Algebraic Groups in Action, First Lecture.

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In any topological investigation of manifolds, compactness[1] is an ultimate tool to have since it enables combinatorial methods. For algebraic varieties[2], the most natural topology is the Zariski topology, which is usually not Hausdorff. Therefore, the ordinary compactness notion should be replaced with a notion of a "completeness." A variety X is called *complete* if for every variety Y, the second projection $X \times Y \to Y$ is a closed mapping in the Zariski topology. In [6] (Géométrie algébrique et géométrie analytique, 1955), Serre shows that an algebraic variety X defined over $\mathbb{C}$ is compact in the classical Hausdorff topology if and only if X is complete[3]. Important examples of complete varieties include all projective varieties[4]. In particular every projective complex algebraic variety is compact. It turns out that all smooth complete algebraic surfaces are projective varieties. However, there exist singular nonprojective complete algebraic surfaces as well as smooth nonprojective complete threefolds; see the references in [4, Remark 4.10.2]. For a simple example that is close to the spirit of this text, see [3, Example 4.2.13]. Later we will see that there are compact complex Lie groups which are not projective.

Many interesting manifolds (resp. algebraic varieties) are naturally equipped with the Lie group (resp. algebraic group) actions on them. An orbit of a Lie group or an algebraic group action is called a *homogeneous space*. Clearly, the group itself is a homogeneous space under each of its left-, right-, or two-sided actions. Other examples of homogeneous spaces include all Grassmann manifolds, all complex tori, all projective spaces, all spaces of constant curvature, such as the euclidean spaces, the spheres, as well as certain finite quotients of (pseudo)orthogonal groups, all symmetric spaces.. While some of these orbit spaces are compact in the Hausdorff topology many of them are not complete. The theory of compactifications of homogeneous spaces of Lie groups is a vast area of geometry with

[1]A locally compact Hausdorff topological space is compact if and only if for every topological space Y, the projection map $X \times Y \to Y$ is a closed map.

[2]For us, an algebraic variety is a reduced and separated scheme of finite type defined over an algebraically closed field.

[3]Very roughly summarized, [6] states that the analytification functor $X \rightsquigarrow X^{\mathrm{an}}$ from proper $\mathbb{C}$-schemes to compact Hausdorff $\mathbb{C}$-analytic spaces is fully faithful.

[4]The vanishing set of a collection of homogeneous polynomials in $\mathbb{P}^n$ is called a projective variety.

connections to almost all known areas of mathematics. In these notes, we will be concerned with a small combinatorial corner of the theory of equivariant completions for only a certain type of algebraic group actions. In fact, most of our discussion will not be specifically about completions of orbits but about their "equivariant embeddings."

Definition 0.1. Let G be a Lie or an algebraic group, and let $H \subseteq G$ be a closed subgroup. A *G-equivariant embedding of G/H* is an irreducible G-variety X with an open orbit isomorphic to G/H.

At the heart of the theory of equivariant embeddings of orbits are the equivariant embeddings of the group itself. If the group itself is compact then it may look at first sight that no new information would be gained from its equivariant embeddings. This is correct to some degree for algebraic groups. Nevertheless, many complex algebraic groups are obtained from compact Lie groups as we will explain later. We note that any complex algebraic group has a unique structure of a complex Lie group. The converse of this statement is not true as it is easily seen from the following example: Let A be denote the image in $\mathbb{C}^2$ of the holomorphic map $s \mapsto (e^s, e^{is})$, $s \in \mathbb{C}$. Then A is a one-dimensional complex Lie group since it is biholomorphically isomorphic to $(\mathbb{C}, +)$. However it is easy to check that there is no nonzero polynomial in two (or more) variables $f(x, y) \in \mathbb{C}[x, y]$ such that $f(e^s, e^{\sqrt{-1}s}) = 0$ for all $s \in \mathbb{C}$. Therefore, A is not a complex algebraic variety. Since not every (complex) Lie group is algebraic, we want to know where the complete Lie groups stand in relation with the algebraic groups. While answering this question, we will explain why the existence of a faithful finite dimensional linear representation of a Lie group is important for algebraicity. Let us proceed with a real example.

The three dimensional unit sphere in $\mathbb{S}^3 \subset \mathbb{R}^4$ can be viewed as the special unitary group via the identification,

$$
\begin{aligned}
\mathbf{SU}_2(\mathbb{C}) := \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} :\ a\bar{a} + b\bar{b} = 1,\ a, b \in \mathbb{C} \right\} &\longrightarrow \mathbb{S}^3 := \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\} \\
\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} &\longmapsto (\mathrm{re}(a), \mathrm{im}(a), \mathrm{re}(b), \mathrm{im}(b)).
\end{aligned}
$$

Hence, it admits the two-sided multiplication action of $\mathbf{SU}_2(\mathbb{C})$. Since $\mathbb{S}^3$ is a compact subset of $\mathbb{R}^4$, the Lie group $\mathbf{SU}_2(\mathbb{C})$ is compact as a real Lie group. But notice that the equation $a\bar{a} + b\bar{b} = 1,\ a, b \in \mathbb{C}$ that defines $\mathbf{SU}_2(\mathbb{C})$ as a matrix group is not really a polynomial equation. Therefore we cannot regard $\mathbf{SU}_2(\mathbb{C})$ as a complex Lie group or as a complex algebraic variety[5] although it is still a "real algebraic group."

[5] A group G equipped with the structure of a differentiable manifold (resp. complex manifold, resp. complex algebraic variety) is said to be a *real Lie group* (resp. a complex Lie group, resp. a complex algebraic group) if the underlying group operations are smooth maps (resp. holomorphic maps, resp. algebraic morphisms). Equivalently, a real Lie group (resp. a complex Lie group, resp. a complex algebraic group) is a group object in the category of differentiable manifolds (resp. category of complex manifolds, resp. category of algebraic varieties).

An *algebraic variety in* $\mathbb{R}^n$[6] is a subset of the form

$$X_S := \{x \in \mathbb{R}^n : \ f(x) = 0 \ \text{ for every } f \in S\},$$

where S is a set of polynomials in $\mathbb{R}[x_1, \dots, x_n]$. Clearly, any real polynomial, that is to say a polynomial with real coefficients, is also a complex polynomial. Therefore, regarding S as a subset of $\mathbb{C}[x_1, \dots, x_n]$ we obtain an affine complex algebraic variety,

$$X_S(\mathbb{C}) := \{x \in \mathbb{C}^n : \ f(x) = 0 \ \text{ for every } f \in S\},$$

called the *complexification of* X_S. Conversely, let $X \subseteq \mathbb{C}^n$ be an affine complex algebraic variety. Then we can view X as an algebraic variety in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. A *real form* X_0 *of* X is an algebraic variety in $\mathbb{R}^{2n}$ such that X_0 is a closed subset of X (as an algebraic variety in $\mathbb{R}^{2n}$), and the complexification of X_0 is isomorphic to X as a complex algebraic variety.

Let $\mathbb{C}[X]$ denote the $\mathbb{C}$-algebra of $\mathbb{C}$-valued polynomial functions on X. Let $z_1, \dots, z_n$ denote the restriction to X of the coordinate functions of $\mathbb{C}^n$. Then $\mathbb{C}[X]$ is generated by $z_1, \dots, z_n$ over $\mathbb{C}$. Let $\overline{\mathbb{C}[X]}$ denote the $\mathbb{C}$-algebra $\mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$, where $\bar{z}_i$ $(i \in \{1, \dots, n\})$ is the complex conjugate of the polynomial function z_i $(i \in \{1, \dots, n\})$. Let Y be another complex affine algebra. A map of complex affine varieties $f : X \to Y$ is called an *anti-holomorphic morphism* if $f^*\mathbb{C}[Y] \subseteq \overline{\mathbb{C}[X]}$. For every real affine variety $X_0 \subset \mathbb{R}^n$ there exists a unique anti-holomorphic automorphism of the complexification $\tau : X_0(\mathbb{C}) \to X_0(\mathbb{C})$ such that $\tau(x_0) = x_0$ for all $x_0 \in X_0$ and $\tau^2(x) = x$ for all $x \in X_0(\mathbb{C})$. It turns out that, with a small restriction, the converse of this observation holds as well: For every anti-holomorphic involutory morphism $\tau : X \to X$, where X is a complex affine algebraic variety, if the fixed point set X_0 of τ contains a smooth point of X, then X_0 is a real form of X. Now let $Y \subseteq \mathbb{C}^n$ be a subset defined by a set of algebraic equation in the coordinate variables $z_1, \dots, z_n$, and their complex conjugates $\bar{z}_1, \dots, \bar{z}_n$. Since these coordinate functions and their conjugates are equivalent to the coordinate functions on $\mathbb{R}^{2n}$ by linear transformations, Y is an algebraic variety in $\mathbb{R}^{2n}$. Our first example, the special unitary group $\mathbf{SU}_2(\mathbb{C})$, is a complete (compact) real form of $\mathbf{SL}_2(\mathbb{C})$, which is a complex hypersurface in $\mathbb{C}^4$.

A compact group is any topological group which is compact and Hausdorff. A compact group called a *compact linear group* if it admits an injective topological group homomorphism into $\mathbf{GL}(V)$, where V is a finite dimensional complex vector space. Thanks to Stone-Weierstrass theorem, it can be shown without much difficulty that every compact linear group has the structure of an algebraic variety in $\mathbb{R}^n$ for some $n \in \mathbb{Z}^+$. In particular, the compact linear groups are real linear Lie groups. Since they are algebraic, every compact linear group K has a complexification $K(\mathbb{C})$, which is a complex affine algebraic group. It turns out that $K(\mathbb{C})$ is a *reductive group*, meaning that the it does not contain any non-trivial normal unipotent subgroup. Furthermore, the assignment $K \rightsquigarrow K(\mathbb{C})$ defines an equivalence of categories between the category of linear compact groups and the category of complex reductive algebraic groups. In summary, we learn from this equivalence of categories that

[6] Strictly speaking, this is the set of "$\mathbb{R}$-rational points" of the complex affine scheme defined by the polynomials in S.

1. for every reductive complex algebraic group G there exists a unique complete (compact) real form K of G such that $K(\mathbb{C}) \cong G$.

2. Every reductive complex algebraic group G is *linear*, that is to say, it admits a faithful algebraic group representation on some finite dimensional complex vector space.

The above discussion shows that the representation theory of compact linear groups is related to the theory of reductive groups in a fundamental way. We note that reductive groups are not complete unless they are zero dimensional. Next, we will consider complete (compact) complex Lie groups. The completeness assumption on such a group puts a very severe restriction on the multiplicative structure.

Let G be a connected complex Lie group, let $e \in G$ be the identity element. For every vector v in the tangent space of G at e, there exists a unique holomorphic one-parameter semigroup homomorphism $\psi_v : \mathbb{C} \to G$ such that $d(\psi_v)_0 : T_0\mathbb{C} \to T_eG$ maps $1 \in T_0\mathbb{C}$ to $v \in T_eG$. Consequently, we get the *exponential map* $\exp : T_eG \to G$ by $\exp(v) := \psi_v(1)$. In particular, we have $\psi_v(t) = \exp(tv)$ (by the uniqueness of ψ_v). Thanks to implicit function theorem, we know that the exponential map exp is a local diffeomorphism. This fact immediately implies that the kernel of exp is a discrete subgroup of the vector space T_eG. We will mention another important property of the exponential map. The uniqueness property of ψ_v also implies that the differential of any complex Lie group automorphism $f : X \to X$ commutes with exp:

$$f(\exp(v)) = \exp(df_e(v)) \qquad (v \in T_eG). \tag{0.2}$$

Let $C_g : G \to G$ ($g \in G$) denote the conjugation map $x \mapsto gxg^{-1}$, $x \in G$. Then we see that $C_g(\exp(v)) = \exp(d(C_g)_e(v))$ for every $v \in T_eG$.

We now proceed with the assumption that G is a complete connected complex Lie group. It follows that G does not admit any non-constant holomorphic map to an affine space, hence, its adjoint representation on T_eG is the trivial representation. In other words the differentials of the conjugation maps, dC_g ($g \in G$), are the identity maps. In particular by (0.2) we have $C_g(\exp(v)) = \exp(v)$ for every $v \in T_eG$. This means that $\exp(v)$ is contained in the center of G for every $v \in T_eG$. But G is connected, so, the image of exp generates G as a group. We conclude from these observations that every complete connected complex Lie group has the structure of an abelian group.

Furthermore, G is isomorphic to a quotient of a vector space T_eG by a discrete subgroup, $\ker \exp$. The induced map $T_eG/\ker \exp \to G$ is a holomorphic map between two equal dimensional complex manifolds. Therefore, it is an isomorphism of complex Lie groups. Also, since lattices are the only discrete subgroups of vector spaces with compact quotient, $\ker \exp$ is a lattice.

Notice that, since a complete complex Lie group G cannot be a compact *linear* group, we cannot apply the easy 'complexification' method to conclude that G is an algebraic group. In fact, this is not true beyond dimension one. The crucial point here is to know whether the complete complex Lie group admits an embedding into a projective space or not. Indeed, the Theorem of Chow states that if X is a complete algebraic variety, and $Y \subseteq X$ is a closed analytic subset of the analytic space structure on X, then Y is Zariski closed in X. In [5], Mumford shows that for a complete complex Lie group of the form $X := V/L$, where V is a g-dimensional complex vector space, $L \subseteq V$ a lattice, the following statements are equivalent:

1. The underlying variety of X is a projective variety.
2. There exist g algebraically independent meromorphic functions on X.
3. There is a positive definite hermitian form H on V such that the image of H on $L \times L$ is integral.

A complete algebraic group is called an *abelian variety*. We just saw that the complete complex Lie groups are closely related to the abelian varieties although there are subtle differences (the existence of a projective embedding, also known as a *polarization*). We also know that the complete real Lie groups lead to the theory of reductive groups, which are affine algebraic groups. Of course, there are many algebraic groups which are neither affine nor projective. But we will not discuss the completions of non-affine algebraic groups beyond this introduction. The reason is a classical theorem of Chevalley (1953), which states that a connected algebraic group is an extension of an abelian variety by a connected affine algebraic group. We will restrict our attention mostly to the equivariant embeddings of the orbits of connected affine algebraic groups. Our aim is to show that this study not only produces new geometric and combinatorial structures related to the actions of affine groups but also reveals new facts about these groups. Furthermore, surprisingly, some equivariant affine embeddings of affine groups appear rather naturally as homogeneous vector bundles on abelian groups see [1, 2]. This already indicates the strong categorical relationship between affine equivariant embeddings and the category of homogeneous vector bundles on abelian varieties.

Some highlights for the reader: The study of the compact linear real Lie groups is equivalent to the study of the complex reductive algebraic groups. A complex reductive algebraic group is not complete unless it is a finite group. The compact complex Lie groups are given by the quotients of vector spaces by sublattices. A compact complex Lie group is an abelian variety if it is polarized, which means that it has an embedding into a projective space by some meromorphic functions. Now an outstanding question that motivates our other lectures is: how do we describe the completions of reductive groups?

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