R : root system of (G, B, T)

$\cup$

S : set of simple roots determined by B.

$\|$

$\{\alpha_1, \cdots, \alpha_\ell\}$

ℓ ← semisimple rank of G = $\dim T_0$

T_0 ← max. torus of the commutator (G,G).

$\downarrow$

$\{\varpi_1, \cdots, \varpi_\ell\}$: fundamental dominant weights

$$\langle \varpi_i, \alpha_j^\vee \rangle := \delta_{ij} \qquad \alpha_j^\vee = \frac{2\alpha_j}{(\alpha_j, \alpha_j)} \in \ddot{O}(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$$

The $\mathbb{Z}^+$-span of $\varpi_1, \cdots, \varpi_\ell$ in $\ddot{O}(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called the dominant chamber.

$$\left(\quad R = R^+ \sqcup R^- \quad \right)$$

$R^+ \supseteq S$, S is a basis for $\ddot{O}(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The Weyl group of R $\left(= \langle s_\alpha \mid \alpha \in R \rangle \right)$ acts on the $\ddot{O}(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the dominant chamber is the domain of transitivity for this action.

HW: $G_0 := SL_3(\mathbb{C}) \supseteq T_0$ = diagonal maximal torus

$B_0 \subseteq G_0$ upper triangular matrices.

① Calculate the "weight lattice" = $\mathbb{Z}$-span of ϖ_1, ϖ_2

② Calculate the W-action on the dominant chamber.

$W = \langle s_{\alpha_1}, s_{\alpha_2} \rangle = \{ e, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} \}$

Let V be a finite dim'l, rational, irreducible representation of G.

Fact: The Borel subgroup $\boxed{B = Q.T.} \subseteq G$ ~~fixes~~ stabilizer a line l in V. (Lie-Kolchin Theorem).

$\Rightarrow$ B acts on l by a character χ.

$\forall\ v \in l^*,\ b \in B, \quad b \cdot v = \underline{\underline{\chi(b)}}\, v$

Fact: Unipotent groups donot have any non-trivial characters.

Therefore U fixes this line l.

$\left(\ddot{O}(B) = \check{O}(T) \right)$

- v is called a B-semiinvariant (if $v \neq 0$).

$G \cdot v \subsetneq V$ spans V.

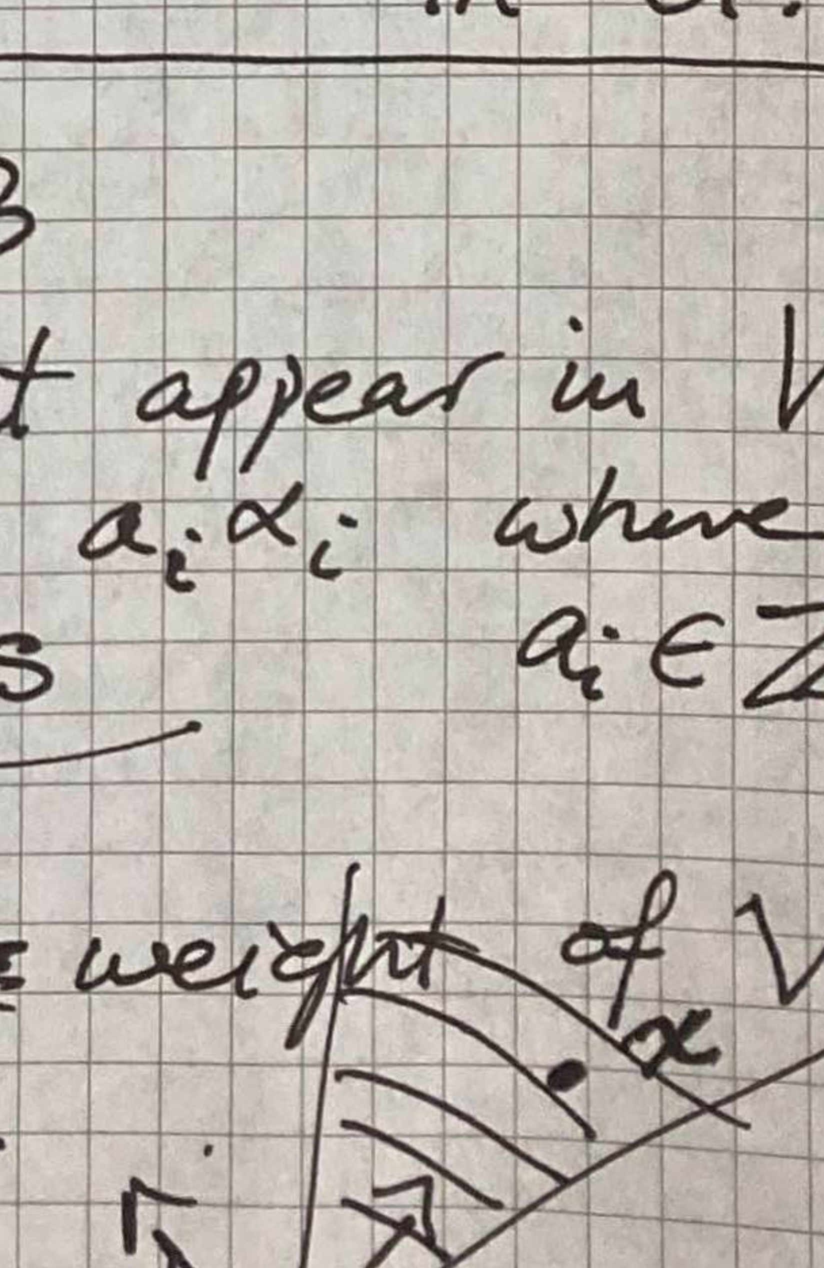
$\boxed{\text{Bruhat-Chevalley: } G = \bigsqcup_{w \in W} U \dot{w} B} \Rightarrow \boxed{U \cdot \bar{B} \text{ is dense in } G.}$

unipotent radical of B

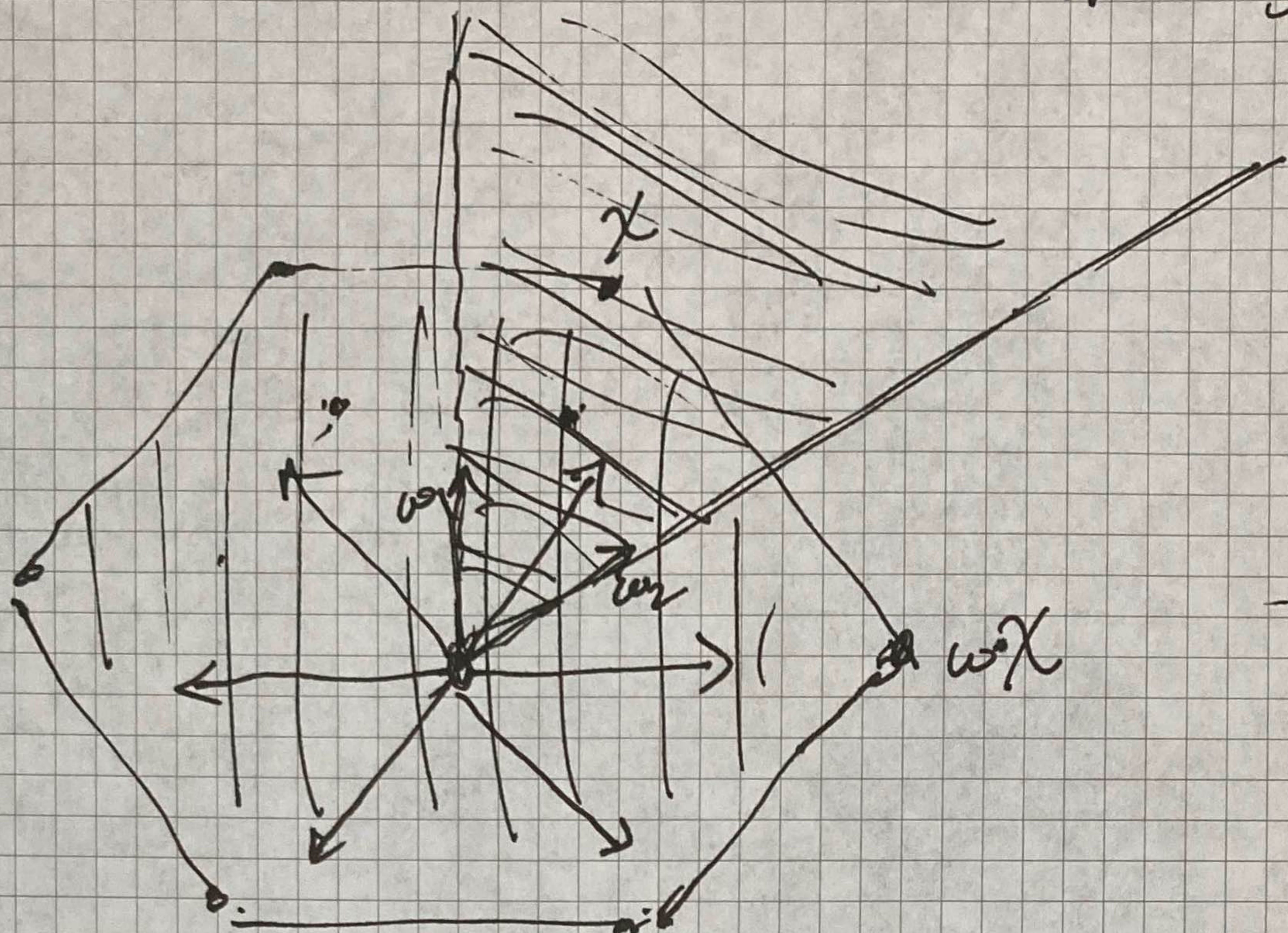
The other B-weights that appear in V are all of the form $\underbrace{\chi - \sum_{\alpha_i \in S} a_i \alpha_i}$ where $a_i \in \mathbb{Z}_{\geq 0}$.

χ is the highest ~~wt~~ weight of V.

$\underline{V = V(\chi)}$.



$W = S_3$



$V = V(\chi)$

dominant weight

$$\boxed{\chi = \sum_{i=1}^{l} a_i \omega_i}$$

$\ddot{O}(B)^{+}$ = monoid of dominant weights

Recall: Bruhat–Chevalley decomp.

$$G = \bigsqcup_{w\in W} BwB$$

if we view G as a $\underset{\displaystyle B\times B}{\overset{\displaystyle G\times G}{\cup\!\!\shortmid}}$-variety,

then the B.C. decomp. shows that G admits an open $B\times B$-orbit.

fact: The orbits of a solvable group on affine variety are affine.

Definition: An irreducible G-variety X is called a spherical G-variety (connected reductive G) if a Borel subgroup B of G has an open orbit in X.

$\underset{(\mathrm{char}\,k=0)}{\Longrightarrow}$ G is a spherical affine $G\times G$-variety.

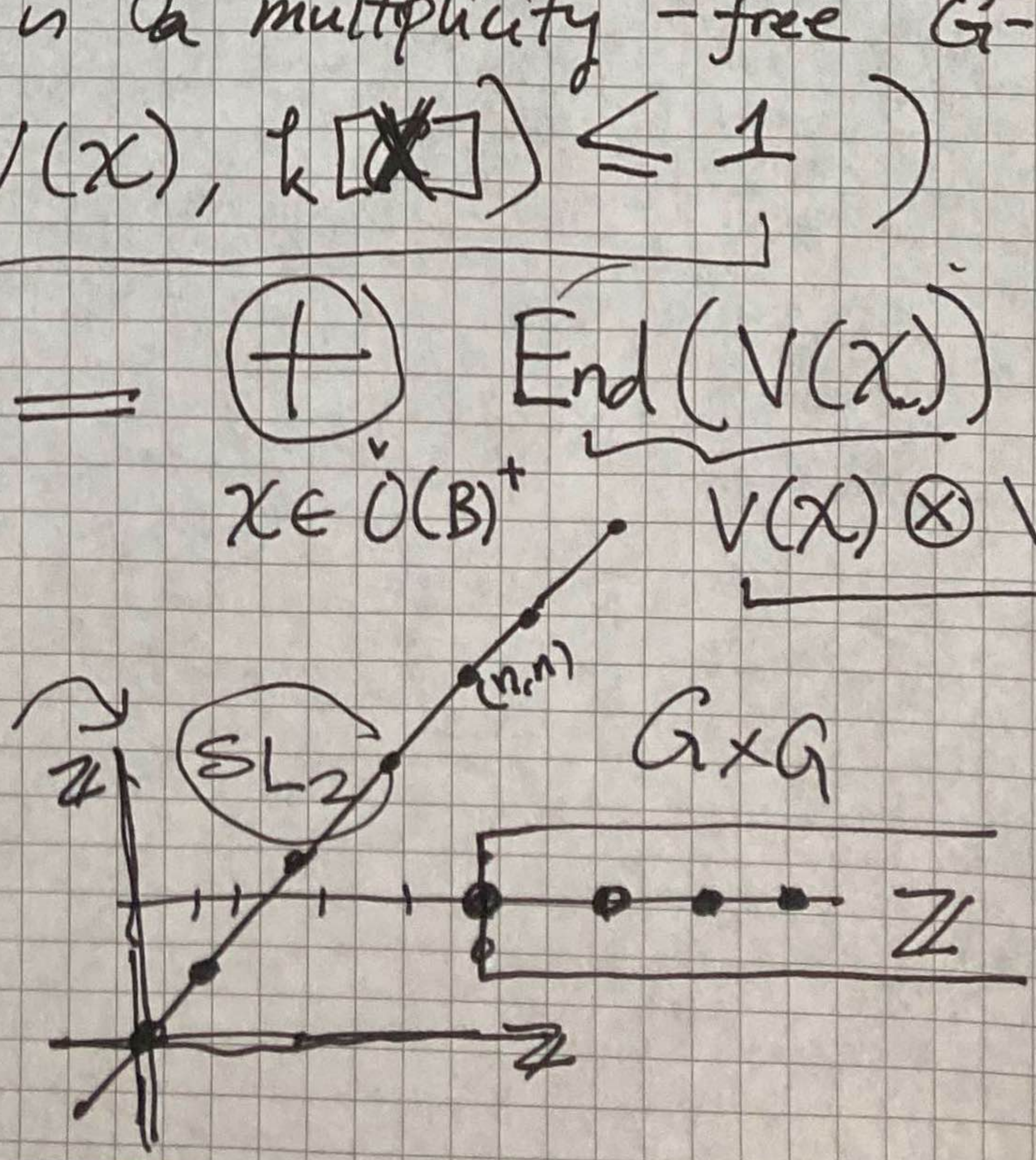
Fact: The coordinate ring of an affine spherical G-variety is a multiplicity-free G-module.

$$\left(\underbrace{\dim \mathrm{Hom}_G(V(\chi), k[X])} \le 1 \right)$$

$$k[G] = \bigoplus_{\chi \in \check{O}(B)^+} \mathrm{End}(V(\chi)) \qquad \underbrace{V(\chi)\otimes V(\chi)^*}$$

$G\times G = SL_2 \times SL_2$

$k[SL_2] \supset$



$\mathbb{Z} = \check{O}(B_0)$

B_0: Borel of SL_2

Let X be an irreducible affine G-variety.

$k[X] \rightsquigarrow k(X)$ = field of rational functions on X.

$\underbrace{k[X]}_{\text{locally finite rep. of } G}$

The B-semiinvariants in $k[X]$ tell us about the irr. sub G-modules.

$f \in k[X]^{(B)} \rightsquigarrow \chi_f$: the character of B ~~...~~ by which B acts on f.

= "the weight of f"

In general, $\frac{1}{f} \notin k[X]^{(B)}$.

notice that $-\chi_f$ is the weight of $\frac{1}{f}$.

So, looking at the B-semiinvariants of $k(X)$ is natural :

$$\begin{aligned} r : k(X)^{(B)} &\longrightarrow \check{O}(B) \\ \text{The map} \qquad \rotatebox[origin=c]{90}{$\in$} \qquad & \qquad \quad \rotatebox[origin=c]{90}{$\in$} \\ f &\longmapsto \chi_f \end{aligned}$$

is a group homom.

The image of r is called the weight lattice of X.

Since $\check{O}(B)$ is a lattice, $r(k(X)^{(B)}) =: \check{O}(X)$. is an honest lattice.

Its rank is called the rank of X.

Consider $\underset{\sim}{G/B}$ as a G-variety.
flag variety.

(This is a spherical variety because of the Bruhat-Chevalley decomposition.)

The $\mathrm{rk}_G(G/B) = 0$.

More generally, if P is any closed subgroup s.t. $B \subseteq P$, then $\mathrm{rk}_G(G/P) = 0$.

<u>Theorem</u>: if the rank (and <u>complexity</u>) of a G-variety X is 0, then X is of the form G/P.

$SL_2 \times SL_2$ - weights that appear in SL_2 are $(n,n) \in \mathbb{N} \times \mathbb{N}$.

$SL_2 \times SL_2 \curvearrowright SL_2$ $\qquad (SL_2 \times SL_2)/_{\mathrm{diag}} SL_2 \cong SL_2$

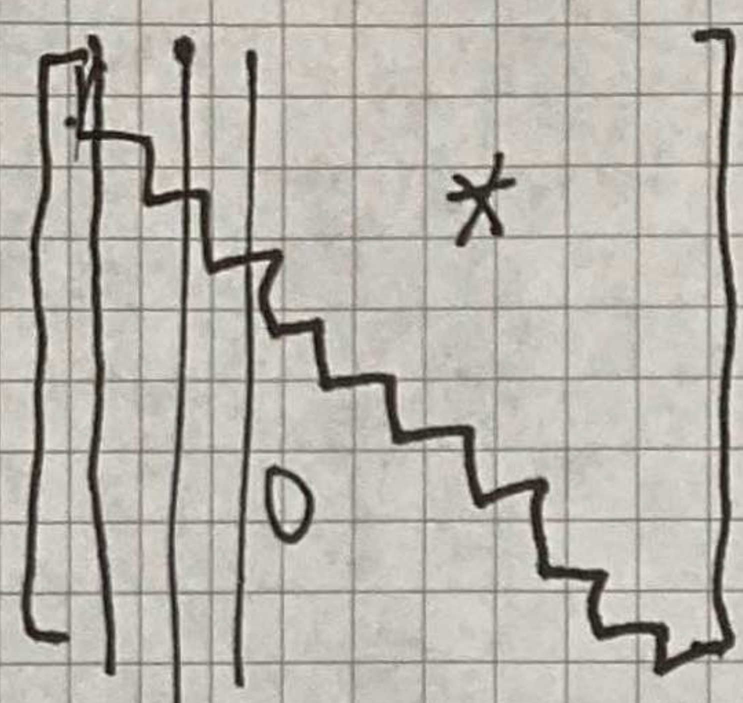
$(g,h) \cdot x = g x h^{-1}$

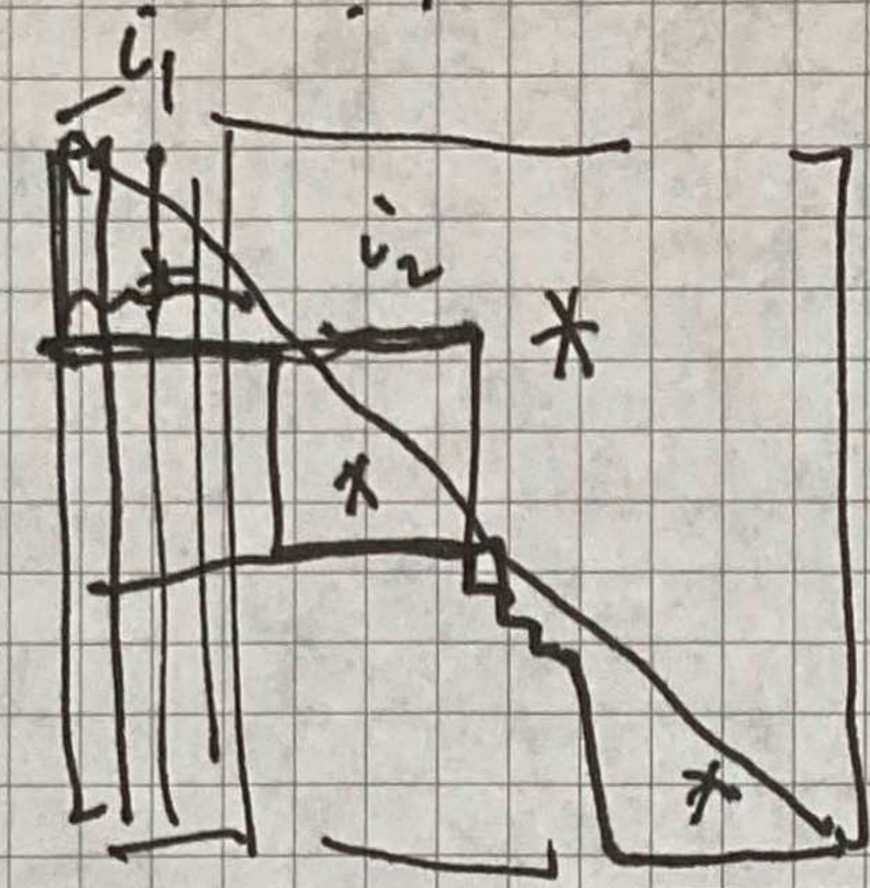
$\underbrace{GL_n/B}$ $= \{ (\underset{\{0\}}{0} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \underset{\mathbb{C}^n}{V_n}) \mid \dim V_i = i \}$

$\boxed{P \supset B}$ $\quad P = P(I)$, $\underline{I \subset S}$.

$GL_n/P = \{ (0 \subseteq V_{i_1} \subseteq \cdots \subseteq V_{i_k} \subseteq V) \mid \{i_1, \ldots, i_k\}$ is determined by $I \}$

GL_n/B $\quad\quad$ GL_n/P





$$k[G] = \boxed{\bigoplus_{\chi \in \check{O}(B)^+} \mathrm{End}(V(\chi))}$$

G is a $\underline{\underline{G \times G}}$-variety.

$$G/H \stackrel{?}{=} \bigoplus_{\chi \in \check{O}(B)^+} V(\chi)$$

as a G-module.

$H = U$

$$T \longrightarrow G/U \longrightarrow G/B$$

$$k[G/U] = \boxed{k[G]^U} \qquad U \times U$$

quasi-affine

✓ [Is it true that for any affine alg. irreducible G-variety X, $\underline{k[X]^U}$ is finitely generated.