Algebraic Groups in Action, Lecture 3.

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July 28, 2021

1 Root Systems, Weyl Groups, Fundamental Weights

Simple algebraic groups are parametrized by the "indecomposable and reduced root systems." There are four infinite families, labeled $\mathtt{A}_\ell$, $\mathtt{B}_\ell$ ($\ell \geqslant 2$), $\mathtt{C}_\ell$ ($\ell \geqslant 3$), and $\mathtt{D}_\ell$ ($\ell \geqslant 4$), and five exceptional types, denoted $\mathtt{E}_6$, $\mathtt{E}_7$, $\mathtt{E}_8$, $\mathtt{F}_4$, $\mathtt{G}_2$ of such root systems. We will briefly review some of the relevant definitions.

We fix a euclidean space E. This means that E is a finite dimensional real vector space together with an inner product $(\cdot,\cdot) : E \times E \to \mathbb{R}$. For each vector $\alpha \in E$, we have the corresponding *reflection operator*, denoted $s_\alpha : E \to E$, that reflects the vectors with respect to the hyperplane that is perpendicular to α. To formulate this more precisely, let us define another 'product' map:

$$\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \qquad (\alpha, \beta \in E,\ \alpha \neq 0).$$

Then for every $\beta \in E$ we have

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

Definition 1.1. Let E be a finite dimensional vector space over $\mathbb{Q}$. A *root system in* E, denoted $\mathtt{R}$, is a finite set of nonzero vectors, called *roots*, in E such that

(1) $\mathtt{R}$ spans E;

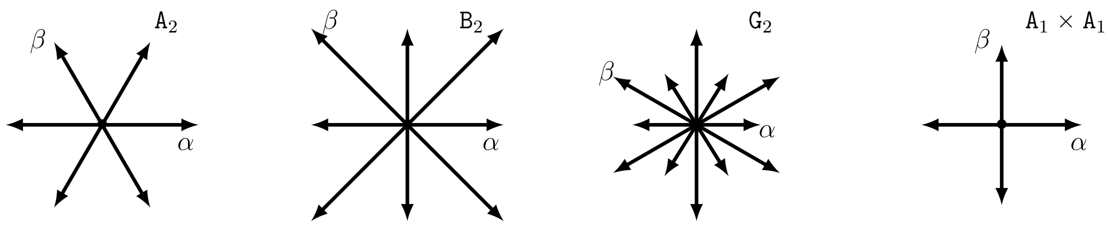
(2) for every root $\alpha \in \mathtt{R}$, the corresponding reflection operator $s_\alpha : E \to E$ permutes $\mathtt{R}$;

(3) for every α and β from $\mathtt{R}$, we have $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

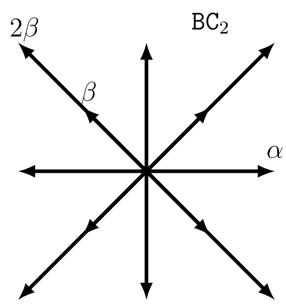
If, in addition, for each $\alpha \in \mathtt{R}$, the condition $c\alpha \in \mathtt{R}$, where $c \in \mathbb{Q}$, implies that $c \in \{-1, +1\}$, then $\mathtt{R}$ is said to be a *reduced root system in* E. In general, if α and $c\alpha$ ($c \in \mathbb{Q}$) are two proportional elements of a root system, then c can be any of the elements of $\{\pm\frac{1}{2}, \pm 1, \pm 2\}$. A root system is called *indecomposable* if it cannot be partitioned into the union of two mutually orthogonal proper subsets. The *rank* of $\mathtt{R}$ is the dimension of E.

Remark 1.2. In the standard textbooks such as [3], where only the root systems associated with reductive algebraic groups are considered, the indecomposable root systems are called the *indecomposable root systems.* Since we will consider the root systems of spherical varieties, which can be reduced, we prefer not to use the word "irreducible" for our root systems.

Indecomposable, reduced. Indecomposable, non-reduced.



Ind., reduced Ind., reduced Ind., reduced Dec., reduced.



Indecomposable but non-reduced.

Figure 1.1: All rank one and rank two root systems. The abbreviations "Ind." and "Dec." stand for the words indecomposable and decomposable, respectively. The simple roots are indicated by α and β.

Let us define the *root system of a reductive group.* Let (G, B, T) be the data of a connected reductive group G, a Borel subgroup B, and a maximal torus T contained in B. An algebraic group homomorphism $\varphi : \mathbb{G}_a \to G$ will be called a *one-parameter subgroup*, abbreviated to 1-psg. We will call a character $\alpha \in \ddot{\mathbf{O}}(T)$ a *root of G relative to T* if there exists a 1-psg $\varphi : \mathbb{G}_a \to G$ such that

$$t\varphi(c)t^{-1} = \varphi(\alpha(t)c) \qquad (t \in T, c \in \mathbb{G}_a). \tag{1.3}$$

We call the image of φ, denoted U_α, a *root subgroup* with respect to T. Although U_α is uniquely determined by α, the 1-psg φ does not need to be unique; we can always change the speed of φ by multiplying it its variable by a nonzero scaler from k^*. The set of all roots of G relative to T is denoted by $\mathtt{R}_{G,T}$. It turns out that G is generated by its maximal torus along with the unipotent subgroups, see [3, Theorem 26.3 (d)].

A closely related construction of the root system $\mathtt{R}_{G,T}$ is obtained via the adjoint representation of G on its tangent space of T_eG. The Lie algebra structure on T_eG will be denoted by $\mathfrak{g}$. Since G is a linear algebraic group, it admits an embedding into a suitable general linear group as a closed subgroup. Then the adjoint representation of G on its Lie algebra Lie(G) is given by the 'conjugation action':

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}(g), \end{aligned}$$

where $\mathrm{Ad}(g)(A) = gAg^{-1}$ for every $A \in \mathfrak{g}$. By differentiating this algebraic group homomorphism, we arrive at the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto \mathrm{ad}\ x$, where $\mathrm{ad}\ x$ is the left translation map $\mathrm{ad}\ x(y) = [x, y]$. By restricting the adjoint representation to the Lie algebra $\mathfrak{t} := \mathrm{Lie}(T)$, which is abelian, we obtain the weight space decomposition,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathtt{R}_{G,T}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the Lie algebra of the root subgroup U_α. The action of $\mathfrak{t}$ on $\mathfrak{g}_\alpha$ is given by

$$[s, x] = (d\alpha(s))x \qquad (s \in \mathfrak{t},\ x \in \mathfrak{g}_\alpha),$$

where $d\alpha : \mathfrak{t} \to k$ is the differential of the character $\alpha \in \ddot{\mathbf{O}}(T)$. Below, we will discuss more general weight space decompositions from the group theory viewpoint.

1.1 Weyl groups

The relative positioning of the root subgroups in a reductive group is controlled by a finite group called the *Weyl group*, which is defined by $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The Weyl group acts as a group of automorphisms of the abelian group $\ddot{\mathbf{O}}(T)$. The action is given as follows. For $w \in W$ and $\alpha \in \ddot{\mathbf{O}}(T)$, the function $w \cdot \chi : T \to \mathbb{G}_m$ is defined by

$$(w \cdot \chi)(t) = \chi(w^{-1}tw) \qquad (t \in T). \tag{1.4}$$

Furthermore, we have $w \cdot (\chi\mu) = (w \cdot \chi)(w \cdot \mu)$ for every $\chi, \mu \in \ddot{\mathbf{O}}(T)$.

Let $\sigma : G \to GL(V)$ be a finite dimensional rational representation of G. We will verify that the action of W in (1.4) descends to a permutation action on a certain set of characters called the "weights of σ." To this end let us view σ (by restriction) as a rational representation of T. Then we have a decomposition $V = \bigoplus_{\chi \in \mathtt{R}(\sigma)} V_\chi$, where V_χ's are the joint eigenspaces for the elements of T:

$$V_\chi := \{v \in V : \ t \cdot v = \chi(t)v \text{ for all } t \in T\}.$$

It is easy to verify that the functions $\chi \in \mathtt{R}(\sigma)$ are actually algebraic characters of the maximal torus. We call them the *weights of the representation* σ. For $\chi \in \mathtt{R}(\sigma)$, the elements of V_χ are called the *weight vectors of weight* χ.

Now let χ be a weight of $\sigma : G \to GL(V)$. Let $v \in V_\chi$ be a weight vector of weight χ. Let w be an element of W. Let $\dot{w}$ be a representative of w in $N_G(T)$. We will check to see where v is sent to via the action of w. To this end, we will compute the action of T on $\sigma(\dot{w})v$:

$$\begin{aligned}\sigma(t)(\sigma(\dot{w})(v)) = (\sigma(t)\sigma(\dot{w}))(v) = \sigma(\dot{w})(\sigma(\dot{w}^{-1}t\dot{w})(v)) &= \sigma(\dot{w})(\chi(\dot{w}^{-1}t\dot{w})\ v) \\ &= \sigma(\dot{w})((w \cdot \chi)(t)\ v) \\ &= (w \cdot \chi)(t)\sigma(\dot{w})(v).\end{aligned}$$

Hence, we conclude that $w \cdot \chi$ is also a weight of σ. In particular, if we use the adjoint representation in place of σ, then we see that $W \cdot \mathtt{R}_{G,T} = \mathtt{R}_{G,T}$.

It is a consequence of the general theory that $\mathtt{R}_{G,T}$ is indeed a reduced root system in the euclidean space $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, where the inner product is chosen so that it is W-invariant. We will not verify this assertion in these notes; most of the details are clearly presented in [1, Ch IV, §14], or [3, Chapter X].

1.2 Terminology of weights

Let G be a connected reductive algebraic group. As usual, let B and T denote a Borel subgroup of G, and a maximal torus contained in B. Let $\mathtt{R}_{G,T}$ denote the root system determined by (G, T). Recall that $\mathtt{R}_{G,T}$ is a finite subset of the character lattice $\ddot{\mathbf{O}}(T)$ of the torus T. The elements of the character lattice $\ddot{\mathbf{O}}(T)$ are sometimes called the *weights of* G *relative to* T.

A root $\alpha \in \mathtt{R}_{G,T}$ is called *positive* if the corresponding root subgroup U_α is contained in B. The set of positive roots will be denoted by $\mathtt{R}^+_{G,T}$. Note that there are several equivalent ways of defining the positive roots. We introduced them by using Borel subgroups to emphasize the combinatorial nature of such subgroups. Similarly to the definition a positive root, an element $\alpha \in \mathtt{R}_{G,T}$ is called *negative* if the corresponding root subgroup U_α is contained in the "opposite" Borel subgroup B^-. The notion of an opposite Borel subgroup will be explained in the next subsection. The set of negative roots will be denoted by $\mathtt{R}^-_{G,T}$. It is easy to check that $\mathtt{R}^-_{G,T} = -\mathtt{R}^+_{G,T}$. (In fact, we could introduce the negative roots this way!) A positive root $\alpha \in \mathtt{R}^+_{G,T}$ is called *simple* if it cannot be written as a sum of two positive roots. The set of all simple roots will be denoted by $\mathtt{S}$.

It is easy to show that $\mathtt{S}$ is a basis for the vector space $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. In general, $\mathtt{S}$ is not a basis for the lattice $\ddot{\mathbf{O}}(T)$ but there is a natural closely related basis for $\ddot{\mathbf{O}}(T)$. To define it, first we label the simple roots as in $\mathtt{S} = \{\alpha_1, \ldots, \alpha_l\}$. For $i \in \{1, \ldots, l\}$, the *fundamental dominant weight* is a character $\varpi_i \in \ddot{\mathbf{O}}(T)$ such that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$, where $\alpha_j^\vee$ is the coroot of α_j. A character $\varpi \in \ddot{\mathbf{O}}(T)$ is said to be a *dominant weight* if it is can be written as a nonnegative integral combination of the fundamental dominant weights. Thus, set of all dominant weights becomes a submonoid of the free abelian group $\ddot{\mathbf{O}}(T)$. The cone span of the monoid of dominant weights in $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called the *fundamental (or dominant) Weyl chamber*. The *anti-dominant Weyl chamber* is the cone spanned by $\{-\alpha_1^\vee, \ldots, -\alpha_l^\vee\}$ in $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The Weyl group acts simply-transitively on the nonzero elements of the vector space $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, it is well-known that the fundamental Weyl chamber is a domain of transitivity for the action of W.

Before we move to our next topic, we want to mention a partial order on the weights. As before, let $\mathtt{S} = \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots for $\mathtt{R}_{G,T}$. Let χ and μ be two weights from $\ddot{\mathbf{O}}(T)$. We define

$$\mu \preceq \chi \iff \chi - \mu = \sum_i a_i \alpha_i,$$

where $a_i \in \mathbb{N}$ for every $i \in \{1, \ldots, l\}$.

1.3 The Bruhat-Chevalley decomposition and Schubert varieties.

In this subsection we will introduce one of the most important results on the structure of a reductive group. We will work with a fixed tripled (G, B, T), where G is a connected reductive group, B is a Borel subgroup, and T is a maximal torus of B. Note that B is naturally isomorphic to the semidirect product of T with the unipotent radical $\mathcal{R}_u(B)$. By our definition of a positive root system $\mathtt{R}^+_{G,T}$, we know that $\mathcal{R}_u(B)$ is directly spanned by the corresponding (positive) root subgroups,

$$\mathcal{R}_u(B) = \prod_{\alpha \in \mathtt{R}^+_{G,T}} U_\alpha,$$

where the product can be taken in any order [3, Proposition 28.1].

Let W denote the Weyl group. Every element $w \in W$ has a representative, denoted $\dot{w}$, in the subgroup $N_G(T) \subset G$. Hereafter, unless there is danger for confusion, we will omit using the dotted notation for w. The *Bruhat-Chevalley decomposition* of G is the decomposition of G into two-sided orbits of B in G:

$$G = \bigsqcup_{w \in W} BwB. \tag{1.5}$$

Since a Borel subgroup is the semidirect product of its maximal unipotent subgroup U and a maximal torus T, and since T normalizes U, the double-coset BwB is canonically isomorphic to UwB. The decomposition in (1.5) descends to a similar decomposition for each *standard parabolic subgroup*, that is, a closed subgroup $P \subseteq G$ such that $B \subseteq P$. More generally, if a closed subgroup of G contains a Borel subgroup (not necessarily equal to B), then it is called a *parabolic subgroup*. A characterizing property of the parabolic subgroups P in G is that G/Q is a projective variety.

Let P be a standard parabolic subgroup. Then $P = \mathcal{R}_u(P) \rtimes L$, where $\mathcal{R}_u(P)$ is the unipotent radical of P and L is a (maximal) reductive subgroup of P that normalizes $\mathcal{R}_u(P)$. Since P is standard, the maximal torus T is contained in L. Then the Weyl group of (L, T), denoted by W_P, is a subgroup of W generated by the simple reflections s_α, where α is a simple root such that the corresponding root subgroup U_α is contained in L. Then we have analogous decomposition

$$P = \bigsqcup_{w \in W_P} BwB. \tag{1.6}$$

Recall that G/P is a projective variety. Since both of G and P have related Bruhat-Chevalley decompositions, the structure of the variety G/P is closely related to the finite quotient set W/W_P.

The projective variety G/P is called a *partial flag variety* of G. If $P = B$, then it will be called the *full flag variety of* G. Thanks to (1.6) every partial flag variety has a decomposition into B-, equivalently, U-orbits,

$$G/P = \bigsqcup_{w \in W^P} BwB/P, \tag{1.7}$$

where $W^P \subseteq W$ is a certain transversal for W_P in W. There is a canonical choice for this transversal. It is defined in terms of a "grading" on the Weyl group,

$$\begin{aligned} \ell : W &\longrightarrow \mathbb{N} \\ w &\longmapsto \dim(BwB/B). \end{aligned} \tag{1.8}$$

Here, $\dim(BwB/B) = \dim(UwB/B)$ is the dimension of the B-orbit of w in G/B. Then, the *canonical transversal* for the subgroup $W_P \subseteq W$ is given by

$$w \in W^P \iff \ell(w) \leqslant \ell(v) \ \text{ for all } v \in wW_P.$$

From now on, W^P will denote this set of minimal length left-coset representatives of W_P in W.

One of the most enticing features of the B-orbits in partial flag varieties is that they carry a deep topological meaning. A proof of the following well-known fact can be found in [1, Ch IV, §14].

Lemma 1.9. *Let P be a parabolic subgroup of G. Then the decomposition of G/P in (1.7) is a cellular decomposition of G/P.*

We are now ready to introduce the Schubert varieties.

Definition 1.10. Let w be an element of W^P. The U-orbit of w in G/P is called a *Schubert cell*; we denote it by C_{wP}. The Zariski closure of UwB/P in G/P is called a *Schubert variety* in G/P; we denote it by X_{wP}.

The Schuber subvarieties of G/B (or of G/P) are partially ordered according to the gluing of their Schubert cells. To make this more precise, next, we introduce the "Bruhat-Chevalley order".

Definition 1.11. The Bruhat-Chevalley order $\leqslant$ on W is the partial order defined by

$$w \leqslant v \iff C_{wB} \subseteq X_{vB} \qquad (w, v \in W).$$

The pair $(W, \leqslant)$ is a ranked poset with respect to the grading function $\ell : W \to \mathbb{Z}$.

A closely related partial order on W, which is also graded by $\ell : W \to \mathbb{Z}$, is the *left weak order*. Let us introduce this partial order by using the combinatorial structure of W. Let $\mathtt{S}$ be a set of simple roots for $\mathtt{R}_{G,T}$. Then the reflections corresponding to the elements of $\mathtt{S}$ generate W. The elements of the generating set, $S := \{s_\alpha : \ \alpha \in \mathtt{S}\}$, are called the *Coxeter generators of W with respect to* $\mathtt{S}$. The *left weak order on* W, denoted by $\leqslant_L$, is the transitive closure of the following covering relations:

$$w \lessdot_L v \iff v = sw \text{ and } \ell(v) = \ell(w) + 1 \qquad (s \in S, v, w \in W). \tag{1.12}$$

Example 1.13. Let us consider $G = GL_3$. We take T as the group of diagonal matrices in G, and we take B as the group of upper triangular matrices in G. The Weyl group W of (G, T) is the symmetric group S_3. The root system of (G, T) is given by $\{\varepsilon_i - \varepsilon_j : \ 1 \leqslant i \neq j \leqslant 3\}$, where $\varepsilon_i : T \to k^*$ is the character of T defined by $\varepsilon_i(\text{diag}(t_1, t_2, t_3)) = t_i$ for $i \in \{1, 2, 3\}$. The set $\mathtt{S} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$ is a set of simple roots for $\mathtt{R}_{G,T}$. Let $s_i \in S_3$ ($i \in \{1, 2\}$) denote the Coxeter generator corresponding to the simple root $\varepsilon_i - \varepsilon_{i+1}$. Then the Hasse diagrams of the weak order and the Bruhat-Chevalley order on S_3 are as depicted in Figure 1.2.

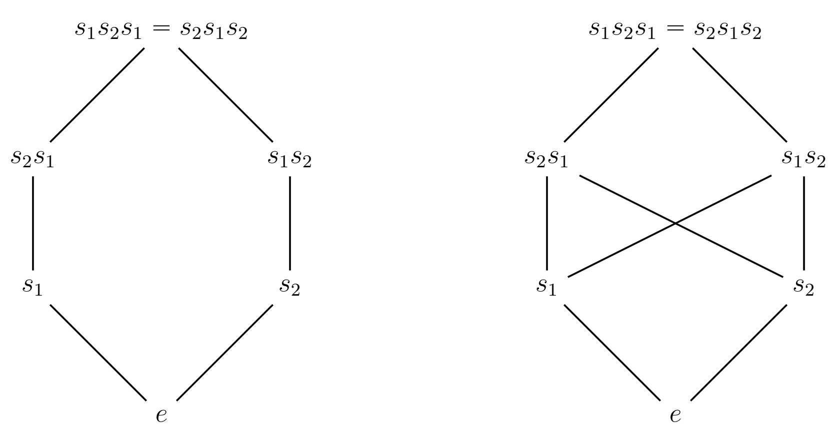


Figure 1.2: $(S_3, \leqslant_L)$ on the left, and $(S_3, \leqslant)$ on the right.

The *right weak order on* W, denoted by $\leqslant_R$, is defined similarly to $\leqslant_L$. The only difference is that one uses the right multiplication by the Coxeter generators instead of the left multiplications. In fact, the left and the right weak orders are isomorphic as posets, $(W, \leqslant_L) \cong (W, \leqslant_R)$.

We close this subsection by a simple application of the Bruhat-Chevalley order. Recall that the Bruhat-Chevalley decomposition of G gives a cellular decomposition of the flag variety G/B. Let us consider the case of $G = GL_2$. Since W is the symmetric group with two elements, $S_2 := \{e, s\}$, we have

$$GL_2/B = BeB/B \cup BsB/B.$$

The cell BsB/B is the one-dimensional affine space $\mathbb{A}^1$, and the cell BeB/B is just a point. Therefore, GL_2/B is the smooth projective variety that is obtained from $\mathbb{A}^1$ by attaching a zero cell to it. There is only one such variety; it is $\mathbb{P}^1$. Of course, there are other more conceptual ways of seeing the isomorphism, $GL_2/B \cong \mathbb{P}^1$.

2 Eigenvectors of an algebraic action

Let S be an affine algebraic group, and let V be an S-module. A nonzero vector $v \in V$ is said to be an S-*eigenvector*, or an S-*semiinvariant*, if there exists a character χ in $\ddot{\mathbf{O}}(S)$ such that $s \cdot v = \chi(s)v$ for all $s \in S$. We set

$$V^{(S)} := \{v \in V : \ v \text{ is an } S\text{-semiinvariant}\}.$$

If $S := T$ is a torus and χ is a character of T, then the T-semiinvariants are precisely the weight vectors of weight χ as we defined before.

Example 2.1. Let T denote the maximal diagonal torus of $GL_2(\mathbb{C})$. We will compute the semiinvariants of the natural action of T on the coordinate ring of $\mathbb{C}^2$, that is, $R := \mathbb{C}[X, Y]$. The action of T on R is given by the restriction of the action in Example **??**. Let $f(X, Y)$ be a T-semiinvariant. Then there exists χ_f in $\ddot{\mathbf{O}}(T)$ such that, for every $A = \mathrm{diag}(a^{-1}, d^{-1}) \in T$,

$$A \cdot f(X, Y) = f(aX, dY) = \chi_f(A) f(X, Y).$$

First by setting $d = 1$ and varying $a \in \mathbb{C}^*$, and then by setting $a = 1$ and varying d in $\mathbb{C}^*$, we see that $f(X, Y)$ is a monomial of the form $\alpha X^s Y^r$ for some $\alpha \in \mathbb{C}^*$ and $r, s \in \mathbb{N}$.

The semiinvariants of a Borel subgroup are contained in the semiinvariants of its maximal tori. More generally, we have the following statement.

Lemma 2.2. *Let S be an affine algebraic group, and let $H \leqslant S$ be a closed subgroup. If V is an S-module, then $V^{(S)} \subseteq V^{(H)}$.*

Proof. Let v be an element of $V^{(S)}$, and let χ_v denote the corresponding S-semiinvariant. Then, the restriction of χ_v to H is a character of H. Therefore, we have $v \in V^{(H)}$. □

Example 2.3. Let B denote the Borel subgroup of upper triangular matrices in $GL_2(\mathbb{C})$. We will compute directly the B-semiinvariants in $R := \mathbb{C}[X, Y]$ for the action that we discussed in Example **??**. Since $R^{(B)} \subseteq R^{(T)}$, we will determine which T-semiinariants are B-semiinvariants. It follows from the argument of Example 2.1 that a B-semiinvariant is a monomial of the form $\alpha X^r Y^s$, where $\alpha \in \mathbb{C}^*$, $r, s \in \mathbb{N}$. Let A^{-1} be an element of B such that $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. Then the action of A on $\alpha X^r Y^s$ is given by

$$A \cdot \alpha X^r Y^s = \alpha a^r X^r (bX + dY)^s. \tag{2.4}$$

Unless $b = 0$ or $s = 0$, the right hand side of (2.4) cannot be a scaler multiple of $\alpha X^r Y^s$. Since for an arbitrary $A \in B$, the entry b can be different than zero, we must have $s = 0$. In other words, $\alpha X^r Y^s$ is a B-semiinvariant if and only if $s = 0$. This argument shows that

$$R^{(B)} = \{\alpha X^r : \ \alpha \in \mathbb{C}^*, \ r \in \mathbb{N}\}.$$

On the other hand, as we observed in Example 2.1, the T-semiinvariants is given by

$$R^{(T)} = \{\alpha X^r Y^s : \ \alpha \in \mathbb{C}^*, \ r, s \in \mathbb{N}\}.$$

2.1 The weight monoid of an action.

Let B be a Borel subgroup of a reductive group. In this section we will introduce some basic representation theoretic invariants of normal B-varieties. Hereafter, by writing $B = TU$, we implicitly assume that T is a maximal torus in B and U is the unipotent radical of B. We begin with reviewing some well-known facts.

The character groups of B and T are canonically isomorphic. This follows from the decomposition $B = TU$ and the fact that the character group of U is trivial. It follows that $\ddot{\mathbf{O}}(B) \cong \ddot{\mathbf{O}}(T)$ is a free abelian group of rank $r := \dim T$ with respect to pointwise multiplication of characters. Note that it is sometimes more convenient to switch to the additive notation for the characters.

Example 2.5. If B is the Borel subgroup of upper triangular matrices in $GL_2(\mathbb{C})$, then $\ddot{\mathbf{O}}(B)$ is freely generated by the characters $\lambda_i : B \to \mathbb{G}_m$ ($i \in \{1, 2\}$) where

$$\lambda_i\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = a_{ii}.$$

If we identify λ_1 (resp. λ_2) with the vector $(1, 0)$ (resp. by $(0, 1)$), then we see that the multiplication of the generators in $\ddot{\mathbf{O}}(B)$ corresponds to the addition of the corresponding integer vectors. Therefore, $\ddot{\mathbf{O}}(B)$ is isomorphic to $\mathbb{Z}^2$.

We proceed with the assumption that G is a connected reductive group such that B is a Borel subgroup of G. It is well-known that the set of all rational simple G-modules is in bijection with a submonoid of $\ddot{\mathbf{O}}(T)$. The geometric result that leads to this parametrization of simple modules is the following statement which follows from [3, Theorem 31.3]. Since its proof uses several important ideas that we will use in the sequel, we sketch its main arguments here.

Lemma 2.6. *Let G be a connected reductive group having $B = TU$ as a Borel subgroup. If V is a simple G-module, then the subspace of U-invariants in V, that is $V^U = \{v \in V : u \cdot v = v \text{ for all } u \in U\}$ contains a one-dimensional subspace on which B acts by a character, $\chi \in \ddot{\mathbf{O}}(B)$. Furthermore, V is uniquely determined by χ up to G-isomorphism.*

Proof. By the Lie-Kolchin Theorem we know that V contains a B-stable one-dimensional weight subspace, $V_\chi \subseteq V$, such that B acts on V_χ by a character χ, see [3, Theorem 17.6]. Let v be a nonzero vector (weight vector) in V_χ. We claim that $U \cdot v = v$. Towards a contradiction, assume otherwise that $U \cdot v \neq v$. Since $b \cdot v = \chi(b)v$ for every $b \in B$, the restriction χ to U gives a nontrivial character for U, which is absurd. This contradiction shows that U fixes the line V_χ point-wise. This proves our first assertion.

For our second assertion, we fix some terminology; let us call the vector v of the previous paragraph a *maximal vector* for V. Let B^- denote the unique Borel subgroup opposite to B. Let U^- denote the maximal unipotent subgroup of B^-. Let $U^- = \prod_{\alpha \in \mathtt{R}^-} U_\alpha$ and $U = \prod_{\alpha\in\mathtt{R}} U_\alpha$ be the root subgroup decompositions of these unipotent subgroups. It is well-known that for all $\alpha \in \mathtt{R}$, the root subgroup U_α maps V_χ into $\sum_{k\in\mathbb{Z}^+} V_{\chi+k\alpha}$, see [3, Proposition 27.3]. Let V' denote the G-submodule of V that is generated by v. Since V is simple, we have $V' = V$. The weights of V' are of the form $\chi - \sum_{\alpha\in\mathtt{R}^+} c_\alpha \alpha$, where $c_\alpha \in \mathbb{Z}^+$, see [3, Proposition 31.2] (which essentially follows from [3, Proposition 27.3]). In other words, χ is the highest weight with respect to the partial order $\preceq$. Let us assume that there exists another 'maximal vector' $w \in V \backslash V_\chi$ so that $U \cdot w = w$ and on which B acts by a weight μ. Then by repeating the previous argument for w, we see that the span of $\chi \precnsim \mu$ which contradicts with the maximality of the weight χ. It follows from the uniqueness of the maximal vector in a simple module that if two simple G-modules V and W possess the same highest weight χ then they must be isomorphic. This finishes the proof of our second assertion. □

Let us strengthen the statement about the space of unipotent invariants in Lemma 2.6.

Corollary 2.7. *Let G be a connected reductive group having $B = TU$ as a Borel subgroup. If V is a simple G-module, then V^U is one-dimensional.*

Proof. Let v and w be two non-proportional vectors in V such that $U \cdot w = w$ and $U \cdot v = v$. Without loss of generality we assume that v is a highest weight vector of weight χ. Then V decomposes as $V' \oplus V_\chi$, where w is an element of V'. Since V' is spanned by the weight spaces V_μ, where $\mu \precnsim \chi$, it is T-stable. In particular, we see that $B \cdot w \subseteq V'$. We now observe that the B^- orbit of w cannot be entirely contained in V'. Otherwise, since B^-B is dense in G, we would have the span of $G \cdot w$ entirely contained in V', which is absurd. Now let $b' \in B^-$ be such that $b'w = v$. Then we have $w = u \cdot w = ub'^{-1}v$ for every $u \in U$. Since both of the spans of B^-v and B^-w are dense in V, the equalities $ub'^{-1}v = b'^{-1}v$ (for all $u \in U$) imply that $b' \in T$. But then the span of w in V is a line on which B acts by the character χ. This contradicts with our assumption that w is a weight vector of weight μ such that $\mu \neq \chi$. □

Notation 2.8. Let G, B, T, U be as in Lemma 2.6. Hereafter, the group $\ddot{\mathbf{O}}(B)$ will be called the *weight lattice* and its elements will be called the *weights*. The highest weight of a simple

G-module as in the proof of Lemma 2.6 is called a *dominant weight* of (G, B, T). The set of all dominant weights will be denoted by $\ddot{\mathbf{O}}(B)^+$. In other words, we set

$$\ddot{\mathbf{O}}(B)^+ := \{\chi \in \ddot{\mathbf{O}}(B) : \ \chi \text{ is the highest weight of a simple } G\text{-module}\}.$$

This set has the structure of a submonoid in $\ddot{\mathbf{O}}(B)$.

Now let X be an irreducible affine G-variety. Then $k(X)$ is the total quotient field of the ring of regular functions $k[X]$. Clearly, $k(X)$ has the structure of a G-module. For the analysis of this representation, the following fact, which was originally proved separately by Hadziev and Grosshans, is essential; it will enable us to define some important combinatorial invariants for the action $G : X$.

Lemma 2.9. *(Hadziev-Grosshans Theorem) The algebra of U-invariant regular functions $k[X]^U$ is a finitely generated k-algebra.*

Remark 2.10. Hadziev-Grosshans Theorem allows us to reduce the analyses of the coordinate rings of many affine varieties to the coordinate rings of toric varieties.

One proof of Hadziev-Grosshans theorem relies on a useful method that is known as the "transfer principle" from a closed subgroup. In our case this principle says that

$$k[X]^U \cong k[X \times G/U]^G \cong (k[X] \otimes k[G]^U)^G.$$

Since G is reductive, the k-algebra of G-invariants of a finitely generated algebra is finitely generated. Note that $k[X]$ is already a finitely generated k-algebra. Therefore, to prove the Hadziev-Grosshans theorem, it suffices to prove the finite generation of $k[G]^U$ only. In the literature, this invariant ring is sometimes called the *flag algebra of G*. It is naturally identified with the coordinate ring of G/U. We note here without proof that for arbitrary affine algebraic group G and a maximal unipotent subgroup $U \leqslant G$, the invariants ring $k[G/U] = k[G]^U$ is finitely generated; see the paragraph after [2, Theorem 5.6].

Lemma 2.11. *Let X be an irreducible quasi-affine S-variety, where S is a connected solvable affine algebraic group. Then any S-invariant in $k(X)$ is the quotient of two S-semiinvariants in $k[X]$ with the same weight. Furthermore, if S is a unipotent group, then $k(X)^S$ is isomorphic to the total quotient field of $k[X]^S$.*

Proof. Let $f = p/q$ be a S-invariant in $k(X)$, where p and q are regular functions. Since any element of the coordinate ring $k[X]$ is contained in a finite dimensional S-stable subspace, we see that the subspace $V := \operatorname{span}\{s \cdot q : \ s \in S\} \subset k[X]$ is finite dimensional. Then by the Lie-Kolchin Theorem, there is a nonzero semiinvariant $h := \sum_{i \in I} c_i s_i q \in V$, where I is a finite set of indices. Then for every $i \in I$ we have $f = s_i \cdot f = \frac{s_i \cdot p}{s_i \cdot q}$. Equivalently, we have the following equations: $(c_i s_i \cdot q) f = (c_i s_i \cdot p)$ for $i \in \{1, \dots, |I|\}$. It follows that f is also equal to $f = \frac{\sum_{i \in I} c_i s_i \cdot p}{\sum_{i \in I} c_i s_i \cdot q}$. But since the denominator is a semiinvariant, and f is invariant, the numerator must also be a semiinvariant with the same weight as the denominator. This finishes the proof of our first assertion. For the proof of our second assertion, it suffices to mention that every semiinvariant of a unipotent group action is an invariant since such a group does not possess any nontrivial characters. $\square$

Remark 2.12. 1. The arguments that we used in the proof of Lemma 2.11 can be used for proving the following statement: any S-semiinvariant in $k(X)$ is the quotient of two S-semiinvariants in $k[X]$.

2. The first assertion of Lemma 2.11 holds true even without the assumption on solvability as long as X is a "factorial" quasi-affine variety. Here, X is *factorial* means that the coordinate ring $k[X]$ is a unique factorization domain.

3. Let U be a unipotent group. If χ is a character of U, then it is the constant function; $\chi(u) = 1$ for every $u \in U$. Therefore, for every rational U-module M, we have $M^{(U)} = M^U$.

We are now going to focus on the B-semiinvariants of a quasi-affine irreducible B-variety X, where B is a Borel subgroup of some connected reductive group. Although initially we do not need to require that X admits an algebraic G-action, let us first assume that this is the case. Recall that the coordinate ring of X is a locally finite representation of G which decomposes as a sum of finite dimensional irreducible G-modules. We further note that the B-highest weights of this representation is closed under addition. In other words, the B-semiinvariants that appear in $k[X]$ all belong to the monoid of dominant weights in the character group $\ddot{\mathbf{O}}(B)$. Since the monoid of dominant weights is a strictly convex rational polyhedral cone in $\ddot{\mathbf{O}}(B) \otimes_{\mathbb{Q}} \mathbb{R}$, the inverse weight $-\chi$ (in additive notation) does not appear as the weight of a regular function in $k[X]$. However, if a nonzero element $f \in k[X]$ is a B-semiinvariant of weight χ, then certainly the rational function $1/f \in k(X)$ is a B-semiinvariant of weight $-\chi$ in $k(X)$. Also, since the pointwise multiplication of two B-semiinvariants in $k(X)^{(B)}$ is another B-semiinvariant in $k(X)$, we see that $k(X)^{(B)}$ is a subgroup of the multiplicative group of the field $k(X)$, and the weights that correspond to the elements of $k(X)^{(B)}$ form a subgroup of $\ddot{\mathbf{O}}(B)$. In other words, the monoid homomorphism $k[X]^{(B)} \to \ddot{\mathbf{O}}(B)^+$, $f \mapsto \chi_f$ extends to a group homomorphism $k(X)^{(B)} \to \ddot{\mathbf{O}}(B)$. We are now ready to ramp up our Notation 2.8.

Definition 2.13. Let X be an irreducible affine (or, quasi-affine) G-variety. The map

$$\begin{aligned} \mathtt{r} : k(X)^{(B)} &\longrightarrow \ddot{\mathbf{O}}(B) \\ f &\longmapsto \chi_f \end{aligned}$$

is a $\mathbb{Z}$-module homomorphism. We call the image of $\mathtt{r}$, that is,

$$\ddot{\mathbf{O}}(X) := \mathtt{r}(k(X)^{(B)}),$$

the *weight lattice of* X. (Some authors call $\ddot{\mathbf{O}}(X)$ the *rank group of* X.) Since $\ddot{\mathbf{O}}(B)$ is a free abelian group, and $\ddot{\mathbf{O}}(X)$ is a subgroup, $\ddot{\mathbf{O}}(X)$ is a free abelian group as well. Its $\mathbb{Z}$-module rank, which we denote by $rk_G(X)$, will be called the *rank of the action* $G : X$.

The above definition of a weight lattice of a variety depends on the coordinate ring. Therefore, for a projective G-variety, some care is need for defining a similar object. In our next example we show that already for an important class of homogeneous spaces, the notion of a weight lattice is not helpful. Nevertheless, we will see later that this class of homogeneous spaces is actually singled out by the fact that their weight lattices are trivial.

Example 2.14. Let P be a parabolic subgroup. We fix a Borel subgroup B in P, and let T be a maximal torus contained in B. By the Bruhat-Chevalley decomposition, we know that B has an open orbit in G/P. Let W denote the Weyl group $N_G(T)/T$, and let W_P denote its subgroup corresponding to the parabolic subgroup P. Let O denote the open orbit of B in G/P. It is well-known that O contains a unique T-fixed point, that is, $z_0 := w_0^P B$, where w_0^P is the longest element of W/W_P. Then we have $O = B \cdot z_0 \cong B/B_{z_0}$, where B_{z_0} is the stabilizer of z_0 in B. Now let $f(x)$ be a B-semiinvariant from $k(O)$, and let χ_f denote the corresponding character. For $b \in B$, let $t \in T$ and $u \in U$ be such that $b = tu$. Then we have $b^{-1} \cdot z_0 = u^{-1} \cdot z_0$. It follows that $b \cdot f(z_0) = f(b^{-1} \cdot z_0) = f(u^{-1} \cdot z_0) = \chi_f(u) \cdot f(z_0) = f(z_0)$. At the same time, as b varies in B, $b^{-1} z_0$ varies over all points of O. In other words, we have $f(O) = f(z_0)$. Since O is open in G/P, we see that $f(x)$ is a constant rational function, hence, $\chi_f = 1$. In conclusion, we see that the weight lattice of G/P is trivial.

The discouraging situation (?) of the previous example is remedied by passing to the affine cones over. Let us demonstrate this by the simplest example in our hands.

Example 2.15. Let X denote $SL_2/B \cong \mathbb{P}^1$. We denote by $\pi : L \to X$ the hyperplane bundle $\mathcal{O}_X(1)$ on X. Clearly, SL_2 acts on the total space of L, and π is a SL_2-equivariant morphism. In other words, L is a G-linearized line bundle for the simply connected semisimple group $G := SL_2$. Let us define the *section ring of* L by

$$R(X, L) := \bigoplus_{n \geqslant 0} H^0(X, L^n). \tag{2.16}$$

Notice that this section ring is nothing but the (graded) coordinate ring of the total space of the dual line bundle $L^\vee$. The fiber of $L^\vee$ at a point $p \in X$ is the line in k^2 passing through the origin and the point p. By definition, the *affine cone associated with the line bundle* $L \to X$ is the affine variety whose coordinate ring is $R(X, L)$. Let $\tilde{X}$ denote this cone. Clearly, in our case, we have $\tilde{X} \cong k^2$. Let us have a closer look at the $SL_2 \times \mathbb{G}_m$-module structure on the coordinate ring $R(X, L) \cong k[x, y]$. The $\mathbb{G}_m$-action gives the usual grading on the polynomial ring $k[x, y]$. In particular, the n-th graded part, that is $H^0(X, L^n)$, is isomorphic to the space of homogeneous polynomials of degree n in $k[x, y]$. Furthermore, for every $n \in \mathbb{N}$, the subspace $H^0(X, L^n)$ is an SL_2-module. It is well-known that this is an irreducible SL_2-module of highest weight $\chi_n := n + 1$, $n \in \mathbb{N}$. Since the semisimple part of $SL_2 \times \mathbb{G}_m$ is SL_2, the weight χ_n is also the highest weight of $H^0(X, L^n)$ as an irreducible rational representation of the reductive group $SL_2 \times \mathbb{G}_m$. In conclusion, the weight lattice of the line bundle $L^\vee \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ as an $SL_2 \times \mathbb{G}_m$-variety is given by

$$\ddot{\mathbf{O}}(\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{Z}.$$

Remark 2.17. It is not difficult to see that the homogeneous space SL_2/U, where U is the unipotent radical of B, is isomorphic to $k^2 \backslash \{0\}$. Thus, the weight lattice of SL_2/U as an SL_2-variety is identical to the weight lattice of k^2 as an $SL_2 \times \mathbb{G}_m$-variety.

2.2 Summary.

Let X be an irreducible quasi-affine G-variety, where G is a connected reductive group. Then coordinate ring $k[X]$ has a natural decomposition into G-submodules as follows:

$$k[X] \cong \bigoplus_{\lambda \in \ddot{\mathbf{O}}(B)^+} (k[X]_\lambda^{(B)}) \otimes V(\lambda),$$

where $V(\lambda)$ is the simple G-module with highest weight λ, and $k[X]_\lambda^{(B)}$ is the G-module

$$k[X]_\lambda^{(B)} := \mathrm{Hom}_G(V(\lambda), k[X]),$$

that is the space of all G-equivariant morphisms from $V(\lambda)$ into $k[X]$. In characteristic 0, $\dim k[X]_\lambda^{(B)}$ is equal to the number of occurrence of $V(\lambda)$ in $k[X]$. To justify the notation, let us note the well-known fact that, for every λ and μ from $\ddot{\mathbf{O}}(B)^+$, the simple module $V(\lambda+\mu)$ occurs exactly once in the tensor product $V(\lambda) \otimes V(\mu)$. Therefore, the B-semiinvariants of simple G-submodules of $k[X]$ form a subsemigroup in $\ddot{\mathbf{O}}(B)$.

Definition 2.18. Let X be an irreducible affine (or, quasi-affine) G-variety. The *weight monoid of* $G : X$, denoted $\ddot{\mathbf{O}}(X)^+$, is the following submonoid of $\ddot{\mathbf{O}}(X)$:

$$\ddot{\mathbf{O}}(X)^+ := \{\lambda \in \ddot{\mathbf{O}}(X) :\ V(\lambda) \text{ occurs in } k[X]\}.$$

We have learned from the discussion preceding Remark 2.12 that the weight monoid $\ddot{\mathbf{O}}(X)^+$ generates the weight lattice $\ddot{\mathbf{O}}(X)$. Furthermore, we observed that $\ddot{\mathbf{O}}(X)^+$ spans a rational polyhedral cone in $\ddot{\mathbf{O}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote the $\mathbb{Q}$-vector space defined by the dual of $\ddot{\mathbf{O}}(X)$ as follows:

$$\mathbf{H}\ddot{\mathbf{O}}(X) := \mathrm{Hom}_{\mathbb{Z}}(\ddot{\mathbf{O}}(X), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathbb{Z}}(\ddot{\mathbf{O}}(X), \mathbb{Q}).$$

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