Algebraic Groups in Action, Lecture 2.

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1 Characters

Most of the results that we review in this section can be found in the standard resources [1, 4, 10] (for algebraic groups) and [3] (for algebraic varieties). However, in many places we use our notation.

Throughout our text, we work with algebraic varieties that are defined over an algebraically closed field. We fix the notation k for such a field. In some places we will assume that $k = \mathbb{C}$. To work over algebraically closed fields different than $\mathbb{C}$, it is convenient to use the following definition of an algebraic variety: An *algebraic variety* is a reduced separated k-scheme of finite type. Notice here that we do not insist on the irreducibility of an algebraic variety unlike in [3]. The reason behind this convention is that, algebraic groups may have more than one connected components. An *algebraic group* is a group scheme over k whose underlying scheme is a smooth algebraic variety. If $k = \mathbb{C}$, then an algebraic group defined over k will be called a *complex algebraic group.*

A morphism of algebraic groups is a rational map which is also a group homomorphism. For example, the map $\mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$, $z \mapsto (1/z, z^2)$ is a morphism of complex algebraic groups. If the underlying variety of an algebraic group G is affine, then G is called an *affine (or linear) algebraic group.* In the rest of this text we will be concerned with the affine algebraic groups only. Note that every finite group is both affine and complete at the same time. These are precisely the zero-dimensional algebraic groups. Also, the only connected, zero-dimensional algebraic group is the trivial group.

The multiplicative group structure on $k\backslash\{0\}$ is often denoted by $\mathbb{G}_m$ while the additive group structure on k is occasionally denoted by $\mathbb{G}_a$. Both of these algebraic groups are connected, one-dimensional, and affine. Any affine algebraic group having these properties is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

1.1 Some algebraic geometry terminology.

We continue to review basic notions from algebraic geometry.

Let X be a variety. First we assume that X is a quasi-affine variety in $\mathbb{A}^n$. This means that X is the intersection of a Zariski-open set with a subvariety in $\mathbb{A}^n$. A function $f : X \to k$ is called *regular at a point* $p \in X$ if the exists an open neighborhood U of p, where f is a rational function of the form $f = h/g$, for some polynomials $h, g \in k[x_1, \dots, x_n]$ where g is nowhere zero on U. If f is regular at every point of X, then f is said to be a *regular function on* X. Next, we assume that X is a quasi-projective variety in $\mathbb{P}^{n-1}$ (where $n \geqslant 1$). This means that X is the intersection of a Zariski-open set with a projective variety in $\mathbb{P}^{n-1}$. The notion of a regular function f on $U \subseteq X$ is defined as before but with the additional requirement that the polynomials h and g are homogeneous and they are of the same degree. If U is an open subset of X, the set of regular functions on U will be denoted by $\mathcal{O}_X(U)$. It is easy to check that $\mathcal{O}_X(U)$ is a ring with respect to point-wise addition and multiplication of regular functions. The assignment $U \mapsto \mathcal{O}_X(U)$ defines a sheaf on X; it is called the *sheaf of regular functions of* X, or the *structure sheaf of* X. For $U = X$, there are several different names and notation for $\mathcal{O}_X(X)$. Sometimes it is denoted by $k[X]$, and sometimes it is denoted by $H^0(X, \mathcal{O}_X)$. In the former notation it is often called the *k-algebra of regular functions of* X; if X is affine, then it is called the *coordinate ring of* X. If the notation $H^0(X, \mathcal{O}_X)$ is used, then it is customarily called the *k-algebra of global sections of the sheaf* $\mathcal{O}_X$. Finally, let us mention that the notation $\mathcal{O}(X)^*$ stands for the group of invertible elements of the multiplicative semigroup of the ring of regular functions $\mathcal{O}(X)$. For any irreducible variety X, we will denote the quotient group $\mathcal{O}(X)^*/k^*$ by $E(X)$.

1.2 Basic abstract representation theory terminology.

Let V be a *k-vector space*, that is, a vector space defined over k. Let S be a group. A *(linear) k-representation* of S is a group homomorphism $\rho : S \to GL(V)$. If such a homomorphism exists, then V is sometimes called an *S-module.* In practice, when a linear k-representation is mentioned one usually writes V or (ρ, V) instead of $\rho : S \to GL(V)$.

Example 1.1. Let S be an abelian group. Let (ρ, V) be a finite dimensional k-representation of S. Since S is abelian, there is a basis $\{v_1, \dots, v_n\}$ of V such that for every $i \in [n]$, v_i is a joint eigenvector for all $g \in S$. By using this basis we identify $GL(V)$ with $GL_n(k)$. Then the image of S under $\rho : S \to GL_n(k)$ consists of diagonal matrices.

Example 1.2. Let S_n denote the symmetric group of permutations of the set $\{1, \dots, n\}$. If σ is a permutation from S_n, then we can "represent" σ by the matrix $\rho(\sigma) = (\sigma_{i,j})_{i,j=1}^n$, where

$$\sigma_{i,j} := \begin{cases} 1 & \text{if } \sigma(i) = j; \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $S_3 = \{123, 213, 132, 231, 312, 321\}$, we have

$$\{\rho(\sigma) : \sigma \in S_3\} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

It is easy to check that the assignment $\sigma \mapsto \rho(\sigma)$, where $\sigma \in S_n$, defines a k-representation of S_n.

Example 1.3. Let H_3 denote the following complex affine algebraic group,

$$H_3 := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \ a, b, c \in \mathbb{C} \right\}.$$

(In a certain context, H_3 is called the *complex Heisenberg group of* 3×3 *matrices.*) We consider the conjugate-matrix multiplication action of H_3 on $\mathbb{C}^3$,

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \bar{a}y + \bar{b}z \\ y + \bar{c}z \\ z \end{bmatrix}.$$

This is an abstract group representation of H_3 on $\mathbb{C}^3$.

Example 1.4. Let G denote the $GL_2(\mathbb{C})$. Let R denote the polynomial ring $R := \mathbb{C}[X, Y]$. Clearly, R is an infinite dimensional vector space. Let A be an element from G such that $A^{-1} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is easy to check that the map $\varphi_A : R \to R$ defined by

$$\varphi_A(f(X, Y)) := f(aX + cY, bX + dY) \qquad (f(X, Y) \in R)$$

is a linear automorphism of R. Furthermore, it is easy to verify that $\varphi_A \circ \varphi_B = \varphi_{AB}$ for every $A, B \in G$. Therefore, the map $\varphi : G \to GL(R)$, $A \mapsto \varphi_A$ defines an infinite dimensional representation of G on R.

1.3 Terminology of reductive groups.

An abstract group G is called 'simple' if it does not possess any normal subgroups other than itself and e. For algebraic groups, this definition is too restrictive in the following sense. A big part of the theory of algebraic groups depends on the theory of Lie algebras. By definition, the *Lie algebra of an algebraic group* G is the tangent space of G at the identity element $e \in G$. However, the passage from G to its Lie algebra does not depend on the finite central subgroups of G. For example, consider the special linear group $G := \mathbf{SL}_n(\mathbb{C})$. The center $Z(G)$ of G is isomorphic to the cyclic group of order n. For any subgroup $Z' \subset Z(G)$, the Lie algebra of the quotient G/Z' is equal to the traceless $n \times n$ matrices with complex entries. Furthermore, among all such quotients, the only simple group is the full quotient, $G/Z(G) = \mathbf{PSL}_n(\mathbb{C})$. In this regard, it is more natural to call an algebraic group G a *simple algebraic group* (or even better, quasi-simple, or almost-simple), if it has no closed normal and connected subgroups other than itself and e. For example, $\mathbf{SL}_2(\mathbb{C})$ is a simple group but not $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})$. The problem with the second example is that it has more than one "simple factor." Later we will see that the simple algebraic groups are parametrized by certain discrete objects called the "indecomposable, reduced root systems."

A positive dimensional algebraic group G is called *semisimple* if it has no closed connected commutative normal subgroups except e. For example, $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})$ is semisimple but not $\mathbf{GL}_2(\mathbb{C})$. The problem with the second example is that it has a one dimensional center. A Lie algebra $\mathfrak{g}$ is said to be a *semisimple Lie algebra* if $\mathfrak{g}$ has no nontrivial commutative ideals. If $\mathfrak{g}$ is the Lie algebra of G, then it is a semisimple Lie algebra if and only if G is a semisimple algebraic group.

Let G be an algebraic group, and let $\{G_i : \ i \in I\}$ be the set of all minimal, closed, connected, and non-commutative subgroups of G such that $\dim G_i > 0$ for $i \in I$. Then G_i's are called the *simple factors* of G. Indeed, they are simple algebraic groups by definition. If G is semisimple, then the indexing set I is finite, and furthermore, there exists a surjective algebraic group homomorphism $\psi : \prod_{i \in I} G_i \to G$ with finite kernel. Such homomorphisms are called *algebraic covering maps*, or *isogenies*. Converse of this statement also holds. Another characterization of the semisimplicity is as follows: A connected algebraic group G is semisimple if and only if G is equal to its commutator subgroup, (G, G).

Next, let us assume momentarily that we know the definition of a reductive group. Let G be a connected reductive group. It follows from [4, Sections 19.5 and 27.5] that G is isomorphic to a quotient of the form $(G_0 \times Z)/Z_0$, where

- $G_0 := (G, G)$;
- Z is a central torus;
- Z_0 is a finite normal central subgroup of $G_0 \times Z$.

This decomposition can be taken as a definition of a connected reductive group. Now let us introduce reductive groups in a proper way.

Definition 1.5. A *reductive algebraic group* is an affine algebraic group G that does not contain a closed normal unipotent subgroup.

For example, $\mathbf{GL}_2(\mathbb{C})$ is a reductive group; it is isomorphic to $(\mathbf{SL}_2(\mathbb{C}) \times Z)^{Z_0}$, where Z is the group of diagonal constant matrices in $\mathbf{GL}_2(\mathbb{C})$, and Z_0 is the subgroup $\{(g, e) \in \mathbf{SL}_2(\mathbb{C}) \times Z : \ g \text{ is diagonal and } g^2 = e\}$. A non-reductive example is $\mathbb{G}_a$ (or any unipotent group).

Let G be an affine algebraic group. A *Borel subgroup* B of G is a maximal closed connected solvable subgroup of G. Let us denote the set of all Borel subgroups of G by $\mathscr{B}(G)$. Then $\cap_{B \in \mathscr{B}(G)} B$ is a normal solvable subgroup of G. The connected component of the identity element in $\cap_{B\in\mathscr{B}(G)}B$ is called the *(solvable) radical* of G; we denote it by $\mathcal{R}(G)$. The maximal closed normal unipotent subgroup of $\mathcal{R}(G)$, denoted $\mathcal{R}_u(G)$, is called the *unipotent radical* of G. In this notation, we have

- G is semisimple if and only if $\mathcal{R}(G)$ is trivial;
- G is reductive if and only if $\mathcal{R}_u(G)$ is trivial.

1.4 The group of characters of an algebraic group.

In this section, we introduce an essential notion for representation theory. Let S be an affine algebraic group.

Definition 1.6. An *algebraic character of* S is an algebraic group homomorphism into $\mathbb{G}_m$.

In other words, an abstract group homomorphism $\chi : S \to k^*$ is a character of S if χ is a nowhere vanishing regular function on S. Equivalently, a group homomorphism $\chi : S \to k^*$ is an algebraic character of S if χ is an element of $\mathcal{O}(S)^*$. Clearly, the set of algebraic characters of S is a group under point-wise multiplication.

Earlier, for an algebraic variety X, we set $E(X)$ to denote $E(X) := \mathcal{O}(X)^*/k^*$. We will show that if S is an algebraic group, then $E(S)$ is nothing but the group algebraic characters of S. This result was discovered by Rosenlicht ([9]) in the mid-1950s. Our exposition closely follows [7, §1].

Proposition 1.7. *Let S be a connected algebraic group. Let f be an element of $\mathcal{O}(S)^*$. (In other words, $f \in k[S]$ and f is nowhere zero on S.) Then f is a character of S if and only if $f(e) = 1$. In particular, the group of algebraic characters of S is isomorphic to the quotient group $E(S)$.*

Proof. The essential ingredient is the following observation: If X and Y are two irreducible varieties, then the map $\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$ is surjective, [7, Proposition 1.1]. In particular, $\mathcal{O}(S)^* \times \mathcal{O}(S)^* \to \mathcal{O}(S \times S)^*$ is surjective. Let $f : S \to k^*$ be a regular function such that $f(e) = 1$. We view f as an invertible regular function on $S \times S$ via $(g_1, g_2) \mapsto f(g_1g_2)$ $(g_1, g_2 \in S)$. Then there exist two invertible regular functions $r_1, r_2 \in \mathcal{O}(S)^*$ such that $f(g_1g_2) = r_1(g_1)r_2(g_2)$ for every $g_1, g_2 \in S$. Notice that, since $f(e) = 1$, the equality $f(g_1g_2) = r_1(g_1)r_2(g_2)$ holds for every $g_1, g_2 \in S$ even if we replace r_i by $r_i/r_i(e)$ for $i \in \{1, 2\}$. But now we see that $f = r_i$ for $i \in \{1, 2\}$. □

In the same reference the authors prove the following important fact.

Proposition 1.8. *Let X be an irreducible variety. Then*

(1) $E(X)$ is a finitely generated free abelian group.

(2) For every algebraic action of a connected affine algebraic group $S : X$, the corresponding canonical action on $E(X)$ is trivial.

In the second part of the proposition, the canonical action of S on $E(X)$ is the one that comes from the action of S on the coordinate ring $\mathcal{O}(X)$.

Proof. We will sketch the idea of the proof of (1). We may assume that X is a normal and quasi-projective variety in some $\mathbb{P}^n$. Let $\overline{X}$ denote the normalization of the Zariski closure of X in $\mathbb{P}^n$. We now consider the principal divisor attached to the elements of $\mathcal{O}(X)^*$,

$$\begin{aligned} \operatorname{div} : \mathcal{O}(X)^* &\longrightarrow \bigoplus_{i=1}^{m} \mathbb{Z}D_i \\ f &\longmapsto \sum_{i=1}^{m} \operatorname{ord}_{D_i}(f) D_i, \end{aligned}$$

where $D_1, \ldots, D_m$ are the irreducible components of $\overline{X} \backslash X$, and ord_{D_i}'s are the corresponding order-of-vanishing functions, defined on the field of rational functions on X. It is easy to check that the constant functions are contained in the kernel of div. Therefore, div descends to an injection on the group $E(X)$. Since the target is a finitely generated abelian group, so is the domain, which is our group $E(X)$.

(2) Let $f \in \mathcal{O}(X)^*$. We will see later that the action of S on $k[X] = \mathcal{O}(X)$ defines a locally finite rational representation. Therefore, the span of the orbit $S \cdot f$ in $\mathcal{O}(X)$ is a finite dimensional vector space. But S is connected, therefore, the S-orbit, which is contained in $\mathcal{O}(X)^*$, is an irreducible algebraic set. The image of this orbit under the quotient $\pi : \mathcal{O}(X)^* \to \mathcal{O}(X)^*/k^*$ is irreducible as well. Since $E(X)$ is a lattice, its irreducible subsets are its points. Therefore, $S \cdot f$ is equal to $k^* f$. In particular, the action of S on the quotient group $E(X) = \mathcal{O}(X)^*/k^*$ is trivial. $\square$

We continue with an important observation.

Lemma 1.9. *Let G be a connected semisimple algebraic group of dimension at least two (or three!). Then G does not possess any nontrivial algebraic character. Consequently, we have $E(G) = 1$, or $\mathcal{O}(G)^* = k^*$.*

Proof. Let $\chi : G \to k^*$ be an algebraic character. Then the restriction of χ to any normal subgroup $G' \subseteq G$ is an algebraic character of G'. Therefore, it suffices to prove our claim for the simple algebraic groups. It is easy to verify that the kernel of an algebraic group homomorphism is an algebraic subgroup. Therefore, if $\chi : G \to k^*$ is an algebraic character of a simple algebraic group G, then $\ker \chi$ is a normal subgroup of G. Let H denote the connected component of the identity in $\ker \chi$. We proceed with the assumption that χ is a nontrivial character; $\ker \chi$ is a proper subgroup of G. Then H is a normal subgroup of finite index in $\ker \chi$, [4, Proposition 7.3]. Since $\ker \chi$ is normal in G, so is H. But G is a simple algebraic group and H is connected. Hence, we see that $H = \{e\}$. Therefore, $\ker \chi$ is a finite group. Hence, we have $\dim G = \dim(G/\ker \chi)$. But since $\chi(G) \subseteq k^*$ and $\chi(G) \cong G/\ker \chi$, we see that G is one-dimensional. This contradiction shows that χ must be trivial. $\square$

We now prove an analogous statement for the unipotent groups.

Lemma 1.10. *Let U be a unipotent group. Then U does not possess any nontrivial algebraic characters. In particular, we have $E(U) = 1$.*

Proof. Since any linear representation of $\mathbb{G}_m$ is diagonalizable, $\mathbb{G}_m$ consists of semisimple elements. At the same time, by the Jordan-Chevalley Decomposition Theorem, we know that the image of a unipotent element is unipotent. Now let $\chi : U \to \mathbb{G}_m$ be an algebraic character of U. The image of χ consists of unipotent elements in $\mathbb{G}_m$. But only unipotent element of $\mathbb{G}_m$ is the identity element. Therefore, χ is the constant (trivial) algebraic character. □

Although semisimple and unipotent groups do not have any nontrivial algebraic characters, reductive groups have algebraic characters. For example, tori are reductive groups; they possess plenty of algebraic characters.

Example 1.11. Let S be the following three dimensional complex torus,

$$S := \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} : \ a_1, a_2, a_3 \in \mathbb{C}^* \right\}.$$

For every $i \in \{1, 2, 3\}$, the map $\lambda_i : S \to \mathbb{G}_m,\ \lambda_i((a_1, a_2, a_3)) = a_i$, is a character of S. The constant map $S \to \mathbb{G}_m$, $(a_1, a_2, a_3) \mapsto 2$ is not a character since it is not a homomorphism.

1.5 Abstract characters vs algebraic characters of algebraic groups.

In the theory of finite group representations, the "trace" function on matrices is used for defining the "character" of a $\mathbb{C}$-representation. More precisely, if G is a finite group and $\rho : G \to GL(V)$ is a representation of G on a finite dimensional $\mathbb{C}$-vector space V, then the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \text{trace}(\rho(g)) = \text{sum of eigenvalues of } \rho(g)$$

is called the *character of* ρ. We want to stress the fact that our definition of a character for affine algebraic groups is not exactly this one! Our definition requires that a character takes nonzero values only, but the trace of a linear operator might be zero. If for all $g \in G$, the value $\chi(g)$ is nonzero, then χ is a linear character as in Definition 1.6.

Example 1.12. The symmetric group S_n has only two linear characters; they are the trivial and the sign representations of S_n.

Next, we will discuss the algebraic characters of diagonalizable groups.

Lemma 1.13. *Let S be an algebraic group. The group of all algebraic characters of S is a linearly independent subset of the space of all k-valued functions on S.*

The proof of this lemma is not difficult but we will not present it here; see [4, §16.1]. We will demonstrate it by a simple example.

Example 1.14. We consider $S = \mathbb{G}_m$. The algebraic characters of S are the regular functions of the form $z \mapsto z^m$ ($z \in \mathbb{C}^*$) where $m \in \mathbb{Z}$. Clearly, each such function is an element of the ring $\mathbb{C}[z, 1/z] = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} z^m$. The linear independence of the algebraic characters is easily verified.

Let us call an affine group D a *diagonalizable group* if it is isomorphic to a closed subgroup of the group of diagonal $n \times n$ matrices with entries from k. An important consequence of the previous lemma is that if D is a diagonalizable group, then $E(D)$ is a vector space basis for $k[D]$. The proof of the next theorem about the structure of a diagonalizable group can be found in [4, §16.2].

Theorem 1.15. *Let D be a diagonalizable group. Then D is isomorphic to a direct product of the form $S \times H$, where S is a torus (always connected!) and H is a finite abelian group.*

1.6 Algebraic actions and rational representations.

Let $f : S \times X \to X$ be an abstract group action of S on X. If, in addition, the map f is a morphism of varieties, then $S : X$ is called an *algebraic S-action*. Also, in this case, X is said to be an *S-variety*. Hereafter, when confusion is unlikely, by the notation $S : X$ we mean an algebraic S-action $S \times X \to X$.

Let $k[X]$ denote the coordinate ring of X. For each algebraic action S on X, there is a corresponding *left translation action* of S on the regular functions [1, Ch I, §1.9]. It is defined as follows. Let s be an element of S. Then we define the linear automorphism $\lambda(s) : k[X] \to k[X]$ by

$$\lambda(s)(g(x)) := g(s^{-1} \cdot x) \qquad (g \in k[X]). \tag{1.16}$$

(The action (1.16) makes perfect sense if we define it as the action of the opposite group G^{op} on $k[X]$. Note that G^{op} is canonically isomorphic to G as an algebraic group.)

A simple but important fact regarding the representation defined by (1.16) is that it is a locally finite representation. More precisely, we have the following well-known fact.

Lemma 1.17. *The k-algebra $k[X]$ is a union of finite dimensional vector subspaces that are S-stable.*

Proof. Let $\alpha : S \times X \to X$ be the morphism that defines the action $S : X$. Then the comorphism $\alpha^* : k[X] \to k[S \times X]$ is defined by $\alpha^*(g)(s, x) = g(\alpha(s, x))$, where $g \in k[X], s \in S$, and $x \in X$. Now let f be a function from $k[X]$. We will construct an S-stable, finite dimensional vector subspace V of $k[X]$ such that $f \in V$.

Since the coordinate ring of $S \times X$ is given by the tensor product, $k[S \times X] = k[S] \otimes k[X]$, we write $\alpha^*(f)$ in the form

$$\alpha^*(f) := \sum_{i=1}^{r} \phi_i \otimes \psi_i$$

for some $\phi_i \in k[S]$ ($i \in \{1, \dots, r\}$) and $\psi_i \in k[X]$ ($i \in \{1, \dots, r\}$). In other words, we have

$$f(s \cdot x) = f(\alpha(s, x)) = \alpha^*(f)(s, x) = \sum_{i=1}^{r} \phi_i(s)\psi_i(x). \tag{1.18}$$

Note that $f(s \cdot x) = (s^{-1} \cdot f)(x)$ for every $s \in S$ and $x \in X$. Thus, we conclude from (1.18) that the set $\{s \cdot f(x) : s \in S, x \in X\}$ is not only S-stable but also contained in the linear span of the functions $\psi_1, \dots, \psi_r$ in $k[X]$. In particular, the linear span $V := \text{span}_k\{s \cdot f(x) : s \in S, x \in X\}$ is a finite dimensional, S-stable subspace of $k[X]$. This finishes the proof. □

Let G be an affine algebraic group. Let M be a k-module, that is, a k-vector space. Let $\sigma : G \to GL(M)$ be an abstract group homomorphism. We have two possibilities for the vector space dimension of M.

(1) First, we assume that $\dim M < \infty$. If, in addition, σ is an algebraic group homomorphism, then M is called a *rational G-module*, and σ is called a *(rational) representation of G*.

(2) Next, we assume that $\dim M = \infty$. If for each element $x \in M$ there exists a finite dimensional $\sigma(G)$-stable linear subspace $V \subset M$ such that $x \in V$, and $\sigma_V : G \to GL(V)$ is a rational G-module as in (1), then we say that M is a *(rational) G-module*. Also in this case, we say that φ is a *rational representation of G*.

Lemma 1.17 shows that if X is an G-variety, then the coordinate ring of X has the structure of a rational G-module. In our first Example 1.3, $\mathbb{C}^3$ is not a rational $H_3(\mathbb{C})$-module, since the action there is not an algebraic action. All of the other examples in this section contain rational modules.

Let us agree on the following abbreviation. When we discuss rational G-modules, if there is no danger for confusion, then let us just write a G-module instead of writing a 'rational G-module'.

Now, a nonzero G-module M is called *simple* if M has no G-submodule other than 0 and itself. In this case we call the representation $\sigma : G \to GL(M)$ *irreducible*. There is no harm in mixing up these terminologies. For a given simple G-module V, the sum of all simple G-submodules of M isomorphic to V is called the *V-isotypic component of M*.

1.7 Algebraic covering homomorphisms.

Let G and $\tilde{G}$ be two connected affine algebraic groups. Let $f : \tilde{G} \to G$ be a surjective algebraic group homomorphism. If the kernel of f is a finite group, then f is called a *covering homomorphism* or an *isogeny*. The group G is called *(algebraically) simply connected* if every isogeny onto G is an isomorphism of algebraic groups. A convenient characterization of simply connected groups is that they are precisely the connected affine groups G such that every finite dimensional representation of the Lie algebra of G is the differential of a rational representation of G.

Let us note that if G is a connected affine complex algebraic group, then viewed as a real Lie group, if G is topologically simply connected, then G is algebraically simply connected. To see this, recall from the Serre's GAGA that the analytification functor is fully faithful. Therefore, the covering homomorphisms of G as an algebraic group are the covering homomorphisms of G as a complex Lie group and vice versa. Therefore, the topological fundamental group of G is trivial if and only if every algebraic covering homomorphism of G is trivial. Another proof of this useful result can be found in [2, Remark 6.3.10].

1.8 A word on the notation.

Let G be an algebraic group. Some authors call an algebraic character of G a *linear character*. Hereafter, when we feel that it will not cause any confusion, we will drop the adjectives like 'linear' or 'algebraic' from our terminology.

In standard textbooks, the set of characters of G is usually denoted by one of the following notations: $X(G)$, $X_*(G)$, $X^*(G)$, $\mathcal{X}_G$, $\mathcal{X}(G)$, $\mathcal{X}_*(G)$, $\mathcal{X}^*(G)$.. Also, we just learned that the quotient group $\mathcal{O}(G)^*/k^*$, denoted by $E(G)$, is isomorphic to the group of characters of G as well. In this text, the character group of an algebraic group will be denoted by $\ddot{\mathbf{O}}(G)$[1]. In the sequel, for a G-variety X, where G is a reductive group and $B \subseteq G$ a Borel subgroup, we will use the notation $\ddot{\mathbf{O}}(X)$ to denote a certain sublattice of the character group of B. We note here that, in general, $E(X) = \mathcal{O}(X)^*/k^*$ is not the same thing as $\ddot{\mathbf{O}}(X)$.

2 Geometric interpretations

In this section we will review some well-known results about the geometric aspects of characters. Our main references for this section are [8, 7, 6]. We proceed with the assumption that k is an algebraically closed field of characteristic 0.

Let X be an algebraic variety. The *Picard group* of X, denoted by $\mathrm{Pic}(X)$, is the group of isomorphism classes of line bundles on X. For homogeneous spaces G/H, there is a close relationship between the Picard group of G/H and the characters of H. In the special case of a Borel subgroup of a reductive group, there is a well-known construction of a line bundle from a character of B; it is sometimes called the "Borel-Weil-Bott construction." We will present the general formulation.

A construction of line bundles. Let G be an affine algebraic group, and let H be a closed subgroup of G. Let $\chi : H \to \mathbb{G}_m$ be a character of $\ddot{\mathbf{O}}(H)$. We consider the following diagonal action of H on $G \times k$:

$$h \cdot (g, x) := (gh^{-1}, \chi(h)x) \qquad (h \in H, g \in G, x \in k).$$

Let us assume that the quotient $(G \times k)/H$, denoted by L_χ, exists as a variety. Then via the first projection map $p : L_\chi \to G/H$, L_χ becomes a line bundle on G/H. We notice that G

[1] We decided to denote the set of algebraic characters of G by $\ddot{\mathbf{O}}(G)$, since the letter O with umlaut is the initial of the Turkish word öz, which could be translated as "one's own."

acts on $L_\chi = (G \times k)/H$ via its action on the first factor. Furthermore, the projection map is equivariant with respect to this action. Therefore, the space of global sections of L_χ has the structure of a G-module. If H is a Borel subgroup of a connected reductive group G, then under some assumptions on the character χ, the vector space $H^0(G/B, L_\chi)$ turns out to be an irreducible representation of G. In fact, the Borel-Weil-Bott theorem states that all irreducible representations of G can be constructed this way, [5, Part II, Ch 5].

It is a natural question to ask if any line bundle on a G-variety has any representation theoretic meaning. The answer is yes under suitable assumptions. To explain the scope of this question, a notion of an 'equivariant Picard group' will be useful.

The equivariant Picard group. As before, let G be an affine algebraic group. Let X be a G-variety, and let $\pi : L \to X$ be a line bundle on X. A *G-linearization of L* is a G-action

$$\varphi : G \times L \to L$$

such that

(1) the natural map $\pi : L \to X$ is G-equivariant;

(2) for every $x \in X, g \in G$, the map $\varphi_x : L_x \to L_{gx}$ is linear. Here, L_x is the fiber of L at $x \in X$.

Definition 2.1. Let X be a G-variety. The *equivariant Picard group of X*, denoted by $\mathrm{Pic}_G(X)$, is the group of isomorphism classes of G-linearized line bundles on X.

The following proposition is useful for computing the (equivariant) Picard groups of G-varieties; it is proved in [7, Proposition 5.1].

Proposition 2.2. *Let G be a reductive group, and let X be an irreducible G-variety which admits a geometric quotient $\pi : X \to X/\!\!/G$. Then we have the following commutative diagram whose rows and columns are exact:*

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & E(X/\!\!/G) \arrow[d] & & \\
 & E(X)^G \arrow[d] & 1 \arrow[d] & \\
 & E(G) \arrow[d] & \mathrm{Pic}(X/\!\!/G) \arrow[d, "\gamma"] & \\
1 \arrow[r] & \mathrm{H}^1_{alg}(G, \mathcal{O}(X)^*) \arrow[r] \arrow[d] & \mathrm{Pic}_G(X) \arrow[r] \arrow[d, "\delta"] & \mathrm{Pic}(X) \\
 & \mathrm{H}^1(G/G^0, E(X)) & \prod_{x \in C} E(G_x) &
\end{tikzcd}$$

In the diagram of the proposition, G^0 stands for the connected component of the identity in G. The product $\prod_{x\in C} E(G_x)$ is taken over a set of representatives of the closed G-orbits in X. For $x \in C$, the group G_x is the stabilizer of x in G. The map δ is obtained by associating to a line bundle L the characters of the isotropy groups on the fibers L_x ($x \in C$). Finally, $\mathrm{H}^1_{alg}(G, \mathcal{O}(X)^*)$ denotes the *group of classes of algebraic cocycles.* (A morphism $\gamma : G \to \mathcal{O}(X)^*$ is said to be an algebraic cocycle if $\gamma(gh) = (g \cdot \gamma(h))\gamma(h)$ holds for every $g, h \in G$.)

Remark 2.3. If G is connected, then we know that $E(X)^G = E(X)$ by Proposition 1.8. Of course, if G is connected, then we have $\mathrm{H}^1(G/G^0, E(X)) = 0$ as well.

The previous proposition is a culmination of several useful theorems that we want to state here.

Theorem 2.4. *Let G be a connected affine group, and let H be a closed subgroup of G. Then there exists a canonical isomorphism,*

$$\delta : \mathrm{Pic}_G(G/H) \to \ddot{\boldsymbol{O}}(H).$$

Identify $\mathrm{Pic}_G(G/H)$ *and* $\ddot{\boldsymbol{O}}(H)$ *via* δ*, and let* ε *be the morphism of "forgetting the* G*-action",*

$$\varepsilon : \mathrm{Pic}_G(G/H) \to \mathrm{Pic}(G/H).$$

If $\mathrm{Pic}\, G = 0$*, then* ε *is surjective and the kernel of* ε *is the subgroup* $res(\ddot{\boldsymbol{O}}(G)) =: \ddot{\boldsymbol{O}}_G(H) \subseteq \ddot{\boldsymbol{O}}(H)$ *of characters that are obtained from* $\ddot{\boldsymbol{O}}(G)$ *by restricting to* $\ddot{\boldsymbol{O}}(H)$*.*

Remark 2.5. For any connected affine algebraic group G, and a closed subgroup $H \subseteq G$, the quotient group $\ddot{\mathbf{O}}(H)/\ddot{\mathbf{O}}_G(H)$ is isomorphic to the character group $\ddot{\mathbf{O}}(H \cap (G,G))$, [8, §4, Lemma 4]. Therefore, if $\mathrm{Pic}\, G = 0$, then by Theorem 2.4, we see that the Picard group of G/H is $\ddot{\mathbf{O}}(H \cap (G,G))$.

Theorem 2.6. *Let G be a connected affine group, and let H be a closed subgroup of G. Let ε be as in Theorem 2.4, and let $\pi : G \to G/H$ be the canonical projection. Then the following statements hold:*

1. *The sequence*

$$\ddot{\boldsymbol{O}}(G) \xrightarrow{res} \ddot{\boldsymbol{O}}(H) \xrightarrow{\varepsilon} \mathrm{Pic}(G/H) \xrightarrow{\pi^*} \mathrm{Pic}(G) \tag{2.7}$$

   is exact.

2. *If H is a connected solvable subgroup of G, then the natural morphism $\pi^* : \mathrm{Pic}(G/H) \to \mathrm{Pic}(G)$ is surjective.*

3. *Let $\mathcal{R}$ denote the radical of G. We assume that $(G,G) \cap \mathcal{R} = 0$ holds. Then $\mathrm{Pic}(G)$ is isomorphic to the fundamental group of $G/\mathcal{R}$. It is also isomorphic to the kernel of the covering homomorphism $\widetilde{G} \to G$, where $\widetilde{G}$ is simply connected.* Important: This statement is not correct if the assumption $(G,G) \cap \mathcal{R} = 0$ is missing.

4. If G is simply connected, then $\mathrm{Pic}(G) = 0$.

The proofs of 1. and 2. are given in [7, Proposition 3.2]. Notice that, in light of Remark 2.5, part 1 can be seen as a restatement of Theorem 2.4. The proofs of 3. and 4. are given in [8]. For part 3, see https://math.stackexchange.com/questions/1999823/picard-groups-and-fundamental-groups-of-connected-algebraic-groups. The statement in part 4 was first proven by Voskserenskii in characteristic 0 [11].

Corollary 2.8. *Let B be a Borel subgroup of a connected reductive group G. Let T be a maximal torus of B. Then we have*

- $\mathrm{Pic}(G/B) \cong \ddot{\boldsymbol{O}}(T_0)$*, where T_0 is the maximal torus in the semisimple part (G, G) of G.*
- $\mathrm{Pic}_G(G/B) \cong \ddot{\boldsymbol{O}}(T)$*.*

Proof. The statement in the second bullet follows directly from Theorem 2.4. We proceed with the proof of the claim in first bullet.

Recall that an isogeny is a surjective algebraic group homomorphism whose kernel is finite. It is easy to verify that the flag varieties are invariant under the morphisms induced by isogenies. Therefore, we can replace G/B by $\tilde{G}/\tilde{B}$, where $\tilde{G}$ is a simply connected cover of G, and $\tilde{B}$ is the preimage of B in $\tilde{G}$. Then by Part 2 of Theorem 2.6, we have $\mathrm{Pic}(G/B) \cong \ddot{\mathbf{O}}(\tilde{B})/res(\ddot{\mathbf{O}}(\tilde{G}))$. We notice that the quotient $\ddot{\mathbf{O}}(\tilde{B})/res(\ddot{\mathbf{O}}(\tilde{G}))$ is isomorphic to the quotient $\ddot{\mathbf{O}}(B)/res(\ddot{\mathbf{O}}(G))$. Finally, we notice that every central torus of G is contained in B; these facts follows from our remark in Subsection 1.3 about the definition of a connected reductive group. Hence, the quotient $\ddot{\mathbf{O}}(B)/res(\ddot{\mathbf{O}}(G))$ is isomorphic to the corresponding quotient $\ddot{\mathbf{O}}(B_0)/res(\ddot{\mathbf{O}}(G_0))$, where G_0 is the semisimple part $G_0 = (G, G)$ and B_0 is the Borel subgroup $B \cap G_0$. Since G_0 is a connected semisimple group, we know that $\ddot{\mathbf{O}}(G_0) = 1$. Therefore, $\mathrm{Pic}(G/B) = \ddot{\mathbf{O}}(B_0) = \ddot{\mathbf{O}}(T_0)$, where T_0 is the maximal torus of B_0. This finishes the proof of our assertion. □

Remark 2.9. Let G be a connected affine group, $\tilde{G} \to G$ be a simply connected cover. If H is a closed subgroup of G, then let $\tilde{H}$ denote preimage of H in $\tilde{G}$. The arguments that we used in the proof of Corollary 2.8 hold in general. First, we have the isomorphism

$$G/H \cong \tilde{G}/\tilde{H}.$$

It follows that G/H is a $\tilde{G}$-variety. Secondly, since $\tilde{G}$ is simply connected, $\mathrm{Pic}(\tilde{G}) = 0$, hence, every line bundle on G/H is $\tilde{G}$-linearizable. This is proved in [6, Proposition 2.4 and its Remark]. We now apply Theorem 2.4 (or part 1. of Theorem 2.6), which gives us

$$\mathrm{Pic}(G/H) = \mathrm{Pic}(\tilde{G}/\tilde{H}) = \ddot{\mathbf{O}}(\tilde{H})/res(\ddot{\mathbf{O}}(\tilde{G})). \tag{2.10}$$

2.1 The Picard group of $\mathbf{GL}_n(\mathbb{C})$.

Next, we want to prove another useful statement, that is,

$$\mathrm{Pic}(\mathbf{GL}_n(\mathbb{C})) = 0. \tag{2.11}$$

We begin with an algebraic reinterpretation of Picard groups.

We begin with recalling some facts from [3, Ch II, §2]. First, we let X be a noetherian, integral, separated scheme which is regular in codimension one. The last condition means that every local ring $\mathcal{O}_x$ of X of dimension one is regular. For example, if X is a normal algebraic variety, then it has these properties.

Let Z be a prime divisor in X. The local ring $\mathcal{O}_{Z,X}$ is a discrete valuation ring in $k(X)$. In particular, the total quotient field of $\mathcal{O}_{Z,X}$ is equal to the function field of X. Let v_Z denote the valuation corresponding to $\mathcal{O}_{Z,X}$. For every element $f \in k(X)$, there are only finitely many prime divisors Z (depending on f) in X such that $v_Z(f) \neq 0$. Thus, we associate a divisor to f as follows:

$$\mathrm{div}(f) := \sum_Z v_Z(f) Z.$$

This is called the *principal divisor* defined by f. The free abelian group that is generated by all divisors is called the *group of Weil divisors*, denoted by $\mathrm{Div}(X)$.

Two divisors D and D' are said to be *linearly equivalent* if $D - D'$ is a principal divisor. Since $(f/g) = (f) - (g)$ for $f, g \in k(X)\backslash\{0\}$, the principal divisors form a subgroup of $\mathrm{Div}(X)$. The quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors is called the *(divisor) class group*, denoted by $\mathrm{Cl}(X)$.

Lemma 2.12. *Let A be a noetherian domain. Then A is a UFD if and only if the affine scheme $X := \mathrm{Spec}\, A$ is normal and $\mathrm{Cl}(X) = 0$.*

Another well-known important property of the class groups is the following:

Lemma 2.13. *Let Z be a proper closed subset of X, and let U denote the open set $X\backslash Z$. If Z is an irreducible subset of codimension one in X, then the sequence*

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where the first map is given by $1 \mapsto 1 \cdot [Z]$, is exact.

A variety X is said to be *locally factorial* if for every $x \in X$ the local ring of x is a unique factorization domain. The relation of the class group to our discussion is the following result:

Lemma 2.14. *Let X be a noetherian, integral, separated scheme. If X is locally factorial, then $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Pic}(X)$.*

Notice that, for a normal, noetherian, integral affine scheme $X = \mathrm{Spec}\, A$, if $\mathrm{Cl}(X) = 0$, then A is a UFD by Lemma 2.12. In particular, X is locally factorial. Hence, by Lemma 2.14, we find that $\mathrm{Pic}(X) = 1$.

Corollary 2.15. *Let n be a positive integer. Then the Picard group of $\mathbf{GL}_n(k)$ is trivial.*

Proof. Since $U := \mathbf{GL}_n(k)$ is nonsingular, it is normal. Furthermore, it is connected, hence it is irreducible. Therefore, $\mathrm{Pic}(U) \cong \mathrm{Cl}(U)$. We now view U as the complement of the hypersurface $\{a \in \mathbf{Mat}_n(k) : \det a = 0\}$ in $\mathbf{Mat}_n(k)$. Since $X := \mathbf{Mat}_n(k)$ is an affine space, we see from Lemma 2.12 that its class group is trivial. It follows from the short exact sequence in Lemma 2.14 that $\mathrm{Cl}(U) = 0$. Since $\mathrm{Cl}(U)$ is isomorphic to $\mathrm{Pic}(U)$, the proof of our assertion is finished. □

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