$\bar{B} r B = w_0 B w_0 r B$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$H \subseteq G$ closed subgroup.

Recall: Let G/H be a spherical homogeneous space. $G \curvearrowright G/H$, $G \supseteq B$ has an open orbit.

$$k(G/H)^{(B)} = \left\{ f \in k(G/H) \;\middle|\; \begin{array}{c} b.f = \chi(b) f \\ \forall b \in B \end{array} \right\}$$

eigenvectors of the B-action on $k(G/H)$.

$$k(G/H)^{(B)} \longrightarrow \ddot{O}(B)$$
$$f \longmapsto \chi_f$$

Note that every B-invariant of $k(G/H)$ is a B-semiinvariant.

Also note that

G/H is spherical $\Longleftrightarrow$ $k(G/H)^B = k$
(Rosenlicht)

$\Rightarrow$ We have the following ses:

$$0 \to k^* \to k(G/B)^{(B)} \longrightarrow \ddot{O}(B) \to 0$$

Note: if D is a G-stable prime divisor, then its v_D is G-invariant.

if $Z \subset X$ a closed G-stable irr. subvariety, then $Bl_Z X$ has a G-stable prime divisor that can be used for defining the G-invariant valuation of Z.

$\mathcal{V}(G/H) :=$ the set of G-valuations.
v is a G-invariant valuation and it comes from a divisor in some model of $k(G/H)$.

$v \in \mathcal{V}(G/H)$, $f \in k(G/H)^{(B)} \subset \check{O}(B)$

$\rho_v : f \longmapsto v(f)$

This way we define a linear functional to v on $\check{O}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$

$$\rho_v \in \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\check{O}(B), \mathbb{Z}) \otimes \mathbb{Q}}_{\mathrm{Hom}_{\mathbb{Z}}(\check{O}(B), \mathbb{Q})} =: \underline{\mathrm{H}\check{O}(X)}$$

Valuations of $k(G/H)$.

Let v be a discrete valuation on $k(G/H)$

$$v : k(G/H) \longrightarrow \mathbb{Q} \cup \{\infty\}$$

- $v(fg) = v(f) + v(g) \quad \forall\, f,g \in k(G/H)^{\times}$
- $v(f+g) \geq \min\{v(f), v(g)\} \quad \forall\, f,g \in k(G/H)^{\times}$, $f+g$
- $v(0) = \infty$

v is called a G-invariant valuation

if $v(g \cdot x) = v(x) \quad \forall\, g \in G,\ x \in G/H$.

Recall: if X is a normal variety, then for every prime divisor D in X, the local ring $\mathcal{O}_{D,X}$ of D is a DVR.

This means that $\mathcal{O}_{D,X}$ is of the form

$$\mathcal{O}_v = \{ \tfrac{f}{g} \in k(X) \mid v(\tfrac{f}{g}) \geq 0 \}$$ for some discrete valuation v of $k(\cancel{X})$.

If D is a prime divisor, we can define

$f \in k(X)$, $v_D(f) :=$ order of vanishing of f on D.

X is called normal if $\forall\, p \in X$ the local ring $\mathcal{O}_{p,X}$ is an integrally closed integral domain.

$$\left.\begin{array}{l} e : \mathcal{V}(G/H) \longrightarrow H\check{\dot{O}}(G/H) \\ \qquad v \longmapsto e_v \\ \text{is injective.} \end{array}\right\} \text{Luna-Vust `Knop'}$$

An irreducible normal G-variety X is called an <u>embedding</u> of G/H if there is an open G-orbit that is isomorphic to G/H.

An embedding X is called a simple embedding of G/H if X has a unique closed G-orbit.

<u>Ex</u>: Let X an affine toric variety. Then X is a simple embedding of its open orbit.

[Let T be a complex (split) torus ($\mathbb{C}^* \times \cdots \times \mathbb{C}^* \cong T$). A normal T-variety X is called a toric variety if there is an open T-orbit.]

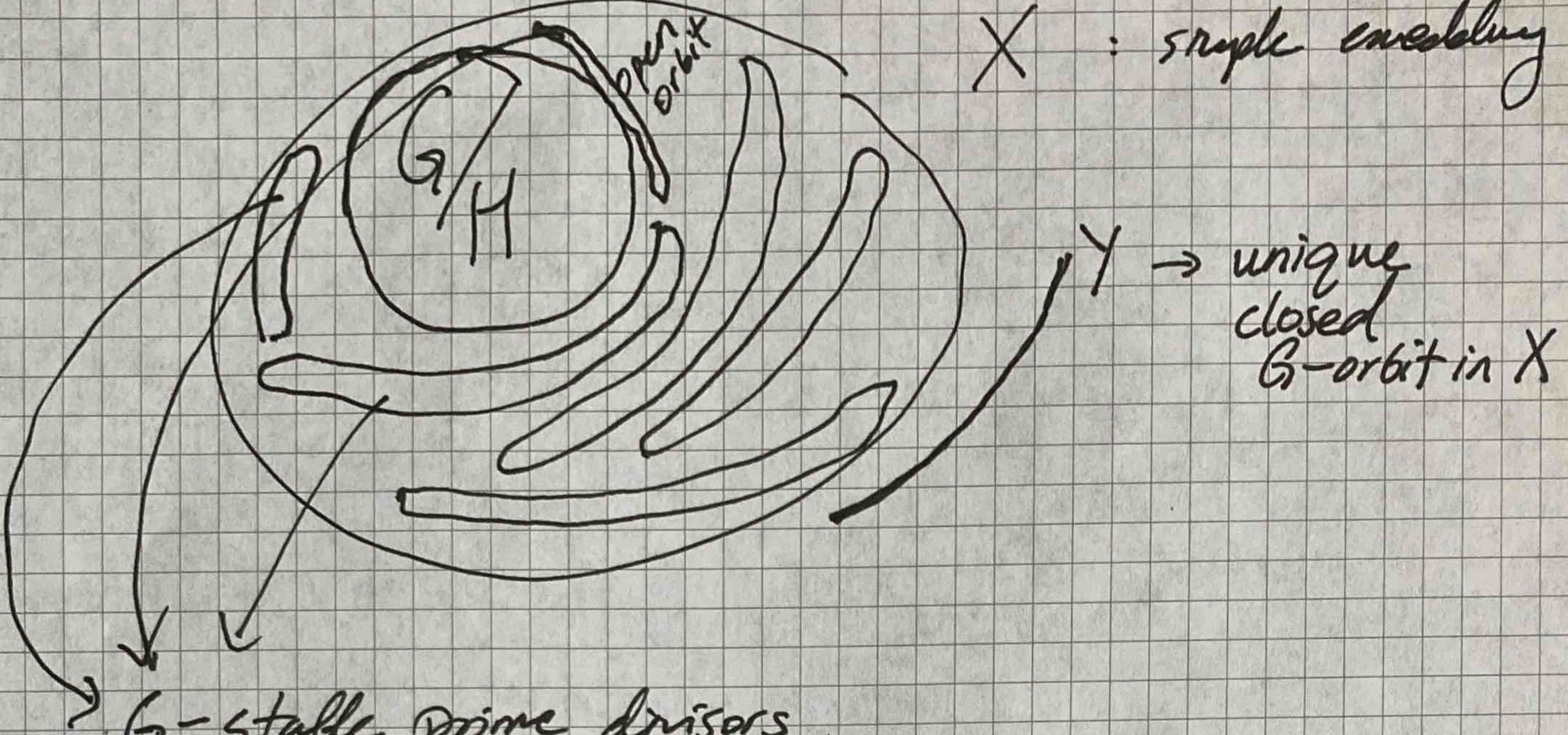
Toric varieties are parametrized by (strongly convex, <u>rational</u>, <u>polyhedral</u>) fans.

the fan of X

<u>note</u>: A toric variety $X = X(\Sigma)$ is complete if and only if Σ is a <u>complete fan</u>.

$$\underbrace{\bigcup_{\sigma \in \Sigma} \operatorname{supp}(\sigma) = \check{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}}$$

Classification of $\boxed{\text{simple}}$ embeddings of G/H :



G-stable prime divisors.

$\underbrace{\qquad}$ each of these gives us a G-valuation in $H\check{O}(G/H)$

In G/H we have an open B-orbit, and we have the B-stable prime divisors.

$\mathcal{D}(X)$: the set of B-stable prime divisors in X

$$\mathcal{F}(X) = \{ D \in \mathcal{D}(X) \mid D \cap \underset{G/H}{\text{open orbit}} \neq \phi \}$$

$$\mathcal{D}_Y(X) := \{ D \in \mathcal{D}(X) \mid Y \subseteq D \}$$

$$\mathcal{F}_Y(X) := \mathcal{D}_Y(X) \cap \mathcal{F}(X).$$

colors of X.

Define $f : \mathcal{F}(X) \longrightarrow H\check{O}(G/H)$

$$D \longmapsto \left(f \mapsto v_D(f) \right)$$

$$\mathcal{C}(X) := \langle \underbrace{f(D)}, \ e(E) \mid D \in \mathcal{F}_Y(X), \ \nu_E \in \mathcal{V}(G/H) \rangle \subseteq \overset{\vee}{H\ddot{O}}(G/H).$$

where E is a prime G-stable divisor.

$$\underset{\text{simple embedding}}{X} \rightsquigarrow (\underbrace{\mathcal{C}(X)}_{\substack{\text{polyhedral}\\ \text{cone in}\\ H\ddot{O}(G/H)}}, \underbrace{\mathcal{F}_Y(X)}_{\text{colors}}) \quad \underline{\text{colored cone of } X}$$

Definition:

A strictly convex colored cone of G/H in a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F} \subset \mathcal{D}$ (B-stable prime divisors) s.t.

1. $\mathcal{C}$ is generated by $f(\mathcal{F})$ and a finite number of G-valuations on $k(G/H)$.

2. the relative interior of $\mathcal{C}$ intersects $e(\mathcal{V}(G/H))$

3. $\mathcal{C}$ is strictly convex and $0 \notin f(\mathcal{F})$.

Thm: The simple embeddings of G/H are in 1-1 correspondence with the strictly convex colored cones of G/H.

$$\mathrm{Mat}_n(\mathbb{C}) \supset GL_n(\mathbb{C})$$

$GL_n \times GL_n \curvearrowright \mathrm{Mat}_n$

$(g, h) \cdot x = g x h^{-1}$.

the open orbit is GL_n.

$B \times B^- \subset GL_n \times GL_n$ has an open orbit.

(LU-decomposition)

$GL_n \times GL_n$-orbits in Mat_n are identified by the ranks of the matrices they contain.

GL_n

|

rank $n-1$ matrices $\longrightarrow$ $\boxed{\text{the unique } GL_n\text{-stable prime divisor.}}$

|

rank $n-2$ matrices

|

⋮

|

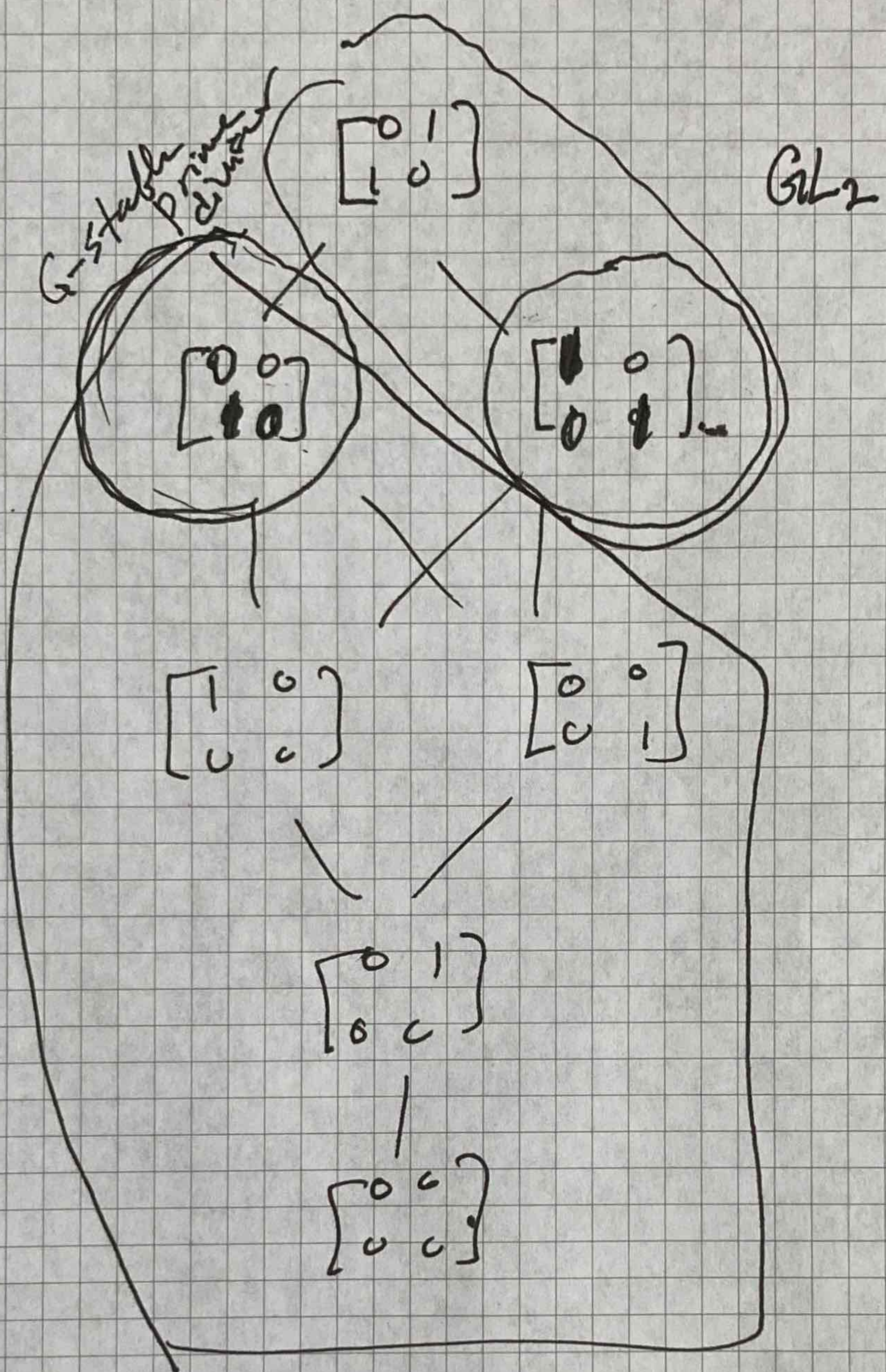
rank 1 matrices

|

$\{0\}$

For $n=2$

there are 7 $B \times B$-orbits



G

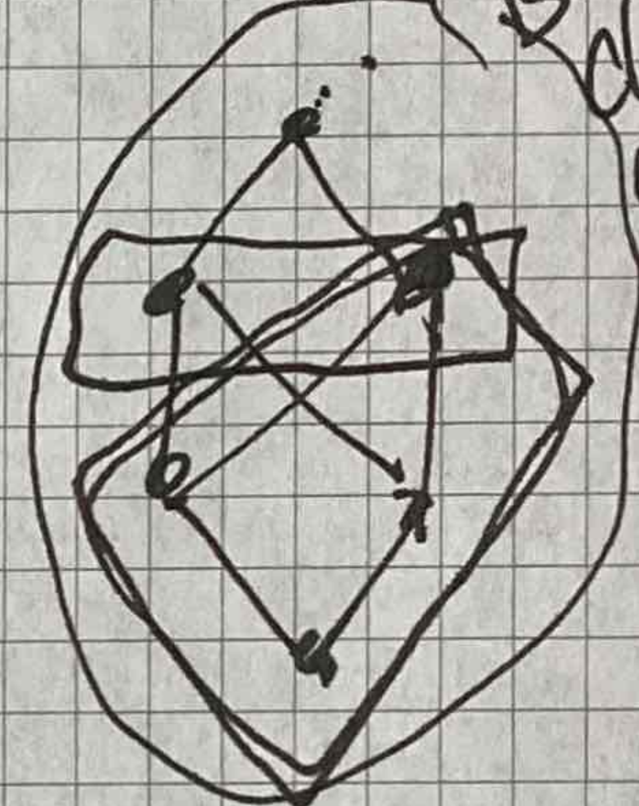
Mat_2

GL_n/B

G/B

S_n

S_3



Bruhat chevalley order on S_3