1. Let $\mathbb{G}_m$ denote the multiplicative group of nonzero complex numbers, $\mathbb{C}^*$.

   (a) Show that $\mathbb{G}_m$ is a complex algebraic group.

   (b) Show directly (without using GAGA) by using the definition of a complete variety that G is not complete.

2. The *n-th complex unitary group*, denoted by $\mathbf{U}_n(\mathbb{C})$, is defined by

$$\mathbf{U}_n(\mathbb{C}) = \{g \in \mathbf{GL}_n(\mathbb{C}) : \ g^{-1} = \bar{g}^\top\},$$

   where the bar on a martrix $g = (g_{i,j})$ means that we are taking the complex-conjugates of the entries of that matrix. (The $\top$ stands for the transpose.)

   (a) Show that $\mathbf{U}(1)$ is a real Lie group.

   (b) Show that $\mathbf{U}(1)$ is not a complex Lie group.

   (c) Show that for every $n \in \mathbb{Z}^+$ the n-th complex unitary group is not a complex Lie group but a compact real Lie group.

   (d) Show that the complexification of $\mathbf{U}_n(\mathbb{C})$ is $\mathbf{GL}_n(\mathbb{C})$.

3. The *n-th complex special unitary group*, denoted by $\mathbf{SU}_n(\mathbb{C})$, is defined by

$$\mathbf{SU}_n(\mathbb{C}) = \{g \in \mathbf{U}_n(\mathbb{C}) : \ \det g = 1\}.$$

   (a) Show that the complexification of $\mathbf{SU}_n(\mathbb{C})$ is $\mathbf{SL}_n(\mathbb{C})$.

   (b) For $n \in \mathbb{Z}^+$, determine the anti-holomorphic involution $\tau : \mathbf{SL}_n(\mathbb{C}) \to \mathbf{SL}_n(\mathbb{C})$ for which the fixed subgroup is $\mathbf{SU}_n(\mathbb{C})$, In other words, show that

$$\mathbf{SL}_n(\mathbb{C})^\tau := \{g \in \mathbf{SL}_n(\mathbb{C}) : \ \tau(g) = g\} = \mathbf{SU}_n(\mathbb{C}).$$

   (c) For $n = 2$, determine the equations of the six dimensional differentiable manifold $\mathbf{SL}_2(\mathbb{C})$ viewed as a submanifold of $\mathbf{Mat}_2(\mathbb{C}) \cong \mathbb{C}^4$.

   (d) Determine the cotangent bundle of $\mathbf{SU}_2(\mathbb{C})$. Compare it with $\mathbf{SL}_2(\mathbb{C})$.

4. Let A be denote the image in $\mathbb{C}^2$ of the holomorphic map $s \mapsto (e^s, e^{is})$, $s \in \mathbb{C}$.

   (a) Show that A is a one-dimensional complex Lie group.

   (b) Show that A is not a complex algebraic variety.

5. Recall that an abelian variety is a complete algebraic group. Recall also that the complexification of every compact real matrix Lie group has the structure of a complex reductive algebraic group. Explain why we cannot readily apply this method to conclude that every compact complex Lie group has the structure of an abelian variety.