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Mirror symmetry and T-duality in the complement of an anticanonical divisor

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Abstract. We study the geometry of complexified moduli spaces of special Lagrangian submanifolds in the complement of an anticanonical divisor in a compact Kähler manifold. In particular, we explore the connections between T-duality and mirror symmetry in concrete examples, and show how quantum corrections arise in this context.

1. Introduction

The Strominger-Yau-Zaslow conjecture [29] asserts that the mirror of a Calabi-Yau manifold can be constructed by dualizing a fibration by special Lagrangian tori. This conjecture has been studied extensively, and the works of Fukaya, Kontsevich and Soibelman, Gross and Siebert, and many others paint a very rich and subtle picture of mirror symmetry as a T-duality modified by "quantum corrections" [15, 20, 21].

On the other hand, mirror symmetry has been extended to the non Calabi-Yau setting, and in particular to Fano manifolds, by considering Landau-Ginzburg models, i.e. non-compact manifolds equipped with a complex-valued function called *superpotential* [18]. Our goal is to understand the connection between mirror symmetry and T-duality in this setting.

For a toric Fano manifold, the moment map provides a fibration by Lagrangian tori, and in this context the mirror construction can be understood as a T-duality, as evidenced e.g. by Abouzaid's work [1, 2]. Evidence in the non-toric case is much scarcer, in spite of Hori and Vafa's derivation of the mirror for Fano complete intersections in toric varieties [18]. The best understood case so far is that of Del Pezzo surfaces [4]; however, in that example the construction of the mirror is motivated by entirely ad hoc considerations. As an attempt to understand the geometry of mirror symmetry beyond the Calabi-Yau setting, we start by formulating the following naive conjecture:

Conjecture 1.1. *Let (X, ω, J) be a compact Kähler manifold, let D be an anticanonical divisor in X, and let Ω be a holomorphic volume form defined over $X \setminus D$. Then a mirror manifold M can be constructed as a moduli space of special Lagrangian tori in $X \setminus D$ equipped with flat $U(1)$ connections over them, with a superpotential $W : M \to \mathbb{C}$ given*

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by Fukaya-Oh-Ohta-Ono's m_0 obstruction to Floer homology. Moreover, the fiber of this Landau-Ginzburg model is mirror to D.

The main goal of this paper is to investigate the picture suggested by this conjecture. Conjecture 1.1 cannot hold as stated, for several reasons. One is that in general the special Lagrangian torus fibration on $X \setminus D$ is expected to have singular fibers, which requires suitable corrections to the geometry of M. Moreover, the superpotential constructed in this manner is not well-defined, since wall-crossing phenomena make m_0 multivalued. In particular it is not clear how to define the fiber of W. These various issues are related to quantum corrections arising from holomorphic discs of Maslov index 0; while we do not attempt a rigorous systematic treatment, general considerations (see §3.2–3.3) and calculations on a specific example (see Section 5) suggest that the story will be very similar to the Calabi-Yau case [15, 21]. Another issue is the incompleteness of M; according to Hori and Vafa [18], this is an indication that the mirror symmetry construction needs to be formulated in a certain renormalization limit (see §4.2). The modifications of Conjecture 1.1 suggested by these observations are summarized in Conjectures 3.10 and 4.4 respectively.

The rest of this paper is organized as follows. In Section 2 we study the moduli space of special Lagrangians and its geometry. In Section 3 we discuss the m_0 obstruction in Floer theory and the superpotential. Then Section 4 is devoted to the toric case (in which the superpotential was already investigated by Cho and Oh [10]), and Section 5 discusses in detail the example of $\mathbb{CP}^2$ with a non-toric holomorphic volume form. Finally, Section 6 explores the relation between the critical values of W and the quantum cohomology of X, and Section 7 discusses the connection to mirror symmetry for the Calabi-Yau hypersurface $D \subset X$.

Finally, a word of warning is in order: in the interest of readability and conciseness, many of the statements made in this paper are not entirely rigorous; in particular, weighted counts of holomorphic discs are always assumed to be convergent, and issues related to the lack of regularity of multiply covered Maslov index 0 discs are mostly ignored. Since the main goal of this paper is simply to evidence specific phenomena and illustrate them by examples, we feel that this approach is not unreasonable, and ask the detail-oriented reader for forgiveness.

2. The complexified moduli space of special Lagrangians

2.1. Special Lagrangians

Let (X, ω, J) be a smooth compact Kähler manifold of complex dimension n, and let $\sigma \in H^0(X, K_X^{-1})$ be a nontrivial holomorphic section of the anticanonical bundle, vanishing on a divisor D. Then the complement $X \setminus D$ carries a nonvanishing holomorphic n-form $\Omega = \sigma^{-1}$. By analogy with the Calabi-Yau situation, for a given $\phi \in \mathbb{R}$ we make the following definition:

Definition 2.1. A Lagrangian submanifold $L \subset X \setminus D$ is *special Lagrangian* with phase ϕ if $\mathrm{Im}\,(e^{-i\phi}\Omega)_{|L} = 0$.

Multiplying Ω by $e^{-i\phi}$ if necessary, in the rest of this section we will consider the case $\phi = 0$. In the Calabi-Yau case, McLean has shown that infinitesimal deformations of special Lagrangian submanifolds correspond to harmonic 1-forms, and that these deformations are unobstructed [22]. (See also [16] and [24] for additional context).

In our case, the restriction to L of $\mathrm{Re}\,(\Omega)$ is a non-degenerate volume form (which we assume to be compatible with the orientation of L), but it differs from the volume form vol_g induced by the Kähler metric g. Namely, there exists a function $\psi \in C^\infty(L, \mathbb{R}_+)$ such that $\mathrm{Re}\,(\Omega)_{|L} = \psi\, vol_g$.

Definition 2.2. A one-form $\alpha \in \Omega^1(L, \mathbb{R})$ is *ψ-harmonic* if $d\alpha = 0$ and $d^*(\psi\alpha) = 0$. We denote by $\mathcal{H}^1_\psi(L)$ the space of ψ-harmonic one-forms.

Lemma 2.3. *Each cohomology class contains a unique ψ-harmonic representative.*

Proof. If $\alpha = df$ is exact and ψ-harmonic, then $\psi^{-1}d^*(\psi\, df) = \Delta f - \psi^{-1}\langle d\psi, df\rangle = 0$. Since the maximum principle holds for solutions of this equation, f must be constant. So every cohomology class contains at most one ψ-harmonic representative.

To prove existence, we consider the elliptic operator $D : \Omega^{\mathrm{odd}}(L, \mathbb{R}) \to \Omega^{\mathrm{even}}(L, \mathbb{R})$ defined by $D(\alpha_1, \alpha_3, \dots) = (\psi^{-1}d^*(\psi\alpha_1), d\alpha_1 + d^*\alpha_3, \dots)$. Clearly the kernel of D is spanned by ψ-harmonic 1-forms and by harmonic forms of odd degree ≥ 3, while its cokernel contains all harmonic forms of even degree ≥ 2 and the function ψ. However D differs from $d + d^*$ by an order 0 operator, so its index is $\mathrm{ind}(D) = \mathrm{ind}(d + d^*) = -\chi(L)$. It follows that $\dim \mathcal{H}^1_\psi(L) = \dim H^1(L, \mathbb{R})$. □

Remark 2.4. Rescaling the metric by a factor of λ^2 modifies the Hodge $*$ operator on 1-forms by a factor of λ^{n-2}. Therefore, if $n \neq 2$, then a 1-form is ψ-harmonic if and only if it is harmonic for the rescaled metric $\tilde{g} = \psi^{2/(n-2)}g$.

Proposition 2.5. *Infinitesimal special Lagrangian deformations of L are in one to one correspondence with ψ-harmonic 1-forms on L. More precisely, a section of the normal bundle $v \in C^\infty(NL)$ determines a 1-form $\alpha = -\iota_v\omega \in \Omega^1(L, \mathbb{R})$ and an $(n-1)$-form $\beta = \iota_v \mathrm{Im}\,\Omega \in \Omega^{n-1}(L, \mathbb{R})$. These satisfy $\beta = \psi *_g \alpha$, and the deformation is special Lagrangian if and only if α and β are both closed. Moreover, the deformations are unobstructed.*

Proof. For special Lagrangian L, we have linear isomorphisms $NL \simeq T^*L \simeq \wedge^{n-1}T^*L$ given by the maps $v \mapsto -\iota_v\omega$ and $v \mapsto \iota_v \mathrm{Im}\,\Omega$. More precisely, given a point $p \in L$, by complexifying a g-orthonormal basis of T_pL we obtain a local frame $(\partial_{x_j}, \partial_{y_j})$ in which ω, J, g are standard at p, and $T_pL = \mathrm{span}(\partial_{x_1}, \dots, \partial_{x_n})$. In terms of the dual basis $dz_j = dx_j + i dy_j$, at the point p we have $\Omega = \psi\, dz_1 \wedge \cdots \wedge dz_n$. Hence, given $v = \sum c_j \partial_{y_j} \in N_pL$, we have $-\iota_v\omega = \sum c_j\, dx_j$ and

$$\iota_v \mathrm{Im}\,\Omega = \psi \sum_j c_j (-1)^{j-1} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n = \psi *_g (-\iota_v\omega).$$

Consider a section of the normal bundle $v \in C^\infty(NL)$, and use an arbitrary metric to construct a family of submanifolds $L_t = j_t(L)$, where $j_t(p) = \exp_p(tv(p))$. Since ω and $\mathrm{Im}\,\Omega$ are closed, we have

$$\frac{d}{dt}_{|t=0}(j_t^*\omega) = L_v\omega = d(\iota_v\omega) \quad \text{and} \quad \frac{d}{dt}_{|t=0}(j_t^*\mathrm{Im}\,\Omega) = L_v\mathrm{Im}\,\Omega = d(\iota_v\mathrm{Im}\,\Omega).$$

Therefore, the infinitesimal deformation v preserves the special Lagrangian condition $\omega_{|L} = \mathrm{Im}\,\Omega_{|L} = 0$ if and only if the forms $\alpha = -\iota_v\omega$ and $\beta = \iota_v\mathrm{Im}\,\Omega$ are closed. Since $\beta = \psi *_g \alpha$, this is equivalent to the requirement that α is ψ-harmonic.

Finally, unobstructedness is proved exactly as in the Calabi-Yau case, by observing that the linear map $v \mapsto (L_v\omega, L_v\mathrm{Im}\,\Omega)$ from normal vector fields to exact 2-forms and exact n-forms is surjective and invoking the implicit function theorem [22]. □

This proposition allows us to consider (at least locally) the moduli space of special Lagrangian deformations of L. This moduli space is a smooth manifold, and carries two natural integer affine structures, obtained by identifying the tangent space to the moduli space with either $H^1(L,\mathbb{R})$ or $H^{n-1}(L,\mathbb{R})$ and considering the integer cohomology lattices.

2.2. The geometry of the complexified moduli space

In this section we study the geometry of the (complexified) moduli space of special Lagrangian submanifolds. In the Calabi-Yau case, our constructions essentially reduce to those in Hitchin's illuminating paper [16].

We now consider pairs (L,∇) consisting of a special Lagrangian submanifold $L \subset X\setminus D$ and a flat unitary connection ∇ on the trivial complex line bundle over L, up to gauge equivalence. (In the presence of a B-field we would instead require ∇ to have curvature $-iB$; here we do not consider B-fields). Allowing L to vary in a given $b_1(L)$-dimensional family $\mathcal{B}$ of special Lagrangian submanifolds (a domain in the moduli space), we denote by M the space of equivalence classes of pairs (L,∇). Our first observation is that M carries a natural integrable complex structure.

Indeed, recall that the gauge equivalence class of the connection ∇ is determined by its holonomy $\mathrm{hol}_\nabla \in \mathrm{Hom}(H_1(L), U(1)) \simeq H^1(L,\mathbb{R})/H^1(L,\mathbb{Z})$. We will choose a representative of the form $\nabla = d + iA$, where A is a ψ-harmonic 1-form on L.

Then the tangent space to M at a point (L,∇) is the set of all pairs $(v,\alpha) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R})$ such that v is an infinitesimal special Lagrangian deformation, and α is a ψ-harmonic 1-form, viewed as an infinitesimal deformation of the flat connection. The map $(v,\alpha) \mapsto -\iota_v\omega + i\alpha$ identifies $T_{(L,\nabla)}M$ with the space $\mathcal{H}^1_\psi(L) \otimes \mathbb{C}$ of complex-valued ψ-harmonic 1-forms on L, which makes M a complex manifold. More explicitly, the complex structure on M is as follows:

Definition 2.6. Given $(v,\alpha) \in T_{(L,\nabla)}M \subset C^\infty(NL) \oplus \Omega^1(L,\mathbb{R})$, we define $J^\vee(v,\alpha) = (a, -\iota_v\omega)$, where a is the normal vector field such that $\iota_a\omega = \alpha$.

The following observation will be useful in Section 3:

Lemma 2.7. *Let $A \in H_2(M, L; \mathbb{Z})$ be a relative homology class with boundary $\partial A \neq 0 \in H_1(L, \mathbb{Z})$. Then the function*

$$z_A = \exp(-\int_A \omega)\, \mathrm{hol}_\nabla(\partial A) : M \to \mathbb{C}^* \tag{2.1}$$

is holomorphic.

Proof. The differential $d \log z_A$ is simply $(v, \alpha) \mapsto \int_{\partial A} -\iota_v \omega + i\alpha$, which is $\mathbb{C}$-linear. □

More precisely, the function z_A is well-defined locally (as long as we can keep track of the relative homology class A under deformations of L), but might be multivalued if the family of special Lagrangian deformations of L has non-trivial monodromy.

If the map $j_* : H_1(L) \to H_1(X)$ induced by inclusion is trivial, then this yields a set of (local) holomorphic coordinates $z_i = z_{A_i}$ on M, by considering a collection of relative homology classes A_i such that ∂A_i form a basis of $H_1(L)$. Otherwise, given a class $c \in H_1(L)$ we can fix a representative γ_c^0 of the class $j_*(c) \in H_1(X)$, and use the symplectic area of a 2-chain in X with boundary on $\gamma_c^0 \cup L$, together with the holonomy of ∇ along the part of the boundary contained in L, as a substitute for the above construction.

Next, we equip M with a symplectic form:

Definition 2.8. Given $(v_1, \alpha_1), (v_2, \alpha_2) \in T_{(L,\nabla)}M$, we define

$$\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) = \int_L \alpha_2 \wedge \iota_{v_1} \mathrm{Im}\, \Omega - \alpha_1 \wedge \iota_{v_2} \mathrm{Im}\, \Omega.$$

Proposition 2.9. *$\omega^\vee$ is a Kähler form on M, compatible with $J^\vee$.*

Proof. First we prove that $\omega^\vee$ is closed and non-degenerate by exhibiting local coordinates on M in which it is standard. Let $\gamma_1, \ldots, \gamma_r$ be a basis of $H_{n-1}(L, \mathbb{Z})$ (modulo torsion), and let $e^1, \ldots, e^r$ be the Poincaré dual basis of $H^1(L, \mathbb{Z})$. Let $\gamma^1, \ldots, \gamma^r$ and $e_1, \ldots, e_r$ be the dual bases of $H^{n-1}(L, \mathbb{Z})$ and $H_1(L, \mathbb{Z})$ (modulo torsion): then $\langle e^i \cup \gamma^j, [L] \rangle = \langle \gamma^j, \gamma_i \rangle = \delta_{ij}$. In particular, for all $a \in H^1(L, \mathbb{R})$ and $b \in H^{n-1}(L, \mathbb{R})$ we have

$$\langle a \cup b, [L] \rangle = \sum_{i,j} \langle a, e_i \rangle \langle b, \gamma_j \rangle \langle e^i \cup \gamma^j, [L] \rangle = \sum_i \langle a, e_i \rangle \langle b, \gamma_i \rangle. \tag{2.2}$$

Fix representatives Γ_i and E_i of the homology classes γ_i and e_i, and consider a point (L', ∇') of M near (L, ∇). L' is the image of a small deformation j' of the inclusion map $j : L \to X$. Consider an n-chain C_i in $X \setminus D$ such that $\partial C_i = j'(\Gamma_i) - j(\Gamma_i)$, and let $p_i = \int_{C_i} \mathrm{Im}\, \Omega$. Also, let θ_i be the integral over E_i of the connection 1-form of ∇' in a fixed trivialization. Then $p_1, \ldots, p_r, \theta_1, \ldots, \theta_r$ are local coordinates on M near (L, ∇), and their differentials are given by $dp_i(v, \alpha) = \langle [\iota_v \mathrm{Im}\, \Omega], \gamma_i \rangle$ and $d\theta_i(v, \alpha) = \langle [\alpha], e_i \rangle$. Using (2.2) we deduce that $\omega^\vee = \sum_{i=1}^r dp_i \wedge d\theta_i$.

Next we observe that, by Proposition 2.5, $\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2))$ can be rewritten as

$$\int_L \alpha_1 \wedge (\psi\ *\iota_{v_2}\omega) - \alpha_2 \wedge (\psi\ *\iota_{v_1}\omega) = \int_L \psi \left(\langle \alpha_1, \iota_{v_2}\omega \rangle_g - \langle \iota_{v_1}\omega, \alpha_2 \rangle_g \right) vol_g.$$

So the compatibility of $\omega^\vee$ with $J^\vee$ follows directly from the observation that

$$\omega^\vee((v_1,\alpha_1), J^\vee(v_2,\alpha_2)) = \int_L \psi\left(\langle\alpha_1,\alpha_2\rangle_g + \langle\iota_{v_1}\omega, \iota_{v_2}\omega\rangle_g\right) vol_g$$

is clearly a Riemannian metric on M. □

Remark 2.10. Consider the projection $\pi : M \to \mathcal{B}$ which forgets the connection, i.e. the map $(L,\nabla) \mapsto L$. Then the fibers of π are Lagrangian with respect to $\omega^\vee$.

If L is a torus, then $\dim M = \dim X = n$ and we can also equip M with a holomorphic volume form defined as follows:

Definition 2.11. Given n vectors $(v_1,\alpha_1),\dots,(v_n,\alpha_n) \in T_{(L,\nabla)}M$, we define

$$\Omega^\vee((v_1,\alpha_1),\dots,(v_n,\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1)\wedge\cdots\wedge(-\iota_{v_n}\omega + i\alpha_n).$$

In terms of the local holomorphic coordinates $z_1,\dots,z_n$ on M constructed from a basis of $H_1(L,\mathbb{Z})$ using the discussion after Lemma 2.7, this holomorphic volume form is simply $d\log z_1 \wedge\cdots\wedge d\log z_n$.

In this situation, the fibers of $\pi : M \to \mathcal{B}$ are special Lagrangian (with phase $n\pi/2$) with respect to $\omega^\vee$ and $\Omega^\vee$. If in addition we assume that ψ-harmonic 1-forms on L have no zeroes (this is automatic in dimensions $n \le 2$ using the maximum principle), then we recover the familiar picture: in a neighborhood of L, (X,J,ω,Ω) and $(M,J^\vee,\omega^\vee,\Omega^\vee)$ carry dual fibrations by special Lagrangian tori.

3. Towards the superpotential

3.1. Counting discs

Thanks to the monumental work of Fukaya, Oh, Ohta and Ono [14], it is now well understood that the Floer complex of a Lagrangian submanifold carries the structure of a *curved* or *obstructed* A_∞-algebra. The key ingredient is the moduli space of J-holomorphic discs with boundary in the given Lagrangian submanifold, together with evaluation maps at boundary marked points. In our case we will be mainly interested in (weighted) counts of holomorphic discs of Maslov index 2 whose boundary passes through a given point of the Lagrangian; in the Fukaya-Oh-Ohta-Ono formalism, this corresponds to the degree 0 part of the obstruction term $\mathfrak{m}_0$. In the toric case it is known that this quantity agrees with the superpotential of the mirror Landau-Ginzburg model; see in particular the work of Cho and Oh [10], and §4 below. In fact, the material in this section overlaps signiicantly with [10], and with §12.7 of [14].

As in §2, we consider a smooth compact Kähler manifold (X,ω,J) of complex dimension n, equipped with a holomorphic n-form Ω defined over the complement of an anticanonical divisor D.

Recall that, given a Lagrangian submanifold L and a nonzero relative homotopy class $\beta \in \pi_2(X,L)$, the moduli space $\mathcal{M}(L,\beta)$ of J-holomorphic discs with boundary on L

representing the class β has virtual dimension $n-3+\mu(\beta)$, where $\mu(\beta)$ is the Maslov index.

Lemma 3.1. *If $L \subset X \setminus D$ is special Lagrangian, then $\mu(\beta)$ is equal to twice the algebraic intersection number $\beta \cdot [D]$.*

Proof. Because the tangent space to L is totally real, the choice of a volume element on L determines a nonvanishing section $\det(TL)$ of $K_X^{-1} = \Lambda^n(TX, J)$ over L. Its square $\det(TL)^{\otimes 2}$ defines a section of the circle bundle $S(K_X^{-2})$ associated to K_X^{-2} over L, independent of the chosen volume element. The Maslov number $\mu(\beta)$ measures the obstruction of this section to extend over a disc Δ representing the class β (see Example 2.9 in [25]).

Recall that D is the divisor associated to $\sigma = \Omega^{-1} \in H^0(X, K_X^{-1})$. Then $\sigma^{\otimes 2}$ defines a section of $S(K_X^{-2})$ over $L \subset X \setminus D$, and since L is special Lagrangian, the sections $\sigma^{\otimes 2}$ and $\det(TL)^{\otimes 2}$ coincide over L (up to a constant phase factor $e^{-2i\phi}$). Therefore, $\mu(\beta)$ measures precisely the obstruction for $\sigma^{\otimes 2}$ to extend over Δ, which is twice the intersection number of Δ with D. □

In fact, as pointed out by M. Abouzaid, the same result holds if we replace the special Lagrangian condition by the weaker requirement that the Maslov class of L vanishes in $X \setminus D$ (i.e., the phase function $\arg(\Omega_{|L})$ lifts to a real-valued function).

Using positivity of intersections, Lemma 3.1 implies that all holomorphic discs with boundary in L have non-negative Maslov index.

We will now make various assumptions on L in order to ensure that the count of holomorphic discs that we want to consider is well-defined:

Assumption 3.2.

(1) there are no non-constant holomorphic discs of Maslov index 0 in (X, L);
(2) holomorphic discs of Maslov index 2 in (X, L) are regular;
(3) there are no non-constant holomorphic spheres in X with $c_1(TX) \cdot [S^2] \leq 0$.

Then, for every relative homotopy class $\beta \in \pi_2(X, L)$ such that $\mu(\beta) = 2$, the moduli space $\mathcal{M}(L, \beta)$ of holomorphic discs with boundary in L representing the class β is a smooth compact manifold of real dimension $n-1$: no bubbling or multiple covering phenomena can occur since 2 is the minimal Maslov index.

We also assume that L is spin (recall that we are chiefly interested in tori), and choose a spin structure. The choice is not important, as the difference between two spin structures is an element of $H^1(L, \mathbb{Z}/2)$ and can be compensated by twisting the connection ∇ accordingly. Then $\mathcal{M}(L, \beta)$ is oriented, and the evaluation map at a boundary marked point gives us an n-cycle in L, which is of the form $n_\beta(L)\,[L]$ for some integer $n_\beta(L) \in \mathbb{Z}$. In simpler terms, $n_\beta(L)$ is the (algebraic) number of holomorphic discs in the class β whose boundary passes through a generic point $p \in L$.

Then, ignoring convergence issues, we can tentatively make the following definition (see also [10], §12.7 in [14], and Section 5b in [27]):

Definition 3.3. $m_0(L,\nabla) = \sum_{\beta,\ \mu(\beta)=2} n_\beta(L)\, \exp(-\int_\beta \omega)\, \mathrm{hol}_\nabla(\partial\beta)$.

If Assumption 3.2 holds for all special Lagrangians in the considered family $\mathcal{B}$, and if the sum converges, then we obtain in this way a complex-valued function on M, which we call *superpotential* and also denote by W for consistency with the literature. In this ideal situation, the integers $n_\beta(L)$ are locally constant, and Lemma 2.7 immediately implies:

Corollary 3.4. *$W = m_0 : M \to \mathbb{C}$ is a holomorphic function.*

An important example is the case of toric fibers in a toric manifold, discussed in Cho and Oh's work [10] and in §4 below: in this case, the superpotential W agrees with Hori and Vafa's physical derivation [18].

Remark 3.5. The way in which we approach the superpotential here is a bit different from that in [14]. Fukaya, Oh, Ohta and Ono consider a single Lagrangian submanifold L, and the function which to a 1-cocycle a associates the degree zero part of $\mathfrak{m}_0 + \mathfrak{m}_1(a) + \mathfrak{m}_2(a,a) + \dots$. However, each of these terms counts holomorphic discs of Maslov index 2 whose boundary passes through a generic point of L, just with different weights. It is not hard to convince oneself that the contribution to $\mathfrak{m}_k(a,a,\dots)$ of a disc in a given class β is weighted by a factor $\frac{1}{k!}\langle a, \partial\beta\rangle^k$ (the coefficient $\frac{1}{k!}$ comes from the requirement that the k input marked points must lie in the correct order on the boundary of the disc). Thus, the series $\mathfrak{m}_0 + \mathfrak{m}_1(a) + \mathfrak{m}_2(a,a) + \dots$ counts Maslov index 2 discs with weights $\exp(\int_{\partial\beta} a)$ (in addition to the weighting by symplectic area). In this sense a can be thought of as a *non-unitary* holonomy (normally with values in the positive part of the Novikov ring for convergence reasons; here we assume convergence and work with complex numbers). Next, we observe that, since the weighting by symplectic area and holonomy is encoded by the complex parameter z_β defined in (2.1), varying the holonomy in a non-unitary manner is equivalent to moving the Lagrangian in such a way that the flux of the symplectic form equals the real part of the connection form. More precisely, this equivalence between a non-unitary connection on a fixed L and a unitary connection on a non-Hamiltonian deformation of L only holds as long as the disc counts n_β remain constant; so in general the superpotential in [14] is the analytic continuation of the germ of our superpotential at the considered point.

Remark 3.6. Condition (3) in Assumption 3.2 can be somewhat relaxed. For example, one can allow the existence of nonconstant J-holomorphic spheres of Chern number 0, as long as all simple (non multiply covered) such spheres are regular, and the associated evaluation maps are transverse to the evaluation maps at interior marked points of J-holomorphic discs of Maslov index 2 in (X,L). Then the union of all holomorphic spheres with Chern number zero is a subset $\mathcal{C}$ of real codimension 4 in X, and the holomorphic discs which intersect $\mathcal{C}$ form a codimension 2 family. In particular, if we choose the point $p \in L$ in the complement of a codimension 2 subset of L then none of the Maslov index 2 discs whose boundary passes through p hits $\mathcal{C}$. This allows us to define $n_\beta(L)$.

Similarly, in the presence of J-holomorphic spheres of negative Chern number, there might exist stable maps in the class β consisting of a disc component of Maslov index > 2 whose boundary passes through the point p together with multiply covered spheres of negative Chern number. The moduli space of such maps typically has excess dimension. However, suitable assumptions on spheres of negative Chern number ensure that these stable maps cannot occur as limits of sequences of honest discs of Maslov index 2 as long as p stays away from a codimension 2 subset in L, which allows us to ignore the issue.

Remark 3.7. In the above discussion we have avoided the use of virtual perturbation techniques. However, at the cost of additional technical complexity we can remove (2) and (3) from Assumption 3.2. Indeed, even if holomorphic discs of Maslov index 2 fail to be regular, as long as there are no holomorphic discs of Maslov index ≤ 0 we can still define $n_\beta(L)$ as a *virtual* count. Namely, the minimality of the Maslov index prevents bubbling of discs, so that when $\mu(\beta) = 2$ the virtual fundamental chain $[\mathcal{M}(L, \beta)]^{vir}$ is actually a cycle, and $n_\beta(L)$ can be defined as the degree of the evaluation map. Moreover, $n_\beta(L)$ is locally constant under Lagrangian isotopies as long are discs of Maslov index ≤ 0 do not occur: indeed, the Lagrangian isotopy induces a cobordism between the virtual fundamental cycles of the moduli spaces.

3.2. Maslov index zero discs and wall-crossing I

In actual examples, condition (1) in Assumption 3.2 almost never holds (with the notable exception of the toric case). Generically, in dimension $n \geq 3$, the best we can hope for is:

Assumption 3.8. All simple (non multiply covered) nonconstant holomorphic discs of Maslov index 0 in (X, L) are regular, and the associated evaluation maps at boundary marked points are transverse to each other and to the evaluation maps at boundary marked points of holomorphic discs of Maslov index 2.

Then simple nonconstant holomorphic discs of Maslov index 0 occur in $(n-3)$-dimensional families, and the set $\mathcal{Z}$ of points of L which lie on the boundary of a nonconstant Maslov index 0 disc has codimension 2 in L. For a generic point $p \in L$, in each relative homotopy class of Maslov index 2 there are finitely many holomorphic discs whose boundary passes through p, and none of them hits $\mathcal{Z}$. We can therefore define an integer $n_\beta(L, p)$ which counts these discs with appropriate signs, and by summing over β as in Definition 3.3 we obtain a complex number $m_0(L, \nabla, p)$.

However, the points p which lie on the boundary of a configuration consisting of two holomorphic discs (of Maslov indices 2 and 0) attached to each other at their boundary form a codimension 1 subset $\mathcal{W} \subset L$. The typical behavior as p approaches such a "wall" is that a Maslov index 2 disc representing a certain class β breaks into a union of two discs representing classes β' and α with $\beta = \beta' + \alpha$, and then disappears altogether (see Figure 1). Thus the walls separate L into various chambers, each of which gives rise to a different value of $m_0(L, \nabla, p)$.

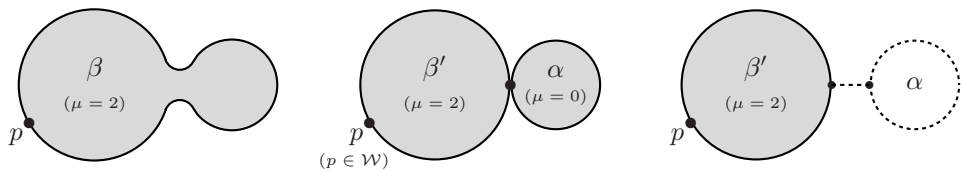

FIGURE 1. Wall-crossing for discs

More conceptually, denote by $\mathcal{M}_k(L,\beta)$ the moduli space of holomorphic discs in (X,L) with k marked points on the boundary representing the class β, and denote by ev_i the evaluation map at the i-th marked point. Then $n_\beta(L,p)$ is the degree at p of the n-chain $(ev_1)_*[\mathcal{M}_1(L,\beta)]$, whose boundary (an $n-1$-chain supported on $\mathcal{W}$) is essentially (ignoring all subtleties arising from multiple covers)

$$\sum_{\substack{\beta=\beta'+\alpha \\ \mu(\alpha)=0 \\ 0<\omega(\alpha)<\omega(\beta)}} (ev_1)_*[\mathcal{M}_2(L,\beta') \underset{ev_2}{\times} \mathcal{M}_1(L,\alpha)],$$

and $m_0(L,\nabla,p)$ is the degree at p of the chain (with complex coefficients)

$$\mathfrak{m}_0 = \sum_\beta \exp(-\textstyle\int_\beta \omega)\, \mathrm{hol}_\nabla(\partial\beta)\, (ev_1)_*[\mathcal{M}_1(L,\beta)].$$

In this language it is clear that these quantities depend on the position of p relatively to the boundary of the chain.

Various strategies can be employed to cancel the boundary and obtain an evaluation cycle, thus leading to a well-defined count $n_\beta(L)$ independently of the point $p \in L$ [11, 14]. For instance, in the cluster approach [11], given a suitably chosen Morse function f on L, one enlarges the moduli space $\mathcal{M}_1(L,\beta)$ by considering configurations consisting of several holomorphic discs connected to each other by gradient flow trajectories of f, with one marked point on the boundary of the component which lies at the root of the tree (which has Maslov index 2, while the other components have Maslov index 0); see Figure 1 (right) for the simplest case.

However, even if one makes the disc count independent of the choice of $p \in L$ by completing the evaluation chain to a cycle, the final answer still depends on the choice of auxiliary data. For example, in the cluster construction, depending on the direction of ∇f relative to the wall, two scenarios are possible: either an honest disc in the class β turns into a configuration of two discs connected by a gradient flow line as p crosses $\mathcal{W}$; or both configurations coexist on the same side of the wall (their contributions to $n_\beta(L,p)$ cancel each other) and disappear as p moves across $\mathcal{W}$. Hence, in the absence of a canonical choice there still isn't a uniquely defined superpotential.

3.3. Maslov index zero discs and wall-crossing II: the surface case

The wall-crossing phenomenon is somewhat different in the surface case ($n = 2$). In dimension 2 a generic Lagrangian submanifold does not bound any holomorphic discs of Maslov index 0, so Assumption 3.2 can be expected to hold for most L, giving rise to a well-defined complex number $m_0(L, \nabla)$. However, in a family of Lagrangians, isolated holomorphic discs of Maslov index 0 occur in codimension 1, leading to wall-crossing discontinuities. The general algebraic and analytic framework which can be used to describe these phenomena is discussed in §19.1 in [14] (see also Section 5c in [27]). Here we discuss things in a more informal manner, in order to provide some additional context for the calculations in Section 5.

Consider a continuous family of (special) Lagrangian submanifolds L_t ($t \in [-\epsilon, \epsilon]$), such that L_t satisfies Assumption 3.2 for $t \neq 0$ and L_0 bounds a unique nontrivial simple holomorphic disc u_α representing a class α of Maslov index 0 (so $\mathcal{M}(L_0, \alpha) = \{u_\alpha\}$). Given a holomorphic disc u_0 representing a class $\beta_0 \in \pi_2(X, L_0)$ of Maslov index 2, we obtain stable maps representing the class $\beta = \beta_0 + m\alpha$ by attaching copies of u_α (or branched covers of u_α) to u_0 at points where the boundary of u_0 intersects that of u_α. These configurations typically deform to honest holomorphic discs either for $t > 0$ or for $t < 0$, but not both.

Using the isotopy to identify $\pi_2(X, L_t)$ with $\pi_2(X, L_0)$, we can consider the moduli space of holomorphic discs with boundary in one of the L_t, representing a given class β, and with k marked points on the boundary, $\tilde{\mathcal{M}}_k(\beta) = \coprod_{t\in[-\epsilon,\epsilon]} \mathcal{M}_k(L_t, \beta)$, and the evaluation maps $ev_i : \tilde{\mathcal{M}}_k(\beta) \to \coprod_t \{t\} \times L_t$. In principle, given a class β with $\mu(\beta) = 2$, the boundary of the corresponding evaluation chain is given by

$$\partial \Big((ev_1)_*[\tilde{\mathcal{M}}_1(\beta)] \Big) = \sum_{m\geq 1} (ev_1)_* \Big[\tilde{\mathcal{M}}_2(\beta - m\alpha) \underset{ev_2}{\times} \tilde{\mathcal{M}}_1(m\alpha) \Big]. \tag{3.1}$$

However, interpreting the right-hand side of this equation is tricky, because of the systematic failure of transversality, even if we ignore the issue of multiply covered discs ($m \geq 2$). Things can be made slightly better by perturbing J to a domain-dependent almost-complex structure (after adding marked points to make the maps stable). Then, as one moves through the one-parameter family of Lagrangians, bubbling of Maslov index 0 discs occurs at different values of t depending on the position at which it takes place along the boundary of the Maslov index 2 component. So, as t varies between $-\epsilon$ and $+\epsilon$ one successively hits several boundary strata, corresponding to bubbling at various points of the boundary; the algebraic number of such elementary wall-crossings is the intersection number $[\partial\beta] \cdot [\partial\alpha]$. However, dealing with multiple covers, or with attachment of more than one bubble at the same point of the boundary of the Maslov index 2 component, requires more sophisticated techniques such as virtual fundamental cycles and multivalued perturbations.

Another approach, which makes some features of the wall-crossing phenomenon more apparent, is to proceed similarly to the construction in Section 19.1 of [14]. As discussed

by Seidel in Section 5c of [27], the following result essentially follows from Fukaya-Oh-Ohta-Ono's construction of A_∞-homomorphisms associated to wall-crossing (we give a sketch of proof for completeness):

Proposition 3.9. *Upon crossing a wall in which L bounds a unique simple Maslov index 0 disc representing a relative class α, the expression of $m_0(L,\nabla)$ as a Laurent series in the variables of Lemma 2.7 is modified by a holomorphic change of variables*

$$z_\beta \mapsto z_\beta\, h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]} \quad \forall \beta \in \pi_2(X,L),$$

where $h(z_\alpha)$ is a power series of the form $1+O(z_\alpha)$ (independent of β).

Proof. Use the flow of a suitable vector field to construct a family of diffeomorphisms $\phi_{t,t'} : L_t \xrightarrow{\sim} L_{t'}$ depending continuously on $t,t' \in [-\epsilon,\epsilon]$ and such that $\phi_{t',t''}\circ\phi_{t,t'} = \phi_{t,t''}$. Then we can define the moduli space $\overrightarrow{\mathcal{M}_1}(\beta)$ of *time-ordered* stable configurations representing the class β, i.e. tuples $(\Sigma,\{t_i\},\{u_i\})$, where:

- $\Sigma = \bigsqcup \Sigma_i/\sim$ is a nodal curve modelled on a rooted tree, with components $\Sigma_i \simeq D^2$ glued together at boundary points;
- the components are labelled by real numbers $t_i \in [-\epsilon,\epsilon]$ such that $t_i \le t_j$ whenever Σ_j lies between Σ_i and the root component (which has the highest label);
- $u_i : (\Sigma_i,\partial\Sigma_i) \to (X, L_{t_i})$ are stable holomorphic maps, of Maslov index 2 for the root component and 0 for all other components, with the following property: if $z \in \Sigma_i \cap \Sigma_j$, then $u_j(z) = \phi_{t_i,t_j}(u_i(z))$,

This moduli space can be equipped with a virtual fundamental chain (or a Kuranishi structure in the language of [14]) constructed by choosing a suitable system of multivalued perturbations. By the same argument as in §19.1 of [14], the boundary cases where bubbling occurs in a component, or where two labels t_i and t_j become equal, exactly cancel each other, so that the boundary of $\overrightarrow{\mathcal{M}_1}(\beta)$ consists of two types of configurations:

(1) the label of a leaf component becomes equal to $-\epsilon$,
(2) the root label becomes equal to ϵ.

Since non-constant Maslov index 0 discs only occur at $t=0$, in the first case there must be a single component, i.e. we have an honest disc in $(X, L_{-\epsilon})$. In the second case we obtain a configuration in which the main component is a Maslov index 2 disc in (X, L_ϵ), and any other non-constant component is a Maslov index 0 disc in (X, L_0). The bubble trees formed by the latter components (together with optional ghost bubbles at labels $t \in [0,\epsilon]$) are attached to the main component via the identification diffeomorphism $\phi_{0,\epsilon} : L_0 \to L_\epsilon$. The upshot is that counts of holomorphic discs in $(X, L_{-\epsilon})$ are equal to (virtual) counts of time-ordered configurations whose root component is a holomorphic disc in (X, L_ϵ).

Fix a generic point $p \in L_\epsilon$, and choose the diffeomorphisms $\phi_{t,t'}$ generically so that the image by $\phi_{0,\epsilon}$ of the boundary of the exceptional disc u_α is transverse to the boundaries of all holomorphic discs in $\mathcal{M}(L_\epsilon, \beta - m\alpha)$ $(m \ge 0)$ whose boundary passes though p (the energy bound and genericity of p imply that there are finitely many such discs). Then the transversality issues that arise when attempting to count time-stable configurations

whose root component passes through p only concern subtrees formed by Maslov index 0 bubbles (ghost bubbles and multiple covers of u_α). For each such subtree, we can choose a system of multivalued perturbations in a manner that only depends on the subtree of Maslov index 0 bubbles itself and possibly on the sign of the intersection between $\phi_{0,\epsilon}(\partial u_\alpha)$ and the boundary of the root disc at the attaching point, but not on any other data concerning the root disc.

Consider a disc $u \in \mathcal{M}(L_\epsilon, \beta - m\alpha)$ with $p \in u(\partial D^2)$, and let $\{q_1, \dots, q_r\} = u(\partial D^2) \cap \phi_{0,\epsilon}(u_\alpha(\partial D^2))$: then the virtual number of time-ordered configurations representing the class β and with root component u is given by

$$\sum_{m_1+\cdots+m_r=m} \left(\prod_{i=1}^{r} \tilde{n}_{m_i\alpha}(q_i) \right),$$

where $\tilde{n}_{m_i\alpha}(q_i) \in \mathbb{Q}$ is a virtual count of all the possible bubble trees representing the total class $m_i\alpha$ that can be attached to the root component at q_i. Determining the actual values of these coefficients (beyond the obvious ones $\tilde{n}_0 = 1$ and $\tilde{n}_\alpha = \pm 1$ depending on orientations) is beyond the scope of this argument, all that matters to us is that for a given exceptional disc u_α they only depend on the total multiplicity m_i and on the sign of the intersection at q_i. Denote by $\tilde{n}^+_{m_i\alpha}$ (resp. $\tilde{n}^-_{m_i\alpha}$) the value of $\tilde{n}_{m_i\alpha}$ for positive (resp. negative) intersections, and consider the generating series $h^\pm_\alpha(q) = 1 + \tilde{n}^\pm_\alpha q + \tilde{n}^\pm_{2\alpha} q^2 + \dots$.

If we encode the disc counts in all classes of the form $\beta_0 + m\alpha$ by generating series

$$F_t(q) = \sum_{m \in \mathbb{Z}} n_{\beta_0 + m\alpha}(L_t)\, q^m,$$

then as explained above $F_{-\epsilon}(q)$ can be determined by counting time-ordered configurations with root component a disc in (X, L_ϵ). If we look only at those configurations where the root component is a given disc $u \in \mathcal{M}(L_\epsilon, \beta_0 + m\alpha)$ whose boundary intersects $\phi_{0,\epsilon}(\partial u_\alpha)$ positively in r_+ points and negatively in r_- points, the resulting contribution to $F_{-\epsilon}(q)$ is $q^m\, h^+_\alpha(q)^{r_+}\, h^-_\alpha(q)^{r_-}$ (while the contribution of u itself to $F_\epsilon(q)$ is just q^m).

Comparing the above construction with the time-reversed version (where labels of components increase away from the root component), or equivalently by considering the family of Lagrangian submanifolds (L_{-t}), it is easy to see that the power series h^+_α and h^-_α must be the inverse of each other. Therefore, the multiplicative factor $h^+_\alpha(q)^{r_+} h^-_\alpha(q)^{r_-}$ simplifies to $h^+_\alpha(q)^{[\partial\beta_0]\cdot[\partial\alpha]}$; this factor is the same for all the discs that contribute to $F_\epsilon(q)$, so in the end $F_{-\epsilon}(q)$ and $F_{+\epsilon}(q)$ differ by a multiplicative factor of $h^+_\alpha(q)^{[\partial\beta_0]\cdot[\partial\alpha]}$.

Finally, the contributions of the discs in the classes $\beta_0 + m\alpha$ to $m_0(L_t, \nabla_t)$ are given by plugging $q = z_\alpha$ (as defined in (2.1)) into $F_t(q)$ and multiplying by z_{β_0}. The values of this expression on either side of $t = 0$ differ from each other by a change of variables, replacing z_{β_0} by $z^*_{\beta_0} = z_{\beta_0} h^+_\alpha(z_\alpha)^{[\partial\beta_0]\cdot[\partial\alpha]}$. These changes of variables are consistent for all classes in $\pi_2(X, L_t)$, in the sense that the new variables still satisfy $z^*_{\beta+\gamma} = z^*_\beta z^*_\gamma$. $\square$

An interesting consequence of Proposition 3.9 (especially in the light of the discussion in §6) is that, while the critical points of the superpotential are affected by the wall-crossing, its critical values are not. However, since the change of variables can map a critical point to infinity (see e.g. Section 5.4 for an example in which this occurs), some critical values may still be lost in the coordinate change.

Finally, we observe that the changes of variables which arise in Proposition 3.9 are formally very similar by the quantum corrections to the complex structure of the mirror proposed by Kontsevich-Soibelman and Gross-Siebert in the Calabi-Yau case [15, 21]. This suggests the following:

Conjecture 3.10. *The mirror to a Kähler surface X (together with an anticanonical divisor D) should differ from the complexified moduli space M of special Lagrangian tori in $X \setminus D$ by "quantum corrections" which, away from the singular fibers, amount to gluing the various regions of M delimited by Maslov index 0 discs according to the changes of variables introduced in Proposition 3.9.*

One difficulty raised by this conjecture is that, whereas the quantum corrections are compatible with the complex structure $J^\vee$, they do not preserve the symplectic form $\omega^\vee$ introduced in Definition 2.8. We do not know how to address this issue, but presumably this means that $\omega^\vee$ should also be modified by quantum corrections.

4. The toric case

In this section, we consider the case where X is a smooth toric variety, and D is the divisor consisting of all degenerate toric orbits. The calculation of the superpotential (Proposition 4.3) is very similar to that in [10], but we provide a self-contained description for completeness. We first recall very briefly some classical facts about toric varieties.

As a Kähler manifold, a toric variety X is determined by its moment polytope $\Delta \subset \mathbb{R}^n$, a convex polytope in which every facet admits an integer normal vector, n facets meet at every vertex, and their primitive integer normal vectors form a basis of $\mathbb{Z}^n$. The moment map $\phi : X \to \mathbb{R}^n$ identifies the orbit space of the T^n-action on X with Δ. From the point of view of complex geometry, the preimage of the interior of Δ is an open dense subset U of X, biholomorphic to $(\mathbb{C}^*)^n$, on which $T^n = (S^1)^n$ acts in the standard manner. Moreover X admits an open cover by affine subsets biholomorphic to $\mathbb{C}^n$, which are the preimages of the open stars of the vertices of Δ (i.e., the union of all the strata whose closure contains the given vertex).

For each facet F of Δ, the preimage $\phi^{-1}(F) = D_F$ is a hypersurface in X; the union of these hypersurfaces defines the toric anticanonical divisor $D = \sum_F D_F$. The standard holomorphic volume form on $(\mathbb{C}^*)^n \simeq U = X \setminus D$, defined in coordinates by $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$, determines a section of K_X with poles along D.

4.1. Toric orbits and the superpotential

Our starting point is the observation that the moment map defines a special Lagrangian torus fibration on $U = X \setminus D$:

Lemma 4.1. *The T^n-orbits in $X \setminus D$ are special Lagrangian (with phase $n\pi/2$).*

Proof. It is a classical fact that the T^n-orbits are Lagrangian; since the T^n-action on $X \setminus D \simeq (\mathbb{C}^*)^n$ is the standard one, in coordinates the orbits are products of circles $S^1(r_1) \times \cdots \times S^1(r_n) = \{(x_1, \ldots, x_n),\ |x_i| = r_i\}$, on which the restriction of $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$ has constant phase $n\pi/2$. □

As above we consider the complexified moduli space M, i.e. the set of pairs (L, ∇) where L is a T^n-orbit and ∇ is a flat $U(1)$ connection on the trivial bundle over L. Recall that L is a product of circles $L = S^1(r_1) \times \cdots \times S^1(r_n) \subset (\mathbb{C}^*)^n \simeq X \setminus D$, and denote the holonomy of ∇ around the j-th factor $S^1(r_j)$ by $\exp(i\theta_j)$. Then the symplectic form introduced in Definition 2.8 becomes $\omega^\vee = (2\pi)^n \sum d\log r_j \wedge d\theta_j$, i.e. up to a constant factor it coincides with the standard Kähler form on $(\mathbb{C}^*)^n \simeq M$. However, as a complex manifold M is not biholomorphic to $(\mathbb{C}^*)^n$:

Proposition 4.2. *M is biholomorphic to $\text{Łog}^{-1}(\operatorname{int}\Delta) \subset (\mathbb{C}^*)^n$, where $\text{Łog} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ is the map defined by $\text{Łog}(z_1, \ldots, z_n) = (-\frac{1}{2\pi}\log|z_1|, \ldots, -\frac{1}{2\pi}\log|z_n|)$.*

Proof. Given a T^n-orbit L and a flat $U(1)$-connection ∇, let

$$z_j(L, \nabla) = \exp(-2\pi\phi_j(L))\,\mathrm{hol}_\nabla(\gamma_j), \tag{4.1}$$

where ϕ_j is the j-th component of the moment map, i.e. the Hamiltonian for the action of the j-th factor of T^n, and $\gamma_j = [S^1(r_j)] \in H_1(L)$ is the homology class corresponding to the j-th factor in $L = S^1(r_1) \times \cdots \times S^1(r_n)$.

Let A_j be a relative homology class in $H_2(X, L)$ such that $\partial A_j = \gamma_j \in H_1(L)$ (it is clear that such a class can be chosen consistently for all T^n-orbits), and consider the holomorphic function z_{A_j} defined by (2.1): then z_j and z_{A_j} differ by a constant multiplicative factor. Indeed, comparing the two definitions the holonomy factors coincide, and given an infinitesimal special Lagrangian deformation $v \in C^\infty(NL)$,

$$d\log|z_{A_j}|(v) = \int_{\gamma_j} -\iota_v\omega = \int_{S^1(r_j)} \omega(X_j, v)\,dt = \int_{S^1(r_j)} -d\phi_j(v)\,dt = d\log|z_j|(v),$$

where X_j is the vector field which generates the action of the j-th factor of T^n (i.e. the Hamiltonian vector field associated to ϕ_j).

Thus $z_1, \ldots, z_n$ are holomorphic coordinates on M, and the $(\mathbb{C}^*)^n$-valued map $(L, \nabla) \mapsto (z_1(L, \nabla), \ldots, z_n(L, \nabla))$ identifies M with its image, which is exactly the preimage by Łog of the interior of Δ. □

Next we study holomorphic discs in X with boundary on a given T^n-orbit L. For each facet F of Δ, denote by $\nu(F) \in \mathbb{Z}^n$ the primitive integer normal vector to F pointing into Δ, and let $\alpha(F) \in \mathbb{R}$ be the constant such that the equation of F is $\langle \nu(F), \phi\rangle + \alpha(F) = 0$. Moreover, given $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we denote by z^a the Laurent monomial $z_1^{a_1} \ldots z_n^{a_n}$, where $z_1, \ldots, z_n$ are the coordinates on M defined by (4.1). Then we have:

Proposition 4.3 (Cho-Oh [10])**.** *There are no holomorphic discs of Maslov index 0 in (X, L), and the discs of Maslov index 2 are all regular. Moreover, the superpotential is*

given by the Laurent polynomial

$$W = m_0(L, \nabla) = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)}\, z^{\nu(F)}. \tag{4.2}$$

Proof. By Lemma 3.1 and positivity of intersection, Maslov index 0 discs do not intersect D, and hence are contained in $X \setminus D \simeq (\mathbb{C}^*)^n$. However, since L is a product of circles $S^1(r_i) = \{|x_i| = r_i\}$ inside $(\mathbb{C}^*)^n$, it follows immediately from the maximum principle applied to $\log x_i$ that $(\mathbb{C}^*)^n$ does not contain any non-constant holomorphic disc with boundary in L.

Next, we observe that a Maslov index 2 disc intersects D at a single point, and in particular it intersects only one of the components, say D_F for some facet F of Δ. We claim that for each facet F there is a unique such disc whose boundary passes through a given point $p = (x_1^0, \dots, x_n^0) \in L \subset (\mathbb{C}^*)^n \simeq X \setminus D$; in terms of the components $(\nu_1, \dots, \nu_n)$ of the normal vector $\nu(F)$, this disc can be parametrized by the map

$$w \mapsto (w^{\nu_1} x_1^0, \dots, w^{\nu_n} x_n^0) \tag{4.3}$$

(for $w \in D^2 \setminus \{0\}$; the point $w = 0$ corresponds to the intersection with D_F).

To prove this claim, we work in an affine chart centered at a vertex v of Δ adjacent to the facet F. Denote by $\eta_1, \dots, \eta_n$ the basis of $\mathbb{Z}^n$ which consists of the primitive integer vectors along the edges of Δ passing through v, oriented away from v, and labelled in such a way that $\eta_2, \dots, \eta_n$ are tangent to F. Associate to each edge vector $\eta_i = (\eta_{i1}, \dots, \eta_{in}) \in \mathbb{Z}^n$ a Laurent monomial $\tilde{x}_i = x^{\eta_i} = x_1^{\eta_{i1}} \dots x_n^{\eta_{in}}$. Then, after the change of coordinates $(x_1, \dots, x_n) \mapsto (\tilde{x}_1, \dots, \tilde{x}_n)$, the affine coordinate chart associated to the vertex v can be thought of as the standard compactification of $(\mathbb{C}^*)^n$ to $\mathbb{C}^n$. In this coordinate chart, L is again a product torus $S^1(\tilde{r}_1) \times \dots \times S^1(\tilde{r}_n)$, where $\tilde{r}_i = r_1^{\eta_i 1} \dots r_n^{\eta_{in}}$, and D_F is the coordinate hyperplane $\tilde{x}_1 = 0$.

Since the complex structure is the standard one, a holomorphic map $u : D^2 \to \mathbb{C}^n$ with boundary in L is given by n holomorphic functions $w \mapsto (u_1(w), \dots, u_n(w))$ such that $|u_i| = \tilde{r}_i$ on the unit circle. Since by assumption the disc hits only D_F, the functions $u_2, \dots, u_n$ have no zeroes, so by the maximum principle they are constant. Moreover the intersection number with D_F is assumed to be 1, so the image of the map u_1 is the disc of radius $\tilde{r}_1$, with multiplicity 1; so, up to reparametrization, $u_1(w) = \tilde{r}_1\, w$. Thus, if we require the boundary of the disc to pass through a given point $p = (\tilde{x}_1^0, \dots, \tilde{x}_n^0)$ of L, then the only possible map (up to reparametrization) is

$$u : w \mapsto (w\, \tilde{x}_1^0, \tilde{x}_2^0, \dots, \tilde{x}_n^0), \tag{4.4}$$

which in the original coordinates is exactly (4.3).

Moreover, it is easy to check (working component by component) that the map (4.4) is regular. In particular, its contribution to the count of holomorphic discs is ± 1, and if we equip L with the trivial spin structure, then the sign depends only on the dimension n, and not on the choice of the facet F or of the T^n-orbit L. Careful inspection of the sign conventions (see e.g. [10, 14, 28]) shows that the sign is $+1$.

The only remaining step in the proof of Proposition 4.3 is to estimate the symplectic area of the holomorphic disc (4.4). For this purpose, we first relabel the toric action so that it becomes standard in the affine chart associated to the vertex v. Namely, observe that the normal vectors $\nu(F_1) = \nu(F), \dots, \nu(F_n)$ to the facets which pass through v form a basis of $\mathbb{Z}^n$ dual to that given by the edge vectors $\eta_1, \dots, \eta_n$. If we precompose the T^n-action with the linear automorphism of T^n induced by the transformation $\sigma \in GL_n(\mathbb{Z})$ which maps the i-th vector of the standard basis to $\nu(F_i)$, then the relabelled action becomes standard in the coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$.

After this relabelling, the moment map becomes $\tilde{\phi} = \sigma^T \circ \phi$, and in a neighborhood of the vertex $\tilde{v} = \sigma^T(v)$ the moment polytope $\tilde{\Delta} = \sigma^T(\Delta)$ is a translate of the standard octant. In particular, denoting by $\tilde{\phi}_1$ the first component of $\tilde{\phi}$, the equation of the facet $\tilde{F} = \sigma^T(F)$ is simply $\tilde{\phi}_1 = -\alpha(F)$. Since u is equivariant with respect to the action of the first S^1 factor, integrating over the unit disc in polar coordinates $w = \rho e^{i\theta}$ we have

$$\int_{D^2} u^*\omega = \iint_{D^2} \omega(\partial_\rho u, \partial_\theta u)\, d\rho\, d\theta = \int_0^{2\pi} \int_0^1 d\tilde{\phi}_1(\partial_\rho u)\, d\rho\, d\theta = 2\pi(\tilde{\phi}_1(L) - \tilde{\phi}_1(u(0))).$$

Since $u(0) \in D_F$, we conclude that

$$\int_{D^2} u^*\omega = 2\pi(\tilde{\phi}_1(L) + \alpha(F)) = 2\pi\langle \nu(F), \phi(L)\rangle + 2\pi\alpha(F).$$

Incorporating the appropriate holonomy factor, we conclude that the contribution of u to the superpotential is precisely $e^{-2\pi\alpha(F)} z^{\nu(F)}$. □

4.2. Comparison with the Hori-Vafa mirror and renormalization

The formula (4.2) is identical to the well-known formula for the superpotential of the mirror to a toric manifold (see Section 5.3 of [18]). However, our mirror is "smaller" than the usual one, because the variables $(z_1, \dots, z_n)$ are constrained to lie in a bounded subset of $(\mathbb{C}^*)^n$. In particular, since the norm of each term in the sum (4.2) is bounded by 1 (as the symplectic area of a holomorphic disc is always positive), in our situation W is always bounded by the number of facets of the moment polytope Δ. While the "usual" mirror could be recovered by analytic continuation from M to all of $(\mathbb{C}^*)^n$ (or equivalently, by allowing the holonomy of the flat connection to take values in $\mathbb{C}^*$ instead of $U(1)$), there are various reasons for not proceeding in this manner, one of them being that the symplectic form $\omega^\vee$ on M blows up near the boundary.

In fact, our description of M resembles very closely one of the intermediate steps in Hori and Vafa's construction (see Section 3.1 of [18]). The dictionary between the two constructions is the following. Given a facet F of Δ, let $y_F = 2\pi\alpha(F) - \log(z^{\nu(F)})$, so that the real part of y_F is the symplectic area of one of the Maslov index 2 discs bounded by L and its imaginary part is (minus) the integral of the connection 1-form along its boundary. Then M is precisely the subset of $(\mathbb{C}^*)^n$ in which $\mathrm{Re}\,(y_F) > 0$ for all facets, and the Kähler form $\omega^\vee$ introduced in Definition 2.8 blows up for $\mathrm{Re}\,(y_F) \to 0$. This is

exactly the same behavior as in equation (3.22) of [18] (which deals with the case of a single variable y).

Hori and Vafa introduce a renormalization procedure which enlarges the mirror and flattens its Kähler metric. While the mathematical justification for this procedure is somewhat unclear, it is interesting to analyze it from the perspective of our construction. Hori and Vafa's renormalization process replaces the inequality $\mathrm{Re}\,(y_F) > 0$ by $\mathrm{Re}\,(y_F) > -k$ for some constant k (see equations (3.24) and (3.25) in [18]), without changing the formula for the superpotential. This amounts to enlarging the moment polytope by $\frac{1}{2\pi}k$ in all directions.

Assuming that X is Fano (or more generally that $-K_X$ is nef), another way to enlarge M in the same manner is to equip X with a "renormalized" Kähler form ω_k (compatible with the toric action) chosen so that $[\omega_k] = [\omega] + k\, c_1(X)$. Compared to Hori and Vafa's renormalization, this operation has the effect of not only extending the domain of definition of the superpotential, but also rescaling it by a factor of e^{-k}; however, if we simultaneously rescale the Kähler form on X and the superpotential, then we obtain a result consistent with Hori and Vafa's. This suggests:

Conjecture 4.4. *The construction of the mirror to a Fano manifold X should be carried out not by using the fixed Kähler form ω, but instead by considering a family of Kähler forms in the classes $[\omega_k] = [\omega] + k\, c_1(X)$, equipping the corresponding complexified moduli spaces of special Lagrangian tori with the rescaled superpotentials $e^k\, W_{(\omega_k)}$, and taking the limit as $k \to +\infty$.*

Of course, outside of the toric setting it is not clear what it means to "take the limit as $k \to +\infty$". A reasonable guess is that one should perform *symplectic inflation* along D, i.e. modify the Kähler form by (1,1)-forms supported in a small neighborhood $\mathcal{V}$ of D, constructed e.g. as suitable smooth approximations of the (1,1)-current dual to D. Special Lagrangians which lie in the complement of $\mathcal{V}$ are not affected by this process: given $L \subset X \setminus \mathcal{V}$, the only effect of the inflation procedure is that the symplectic areas of the Maslov index 2 discs bounded by L are increased by k; this is precisely compensated by the rescaling of the superpotential by a multiplicative factor of e^k. On the other hand, near D the change of Kähler form should "enlarge" the moduli space of special Lagrangians.

In the non-Fano case (more specifically when $-K_X$ is not nef), it is not clear how renormalization should be performed, or even whether it should be performed at all. For example, consider the Hirzebruch surface $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ for $m > 2$, as studied in §5.2 of [3]. The superpotential is given by

$$W = z_1 + z_2 + \frac{e^{-A}}{z_1 z_2^m} + \frac{e^{-B}}{z_2},$$

where A and B are the symplectic areas of the zero section (of square $+m$) and the fiber respectively, satisfying $A > mB$. An easy calculation shows that W has $m+2$ critical points in $(\mathbb{C}^*)^2$; the corresponding vanishing cycles generate the Fukaya category of this

Landau-Ginzburg model. As explained in [3], this is incorrect from the point of view of homological mirror symmetry, since the derived category of coherent sheaves of $\mathbb{F}_m$ is equivalent to a subcategory generated by only four of these $m+2$ vanishing cycles. An easy calculation shows that the z_2 coordinates of the critical points are the roots of the equation

$$z_2^{m-2}(z_2^2 - e^{-B})^2 - m^2 e^{-A} = 0.$$

Provided that $A > mB + O(\log m)$, one easily shows that only four of the roots lie in the range $e^{-B} < |z_2| < 1$ (and these satisfy $|z_1| < 1$ and $|e^{-A}/z_1 z_2^m| < 1$ as needed). This suggests that one should only consider $M = \text{Łog}^{-1}(\operatorname{int}\Delta) \subset (\mathbb{C}^*)^2$ rather than all of $(\mathbb{C}^*)^2$. (Note however that the behavior is not quite the expected one when A is too close to mB, for reasons that are not entirely clear).

Perhaps a better argument against renormalization (or analytic continuation) in the non-Fano case can be found in Abouzaid's work [1, 2]. Abouzaid's approach to homological mirror symmetry for toric varieties is to consider admissible Lagrangians which occur as sections of the Łog map with boundary in a suitable tropical deformation of the fiber of the Landau-Ginzburg model. More specifically, the deformed fiber lies near $\text{Łog}^{-1}(\Pi)$, where Π is the tropical hypersurface in $\mathbb{R}^n$ associated to a rescaling of the Laurent polynomial W; the interior of Δ is a connected component of $\mathbb{R}^n \setminus \Pi$, and Abouzaid only considers admissible Lagrangian sections of the Łog map over this connected component. Then the results in [1, 2] establish a correspondence between these sections and holomorphic line bundles over X.

When $-K_X$ is not nef (for example, for the Hirzebruch surface $\mathbb{F}_m$ with $m > 2$), $\mathbb{R}^n \setminus \Pi$ has more than one bounded connected component, and the other components also give rise to some admissible Lagrangians; however Abouzaid's work shows that those are not relevant to mirror symmetry for X, and that one should instead focus exclusively on those Lagrangians which lie in the bounded domain $M \subset (\mathbb{C}^*)^n$.

5. A non-toric example

The goal of this section is to work out a specific example in the non-toric setting, in order to illustrate some general features which are not present in the toric case, such as wall-crossing phenomena and quantum corrections. Let $X = \mathbb{CP}^2$, equipped with the standard Fubini-Study Kähler form (or a multiple of it), and consider the anticanonical divisor $D = \{(x : y : z),\ (xy - \epsilon z^2)z = 0\}$ (the union of the conic $xy = \epsilon z^2$ and the line $z = 0$), for some $\epsilon \neq 0$. We equip $\mathbb{CP}^2 \setminus D$ with the holomorphic (2,0)-form which in the affine coordinate chart $\{(x : y : 1),\ (x, y) \in \mathbb{C}^2\}$ is given by

$$\Omega = \frac{dx \wedge dy}{xy - \epsilon}.$$

5.1. A family of special Lagrangian tori

The starting point of our construction is the pencil of conics defined by the rational map $f : (x : y : z) \mapsto (xy : z^2)$. We will mostly work in affine coordinates, and think of f as

the map from $\mathbb{C}^2$ to $\mathbb{C}$ defined by $f(x,y) = xy$, suitably extended to the compactification. The fiber of f above any non-zero complex number is a smooth conic, while the fiber over 0 is the union of two lines (the x and y coordinate axes), and the fiber over ∞ is a double line.

The group S^1 acts on each fiber of f by $(x,y) \mapsto (e^{i\theta}x, e^{-i\theta}y)$. We will consider Lagrangian tori which are contained in $f^{-1}(\gamma)$ for some simple closed curve $\gamma \subset \mathbb{C}$, and consist of a single S^1-orbit inside each fiber a point of γ. Recall that the symplectic fibration f carries a natural horizontal distribution, given at every point by the symplectic orthogonal to the fiber. Parallel transport with respect to this horizontal distribution yields symplectomorphisms between smooth fibers, and $L \subset f^{-1}(\gamma)$ is Lagrangian if and only if it is invariant by parallel transport along γ.

Each fiber of f is foliated by S^1-orbits, and contains a distinguished orbit that we call the *equator*, namely the set of points where $|x| = |y|$. We denote by $\delta(x,y)$ the signed symplectic area of the region between the S^1-orbit through (x,y) and the equator in the fiber $f^{-1}(xy)$, with the convention that $\delta(x,y)$ is positive if $|x| > |y|$ and negative if $|x| < |y|$. Since S^1 acts by symplectomorphisms, parallel transport is S^1-equivariant. Moreover, the symplectic involution $(x,y) \mapsto (y,x)$ also preserves the fibers of f, and so parallel transport commutes with it. This implies that parallel transport maps equators to equators, and maps other S^1-orbits to S^1-orbits in a δ-preserving manner.

Definition 5.1. Given a simple closed curve $\gamma \subset \mathbb{C}$ and a real number $\lambda \in (-\Lambda, \Lambda)$ (where $\Lambda = \int_{\mathbb{CP}^1} \omega$ is the area of a line), we define

$$T_{\gamma,\lambda} = \{(x,y) \in f^{-1}(\gamma),\ \delta(x,y) = \lambda\}.$$

By construction $T_{\gamma,\lambda}$ is an embedded Lagrangian torus in $\mathbb{CP}^2$, except when $0 \in \gamma$ and $\lambda = 0$ (in which case it has a nodal singularity at the origin).

Moreover, when $0 \notin \gamma$, we say that $T_{\gamma,\lambda}$ is of *Clifford type* if γ encloses the origin, and of *Chekanov type* otherwise.

This terminology is motivated by the observation that the product tori $S^1(r_1) \times S^1(r_2) \subset \mathbb{C}^2$ (among which the Clifford tori) are of the form $T_{\gamma,\lambda}$ where γ is the circle of radius $r_1 r_2$ centered at the origin, whereas one way to define the so-called *Chekanov torus* [6, 12] is as $T_{\gamma,0}$ for γ a loop that does not enclose the origin (see [12]).

Recall that the anticanonical divisor D is the union of the fiber $f^{-1}(\epsilon)$ and the line at infinity. The following proposition motivates our interest in the tori $T_{\gamma,\lambda}$ in the specific case where $\gamma = \gamma(r)$ is a circle of radius r centered at ϵ.

Proposition 5.2. *The tori* $T_{\gamma(r),\lambda} = \{(x,y),\ |xy - \epsilon| = r,\ \delta(x,y) = \lambda\}$ *are special Lagrangian with respect to* $\Omega = (xy - \epsilon)^{-1}\, dx \wedge dy$.

Proof. Let $H(x,y) = |xy - \epsilon|^2$, and let X_H be the corresponding Hamiltonian vector field, i.e. the vector field such that $\iota_{X_H}\omega = dH$. We claim that X_H is everywhere tangent to $T_{\gamma(r),\lambda}$. In fact, H is constant over each fiber of f, so X_H is symplectically orthogonal to the fibers, i.e. it lies in the horizontal distribution. Moreover, X_H is tangent to the level

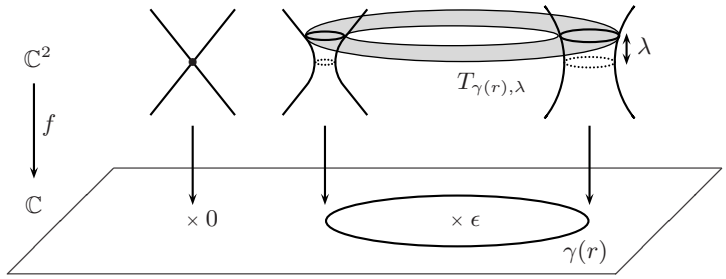

FIGURE 2. The special Lagrangian torus $T_{\gamma(r),\lambda}$

sets of H; so, up to a scalar factor, X_H is in fact the horizontal lift of the tangent vector to $\gamma(r)$, and thus it is tangent to $T_{\gamma(r),\lambda}$.

The tangent space to $T_{\gamma(r),\lambda}$ is therefore spanned by X_H and by the vector field generating the S^1-action, $\xi = (ix, -iy)$. However, we observe that

$$\iota_\xi \Omega = \frac{ix\,dy + iy\,dx}{xy - \epsilon} = i\,d\log(xy - \epsilon).$$

It follows that $\operatorname{Im}\Omega(\xi, X_H) = d\log|xy - \epsilon|\,(X_H)$, which vanishes since X_H is tangent to the level sets of H. Hence $T_{\gamma(r),\lambda}$ is special Lagrangian. □

Thus $\mathbb{CP}^2 \setminus D$ admits a fibration by special Lagrangian tori $T_{\gamma(r),\lambda}$, with a single nodal fiber $T_{\gamma(|\epsilon|),0}$. For $r < |\epsilon|$ the tori $T_{\gamma(r),\lambda}$ are of Chekanov type, while for $r > |\epsilon|$ they are of Clifford type. We shall now see that wall-crossing occurs for $r = |\epsilon|$, thus separating the moduli space into two chambers $r < |\epsilon|$ and $r > |\epsilon|$. We state the next two lemmas in a slightly more general context.

Lemma 5.3. *If $\gamma \subset \mathbb{C}$ is a simple closed loop and $w \in \mathbb{C}$ lies in the interior of γ, then for any class $\beta \in \pi_2(\mathbb{CP}^2, T_{\gamma,\lambda})$, the Maslov index is $\mu(\beta) = 2(\beta \cdot [f^{-1}(w)] + \beta \cdot [\mathbb{CP}^1_\infty])$, where $\mathbb{CP}^1_\infty$ is the line at infinity in $\mathbb{CP}^2$.*

Proof. If γ is a circle centered at w, then Proposition 5.2 implies that $T_{\gamma,\lambda}$ is special Lagrangian for $\Omega = (xy - w)^{-1} dx \wedge dy$, and the result is then a direct consequence of Lemma 3.1. The general case follows by continuously deforming γ to such a circle, without crossing w nor the origin, and keeping track of relative homotopy classes through this Lagrangian deformation, which affects neither the Maslov index nor the intersection numbers with $f^{-1}(w)$ and $\mathbb{CP}^1_\infty$. □

Using positivity of intersection, this lemma precludes the existence of holomorphic discs with negative Maslov index. Moreover:

Lemma 5.4. *The Lagrangian torus $T_{\gamma,\lambda}$ bounds a nontrivial holomorphic disc of Maslov index 0 if and only if $0 \in \gamma$.*

Proof. Assume there is a non-trivial holomorphic map $u : (D^2, \partial D^2) \to (\mathbb{CP}^2, T_{\gamma,\lambda})$ representing a class of Maslov index 0, and choose a point $w \in \mathbb{C}$ inside the region delimited by γ. By positivity of intersection and Lemma 5.3, the image of u must be disjoint from $f^{-1}(w)$ and from the line at infinity. The projection $f \circ u$ is therefore a well-defined holomorphic map from $(D^2, \partial D^2)$ to $(\mathbb{C}, \gamma)$, whose image avoids w.

It follows that $f \circ u$ is constant, i.e. the image of u is contained in the affine part of a fiber of f, say $f^{-1}(c)$ for some $c \in \gamma$. However, for $c \neq 0$ the affine conic $xy = c$ is topologically a cylinder $S^1 \times \mathbb{R}$, intersected by $T_{\gamma,\lambda}$ in an essential circle, which does not bound any nontrivial holomorphic disc. Therefore $c = 0$, and $0 \in \gamma$.

Conversely, if $0 \in \gamma$, we observe that $f^{-1}(0)$ is the union of two complex lines (the x and y coordinate axes), and its intersection with $T_{\gamma,\lambda}$ is a circle in one of them (depending on the sign of λ). Excluding the degenerate case $\lambda = 0$, it follows that $T_{\gamma,\lambda}$ bounds a holomorphic disc of area $|\lambda|$, contained in one of the coordinate axes; by Lemma 5.3 its Maslov index is 0. □

5.2. The superpotential

We now consider the complexified moduli space M associated to the family of special Lagrangian tori constructed in Proposition 5.2. The goal of this section is to compute the superpotential; by Lemma 5.4, the cases $r < |\epsilon|$ and $r > |\epsilon|$ should be treated separately.

We start with the Clifford case ($r > |\epsilon|$). By deforming continuously $\gamma(r)$ into a circle centered at the origin without crossing the origin, we obtain a Lagrangian isotopy from $T_{\gamma(r),\lambda}$ to a product torus $S^1(r_1) \times S^1(r_2) \subset \mathbb{C}^2$, with the property that the minimal Maslov index of a holomorphic disc remains at least 2 throughout the deformation. Therefore, by Remark 3.7, for each class β of Maslov index 2, the disc count $n_\beta(L)$ remains constant throughout the deformation. The product torus corresponds to the toric case considered in Section 4, so we can use Proposition 4.3.

Denote by z_1 and z_2 respectively the holomorphic coordinates associated to the relative homotopy classes β_1 and β_2 of discs parallel to the x and y coordinate axes in $(\mathbb{C}^2, S^1(r_1) \times S^1(r_2))$ via the formula (2.1). Then Proposition 4.3 implies:

Proposition 5.5. *For $r > |\epsilon|$, the superpotential is given by*

$$W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2}. \tag{5.1}$$

The first two terms in this expression correspond to sections of f over the disc Δ of radius r centered at ϵ (the first one intersecting $f^{-1}(0)$ at a point of the y-axis, while the second one hits the x-axis), whereas the last term corresponds to a disc whose image under f is a double cover of $\mathbb{CP}^1 \setminus \Delta$ branched at infinity.

Next we consider the case $r < |\epsilon|$, where $\gamma = \gamma(r)$ does not enclose the origin. We start with the special case $\lambda = 0$, which is the one considered by Chekanov and Eliashberg-Polterovich [6, 12].

The fibration f is trivial over the disc Δ bounded by γ, and over Δ it admits an obvious holomorphic section with boundary in $T_{\gamma,0}$, given by the portion of the line $y = x$ for which $x \in \sqrt{\Delta}$ (one of the two preimages of Δ under $z \mapsto z^2$). More generally, by considering the portion of the line $y = e^{2i\theta}x$ where $x \in e^{-i\theta}\sqrt{\Delta}$, and letting $e^{i\theta}$ vary in S^1, we obtain a family of holomorphic discs of Maslov index 2 with boundary in $T_{\gamma,0}$. One easily checks that these discs are regular, and that they boundaries sweep out $T_{\gamma,0}$ precisely once; we denote their class by β.

Other families of Maslov index 2 discs are harder to come by; the construction of one such family is outlined in an unfinished manuscript of Blechman and Polterovich, but the complete classification has only been carried out recently by Chekanov and Schlenk [7]. In order to state Chekanov and Schlenk's results, we need one more piece of notation. Given a line segment which joins the origin to a point $c = \rho e^{i\theta} \in \gamma$, consider the Lefschetz thimble associated to the critical point of f at the origin, i.e. the Lagrangian disc with boundary in $T_{\gamma,0}$ formed by the collection of equators in the fibers of f above the segment $[0, c]$; this is just a disc of radius $\sqrt{\rho}$ in the line $y = e^{i\theta}\bar{x}$. We denote by $\alpha \in \pi_2(\mathbb{CP}^2, T_{\gamma,0})$ the class of this disc; one easily checks that α, β, and $H = [\mathbb{CP}^1]$ form a basis of $\pi_2(\mathbb{CP}^2, T_{\gamma,0})$.

Lemma 5.6 (Chekanov-Schlenk [7])**.** *The only classes in $\pi_2(\mathbb{CP}^2, T_{\gamma,0})$ which may contain holomorphic discs of Maslov index 2 are β and $H - 2\beta + k\alpha$ for $k \in \{-1, 0, 1\}$.*

Proof. We compute the intersection numbers of α, β and H with the x-axis, the y-axis, and the fiber $f^{-1}(\epsilon)$, as well as their Maslov indices (using Lemma 5.3):

class	x-axis	y-axis	$f^{-1}(\epsilon)$	μ
α	-1	1	0	0
β	0	0	1	2
H	1	1	2	6

A class of Maslov index 2 is of the form $\beta + m(H - 3\beta) + k\alpha$ for $m, k \in \mathbb{Z}$; the constraints on m and k come from positivity of intersections. Considering the intersection number with $f^{-1}(\epsilon)$, we must have $m \leq 1$; and considering the intersection numbers with the x-axis and the y-axis, we must have $m \geq |k|$. It follows that the only possibilities are $m = k = 0$ and $m = 1$, $|k| \leq 1$. □

Proposition 5.7 (Chekanov-Schlenk [7])**.** *The torus $T_{\gamma,0}$ bounds a unique S^1-family of holomorphic discs in each of the classes β and $H - 2\beta + k\alpha$, $k \in \{-1, 0, 1\}$. These discs are regular, and the corresponding evaluation maps have degree 2 for $H - 2\beta$ and 1 for the other classes.*

Sketch of proof. We only outline the construction of the holomorphic discs in the classes $H - 2\beta + k\alpha$, following Blechman-Polterovich and Chekanov-Schlenk. The reader is referred to [7] for details and for proofs of uniqueness and regularity.

Let φ be a biholomorphism from the unit disc D^2 to the complement $\mathbb{CP}^1 \setminus \Delta$ of the disc bounded by γ, parametrized so that $\varphi(0) = \infty$ and $\varphi(a^2) = 0$ for some $a \in (0,1)$, and consider the double branched cover $\psi(z) = \varphi(z^2)$, which has a pole of order 2 at the origin and simple roots at $\pm a$. We will construct holomorphic maps $u : (D^2, \partial D^2) \to (\mathbb{CP}^2, T_{\gamma,0})$ such that $f \circ u = \psi$. Let

$$\tau_a(z) = \frac{z-a}{1-az}, \quad \tau_{-a}(z) = \frac{z+a}{1+az}, \quad \text{and} \quad g(z) = \frac{z^2\,\psi(z)}{\tau_a(z)\,\tau_{-a}(z)}.$$

Since $\tau_{\pm a}$ are biholomorphisms of the unit disc mapping $\pm a$ to 0, the map g is a nonvanishing holomorphic function over the unit disc, and hence we can choose a square root $\sqrt{g}$. Then for any $e^{i\theta} \in S^1$ we can consider the holomorphic maps

$$z \mapsto \big(e^{i\theta}\,\tau_a(z)\,\tau_{-a}(z)\,\sqrt{g(z)} : e^{-i\theta}\,\sqrt{g(z)} : z\big), \tag{5.2}$$

$$z \mapsto \big(e^{i\theta}\,\tau_a(z)\,\sqrt{g(z)} : e^{-i\theta}\,\tau_{-a}(z)\,\sqrt{g(z)} : z\big), \tag{5.3}$$

$$z \mapsto \big(e^{i\theta}\,\sqrt{g(z)} : e^{-i\theta}\,\tau_a(z)\,\tau_{-a}(z)\,\sqrt{g(z)} : z\big). \tag{5.4}$$

Letting u be any of these maps, it is easy to check that $f \circ u = \psi$, and that the first two components of u have equal norms when $|z| = 1$ (using the fact that $|\tau_a(z)| = |\tau_{-a}(z)| = 1$ for $|z| = 1$). So in all cases ∂D^2 is mapped to $T_{\gamma,0}$. One easily checks (e.g. using intersection numbers with the coordinate axes) that the classes represented by these maps are $H - 2\beta + k\alpha$ with $k = 1, 0, -1$ respectively for (5.2)–(5.4).

Chekanov and Schlenk show that these maps are regular, and that this list is exhaustive [7]. (In fact, since they enumerate discs whose boundary passes through a given point of $T_{\gamma,0}$, they also introduce a fourth map which differs from (5.3) by swapping τ_a and τ_{-a}; however this is equivalent to reparametrizing by $z \mapsto -z$).

Finally, the degrees of the evaluation maps are easily determined by counting the number of values of $e^{i\theta}$ for which the boundary of u passes through a given point of $T_{\gamma,0}$; however it is important to note here that, for the maps (5.2) and (5.4), replacing θ by $\theta + \pi$ yields the same disc up to a reparametrization ($z \mapsto -z$). □

By Lemma 5.4 and Remark 3.7, the disc counts remain the same in the general case (no longer assuming $\lambda = 0$), since deforming λ to 0 yields a Lagrangian isotopy from $T_{\gamma,\lambda}$ to $T_{\gamma,0}$ in the complement of $f^{-1}(0)$. Therefore, denoting by u and w the holomorphic coordinates on M associated to the classes β and α respectively, we have:

Proposition 5.8. *For $r < |\epsilon|$, the superpotential is given by*

$$W = u + \frac{e^{-\Lambda}}{u^2 w} + 2\,\frac{e^{-\Lambda}}{u^2} + \frac{e^{-\Lambda} w}{u^2} = u + \frac{e^{-\Lambda}(1+w)^2}{u^2 w}. \tag{5.5}$$

5.3. Wall-crossing, quantum corrections and monodromy

In this section, we compare the two formulas obtained for the superpotential in the Clifford and Chekanov cases (Propositions 5.5 and 5.8), in terms of wall-crossing at $r = |\epsilon|$. We start with a simple observation:

Lemma 5.9. *The expressions (5.1) and (5.5) are related by the change of variables* $u = z_1 + z_2$, $w = z_1/z_2$.

To see how this fits with the general discussion of wall-crossing in §3.3 and Proposition 3.9, we consider separately the two cases $\lambda > 0$ and $\lambda < 0$. We use the same notations as in the previous section concerning relative homotopy classes (β_1, β_2 on the Clifford side, β, α on the Chekanov side) and the corresponding holomorphic coordinates on M (z_1, z_2 and u, w).

First we consider the case where $\lambda > 0$, i.e. $T_{\gamma(r),\lambda}$ lies in the region where $|x| > |y|$. When $r = |\epsilon|$, $T_{\gamma(r),\lambda}$ intersects the x-axis in a circle, which bounds a disc u_0 of Maslov index 0. In terms of the basis used on the Clifford side, the class of this disc is $\beta_1 - \beta_2$; on the Chekanov side it is α.

As r decreases through $|\epsilon|$, two of the families of Maslov index 2 discs discussed in the previous section survive the wall-crossing: namely the family of holomorphic discs in the class β_2 on the Clifford side becomes the family of discs in the class β on the Chekanov side, and the discs in the class $H - \beta_1 - \beta_2$ on the Clifford side become the discs in the class $H - 2\beta - \alpha$ on the Chekanov side. This correspondence between relative homotopy classes determines the change of variables between the coordinate systems (z_1, z_2) and (u, w) of the two charts on M along the $\lambda > 0$ part of the wall:

$$\left\{ \begin{array}{rcl} \alpha & \leftrightarrow & \beta_1 - \beta_2 \\ \beta & \leftrightarrow & \beta_2 \\ H - 2\beta - \alpha & \leftrightarrow & H - \beta_1 - \beta_2 \end{array} \qquad \begin{array}{rcl} w & \leftrightarrow & z_1/z_2 \\ u & \leftrightarrow & z_2 \\ e^{-\Lambda}/u^2 w & \leftrightarrow & e^{-\Lambda}/z_1 z_2 \end{array} \right. \tag{5.6}$$

However, with this "classical" interpretation of the geometry of M the formulas (5.1) and (5.5) do not match up, and the superpotential presents a wall-crossing discontinuity, corresponding to the contributions of the various families of discs that exist only on one side of the wall. As r decreases through $|\epsilon|$, holomorphic discs in the class β_1 break into the union of a disc in the class $\beta = \beta_2$ and the exceptional disc u_0, and then disappear entirely. Conversely, new discs in the classes $H - 2\beta$ and $H - 2\beta + \alpha$ are generated by attaching u_0 to a disc in the class $H - \beta_1 - \beta_2 = H - 2\beta - \alpha$ at one or both of the points where their boundaries intersect. Thus the correspondence between the two coordinate charts across the wall should be corrected to:

$$\left\{ \begin{array}{rcl} \beta & \leftrightarrow & \{\beta_1, \beta_2\} \\ H - 2\beta + \{-1, 0, 1\}\alpha & \leftrightarrow & H - \beta_1 - \beta_2 \end{array} \qquad \begin{array}{rcl} u & \leftrightarrow & z_1 + z_2 \\ \dfrac{e^{-\Lambda}(1+w)^2}{u^2 w} & \leftrightarrow & \dfrac{e^{-\Lambda}}{z_1 z_2} \end{array} \right. \tag{5.7}$$

This corresponds to the change of variables $u = z_1 + z_2$, $w = z_1/z_2$ as suggested by Lemma 5.9; the formula for w is the same as in (5.6), but the formula for u is affected by a multiplicative factor $1 + w$, from $u = z_2$ to $u = z_1 + z_2 = (1 + w)z_2$. This is precisely the expected behavior in view of Proposition 3.9.

Remark 5.10. Given $c \in \gamma$, the class $\alpha = \beta_1 - \beta_2 \in \pi_2(\mathbb{CP}^2, T_{\gamma,\lambda})$ can be represented by taking the portion of $f^{-1}(c)$ lying between $T_{\gamma,\lambda}$ and the equator, which has symplectic

area λ, together with a Lagrangian thimble. Therefore $|w| = \exp(-\lambda)$. In particular, for $\lambda \gg 0$ the correction factor $1 + w$ is $1 + o(1)$.

The case $\lambda < 0$ can be analyzed in the same manner. For $r = |\epsilon|$ the Lagrangian torus $T_{\gamma(r),\lambda}$ now intersects the y-axis in a circle; this yields a disc of Maslov index 0 representing the class $\beta_2 - \beta_1 = -\alpha$. The two families of holomorphic discs that survive the wall-crossing are those in the classes β_1 and $H - \beta_1 - \beta_2$ on the Clifford side, which become β and $H - 2\beta + \alpha$ on the Chekanov side. Thus, the coordinate change along the $\lambda < 0$ part of the wall is

$$\left\{ \begin{array}{rcl} -\alpha & \leftrightarrow & \beta_2 - \beta_1 \\ \beta & \leftrightarrow & \beta_1 \\ H - 2\beta + \alpha & \leftrightarrow & H - \beta_1 - \beta_2 \end{array} \qquad \begin{array}{rcl} w^{-1} & \leftrightarrow & z_2/z_1 \\ u & \leftrightarrow & z_1 \\ e^{-\Lambda} w/u^2 & \leftrightarrow & e^{-\Lambda}/z_1 z_2 \end{array} \right. \tag{5.8}$$

However, taking wall-crossing phenomena into account, the correspondence should be modified in the same manner as above, from (5.8) to (5.7), which again leads to the change of variables $u = z_1 + z_2$, $w = z_1/z_2$; this time, the formula for u is corrected by a multiplicative factor $1 + w^{-1}$, from $u = z_1$ to $u = z_1 + z_2 = (1 + w^{-1})z_1$.

Remark 5.11. The discrepancy between the gluing formulas (5.6) and (5.8) is due to the monodromy of the family of special Lagrangian tori $T_{\gamma(r),\lambda}$ around the nodal fiber $T_{\gamma(|\epsilon|),0}$. The vanishing cycle of the nodal degeneration is the loop $\partial\alpha$, and in terms of the basis $(\partial\alpha, \partial\beta)$ of $H_1(T_{\gamma(r),\lambda}, \mathbb{Z})$, the monodromy is the Dehn twist

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This induces monodromy in the affine structures on the moduli space $\mathcal{B} = \{(r, \lambda)\}$ and its complexification M. Namely, M carries an integral (complex) affine structure given by the coordinates $(\log z_1, \log z_2)$ on the Clifford chamber $|r| > \epsilon$ and the coordinates $(\log w, \log u)$ on the Chekanov chamber $|r| < \epsilon$. Combining (5.6) and (5.8), moving around $(r, \lambda) = (|\epsilon|, 0)$ induces the transformation $(w, u) \mapsto (w, uw)$, i.e.

$$(\log w, \log u) \mapsto (\log w, \log u + \log w).$$

Therefore, in terms of the basis $(\partial_{\log u}, \partial_{\log w})$ of TM, the monodromy is given by the transpose matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Taking quantum corrections into account, the discrepancy in the coordinate transformation formulas disappears (the gluing map becomes (5.7) for both signs of λ), but the monodromy remains the same. Indeed, the extra factors brought in by the quantum corrections, $1 + w$ for $\lambda > 0$ and $1 + w^{-1}$ for $\lambda < 0$, are both of the form $1 + o(1)$ for $|\lambda| \gg 0$.

5.4. Another example: $\mathbb{CP}^1 \times \mathbb{CP}^1$

We now briefly discuss a related example: consider $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, equipped with a product Kähler form such that the two factors have equal areas, let D be the union of the two lines at infinity and the conic $\{xy = \epsilon\} \subset \mathbb{C}^2$, and consider the 2-form $\Omega = (xy - \epsilon)^{-1}\, dx \wedge dy$ on $X \setminus D$.

The main geometric features remain the same as before, the main difference being that the fiber at infinity of $f : (x, y) \mapsto xy$ is now a union of two lines $L^1_\infty = \mathbb{CP}^1 \times \{\infty\}$ and $L^2_\infty = \{\infty\} \times \mathbb{CP}^1$; apart from this, we use the same notations as in §5.1–5.3. In particular, it is easy to check that Proposition 5.2 still holds. Hence we consider the same family of special Lagrangian tori $T_{\gamma(r),\lambda}$ as above. Lemmas 5.3 and 5.4 also remain valid, except that the Maslov index formula in Lemma 5.3 becomes

$$\mu(\beta) = 2(\beta \cdot [f^{-1}(w)] + \beta \cdot [L^1_\infty] + \beta \cdot [L^2_\infty]). \tag{5.9}$$

In the Clifford case ($r > |\epsilon|$), the superpotential can again be computed by deforming to the toric case. Denoting again by z_1 and z_2 the holomorphic coordinates associated to the relative classes β_1 and β_2 parallel to the x and y coordinate axes in $(\mathbb{C}^2, S^1(r_1) \times S^1(r_2))$, we get

$$W = z_1 + z_2 + \frac{e^{-\Lambda_1}}{z_1} + \frac{e^{-\Lambda_2}}{z_2}, \tag{5.10}$$

where Λ_i are the symplectic areas of the two $\mathbb{CP}^1$ factors. (For simplicity we are only considering the special case $\Lambda_1 = \Lambda_2$, but we keep distinct notations in order to hint at the general case).

On the Chekanov side ($r < |\epsilon|$), we analyze holomorphic discs in $(\mathbb{CP}^1 \times \mathbb{CP}^1, T_{\gamma,0})$ similarly to the case of $\mathbb{CP}^2$. We denote again by β the class of the trivial section of f over the disc Δ bounded by γ, and by α the class of the Lefschetz thimble; and we denote by $H_1 = [\mathbb{CP}^1 \times \{pt\}]$ and $H_2 = [\{pt\} \times \mathbb{CP}^1]$. Then we have:

Proposition 5.12. *The only classes in $\pi_2(\mathbb{CP}^1 \times \mathbb{CP}^1, T_{\gamma,0})$ which may contain holomorphic discs of Maslov index 2 are β, $H_1 - \beta - \alpha$, $H_1 - \beta$, $H_2 - \beta$, and $H_2 - \beta + \alpha$. Moreover, $T_{\gamma,0}$ bounds a unique S^1-family of holomorphic discs in each of these classes, and the corresponding evaluation maps all have degree 1.*

Proof. We compute the intersection numbers of α, β, H_1 and H_2 with the coordinate axes, the fiber $f^{-1}(\epsilon)$, and the lines at infinity:

class	x-axis	y-axis	L^1_∞	L^2_∞	$f^{-1}(\epsilon)$	μ
α	-1	1	0	0	0	0
β	0	0	0	0	1	2
H_1	0	1	0	1	1	4
H_2	1	0	1	0	1	4

The Maslov index formula (5.9) and positivity of intersections with $f^{-1}(\epsilon)$, L^1_∞ and L^2_∞ imply that a holomorphic disc of Maslov index 2 must represent one of the classes $\beta + k\alpha$, $H_1 - \beta + k\alpha$, or $H_2 - \beta + k\alpha$, for some $k \in \mathbb{Z}$. Positivity of intersections with the x and y axes further restricts the list to the five possibilities mentioned in the statement of the proposition.

Discs in the class β are sections of f over the disc Δ bounded by γ; since they are contained in $\mathbb{C}^2$, they are the same as in the case of $\mathbb{CP}^2$. Discs in the other classes are sections of f over the complement $\mathbb{CP}^1 \setminus \Delta$. Denote by φ the biholomorphism from D^2 to $\mathbb{CP}^1 \setminus \Delta$ such that $\varphi(0) = \infty$ and $\varphi(a) = 0$ for some $a \in (0,1)$: we are looking for holomorphic maps $u : D^2 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ such that $f \circ u = \varphi$ and $u(\partial D^2) \subset T_{\gamma,0}$. Considering the map $q : (x,y) \mapsto x/y$, we see that $q \circ u$ has either a pole or a zero at 0 and at a, depending on the class represented by u, and takes non-zero complex values everywhere else; moreover it maps the unit circle to itself. It follows that $q \circ u$ has degree 2 and can be expressed as $q \circ u(z) = e^{2i\theta}\, z^{\pm 1} \tau_a(z)^{\pm 1}$, where $e^{i\theta} \in S^1$ and $\tau_a(z) = (z-a)/(1-az)$. Choosing a square root $\sqrt{h}$ of $h(z) = z\,\varphi(z)/\tau_a(z)$, we conclude that u is one of

$$z \mapsto (e^{i\theta}\, z^{-1}\, \tau_a(z)\, \sqrt{h(z)},\, e^{-i\theta} \sqrt{h(z)}), \qquad z \mapsto (e^{i\theta}\, \tau_a(z)\, \sqrt{h(z)},\, e^{-i\theta}\, z^{-1} \sqrt{h(z)}),$$
$$z \mapsto (e^{i\theta}\, z^{-1} \sqrt{h(z)},\, e^{-i\theta}\, \tau_a(z)\, \sqrt{h(z)}), \qquad z \mapsto (e^{i\theta} \sqrt{h(z)},\, e^{-i\theta}\, z^{-1}\, \tau_a(z) \sqrt{h(z)}).$$

□

As before, this implies:

Corollary 5.13. *For $r < |\epsilon|$, the superpotential is given by*

$$W = u + \frac{e^{-\Lambda_1}(1+w)}{uw} + \frac{e^{-\Lambda_2}(1+w)}{u}, \tag{5.11}$$

where u and w are the coordinates associated to the classes β and α respectively.

Comparing the formulas (5.10) and (5.11), we see that they are related by the change of variables

$$u = z_1 + z_2, \quad w = z_1/z_2. \tag{5.12}$$

As in the case of $\mathbb{CP}^2$, this can be understood in terms of wall-crossing and quantum corrections; the discussion almost identical to that in §5.3 and we omit it.

However, we would like to point out one a slightly disconcerting feature of this example. Since we have assumed that $\Lambda_1 = \Lambda_2 = \Lambda$, the right-hand side of (5.11) simplifies to $u + e^{-\Lambda}(1+w)^2/uw$; this Laurent polynomial has only two critical points, instead of four for the right-hand side of (5.10) ($z_1 = \pm e^{-\Lambda/2}$, $z_2 = \pm e^{-\Lambda/2}$). In particular, the critical value 0 is lost in the change of variables, which is unexpected considering the discussion after Proposition 3.9. The reason is of course that the change of variables (5.12) does not quite map $(\mathbb{C}^*)^2$ to itself, and the critical points where $z_1 + z_2 = 0$ are missing in the (u,w) picture.

6. Critical values and quantum cohomology

The goal of this section is to discuss a folklore result which asserts that the critical values of the mirror superpotential are the eigenvalues of quantum multiplication by $c_1(X)$. The argument we present is known to various experts in the field (Kontsevich, Seidel, ...), but to our knowledge it has not appeared in the literature. We state the result specifically in the toric case; however, there is a more general relation between the superpotential and $c_1(X)$, see Proposition 6.8 below.

Theorem 6.1. *Let X be a smooth toric Fano variety, and let $W : M \to \mathbb{C}$ be the mirror Landau-Ginzburg model. Then all the critical values of W are eigenvalues of the linear map $QH^*(X) \to QH^*(X)$ given by quantum cup-product with $c_1(X)$.*

6.1. Quantum cap action on Lagrangian Floer homology

The key ingredient in the proof of Theorem 6.1 is the *quantum cap action* of the quantum cohomology of X on Lagrangian Floer homology. While the idea of quantum cap action on Floer homology of symplectomorphisms essentially goes back to Floer [13], its counterpart in the Lagrangian setting has been much less studied; it can be viewed as a special case of Seidel's construction of open-closed operations on Lagrangian Floer homology (see e.g. Section 4 of [26]). We review the construction, following ideas of Seidel and focusing on the specific setting that will be relevant for Theorem 6.1.

The reader is also referred to Biran and Cornea's work [5], which gives a very detailed and careful account of this construction using a slightly different approach.

Let L be a compact oriented, relatively spin Lagrangian submanifold in a compact symplectic manifold (X^{2n}, ω) equipped with an almost-complex structure J, and let ∇ be a flat $U(1)$-connection on the trivial bundle over L. We start by describing the operation at the chain level. Following Fukaya-Oh-Ohta-Ono [14], we use singular chains as the starting point for the Floer complex, except we use complex coefficients and assume convergence of all power series. Moreover, for simplicity we quotient out by those chains whose support is contained in that of a lower-dimensional chain (this amounts to treating pseudocycles as honest cycles, and allows us to discard irrelevant terms even when working at the chain level).

Given a class $\beta \in \pi_2(X, L)$, we denote by $\hat{\mathcal{M}}(L, \beta)$ the space of J-holomorphic maps from $(D^2, \partial D^2)$ to (X, L) representing the class β (without quotienting by automorphisms of the disc). We denote by $\hat{ev}_{\beta,\pm 1} : \hat{\mathcal{M}}(L, \beta) \to L$ and $\hat{ev}_{\beta,0} : \hat{\mathcal{M}}(L, \beta) \to X$ the evaluation maps at the boundary points $\pm 1 \in \partial D^2$ and the interior point $0 \in D^2$. (So in fact we think of $\hat{\mathcal{M}}(L, \beta)$ as a moduli space of pseudoholomorphic discs with two marked points on the boundary and one marked point in the interior, constrained to lie on the geodesic between the two boundary marked points).

We will assume throughout this section that the spaces $\hat{\mathcal{M}}(L, \beta)$ carry well-defined fundamental chains (of dimension $n + \mu(\beta)$), and that the evaluation maps are transverse to the chains in L and X that we consider; typically it is necessary to introduce suitable

perturbations in order for these assumptions to hold, but none will be needed for the application that we have in mind.

Definition 6.2. Let $C \in C_*(L)$ and $Q \in C_*(X)$ be chains in L and X respectively, such that $C \times Q$ is transverse to the evaluation maps $\hat{ev}_{\beta,1} \times \hat{ev}_{\beta,0} : \hat{\mathcal{M}}(L,\beta) \to L \times X$. Then we define

$$Q \cap C = \sum_{\beta \in \pi_2(X,L)} z_\beta \; Q \cap_\beta C \in C_*(L), \tag{6.1}$$

where $z_\beta = \exp(-\int_\beta \omega)\mathrm{hol}_\nabla(\partial\beta)$ and

$$Q \cap_\beta C = (\hat{ev}_{\beta,-1})_*(\hat{ev}_{\beta,1} \times \hat{ev}_{\beta,0})^*(C \times Q).$$

In terms of the cohomological degrees $\deg(C) = n - \dim C$ and $\deg(Q) = 2n - \dim Q$, the term $Q \cap_\beta C$ has degree $\deg(C) + \deg(Q) - \mu(\beta)$.

Recall that the Floer differential $\delta = \mathfrak{m}_1 : C_*(L) \to C_*(L)$ is defined in terms of the moduli spaces of pseudoholomorphic discs with two marked points on the boundary, $\mathcal{M}_2(L,\beta) = \hat{\mathcal{M}}(L,\beta)/\mathbb{R}$ (where $\mathbb{R}$ is the stabilizer of $\{\pm 1\}$), and the corresponding evaluation maps $ev_{\beta,\pm 1} : \mathcal{M}_2(L,\beta) \to L$, by the formula

$$\delta(C) = \partial C + \sum_{\beta \neq 0} z_\beta \, \delta_\beta(C), \quad \text{where} \quad \delta_\beta(C) = (ev_{\beta,-1})_*(ev_{\beta,1})^*(C).$$

We denote by $\delta'(C) = \sum_{\beta \neq 0} z_\beta \delta_\beta(C)$ the "quantum" part of the differential.

Assuming there is no obstruction to the construction of Floer homology, the cap product (6.1) descends to a well-defined map

$$\cap : H_*(X) \otimes HF(L,L) \to HF(L,L).$$

In general, the failure of the cap product to be a chain map is encoded by a higher order operation defined as follows. Let $\hat{\mathcal{M}}_3^+(L,\beta) \simeq \hat{\mathcal{M}}(L,\beta) \times \mathbb{R}$ be the moduli space of J-holomorphic maps from $(D^2, \partial D^2)$ to (X,L), with one interior marked point at 0 and three marked points on the boundary at ± 1 and at $q = \exp(i\theta)$, $\theta \in (0,\pi)$. We denote by $\hat{ev}^+_{\beta,q} : \hat{\mathcal{M}}_3^+(L,\beta) \to L$ the evaluation map at the extra marked point. Define similarly the moduli space $\hat{\mathcal{M}}_3^-(L,\beta)$ of pseudoholomorpic discs with an extra marked point at $q = \exp(i\theta)$, $\theta \in (-\pi, 0)$, and the evaluation map $\hat{ev}^-_{\beta,q}$. Then given chains $C, C' \in C_*(L)$ and $Q \in C_*(X)$ in transverse position, we define

$$\mathfrak{h}^\pm(C,C',Q) = \sum_{\beta \in \pi_2(X,L)} z_\beta \; \mathfrak{h}^\pm_\beta(C,C',Q),$$

where $$\mathfrak{h}^\pm_\beta(C,C',Q) = (\hat{ev}_{\beta,-1})_*(\hat{ev}_{\beta,1} \times \hat{ev}^\pm_{\beta,q} \times \hat{ev}_{\beta,0})^*(C \times C' \times Q).$$

Note that the term $\mathfrak{h}^\pm_\beta(C,C',Q)$ has degree $\deg(C) + \deg(C') + \deg(Q) - \mu(\beta) - 1$. Also recall from §3 that the obstruction $\mathfrak{m}_0 \in C_*(L)$ is defined by

$$\mathfrak{m}_0 = \sum_{\beta \neq 0} z_\beta \; (ev_{\beta,1})_*[\mathcal{M}_1(L,\beta)],$$

where $\mathcal{M}_1(L,\beta)$ is the moduli space of pseudoholomorphic discs in the class β with a single boundary marked point.

Proposition 6.3. *Assume that all the chains are transverse to the appropriate evaluation maps. Then up to signs we have*

$$\delta(Q\cap C) = \pm(\partial Q)\cap C \pm Q\cap\delta(C) \pm \mathfrak{h}^+(C,\mathfrak{m}_0,Q) \pm \mathfrak{h}^-(C,\mathfrak{m}_0,Q). \tag{6.2}$$

Sketch of proof. The boundary $\partial(Q\cap C)$ of the chain $Q\cap C$ consists of several pieces, corresponding to the various possible limit scenarios:

(1) One of the two input marked points is mapped to the boundary of the chain on which it is constrained to lie. The corresponding terms are $(\partial Q)\cap C$ and $Q\cap(\partial C)$.

(2) Bubbling occurs at one of the boundary marked points (equivalently, after reparametrizing this corresponds to the situation where the interior marked point which maps to Q converges to one of the boundary marked points). The case where the bubbling occurs at the incoming marked point $+1$ yields a term $Q\cap\delta'(C)$, while the case where the bubbling occurs at the outgoing marked point -1 yields a term $\delta'(Q\cap C)$.

(3) Bubbling occurs at some other point of the boundary of the disc, i.e. we reach the boundary of $\hat{\mathcal{M}}(L,\beta)$; the resulting contributions are $\mathfrak{h}^+(C,\mathfrak{m}_0,Q)$ when bubbling occurs along the upper half of the boundary, and $\mathfrak{h}^-(C,\mathfrak{m}_0,Q)$ when it occurs along the lower half. □

We will consider specifically the case where L does not bound any non-constant pseudoholomorphic discs of Maslov index less than 2; then the following two lemmas show that Floer homology and the quantum cap action are well-defined. (In fact, it is clear from the arguments that the relevant property is the fact that $\mathfrak{m}_0$ is a scalar multiple of the fundamental class $[L]$). The following statement is part of the machinery developed by Fukaya, Oh, Ohta and Ono [14] (see also [8]):

Lemma 6.4. *Assume that L does not bound any non-constant pseudoholomorphic discs of Maslov index less than 2. Then:*

(1) *$\mathfrak{m}_0$ is a scalar multiple of the fundamental class $[L]$;*
(2) *the Floer cohomology $HF(L,L)$ is well-defined, and $[L]$ is a Floer cocycle;*
(3) *the chain-level product $\mathfrak{m}_2$ defines a well-defined associative product on $HF(L,L)$, for which $[L]$ is a unit.*

Sketch of proof. (1) The virtual dimension of $\mathcal{M}_1(L,\beta)$ is $n-2+\mu(\beta)$, so for degree reasons the only non-trivial contributions to $\mathfrak{m}_0$ come from classes of Maslov index 2, and $\mathfrak{m}_0$ is an n-chain; moreover, minimality of the Maslov index precludes bubbling, so that $\mathfrak{m}_0$ is actually a cycle, i.e. a scalar multiple of the fundamental class $[L]$.

(2) It is a well-known fact in Floer theory (see e.g. [14]) that the operations $(\mathfrak{m}_k)_{k\geq 0}$ satisfy the A_∞ equations. In particular, for all $C\in C_*(L)$ we have

$$\mathfrak{m}_1(\mathfrak{m}_1(C)) + \mathfrak{m}_2(C,\mathfrak{m}_0) + (-1)^{\deg(C)+1}\mathfrak{m}_2(\mathfrak{m}_0,C) = 0. \tag{6.3}$$

To prove that $\mathfrak{m}_1$ $(= \delta)$ squares to zero, it is enough to show that

$$\mathfrak{m}_2(C, [L]) = C \quad \text{and} \quad \mathfrak{m}_2([L], C) = (-1)^{\deg(C)}\, C, \tag{6.4}$$

since it then follows that the last two terms in (6.3) cancel each other.

Recall that the products $\mathfrak{m}_2(C, [L])$ and $\mathfrak{m}_2([L], C)$ are defined by considering for each class $\beta \in \pi_2(X, L)$ the moduli space of J-holomorphic discs with three boundary marked points in the class β, requiring two of the marked points to map to C and $[L]$ respectively, and evaluating at the third marked point. However, the incidence condition corresponding to the chain $[L]$ is vacuous; so, provided that $\beta \neq 0$, by forgetting the unnecessary marked point we see that the construction yields a chain whose support is contained in that of $\delta_\beta(C)$, which has dimension one less. It follows that the only nontrivial contribution comes from constant discs; that contribution is precisely C, up to a sign factor left to the reader.

The fact that $[L]$ is a Floer cocycle follows from the observation that, for any relative class $\beta \neq 0$ containing holomorphic discs, $\delta_\beta([L])$ is a chain of dimension $n - 1 + \mu(\beta) \geq n + 1$ in L, and hence trivial. It follows that $\delta([L]) = \partial[L] = 0$.

(3) In the unobstructed case, the compatibility of $\mathfrak{m}_2$ with the Floer differential and its associativity at the level of cohomology follow from the A_∞ equations. When $\mathfrak{m}_0$ is nonzero, terms of the form $\mathfrak{m}_k(\dots, \mathfrak{m}_0, \dots)$ appear in these equations and make them harder to interpret geometrically. However, we observe that $\mathfrak{m}_k(\dots, [L], \dots) \equiv 0$ for all $k \geq 3$. Indeed, $\mathfrak{m}_k(a_{k-1}, \dots, a_i, [L], a_{i-1}, \dots, a_1)$ counts pseudoholomorphic discs with $k + 1$ boundary marked points, where the incidence condition at one of the marked points is given by the fundamental cycle $[L]$ and hence vacuous; as above, deleting this extraneous marked point shows that the contribution of each relative class $\beta \in \pi_2(X, L)$ to $\mathfrak{m}_k(a_{k-1}, \dots, a_i, [L], a_{i-1}, \dots, a_1)$ has support contained in that of the corresponding contribution to $\mathfrak{m}_{k-1}(a_{k-1}, \dots, a_1)$, which has dimension one less. Hence we can ignore all the terms involving $\mathfrak{m}_0$, and the properties of $\mathfrak{m}_2$ are the same as in the unobstructed case.

Finally, the fact that $[L]$ is a unit for the product in Floer cohomology follows directly from (6.4) (recalling that the sign conventions in A_∞-algebras are different from those of usual differential graded algebras, see e.g. [14, 28]). □

Lemma 6.5. *Assume that L does not bound any non-constant pseudoholomorphic discs of Maslov index less than 2. Then the cap product descends to a well-defined map $\cap : H_*(X) \otimes HF(L, L) \to HF(L, L)$.*

Proof. By the previous lemma, $\mathfrak{m}_0$ is a scalar multiple of $[L]$. Next we observe that, in the construction of $\mathfrak{h}^\pm(C, [L], Q)$, the incidence constraint at the extra marked point $q = \exp(i\theta)$ is vacuous. So the support of each chain $\mathfrak{h}^\pm_\beta(C, [L], Q)$ is contained in that of the chain $Q \cap_\beta C$, which has dimension one less. This allows us to discard the terms $\mathfrak{h}^\pm(C, \mathfrak{m}_0, Q)$ in (6.2).

Therefore, Proposition 6.3 implies that, if Q is a cycle ($\partial Q = 0$) and C is a Floer cocycle ($\delta(C) = 0$), then $Q \cap C$ determines a Floer cocycle, whose class depends only on the classes of Q and C. $\square$

Next we show that the cap product makes $HF(L, L)$ a module over the quantum cohomology ring of X. We denote by $*$ the quantum cup-product on $QH^*(X) = H^*(X, \mathbb{C})$, working again with complex coefficients, i.e., specializing the Novikov parameters appropriately so that J-holomorphic spheres in a class $A \in H_2(X)$ are counted with a coefficient $\exp(-\int_A \omega)$, and assuming convergence as usual. Moreover, we use Poincaré duality and work with homology instead of cohomology.

Proposition 6.6. *Assume that $\mathfrak{m}_0$ is a multiple of $[L]$, so that the cap product $\cap$: $H_*(X) \otimes HF(L, L) \to HF(L, L)$ is well-defined. Then for any $[C] \in HF(L, L)$,*

$$[X] \cap [C] = [C],$$

and for any $[Q_1], [Q_2] \in H_*(X)$,

$$[Q_1] \cap ([Q_2] \cap [C]) = ([Q_1] * [Q_2]) \cap [C].$$

Sketch of proof. We first show that $[X]$ acts by identity. Observe that, in the definition of the cap product, the incidence constraint at the interior marked point $0 \in D^2$ is vacuous when $Q = [X]$. So for $\beta \neq 0$ the support of the chain $[X] \cap_\beta C$ is contained in that of the chain $\delta_\beta(C)$, which has dimension one less. Hence, nonconstant holomorphic discs contribute trivially to $[X] \cap [C]$. On the other hand the contribution of constant discs is just the classical intersection of chains, so that $[X] \cap [C] = [C]$.

To prove the second part of the proposition, consider the moduli space $\hat{\mathcal{M}}^{(2)}(L, \beta)$ of J-holomorphic maps from $(D^2, \partial D^2)$ to (X, L) with two boundary marked points at ± 1 and two interior marked points on the real axis, at $-1 < q_1 < q_2 < 1$ (up to simultaneous translation). Denote by $\hat{ev}_{\beta,q_1}$ and $\hat{ev}_{\beta,q_2}$ the evaluation maps at the interior marked points, and define

$$\Theta(Q_1, Q_2, C) = \sum_{\beta \in \pi_2(X,L)} z_\beta \ (\hat{ev}_{\beta,-1})_*(\hat{ev}_{\beta,1} \times \hat{ev}_{\beta,q_1} \times \hat{ev}_{\beta,q_2})^*(C \times Q_1 \times Q_2).$$

Given representatives Q_1, Q_2 of the given classes $[Q_1], [Q_2]$ and a chain $C \in C_*(L)$, assuming transversality as usual, a case-by-case analysis similar to the proof of Proposition 6.3 shows that

$$\delta(\Theta(Q_1, Q_2, C)) = \pm\Theta(Q_1, Q_2, \delta(C)) \pm Q_1 \cap (Q_2 \cap C) \pm (Q_1 * Q_2) \cap C \pm (\mathfrak{m}_0\text{-terms}).$$

More precisely, the boundary of the chain $\Theta(Q_1, Q_2, C)$ consists of:

(1) $\Theta(Q_1, Q_2, \partial C)$, corresponding to the situation where the input marked point at $+1$ is mapped to the boundary of the chain C. (Note that since Q_1 and Q_2 are cycles, we do not include the two terms $\Theta(\partial Q_1, Q_2, C)$ and $\Theta(Q_1, \partial Q_2, C)$ which would be present in the general case.)

(2) $\delta'(\Theta(Q_1, Q_2, C))$ and $\Theta(Q_1, Q_2, \delta'(C))$, corresponding to bubbling at one of the boundary marked points ± 1 (or equivalently after reparametrization, the situation where the interior marked points q_1, q_2 both converge to ± 1).

(3) $Q_1 \cap (Q_2 \cap C)$, corresponding to the situation where $q_1 \to -1$ (or equivalently up to reparametrization, $q_2 \to 1$), resulting in a two-component map with one interior marked point in each component.

(4) $(Q_1 * Q_2) \cap C$, corresponding to the situation where the two marked points q_1 and q_2 come together, leading to the bubbling of a sphere component which carries both marked points, attached to the disc at a point on the real axis.

(5) terms involving $\mathfrak{m}_0$ and higher order operations defined analogously to $\mathfrak{h}^{\pm}$, involving moduli spaces of discs with three marked points on the boundary and two in the interior; these occur as in the proof of Proposition 6.3 when bubbling occurs at a point of $\partial D^2 \setminus \{\pm 1\}$.

By the same argument as in the proof of Lemma 6.5, when $\mathfrak{m}_0$ is a multiple of $[L]$ we can safely ignore the last set of terms because the corresponding chains are supported on lower-dimensional subsets. Thus, if we assume that C is a Floer cocycle (i.e., $\delta(C) = 0$), the above formula implies that the Floer cocycles $Q_1 \cap (Q_2 \cap C)$ and $(Q_1 * Q_2) \cap C$ represent the same Floer cohomology class (up to signs, which are left to the reader). □

6.2. Cap product by $c_1(X)$ and proof of Theorem 6.1

Let (X, ω, J) be a smooth compact Kähler manifold of complex dimension n, equipped with a holomorphic n-form Ω defined over the complement of an anticanonical divisor D. Let $L \subset X \setminus D$ be a special Lagrangian submanifold, or more generally a Lagrangian submanifold whose Maslov class vanishes in $X \setminus D$ (so that Lemma 3.1 holds), and let ∇ be a flat $U(1)$-connection on the trivial line bundle over L. We assume that L does not bound any nonconstant holomorphic disc of Maslov index less than 2, so that the Floer obstruction $\mathfrak{m}_0$ is a constant multiple of the fundamental class, $\mathfrak{m}_0 = m_0(L, \nabla)\,[L]$, and Floer homology and the quantum cap product are well-defined.

Lemma 6.7. $c_1(X) \cap [L] = m_0(L, \nabla)\,[L]$.

Proof. We actually compute the cap product $[D] \cap [L]$. Since $[D] \cap_\beta [L]$ is a chain of dimension $n - 2 + \mu(\beta)$ in L, the only contributions to $[D] \cap [L]$ come from classes of Maslov index at most 2. Moreover, since $L \subset X \setminus D$, there are no contributions from constant discs, so we only need to consider classes with $\mu(\beta) = 2$.

By Lemma 3.1 every holomorphic map $u : (D^2, \partial D^2) \to (X, L)$ of Maslov index 2 intersects D in a single point $u(z_0)$, $z_0 \in D^2$. Moreover, for every $q = e^{i\theta} \in \partial D^2$ there exists a unique automorphism of D^2 which maps z_0 to 0 and q to -1. It follows that $[D] \cap_\beta [L]$ is the chain consisting of all boundary points of all holomorphic discs in the class β, i.e. $[D] \cap_\beta [L] = (ev_{\beta,1})_*[\mathcal{M}_1(L, \beta)]$. Summing over β, we conclude that $[D] \cap [L] = \mathfrak{m}_0 = m_0(L, \nabla)\,[L]$. □

Lemma 6.7 implies the following proposition, which is the core of Theorem 6.1.

Proposition 6.8. *If $HF(L,L) \neq 0$ then $m_0(L,\nabla)$ is an eigenvalue of the linear map $\Lambda : QH^*(X) \to QH^*(X)$ defined by $\Lambda(\alpha) = \alpha * c_1(X)$.*

Proof. Since $[L]$ is the unit for the product on Floer cohomology, the assumption that $HF(L,L) \neq 0$ implies that $[L]$ is a nonzero element of $HF(L,L)$. Lemma 6.7 states that $(c_1(X) - m_0(L,\nabla)) \cap [L] = 0$. But then Proposition 6.6 implies that quantum cup-product by $c_1(X) - m_0(L,\nabla)$ is not invertible. □

The only remaining ingredient of the proof of Theorem 6.1 is to show that critical points of the superpotential correspond to special Lagrangians with nonzero Floer homology. This follows from a general interpretation of the k-th derivatives of the superpotential in terms of $\mathfrak{m}_k$, at least in the toric Fano case (see the works of Cho and Oh [10, 9]). Theorem 10.1 in [10] states that, if X is a toric Fano variety and L is a toric fiber, then $HF(L,L)$ is non-zero if and only if the contributions of Maslov index 2 classes to $\delta([pt])$ cancel out in $H_1(L)$. In our terminology, the statement is:

Proposition 6.9 (Cho-Oh [10])**.** *Let L be a toric fiber in a toric Fano variety, equipped with a flat $U(1)$ connection ∇. Then $HF(L,L) \neq 0$ if and only if*

$$m_1(L,\nabla) := \sum_{\mu(\beta)=2} n_\beta(L) \exp(-\textstyle\int_\beta \omega) \mathrm{hol}_\nabla(\partial\beta)\, [\partial\beta] = 0 \in H_1(L,\mathbb{C}). \tag{6.5}$$

The "only if" part actually holds even in the non-toric case, assuming the minimal Maslov index of a holomorphic disc to be 2. Indeed, it is easy to check that if C is a codimension 1 cycle in L (i.e. $\partial C = 0$) then

$$\delta(C) = \pm([C] \cdot m_1(L,\nabla))\, [L] \tag{6.6}$$

(classes of Maslov index > 2 do not contribute to $\delta(C)$ for dimension reasons), so that when $m_1(L,\nabla) \neq 0$ the fundamental class $[L]$ can be realized as a Floer coboundary. However, to our knowledge the "if" part of the statement has not been proved outside of the toric case; the argument in [10] relies on the specific features of holomorphic discs in toric varieties to show that classes of Maslov index > 2 never contribute to the Floer differential (Proposition 7.2 in [10]), so that (6.6) holds for cycles of any dimension, and vanishing of $m_1(L,\nabla)$ implies nontriviality of the Floer homology.

Finally, recall from Section 2 that $T_{(L,\nabla)}M \simeq \mathcal{H}^1_\psi(L) \otimes \mathbb{C} \simeq H^1(L,\mathbb{C})$, by mapping $(v,\alpha) \in T_{(L,\nabla)}M \subset C^\infty(NL) \oplus \Omega^1(L,\mathbb{R})$ to $[-\iota_v\omega + i\alpha]$. Then we have:

Lemma 6.10. *The differential of $W = m_0 : M \to \mathbb{C}$ is*

$$dW_{(L,\nabla)}(v,\alpha) = \langle [-\iota_v\omega + i\alpha], m_1(L,\nabla)\rangle.$$

Proof. Let $z_\beta = \exp(-\int_\beta \omega)\mathrm{hol}_\nabla(\partial\beta)$, and observe as in the proof of Lemma 2.7 that $d\log z_\beta(v,\alpha) = \langle [-\iota_v\omega + i\alpha], [\partial\beta]\rangle$ (by Stokes' theorem). Hence, the differential of $W = \sum n_\beta(L)\, z_\beta$ is $dW(v,\alpha) = \sum n_\beta(L)\, z_\beta\, \langle [-\iota_v\omega + i\alpha], [\partial\beta]\rangle$. □

Theorem 6.1 now follows from Proposition 6.8, Proposition 6.9 and Lemma 6.10: if (L,∇) is a critical point of W then by Lemma 6.10 it satisfies $m_1(L,\nabla) = 0$, and hence by

Proposition 6.9 the Floer cohomology $HF(L, L)$ is nontrivial. Proposition 6.8 then implies that the critical value $W(L, \nabla) = m_0(L, \nabla)$ is an eigenvalue of quantum multiplication by $c_1(X)$.

7. Admissible Lagrangians and the reference fiber

In this section we give a brief, conjectural discussion of the manner in which the mirror construction discussed in the preceding sections relates to mirror symmetry for the Calabi-Yau hypersurface $D \subset X$. For simplicity, unless otherwise specified we assume throughout this section that D is smooth.

7.1. The boundary of M and the reference fiber

Denote by $\sigma \in H^0(X, K_X^{-1})$ the defining section of D, and identify a tubular neighborhood U of D with a neighborhood of the zero section in the normal bundle $N_D \simeq (K_X^{-1})_{|D}$, denoting by $p : U \to D$ the projection. Then we have:

Lemma 7.1. *D carries a nonvanishing holomorphic $(n-1)$-form Ω_D, called the* residue *of Ω along D, such that, in a neighborhood of D,*

$$\Omega = \sigma^{-1} d\sigma \wedge p^* \Omega_D + O(1). \tag{7.1}$$

Note that, even though $\sigma^{-1} d\sigma$ depends on the choice of a holomorphic connection on K_X^{-1} (one can e.g. use the Chern connection), it only does so by a bounded amount, so this ambiguity has no incidence on (7.1). The choice of the projection $p : U \to D$ does not matter either, for a similar reason.

Proof. Near any given point $q \in D$, choose local holomorphic coordinates $(x_1, \dots, x_n)$ on X such that D is the hypersurface $x_1 = 0$. Then locally we can write

$$\Omega = x_1^{-1} h(x_1, x_2, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

for some nonvanishing holomorphic function h. We set $\Omega_D = h(0, x_2, \dots, x_n)\, dx_2 \wedge \cdots \wedge dx_n$; in other terms, $\Omega_D = (x_1 \iota_{\partial_{x_1}} \Omega)_{|D}$.

If we change the coordinate system to a different one $(y_1, \dots, y_n)$ for which D is again defined by $y_1 = 0$, then $x_1 = y_1\, \phi(y_1, \dots, y_n)$ for some nonvanishing holomorphic function ϕ, so that $x_1^{-1}\, dx_1 = y_1^{-1}\, dy_1 + d(\log \phi)$. Therefore, denoting by $\mathcal{J}$ the Jacobian of the change of variables $(x_2, \dots, x_n) \mapsto (y_2, \dots, y_n)$ on D, we have $x_1^{-1} h\, dx_1 \wedge \cdots \wedge dx_n = (y_1^{-1} h\, \mathcal{J} + O(1))\, dy_1 \wedge \cdots \wedge dy_n$. Hence Ω_D is well-defined.

Finally, equation (7.1) follows by considering a coordinate system in which the first coordinate is exactly σ in a local trivialization of K_X^{-1} and the other coordinates are pulled back from D by the projection p. □

Lemma 7.1 shows that D (equipped with the restricted complex structure and symplectic form, and with the volume form Ω_D) is a Calabi-Yau manifold.

Remark 7.2. If D has normal crossing singularities, then the same construction yields a holomorphic $(n-1)$-form Ω_D which has poles along the singular locus of D.

Assume that Λ is a special Lagrangian submanifold in (D, Ω_D): then, in favorable cases, we can try to look for special Lagrangian submanifolds in $X \setminus D$ which are S^1-fibered over Λ. For example, if we are in a product situation, i.e. locally $X = \mathbb{C} \times D$, with the product complex structure and Kähler form, and $\Omega = x^{-1}dx \wedge \Omega_D$, then $S^1(r) \times \Lambda$ is special Lagrangian in $(X \setminus D, \Omega)$.

In general, the classical symplectic neighborhood theorem implies that U is symplectomorphic to a neighborhood of the zero section in a symplectic vector bundle over D. It is then easy to see that the preimage of Λ is foliated by Lagrangian submanifolds (which intersect each fiber in concentric circles). Due to the lack of compatibility between the standard symplectic chart and the holomorphic volume form, these submanifolds are not special Lagrangian in general, but Lemma 7.1 implies that, as the radius of the circle in the fiber tends to zero (i.e., as the submanifolds get closer to D), they become closer and closer to being special Lagrangian. Thus it is reasonable to hope for the existence of a nearby family of special Lagrangian submanifolds.

In terms of the moduli space M of pairs (L, ∇) consisting of a special Lagrangian submanifold of $(X \setminus D, \Omega)$ and a flat $U(1)$-connection on the trivial bundle over L, this suggests the following (somewhat optimistic) conjecture:

Conjecture 7.3. *Near its boundary, M consists of pairs (L, ∇) such that the Lagrangian submanifold $L \subset U \cap (X \setminus D)$ is a circle bundle over a special Lagrangian submanifold of D, with the additional property that every fiber bounds a holomorphic disc of Maslov index 2 contained in U.*

The main evidence for this conjecture is given by the examples in Sections 4 and 5 above. In those examples D has normal crossing singularities, along which Ω_D has poles, so special Lagrangian submanifolds of D are defined as in Section 2. In this setting, the moduli space M also has corners, corresponding to the situation where L lies close to the singular locus of D. Apart from this straightforward adaptation, the boundary structure of the moduli space of special Lagrangians is exactly as described by the conjecture.

More precisely, as one approaches the boundary of M, the special Lagrangian submanifold L collapses to a special Lagrangian submanifold Λ of D, and the collapsing makes L a (topologically trivial) S^1-bundle over Λ. Moreover, each circle fiber bounds a small holomorphic disc which intersects D transversely in a single point; we denote by $\delta \in \pi_2(X, L)$ the homotopy class of these discs. As L collapses onto Λ, the symplectic area $\int_\delta \omega$ shrinks to zero; in terms of the variable $z_\delta = \exp(-\int_\delta \omega)\mathrm{hol}_\nabla(\partial\delta)$, we get $|z_\delta| \to 1$. In other terms, Conjecture 7.3 implies that ∂M is defined by the equation $|z_\delta| = 1$.

Among the points of ∂M, those where $z_\delta = 1$ stand out, because they correspond to the situation where the holonomy of ∇ is trivial along the fiber of the S^1-bundle $L \to \Lambda$, i.e. ∇ is lifted from a connection on the trivial bundle over Λ. The set of such points can therefore be identified with a moduli space M_D of pairs of special Lagrangian submanifolds in D and flat $U(1)$-connections over them.

Conjecture 7.4. *The subset $M_D = \{z_\delta = 1\} \subset \partial M$ is the Strominger-Yau-Zaslow mirror of D.*

Assuming these conjectures, it is tempting to think of M_D as the *reference fiber* (or "fiber at infinity") of the Landau-Ginzburg model $W : M \to \mathbb{C}$. Of course, M_D is not actually a fiber of W; but the contributions of other relative homotopy classes to the superpotential are negligible compared to z_δ, at least in the rescaling limit suggested by Conjecture 4.4. So, near ∂M, we expect to have $W = z_\delta + o(1)$, and the fiber $W^{-1}(1)$ (when it is well-defined, which is not always the case considering wall-crossing and boundary phenomena) can essentially be identified with M_D.

Moreover, we expect the boundary of M to fiber over S^1. Indeed, we can set the holonomy of ∇ along the fiber of the S^1-bundle $L \to \Lambda$ to equal any given unit complex number, rather than 1. Thus we have:

Proposition 7.5. *Assuming Conjecture 7.3, the map $z_\delta : \partial M \to S^1$ is a fibration with fiber M_D.*

A hasty look at the situation might lead one to conclude, incorrectly, that the fibration $z_\delta : \partial M \to S^1$ is trivial. In fact, the symplectic monodromy of this fibration, viewed as an autoequivalence of the Fukaya category of M_D, is expected to be mirror to the autoequivalence of $D^b Coh(D)$ induced by $\mathcal{E} \mapsto (K_X)_{|D} \otimes \mathcal{E}$.

For example, consider the case where $X = \mathbb{CP}^2$ and D is a smooth elliptic curve obtained by a small generic deformation of the union of the coordinate lines. While we do not have an explicit description of M in this setting, one can use the toric model as a starting point, and the example in Section 5 as an indication of how the smoothing of D affects the geometry of M near the corners of the moment polytope. Let L be a special Lagrangian torus which lies close to the portion of D located along the x and y coordinate axes: then δ is the relative homotopy class called β in §5; using the same notations as in §5, $\partial\beta$ is the fiber of the S^1-bundle $L \to \Lambda$, while $\partial\alpha$ is a section. If we move Λ by translations all around the elliptic curve D, and look at the corresponding family of special Lagrangians $L \subset X \setminus D$, a short calculation shows that the monodromy acts on $H_1(L)$ by

$$\partial\beta \mapsto \partial\beta, \quad \partial\alpha \mapsto \partial\alpha + 9\,\partial\beta. \tag{7.2}$$

Observe that $M_{D,\theta} = \{z_\beta = e^{i\theta}\} \subset \partial M$ is an S^1-bundle over S^1, where the base corresponds to the moduli space $\mathcal{B}_D$ of special Lagrangians $\Lambda \subset D$ (occurring as collapsed limits of special Lagrangian tori $L \subset X \setminus D$), and the fiber corresponds to the holonomy of the flat connection over L with the constraint $\mathrm{hol}_\nabla(\partial\beta) = e^{i\theta}$. For $\theta = 0$, a section of the S^1-bundle $M_{D,0} = M_D \to \mathcal{B}_D$ is given by fixing the holonomy along $\partial\alpha$, e.g. requiring $\mathrm{hol}_\nabla(\partial\alpha) = 1$; since $\mathrm{hol}_\nabla(\partial\beta) = 1$ this constraint is compatible with the monodromy (7.2). Now, increase θ from 0 to 2π and deform this section into each $M_{D,\theta} = \{z_\beta = e^{i\theta}\}$: it follows from (7.2) that we get a section of the S^1-bundle $M_{D,\theta} \to \mathcal{B}_D$ along which the holonomy $\mathrm{hol}_\nabla(\partial\alpha)$ varies by $9\,\theta$. So, when we return to M_D as θ reaches 2π, the homotopy class of the section has changed by 9 times the fiber of the S^1-bundle $M_D \to \mathcal{B}_D$: i.e., in this example the monodromy of the fibration $z_\beta : \partial M \to S^1$ is given by

$$\begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix}.$$

7.2. Fukaya categories and restriction functors

We now return to the general case, and discuss briefly the Fukaya category of the Landau-Ginzburg model $W : M \to \mathbb{C}$, assuming that Conjectures 7.3 and 7.4 hold. The general idea, which goes back to Kontsevich [19] and Hori-Iqbal-Vafa [17], is to allow as objects *admissible Lagrangian submanifolds* of M, i.e. potentially non-compact Lagrangian submanifolds which, outside of a compact subset, are invariant under the gradient flow of $-\mathrm{Re}(W)$. The case of Lefschetz fibrations (i.e., when the critical points of W are nondegenerate) has been studied in great detail by Seidel; in this case, which is by far the best understood, the theory can be formulated in terms of the vanishing cycles at the critical points (see e.g. [28]).

The formulation which is the most relevant to us is the one which appears in Abouzaid's work [1, 2]: in this version, one considers Lagrangian submanifolds of M with boundary contained in a given fiber of the superpotential, and which near the reference fiber are mapped by W to an embedded curve $\gamma \subset \mathbb{C}$. In our case, using the fact that near ∂M the superpotential is $W = z_\delta + o(1)$, we consider Lagrangian submanifolds with boundary in $M_D = \{z_\delta = 1\}$:

Definition 7.6. A Lagrangian submanifold $L \subset M$ with (possibly empty) boundary $\partial L \subset M_D$ is *admissible with slope 0* if the restriction of z_δ to L takes real values near the boundary of L.

Similarly, we say that L is admissible with slope $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if $\partial L \subset M_D$ and, near ∂L, z_δ takes values in the half-line $1 - e^{i\theta}\mathbb{R}_+$. The definition of Floer homology for admissible Lagrangians is the usual one in this context: to determine $HF(L_1, L_2)$, one first deforms L_2 (rel. its boundary) to an admissible Lagrangian L_2^+ whose slope is greater than that of L_1, and one computes Floer homology for the pair of Lagrangians (L_1, L_2^+) inside M (ignoring boundary intersections).

We denote by $\mathcal{F}(M, M_D)$ the Fukaya category constructed in this manner. Replacing the superpotential by z_δ in the definition has two advantages: on one hand it makes admissibility a much more geometric condition, and on the other hand it eliminates difficulties associated with wall-crossing and definition of the superpotential. In particular, when comparing the B-model on X and the A-model on M this allows us to entirely eliminate the superpotential from the discussion. Since in the rescaling limit of Conjecture 4.4 the superpotential is expected to be $W = z_\delta + o(1)$, we conjecture that $\mathcal{F}(M, M_D)$ is derived equivalent to the physically relevant category of Lagrangian submanifolds.

Finally, we conclude the discussion by observing that, by construction, the boundary of an admissible Lagrangian in M is a Lagrangian submanifold of M_D (possibly empty, and not necessarily connected). We claim that there is a well-defined *restriction functor* $\rho : \mathcal{F}(M, M_D) \to \mathcal{F}(M_D)$ from the Fukaya category of M to that of M_D, which at the level of objects is simply $(L, \nabla) \mapsto (\partial L, \nabla_{|\partial L})$. At the level of morphisms, the restriction functor essentially projects to the part of the Floer complex generated by the intersection points near the boundary. More precisely, given an intersection point $p \in \mathrm{int}(L_1) \cap \mathrm{int}(L_2^+)$,

$\rho(p)$ is a linear combination of intersection points in which the coefficient of $q \in \partial L_1 \cap \partial L_2$ counts the number of holomorphic strips connecting p to q in $(M, L_1 \cup L_2^+)$.

This suggests the following conjecture, which can be thought of as "relative homological mirror symmetry" for the pair (X, D):

Conjecture 7.7. *There is a commutative diagram*

$$\begin{array}{ccc} D^b Coh(X) & \xrightarrow{restr} & D^b Coh(D) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ D^\pi \mathcal{F}(M, M_D) & \xrightarrow{\rho} & D^\pi \mathcal{F}(M_D) \end{array}$$

In this diagram, the horizontal arrows are the restriction functors, and the vertical arrows are the equivalences predicted by homological mirror symmetry.

Some evidence for Conjecture 7.7 is provided by the case of Del Pezzo surfaces [4]. Even though it is not clear that the construction of the mirror in [4] corresponds to the one discussed here, it is striking to observe how the various ingredients fit together in that example. Namely, by comparing the calculations for Del Pezzo surfaces in [4] with Polishchuk and Zaslow's work on mirror symmetry for elliptic curves [23], it is readily apparent that:

- the fiber of the Landau-Ginzburg model $W : M \to \mathbb{C}$ is mirror to an elliptic curve E in the anticanonical linear system $|K_X^{-1}|$;
- the Fukaya category of the superpotential admits an exceptional collection consisting of Lefschetz thimbles; under mirror symmetry for elliptic curves, their boundaries, which are the vanishing cycles of the critical points of W, correspond exactly to the restrictions to E of the elements of an exceptional collection for $D^b Coh(X)$;
- the behavior of the restriction functors on these exceptional collections is exactly as predicted by Conjecture 7.7.

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