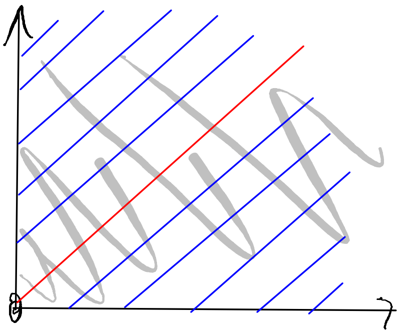


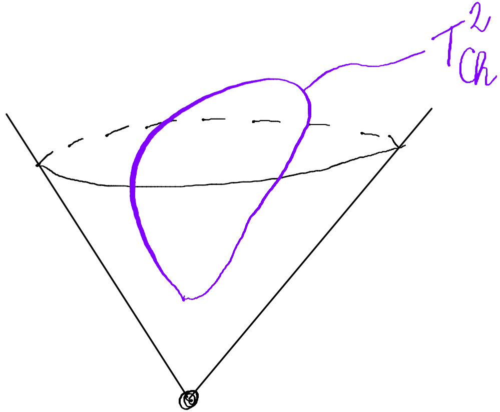
Now for S_{L^1} :

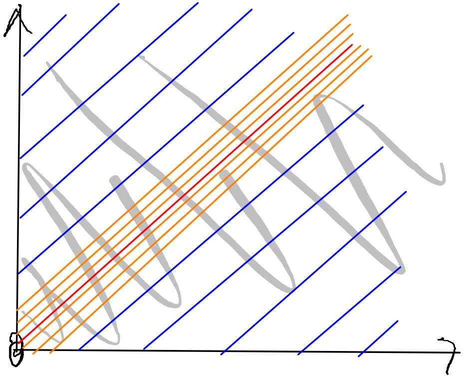
① Find an identification:

$$H^1(L^1;\mathbb{R}) \cong \mathbb{R}^2 \supseteq \mathcal{U}^1 \longrightarrow \{\text{Lagrangians}\}$$

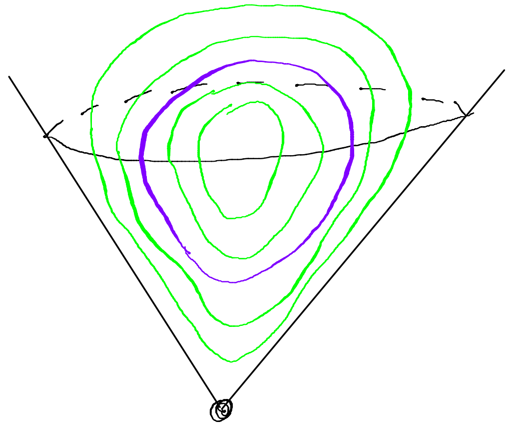
Recall the construction:







Deformation parameter $\delta \in \mathbb{R}$



Deformation parameter $t \in \mathbb{R}$

② Displacement energies.

Remark: The reduced space is singular only for $\delta = 0$, hence $\forall s \neq 0$,

(versal deformation) $\longrightarrow$ $L'(t,s) \cong T(x,y)$

for some (x,y), but we know their displacement energies!

Actually, $$L'(t,s) \simeq \begin{cases} T(1+t+s,\ 1+t) & s>0 \\ T(1+t,\ 1+t-s) & s<0. \end{cases}$$

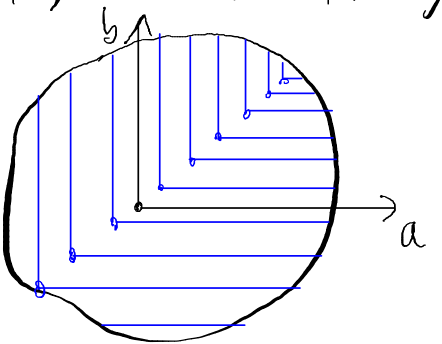
$$\Rightarrow S_{L'}(t,s) = e(L'(t,s)) = \begin{cases} \min\{1+t+s,\ 1+t\} & s>0 \\ \min\{1+t,\ 1+t-s\} & s<0 \end{cases}$$

$$= 1+t. \qquad \forall s \neq 0$$

$L = T(1,1) =$ product torus.

$$S_L(a,b) = \min\{1+a,\ 1+b\}$$

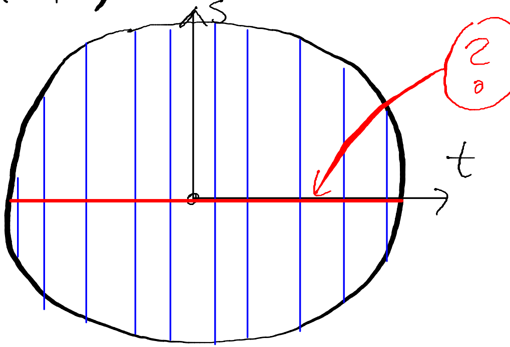
level sets $S_L^{-1}(c)$



$L' =$ Chekanov torus.

$$S_{L'}(t,s) = 1+t \quad \forall \neq 0$$

level sets $S_{L'}^{-1}(c)$



Two directions of generalization:

$\boxed{T_{Ch} \subseteq \mathbb{C}^2}$ $\xrightarrow{\text{more tori}}$ $T_1, T_2, \dots \subseteq \mathbb{C}^3$ Auroux
$\subseteq \mathbb{CP}^2$ Vianna, Galkin – Mikhalkin, ...

(Felix's talk ...)

$\downarrow$ more ambient spaces

$T_{Ch} \subseteq (M, \omega)$ for certain toric (M, ω)

B. '20

↳ 1) Our construction is purely toric!
2) Need to know displacement energies of toric fibres.

Thank you for the invitation &
your attention!

Exercises: (Hints at the end →)

① Here are some components of the proof of the Marsden–Weinstein symplectic reduction theorem:

i) Linear reduction: Let (V,ω) be a symplectic vector space and $E \subseteq V$ a coisotropic subspace, i.e. $E^{\omega} \subseteq E$ for $E^{\omega} := \{ v \in V \mid \omega(v,e) = 0 \ \forall e \in E \}$. Show that E/E^{ω} carries an induced sympl. form $\tilde{\omega}$:

$$\begin{array}{ccc} E & \hookrightarrow & (V,\omega) \\ \downarrow \pi & & \\ (E/E^{\omega}, \tilde{\omega}) & & \end{array} \qquad \pi^{*}\tilde{\omega} = \omega|_{E}.$$

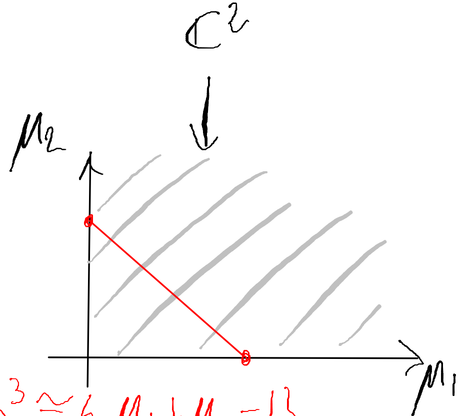
ii) Show that $T_x \mu^{-1}(c)^{\omega} = T_x \mathcal{O}_x$, where:

* $\mu : M \to \mathbb{R}^k$ is a moment map, $c \in \mathbb{R}^k$ a regular value,
* $\mathcal{O}_x = T^k \cdot x$ orbit of the Ham. action.

(Point ii) is hard if you are not familiar with moment maps....
Use the fact that
$$d\langle \mu, \xi \rangle = \iota(X_\xi)\omega$$
where: $\xi \in \mathrm{Lie}(T^k) = \mathfrak{t}^k = \mathbb{R}^k$ and X_ξ the infinitesimal vector field of the T^k-action in direction $\xi \in \mathbb{R}^k$. This is the equivalent of Hamilton's equation:
$$dH = \iota(X_H)\omega$$
from the first lecture. In other words, $\langle \mu, \xi \rangle \in \mathcal{C}^\infty(M)$ is the Hamiltonian generating the flow of $\exp(t\xi) \in T^n$)

② Carry out the following symplectic reduction explicitly:

Starting with the toric system: $\mu : \mathbb{C}^2 \longrightarrow \mathbb{R}^2_{\geq 0}$
$(z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$
and the red level set $\{\mu_1 + \mu_2 = 1\}$,



i) identify the group action generated by $H := \mu_1 + \mu_2$

ii) Show that the symplectic reduction thm. applies, i.e. that the action from i) preserves $H^{-1}(1) = \{\mu_1 + \mu_2 = 1\}$ and acts freely on this set.

iii) Identify the quotient.

3. (Real Lagrangian submanifolds)

A nice source of examples for Lagrangian submanifolds are fixed point sets of antisymplectic involutions.

An antisymplectic involution is a smooth map:

$\sigma : (M,\omega) \to (M,\omega)$ with 1) $\sigma \circ \sigma = \mathrm{id}$, 2) $\sigma^* \omega = -\omega$

i) Show that $\mathrm{Fix}\,\sigma = \{x \in M \mid \sigma(x) = x\}$ is Lagrangian (if it is non-empty).

ii) Show that $\mathbb{RP}^n \subseteq \mathbb{CP}^n$ is Lagrangian.

4. (Toric fibres)

Show that toric fibres are Lagrangian.

5. (Lifting of Lagrangians)

Let
$$\begin{tikzcd} \mu^{-1}(c) \arrow[r, hook] \arrow[d, "\pi"'] & (M,\omega) \\ (\tilde{M},\tilde{\omega}) & \end{tikzcd}$$
be a symplectic reduction.

i) Let $\tilde{L} \subseteq \hat{M}$ be a Lagrangian.
Show that $\pi^{-1}(\tilde{L}) =: L \subseteq M$ is Lagrangian.

There is a converse statement:

ii) Let $L \subseteq \mu^{-1}(c)$ a Lagrangian. Show that L is automatically preserved by the action of T^k generated by μ.
iii) Show that $\tilde{L} := L/T^k \subseteq \hat{M} = \mu^{-1}(c)/T^k$ is Lagrangian.

1. i) If you are stuck, see McDuff–Salamon "Introduction to sympl. topology".
   ii) Use $T_x \mu^{-1}(c) = \ker(d\mu|_x)$ together with Hamilton's equation $d\langle \mu, \xi \rangle = \iota(X_\xi)\omega$

2. /

3. i) Look at $\sigma_*|_x \circlearrowright T_x M$ for $x \in \mathrm{Fix}\,\sigma$.
   ii) /

4. Use invariance of the moment map:
$$\mu(t.x) = \mu(x) \quad \forall t \in T^n, x \in M$$
and differentiate.

5. ii) Show that any Lagrangian $L \subseteq H^{-1}(c)$ in the level set of a Hamiltonian H is invariant under ϕ_H^t.