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Dirac operators on manifolds with periodic ends

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Abstract. This paper studies Dirac operators on end-periodic spin manifolds of dimension at least 4. We provide a necessary and sufficient condition for such an operator to be Fredholm for a generic end-periodic metric. We make use of end-periodic Dirac operators to give an analytical interpretation of an invariant of non-orientable smooth 4-manifolds due to Cappell and Shaneson. From this interpretation we show that some exotic non-orientable 4-manifolds do not admit a metric of positive scalar curvature.

1. Introduction

Let M be a connected smooth spin n–manifold with $n \geq 4$. With any choice of Riemannian metric g on M one associates the Dirac operator $D(M,g) : C^\infty(M,S_g) \to C^\infty(M,S_g)$ on the sections of the spinor bundle S_g. This is a self–adjoint elliptic differential operator of order one. Deciding whether this operator is Fredholm is an important but difficult problem if M is not compact.

In this paper, we study this problem for non–compact manifolds M whose ends are periodic in the sense of Taubes [32]. In the first part of the paper, we prove that certain conditions on the topology of M are sufficient for the Dirac operator $D(M,g)$ to be Fredholm for a generic periodic metric g. The second part (Sections 5 and 6) is concerned with applications of such Dirac operators to 4-dimensional topology. We define a metric-dependent invariant of a homology $S^1 \times S^3$ which lifts its Rohlin invariant [28]. This leads to a new analytic interpretation of an invariant defined by Cappell and Shaneson for some non-orientable 4-manifolds in [10] (a different analytic interpretation can be found in [31]). We then use this interpretation to show that some of the non-orientable 4-manifolds constructed in [10] and [1] do not admit a Riemannian metric of positive scalar curvature. It appears that this phenomenon is special to dimension four.

A different approach to analyzing the Dirac operator on non–compact manifolds M is due to Gromov and Lawson [13] who showed that $D(M,g)$ is Fredholm if the metric g has positive scalar curvature at the end of M. This analysis is sufficient for some applications discussed in the second part of our paper; unfortunately, it does not apply in many situations of geometric interest. In fact, we were led to study Dirac operators

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on 4-manifolds with periodic ends by our earlier studies [27, 28, 29] of gauge theory on manifolds with first cohomology isomorphic to $\mathbb{Z}$; this paper provides sufficient conditions, valid in all dimensions, for these operators to be Fredholm.

To state the result more precisely, recall that any smooth closed connected spin n–manifold X endowed with a smooth map $f : X \to S^1$ defines an element $[X] \in \Omega_n^{\mathrm{spin}}(S^1)$. Choose a Riemannian metric g on X and let $\alpha(X) \in KO_n(S^1)$ be the image of $[X]$ under the homomorphism $\alpha : \Omega_n^{\mathrm{spin}}(S^1) \to KO_n(S^1)$ described in detail in Section 4. Suppose that $f : X \to S^1$ induces an epimorphism $f_* : \pi_1 X \to \mathbb{Z}$ and let $\tilde{X}$ be the infinite cyclic cover of X determined by f_* with the induced metric and spin structure. If M is a spin manifold with periodic end modeled on $\tilde{X}$ (i.e. diffeomorphic to $\tilde{X}$ outside of a compact set – see Section 5 for the precise definition), any metric on M which agrees with the metric induced by g over the end will again be called g.

Theorem 1. *Let M be a smooth connected spin n–manifold with periodic end modeled on $\tilde{X}$. Then $\alpha(X) = 0$ if and only if, for a generic metric g on X, the Dirac operator $D(M, g) : L^2_1(M, S_g) \to L^2(M, S_g)$ is Fredholm.*

The precise meaning of the word "generic" is explained in Remark 3.2. Theorem 1 is deduced from Taubes' fundamental work on end–periodic differential operators [32] and the following result. For any $c \in \mathbb{R}$, define the twisted Dirac operator

$$D^c = D + ic\, f^*(d\theta) : C^\infty(X, S_g) \to C^\infty(X, S_g)$$

where $d\theta$ is the volume form on S^1, and its pull back $f^*(d\theta)$ acts on the sections of S_g via Clifford multiplication.

Theorem 2. *Let X be a smooth closed connected spin n–manifold, $n \geq 4$, and $f : X \to S^1$ a smooth map such that $f_* : \pi_1 X \to \mathbb{Z}$ is onto. Then $\alpha(X) \in KO_n(S^1)$ vanishes if and only if, for a generic metric g on X, we have $\ker D^c(X, g) = 0$ for all $c \in \mathbb{R}$.*

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2. Deducing Theorem 1 from Theorem 2

Let X be a connected closed smooth spin n–manifold with $n \geq 4$. Choose a Riemannian metric g on X and associate with it the self–adjoint Dirac operator

$$D(X, g) : C^\infty(X, S_g) \to C^\infty(X, S_g)$$

on the sections of the spinor bundle S_g.

Given a smooth map $f : X \to S^1$ inducing an epimorphism $f_* : \pi_1 X \to \mathbb{Z}$, consider the infinite cyclic covering $\tilde{X} \to X$ associated with f_*. Then $\tilde{X}$ inherits a Riemannian metric from X, called again g, and a spin structure, which in turn give rise to the periodic Dirac operator $D(\tilde{X},g) : C^\infty(\tilde{X},S_g) \to C^\infty(\tilde{X},S_g)$. We wish to prove that, for a generic metric g on X, the L^2–completion of this operator, $D(\tilde{X},g) : L^2_1(\tilde{X},S_g) \to L^2(\tilde{X},S_g)$, is Fredholm. The Fredholmness of more general Dirac operators claimed in Theorem 1 will then follow by the excision principle, see Lockhart–McOwen [19]. The following construction is due to Taubes [32].

Observe that $S_g \to \tilde{X}$ is a pull back to $\tilde{X}$ of the spinor bundle $S_g \to X$. Let $\tau : \tilde{X} \to \tilde{X}$ be a covering translation and use the same symbol τ to denote its lift to an automorphism of the bundle $S_g \to \tilde{X}$. Choose a smooth map $\tilde{f} : \tilde{X} \to \mathbb{R}$ lifting $f : X \to S^1$.

Given $\psi \in C_0^\infty(\tilde{X},S_g)$ and $z \in \mathbb{C}^*$, denote by $\tau^*(\psi)$ the pull back of ψ and define

$$\hat{\psi}_z = z^{\tilde{f}} \cdot \sum_{n=-\infty}^{\infty} z^n \cdot (\tau^*)^n(\psi),$$

for a fixed choice of branch of $\ln z$. Then $\hat{\psi}_z$ is a smooth section of $S_g \to \tilde{X}$ such that

$$\tau^*(\hat{\psi}_z) = z^{\tilde{f}+1} \cdot \sum_n z^n \cdot (\tau^*)^{n+1}(\psi) = z^{\tilde{f}} \cdot \sum_n z^{n+1}(\tau^*)^{n+1}(\psi) = \hat{\psi}_z,$$

therefore, $\hat{\psi}_z$ defines a smooth section of $S_g \to X$. Taubes calls this operation the Fourier–Laplace transform; it is also known as the modified Z-transform [17]. Conjugating the Dirac operator $D(\tilde{X},g)$ by the Fourier–Laplace transform and its inverse [32, equation (4.5)] converts $D(\tilde{X},g)$ into the family of operators

$$D_z(X,g) = z^{\tilde{f}} \cdot D(X,g) \cdot z^{-\tilde{f}} = D(X,g) + z^{\tilde{f}} \cdot [D(X,g), z^{-\tilde{f}}],$$

where $D(X,g)$ is the Dirac operator on X, and $z^{\tilde{f}} \cdot [D(X,g), z^{-\tilde{f}}]$ is a zero order operator (a bundle automorphism of S_g), compare with [32]. The following is a special case of the more general result of Taubes [32, Lemma 4.3].

Theorem 2.1. *The operator $D(\tilde{X},g) : L^2_1(\tilde{X},S_g) \to L^2(\tilde{X},S_g)$ is Fredholm if and only if $\ker D_z(X,g) = 0$ for all $z \in \mathbb{C}^*$ with $|z| = 1$.*

A direct calculation shows that, if $z = e^{-ic}$ for $c \in \mathbb{R}$, then $D_z(X,g) = D(X,g) + ic\, f^*(d\theta)$, where $d\theta$ is the standard volume form on S^1. These are precisely the twisted Dirac operators $D^c(X,g)$ of Theorem 2.

3. Transporting invertibility

Let X and X' be connected closed smooth spin n–manifolds with $n \geq 4$. We will say that $f : X \to S^1$ is *spin cobordant to* $f' : X' \to S^1$ *over* S^1 if they define the same element in the group $\Omega_n^{\text{spin}}(S^1)$. This means that one can find an oriented smooth spin cobordism W from X to X' and a smooth function $F : W \to S^1$ restricting to f and f' respectively at the two boundary components. The main technical step in proving Theorem 2 is the following result.

Theorem 3.1. *Suppose that $f : X \to S^1$ is spin cobordant to $f' : X' \to S^1$ over S^1 via a cobordism admitting a handle decomposition (starting at X) with handles of index at most $n-1$. If there is a metric g on X such that $\ker D^c(X,g) = 0$ for all c, then one can find a metric g' on X' such that $\ker D^c(X',g') = 0$ for all c.*

Remark 3.2. Finding just one metric g as in the above theorem is sufficient to show that such metrics form a non–empty subset of the space of Riemannian metrics on X which is open in C^1–topology and dense in all C^k–topologies, $k \geq 1$, see Maier [20, Proposition 3.1]. The metrics in the above subset are referred to as "generic".

The proof of Theorem 3.1 closely follows the proof of Theorem 1.2 in the paper of Ammann, Dahl, and Humbert [2], which is a stronger assertion for the (untwisted) Dirac operator $D(X,g)$: they prove existence of a metric g' on X' such that $\dim\ker D(X',g') \leq \dim\ker D(X,g)$ even when $\ker D(X,g) \neq 0$. The analytical estimates that are at the heart of the proof of Theorem 3.1 are very similar to those which lie behind [2, Theorem 1.2], except that one has to pay extra attention to the additional parameter c. For this reason, this section will provide a sketch that consists of pointers to the relevant lemmas in [2] together with comments on the modifications necessary to take account of c. We will be happy to supply full details to the interested reader.

The proof of Theorem 3.1 will proceed by induction on the number of handles in the spin cobordism. Adding a 0–handle amounts to taking the disjoint union with an $(n+1)$–ball, which does not raise $\dim\ker D^c(X,g)$ since the n-sphere has positive scalar curvature. So we may assume in our induction that we are adding handles of index at least 1.

Let W be a spin cobordism over S^1 from $f : X \to S^1$ to $f' : X' \to S^1$ with just one handle of index $k+1$, $0 \leq k \leq n-2$, so that

$$W = ([0,1] \times M) \cup (D^{\,k+1} \times D^{\,n-k}),$$

where the handle is glued along $S^k \times D^{n-k} \subset X$. Then X' can be obtained from X by cutting out $S^k \times D^{n-k}$ and gluing back $D^{k+1} \times S^{n-k-1}$:

$$X' = (X - S^k \times D^{\,n-k}) \cup (D^{\,k+1} \times S^{n-k-1}). \tag{1}$$

In this n–dimensional description, the fact that W is a spin cobordism over S^1 means that the spin–structure on X' and the smooth function $f' : X' \to S^1$ agree with the spin structure on X and the function $f : X \to S^1$ over

$$X - (S^k \times D^{\,n-k}) = X' - (D^{\,k+1} \times S^{n-k-1}).$$

Theorem 3.1 will follow as soon as we show that, if X' is obtained from X by operation (1) and $\ker D^c(X,g) = 0$ for all c, then there exists a metric g' on X' such that $\ker D^c(X',g') = 0$ for all c.

3.1. Preliminaries

Let X be a connected closed smooth spin n–manifold equipped with a smooth function $f : X \to S^1$, and let $D^c(X,g) = D(X,g) + ic\, f^* d\theta$, $c \in \mathbb{R}$, be the twisted Dirac operators.

Lemma 3.3. *For any real number c, the spectra of $D^{c+1}(X,g)$ and $D^c(X,g)$ coincide. In particular,* $\ker D^{c+1}(X,g) = \ker D^c(X,g)$.

Proof. View θ as a multivalued function on S^1 and observe that $e^{if^*\theta}$ is a smooth function on X. Then

$$\begin{aligned} e^{-if^*\theta} D^c(X,g)\,(e^{if^*\theta}\psi) &= e^{-if^*\theta}(de^{if^*\theta}\psi + e^{if^*\theta} D^c(X,g)\psi) \\ &= if^*(d\theta)\psi + D^c(X,g)\psi \\ &= D^{c+1}(X,g)\psi \end{aligned}$$

so that $D^{c+1}(X,g)$ and $D^c(X,g)$ have the same spectrum. □

Lemma 3.4. *If the form $f^*d\theta$ is exact then, for any real c, the spectrum of $D^c(X,g)$ equals the spectrum of $D(X,g)$.*

Proof. Write $f^*d\theta = du$ for some smooth function $u : X \to \mathbb{R}$. Then

$$D^c(X,g) = D(X,g) + ic\,du = e^{-icu}\,D(X,g)\,e^{icu},$$

therefore, the spectra of $D^c(X,g)$ and $D(X,g)$ are the same for all $c \in \mathbb{R}$. □

In other language, these lemmas say that the operators D^c and D^{c+1} (resp. D and D^c when $f^*d\theta$ is exact) are gauge equivalent.

Corollary 3.5. *Let S^ℓ, $\ell \geq 1$, be an ℓ–dimensional sphere with round metric g^{rd} of radius one and bounding spin structure, and $M = N \times S^\ell$ a closed spin manifold with a product spin structure and a product metric $g = g_N + g^{\mathrm{rd}}$. For any $f : M \to S^1$ such that $f^*d\theta$ is exact, the spectrum of $D^c(M,g)^2$ is bounded from below by $\ell^2/4$.*

Proof. Let $f^*d\theta = du$ then $D^c(M,g) = e^{-icu}D(M,g)\,e^{icu}$ as in the proof of Lemma 3.4, therefore, $D^c(M,g)^2 = e^{-icu}\,D(M,g)^2\,e^{icu}$ and $D^c(M,g)^2$ and $D(M,g)^2$ have the same spectrum. The rest of the proof is as in [2, Proposition 2.5], using estimates on the spectrum of $D(S^\ell, g^{\mathrm{rd}})^2$. If $\ell \geq 2$, this spectrum is bounded from below by $\ell^2/4$, see for instance [3]. If $\ell = 1$ and the spin structure extends over D^2, we have $D(S^1, g^{\mathrm{rd}}) = i\,d/d\theta + 1/2$, which obviously gives the desired estimate. (Note that, with respect to the non–bounding spin structure, the Dirac operator is of the form $D(S^1, g^{\mathrm{rd}}) = i\,d/d\theta$ and hence has a non–zero kernel.) □

In the analysis of the Dirac operator on a spin manifold, a key role is played by the Lichnerowicz formula for its square. For a twisted Dirac operator, this formula has an extra term involving the curvature of the twisting connection, see for instance [6, page 134]. In our case the twisting connection $icf^*d\theta$ is flat, so the formula becomes

$$D^c(X,g)^2 = (\bar\nabla^{g,c})^*\,\bar\nabla^{g,c} \; + \; \frac{\kappa}{4}, \tag{2}$$

where κ is the scalar curvature of (X,g) and $\bar\nabla^{g,c}$ is the induced (twisted) connection on the spin bundle S_g.

Our final observation is that, for any metrics g and g' on X, there exists a unique automorphism $b : TX \to TX$ which is positive, symmetric with respect to g, and has the property that $g(\xi,\eta)=g'(b(\xi),b(\eta))$. The map on orthonormal frames induced by b gives rise to a map $\beta^g_{g'}:S_g\to S_{g'}$ of spinor bundles associated with the metrics g and g'. Note that this map preserves the fiberwise length of spinors and that $\beta^{g'}_g\circ\beta^g_{g'}=1$. The diagram

$$\begin{array}{ccc} C^\infty(X,S_g) & \xrightarrow{\beta^g_{g'}} & C^\infty(X,S_{g'}) \\ {\scriptstyle D^c(X,g)}\downarrow & & \downarrow{\scriptstyle D^c(X,g')} \\ C^\infty(X,S_g) & \xrightarrow{\beta^g_{g'}} & C^\infty(X,S_{g'}). \end{array}$$

need not commute; however, there is an explicit formula relating the operators $D^c(X,g)$ and $(\beta^g_{g'})^{-1}\circ D^c(X,g')\circ\beta^g_{g'}:C^\infty(X,S_g)\to C^\infty(X,S_g)$. All we need is its rather general form; compare with [7] or [20],

$$((\beta^g_{g'})^{-1}\circ D^c(X,g')\circ\beta^g_{g'})(\psi)=D^c(X,g)(\psi)+A(\bar\nabla^{g,c}\psi)+B(\psi), \tag{3}$$

where $A:T^*X\otimes S_g\to S_g$ and $B:S_g\to S_g$ are bundle maps independent of c such that

$$|A|\le C\,|g-g'|_g \quad\text{and}\quad |B|\le C\,(|g-g'|_g+|\nabla^g(g-g')|_g) \tag{4}$$

for some constant C. In the above formulas, ∇^g is the Levi–Civita connection on TX associated with metric g, and $\bar\nabla^{g,c}$ is the induced (twisted) connection on S_g.

3.2. Approximating by a product metric

Let g be a metric on X and denote by h the induced metric on $S^k\subset X$, $0\le k\le n-2$. Use the exponential map to identify a tubular neighborhood $U(R)$ of S^k of radius $R>0$ with the product $S^k\times D^{n-k}$. This neighborhood has two metrics: one is the original metric g and the other the product metric $h+g^{\mathrm{fl}}$, where g^{fl} is the flat metric on D^{n-k}.

Choose a small real number $\delta>0$ and a smooth cut–off function $\eta:X\to\mathbb{R}$ such that $0\le\eta\le 1$, $\eta=1$ on $U(\delta)\subset X$, $\eta=0$ on $X-U(2\delta)$, and $|d\eta|_g\le 2/\delta$. Consider the metric

$$g_\delta=\eta\,(h+g^{\mathrm{fl}})+(1-\eta)\,g,$$

which is a product metric in $U(\delta)$. The proof of the following proposition, which is along the lines of [2, Proposition 3.3], will be sketched in the remainder of this subsection.

Proposition 3.6. *Let g be a metric on X such that* $\ker D^c(X,g)=0$ *for all c. Then, if $\delta>0$ is sufficiently small,* $\ker D^c(X,g_\delta)=0$ *for all c.*

Suppose this is not true. Then one can find a sequence $\delta_m\to 0$ and a sequence c_m such that $\ker D^{c_m}(X,g_{\delta_m})\ne 0$. Lemma 3.3 implies that there is no loss of generality in assuming that $0\le c_m\le 1$ for all m. Therefore, passing to a subsequence if necessary, we will assume that c_m is a convergent sequence and denote its limit by c.

For the sake of simplicity, we will use the notations $g_m = g_{\delta_m}$ and $\beta_g^{g_m} = \beta_m$. Choose a non–zero spinor $\varphi_m \in \ker D^{c_m}(X, g_m)$ and normalize it so that

$$\int_X |\varphi_m|^2 \, dv^{g_m} = 1. \tag{5}$$

Lemma 3.7. *The sequence $\beta_m \varphi_m$ is bounded in $L^2_1(X, g)$.*

Proof. Observe that the estimates (3) are uniform in c, hence the proof of [2, Lemma 3.4] generalizes to our case word for word to show that the sequences $\beta_m \varphi_m$ and $\bar\nabla^{g,c_m}(\beta_m \varphi_m)$ are bounded in $L^2(X, g)$. To finish the proof, observe that

$$\bar\nabla^{g,c_m}(\beta_m \varphi_m) = \bar\nabla^g(\beta_m \varphi_m) + ic_m f^* d\theta \otimes \beta_m \varphi_m$$

hence

$$|\bar\nabla^g(\beta_m \varphi_m)|^2_g \quad \le \quad 2\, |\bar\nabla^{g,c_m}(\beta_m \varphi_m)|^2_g \; + \; 2\, c_m^2\, |f^* d\theta|^2_g\, |\beta_m \varphi_m|^2$$

with $c_m^2 \le 1$. Therefore,

$$\int_X |\bar\nabla^g(\beta_m \varphi_m)|^2_g \, dv^g \; \le \; C \int_X |\bar\nabla^{g,c_m}(\beta_m \varphi_m)|^2_g \, dv^g \; + \; C \int_X |\beta_m \varphi_m|^2 \, dv^g,$$

and the sequence

$$\int_X |\bar\nabla^g(\beta_m \varphi_m)|^2_g \, dv^g$$

is bounded. □

Since $\beta_m \varphi_m$ is bounded in $L^2_1(X, g)$, after passing to a subsequence if necessary, we may assume that $\beta_m \varphi_m$ converges to a spinor φ weakly in $L^2_1(X, g)$ and strongly in $L^2(X, g)$. Because of the strong convergence in $L^2(X, g)$, the normalization (5) is preserved in the limit; in particular, $\varphi \ne 0$. To continue, we will need the following extension of [2, Lemma 2.2].

Lemma 3.8. *Let (M, g) be a Riemannian manifold, not necessarily compact, and $f : M \to S^1$ a smooth map. Suppose that $D^c(M, g)(\varphi) = 0$ for some $c \in [0, 1]$. Then, for any compact set $K \subset M$, there exists a constant $C = C(K, M, g)$ independent of $c \in [0, 1]$ such that*

$$\|\varphi\|_{C^2(K,g)} \; \le \; C\, \|\varphi\|_{L^2(M,g)}.$$

Proof. This follows by applying a standard bootstrapping argument to the equation $D^c(M, g)(\varphi) = 0$. More precisely, write this equation in the form

$$D(M, g)(\varphi) = -icf^*(d\theta)\, \varphi. \tag{6}$$

Since $0 \le c \le 1$, the $L^2(M, g)$–norm of the right hand side can be estimated from above by $C\, \|\varphi\|_{L^2(M,g)}$, where C does not depend on c. Therefore, for any compact set $K_1 \subset M$, there is a constant C independent of c such that

$$\|\varphi\|_{L^2_1(K_1,g)} \; \le \; C\, \|\varphi\|_{L^2(M,g)}.$$

Apply $D(M,g)$ to both sides of (6) to express $D(M,g)^2(\varphi)$ as a linear combination of φ and its first derivative with coefficients depending polynomially on c and the first two derivatives of f. Therefore, there is a constant C independent of c such that

$$\|\varphi\|_{L^2_2(K_1,g)} \;\le\; C\,\|\varphi\|_{L^2(M,g)}.$$

Repeat the argument sufficiently many times to obtain the estimate

$$\|\varphi\|_{L^2_k(K_1,g)} \;\le\; C\,\|\varphi\|_{L^2(M,g)} \quad \text{with} \quad k > 1 + n/2.$$

If the boundary of K_1 is smooth, then we have the Sobolev embedding

$$L^2_k(K_1,g) \to C^1(K_1,g)$$

so that $\|\varphi\|_{C^1(K_1,g)} \;\le\; C\,\|\varphi\|_{L^2(X,g)}$ with a constant C independent of c. Now one can use Schauder's estimates as in [2, Lemma 2.2] to obtain the C^2–estimate. □

Apply the above lemma to the sequence $\beta_m\varphi_m$ to conclude that, for any sufficiently small $\varepsilon > 0$, the sequence $\beta_m\varphi_m$ is bounded in $C^2(X - U(\varepsilon))$. The Ascoli Theorem then implies that a subsequence of $\beta_m\varphi_m$ converges in $C^1(X - U(\varepsilon))$. The limit φ is in $C^1_{\text{loc}}(X - S)$ and satisfies the equation $D^c(X,g)(\varphi) = 0$ on $X - S$. The removal of singularities lemma [2, Lemma 2.4] says that φ extends to a smooth spinor on X such that $D^c(X,g)(\varphi) = 0$. This contradicts the assumption that $\ker D^c(X,g) = 0$ for all c, and proves Proposition 3.6.

3.3. Metrics which are product in the surgery region

Let X' be obtained from X by surgery along an embedded sphere S^k as in (1). According to Proposition 3.6, we may assume that the metric on X is a product on a tubular neighborhood $U(R)$ of S^k of radius $R > 0$. For any sufficiently small number $\rho > 0$, Ammann, Dahl and Humbert [2, Section 3.2] construct a metric g_ρ on X' which coincides with g on $X - U(R)$, is conformally equivalent to g on the entire region $X - U(2\rho)$, and is a product metric $\gamma_\rho + g^{\text{rd}}$ on the region $D^{k+1} \times S^{n-k-1}$ glued into $X - U(\rho/2)$ by the surgery.

Proposition 3.9. *Suppose that* $\ker D^c(X,g) = 0$ *for all* c. *Then, for any metric* g_ρ *with sufficiently small* $\rho > 0$, *we have* $\ker D^c(X', g_\rho) = 0$ *for all* c.

Our proof follows rather closely the proof in [2, Section 3.2], with some additional observations. Suppose that the proposition is not true. Then one can find a sequence $\rho_m > 0$ converging to zero and a sequence $c_m \in \mathbb{R}$ such that $\ker D^{c_m}(X', g_{\rho_m}) \neq 0$ for all m.

Lemma 3.3 implies that there is no loss of generality in assuming that $0 \le c_m \le 1$ for all m. Therefore, passing to a subsequence if necessary, we will assume that c_m is a convergent sequence and denote its limit by c.

For each m, choose a spinor $\psi_m \in \ker D^{c_m}(X', g_{\rho_m})$ and view it as a spinor on $X' - U'(2\rho_m) = X - U(2\rho_m)$, two manifolds with conformally equivalent metrics. Since the kernel of a (twisted) Dirac operator is conformally invariant, ψ_m gives rise to a spinor

φ_m in the kernel of $D^{c_m}(X,g)$ on $X - U(2\rho_m)$. Fix a real number s_m slightly greater than $2\rho_m$ and normalize φ_m so that

$$\int_{X-U(s_m)} |\varphi_m|^2\, dv^g = \int_{X'-U'(s_m)} |\varphi_m|^2\, dv^g = 1. \tag{7}$$

Lemma 3.10. *The sequence φ_m is bounded in $L^2(X-U(\delta))$ for any choice of $\delta \in (0,R)$ by a constant independent of both δ and c_m.*

Proof. In the untwisted case, this is derived in [2] as follows. The metric g_ρ is a product metric on $U'(2s_m) = D^{k+1}\times S^{n-k-1}$ of the form $h_\rho + g^{\mathrm{rd}}$. Embed the disk (D^{k+1},h_ρ) isometrically into a closed spin Riemannian manifold (N,g_N) of dimension $k+1$, and let $M = N\times S^{n-k-1}$ with product metric $g_M = g_N + g^{\mathrm{rd}}$ and the bounding spin structure on S^{n-k-1}. The spectrum of $D(M,g_M)^2$ is bounded from below by $(n-k-1)^2/4$, which is positive because $k\le n-2$. This leads to the desired estimate on the L^2 norm of φ_m through the use of the Rayleigh quotients for the operator $D(M,g_M)^2$ over $U'(2s_m)\subset M$, see Proposition 3.6 and estimate (17) in [2].

The same approach works for the operators $D^{c_m}(M,g_M)^2$ to give an estimate which is uniform in c_m as long as we can apply Corollary 3.5, or in other words, as long as we know that $(f')^*(d\theta)\in\Omega^1(U'(2s_m))$ is the restriction of an exact 1–form on M. The latter can be seen as follows. The restriction of the map $f' : X'\to S^1$ to $U'(2s_m)$ can be included into the following commutative diagram

$$\begin{array}{ccc} S^{n-k-1} & \xrightarrow{\ i'\ } & U'(2s_m) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ D^{n-k} & \xrightarrow{\ f\ } & S^1 \end{array}$$

where the maps i and i' are induced by the inclusions of $S^{n-k-1} = \partial D^{n-k}\subset S^k\times D^{n-k} = U(\rho_m/2)$ and $S^{n-k-1}\subset D^{k+1}\times S^{n-k-1} = U'(\rho_m/2)$, respectively. The map $f\circ i : S^{n-k-1}\to S^1$ factors through the contractible D^{n-k} and hence is homotopic to zero. Together with the fact that $(i')^* : H^1(U'(2s_m))\to H^1(S^{n-k-1})$ is an isomorphism, this implies that $(f')^* : H^1(S^1)\to H^1(U'(2s_m))$ is zero. In particular, $(f')^*(d\theta)$ on $U'(2s_m)$ is exact (of course, this form is automatically exact unless $k=n-2$). We let $(f')^*(d\theta) = du'$ for some $u' : U'(2s_m)\to\mathbb{R}$ and extend u' to a smooth function u on $M = N\times S^{n-k-1}$. □

Let N be a sufficiently large positive integer. Then the sequence φ_m is bounded in $L^2(X-U(1/N))$ and, by Lemma 3.8, also in $C^2(X-U(2/N))$. By Ascoli's Theorem, there is a subsequence φ_m^0 which converges in $C^1(X-U(2/N))$ to a spinor φ^0. By induction construct, for each $i\ge 1$, a subsequence φ_m^i of φ_m^{i-1} converging to a spinor φ^i in $C^1(X-U(2/(N+i)))$. Obviously, φ^i is an extension of φ^{i-1} for $i\ge 1$. Use the diagonal algorithm to construct a sequence φ'_m converging in $C^1_{\mathrm{loc}}(X-S^k)$ to a spinor φ'.

As $D^{c_m}(X,g)(\varphi'_m) = 0$ over $X - U(2\rho_m)$ with $\lim \rho_m = 0$ and $\lim c_m = c$, the $C^1_{\text{loc}}(X - S^k)$ convergence implies that $D^c(X,g)(\varphi') = 0$ on $X - S^k$. The removal of singularities lemma [2, Lemma 2.4] says that φ' extends to a smooth spinor on all of X such that $D^c(X,g)(\varphi') = 0$. Moreover, the normalization (7) is preserved in the limit thus implying that $\varphi' \neq 0$. This contradicts the assumption that $\ker D^c(X,g) = 0$ for all c. Proposition 3.9 follows.

4. Proof of Theorem 2

In this section, we first give a detailed description of the invariant α, and then use the results of Section 3 on transporting the invertibility of the twisted Dirac operators across a cobordism to prove Theorem 2.

4.1. The invariant α

Every smooth closed spin n–manifold X equipped with a map $f : X \to B\pi$ to the classifying space of a discrete group π defines an element $[X]$ in the spin cobordism group $\Omega_n^{\text{spin}}(B\pi)$. By definition, $\alpha(X) \in KO_n(B\pi)$ is the image of $[X]$ under the natural transformation of homology theories $\alpha : \Omega_*^{\text{spin}} \to KO_*$ defined in Milnor [21].

The only two cases we will be interested in are those of $\pi = \mathbb{Z}$ (corresponding to spin manifolds over a circle) and $\pi = \{1\}$ (corresponding to spin manifolds). As a matter of convenience, the respective α–invariants will be denoted by $\alpha(X) \in KO_n(S^1)$ and $\alpha_n(X) \in KO_n$. They can be described explicitly as follows, compare with Bär–Dahl [4, Section 3].

A straightforward calculation shows that $KO_n(S^1) = KO_n \oplus KO_{n-1}$. Then, for any spin n–manifold X endowed with a smooth map $f : X \to S^1$,

$$\alpha(X) = \alpha_n(X) + \alpha_{n-1}(Y) \in KO_n \oplus KO_{n-1},$$

where $Y = f^{-1}(p)$ for any choice of a regular value $p \in S^1$. Moreover, $KO_k = \mathbb{Z}$ if $k = 0, 4 \pmod 8$, $KO_k = \mathbb{Z}/2$ if $k = 1, 2 \pmod 8$, $KO_k = 0$ otherwise, and

$$\alpha_k(X) = \begin{cases} \operatorname{ind} D^+(X,g) & \text{if } n = 0 \pmod 8, \\ 1/2 \operatorname{ind} D^+(X,g) & \text{if } n = 4 \pmod 8, \\ \dim\ker D(X,g) \pmod 2 & \text{if } n = 1 \pmod 8, \\ \dim\ker D^+(X,g) \pmod 2 & \text{if } n = 2 \pmod 8, \end{cases}$$

where $D^+(X,g) : C^\infty(X, S_g^+) \to C^\infty(X, S_g^-)$ is the chiral Dirac operator. In particular, $\alpha_4(X) = -\operatorname{sign}(X)/16$ for any spin 4-manifold X.

It is clear from the above that the invertibility of $D(X,g)$ on a spin n–manifold X implies that $\alpha_n(X) = 0 \in KO_n$.

4.2. Vanishing of $\alpha(X)$ is necessary

We wish to prove that, if X is a spin manifold of dimension $n \geq 4$ over a circle which admits a metric g such that the operators $D^c(X,g)$ are invertible for all $c \in \mathbb{R}$, then $\alpha(X) = 0 \in KO_n(S^1)$.

This is certainly true for $n = 0, 4, 6, 7 \pmod 8$ because, in these dimensions, $KO_{n-1} = 0$ so that $\alpha(X) = \alpha_n(X)$, and the latter vanishes for invertible $D(X,g)$ (in particular, the invertibility of all $D^c(X,g)$ is not really needed). The proof of the general case sketched below was suggested to us by Ulrich Bunke.

Let $C^*_{\mathbb{R}}(\mathbb{Z})$ be the real group C^*–algebra of $\mathbb{Z}$. It follows from the positive solution to the real version of the Baum–Connes conjecture for group $\mathbb{Z}$ that the assembly map $A : KO_n(S^1) \to KO_n(C^*_{\mathbb{R}}(\mathbb{Z}))$ is injective. Therefore, it is sufficient for us to show that $A(\alpha(X))$ vanishes.

The class $A(\alpha(X)) \in KO_n(C^*_{\mathbb{R}}(\mathbb{Z}))$ admits the following description. The involution $c \to -c$ combined with the real structure on $D(X,g)$ makes operators $D^c(X,g)$ into a real family $\mathcal{D}$ acting on a Hilbert bundle over S^1. The space of sections of this bundle is naturally a Hilbert module over $C(S^1)$. The Fourier transform makes it into a module over $C^*(\mathbb{Z})$ and further into a real module over $C^*_{\mathbb{R}}(\mathbb{Z})$ if one takes into account the above real structure, as in the paper of Bunke–Schick [9]. The K–theoretic index of the family $\mathcal{D}$ is then $A(\alpha(X)) \in KO_n(C^*_{\mathbb{R}}(\mathbb{Z}))$.

In this interpretation, the invertibility of all $D^c(X,g)$ means that the family $\mathcal{D}$ is invertible on each fiber over S^1. Since S^1 is compact, one can find a positive uniform bound from below on the spectrum of $D^c(X,g)^2$, which makes the family of inverses uniformly bounded. This implies of course that the index of $\mathcal{D}$ is zero so that $A(\alpha(X))$ vanishes.

4.3. Vanishing of $\alpha(X)$ is sufficient in dimension 4

A direct calculation (using for instance the Atiyah–Hirzebruch spectral sequence) shows that $\Omega_4^{\text{spin}}(S^1) = \Omega_4^{\text{spin}} \oplus \Omega_3^{\text{spin}}$, and it is well known that $\Omega_3^{\text{spin}} = 0$ and Ω_4^{spin} is isomorphic to $\mathbb{Z}$ via $[X] \to \operatorname{sign}(X)/16$. Therefore, if $\alpha(X) = -\operatorname{sign}(X)/16 = 0$, there is a spin cobordism over a circle from S^4 to X. The sphere S^4 has a metric g of positive scalar curvature, hence all operators $D^c(S^4,g)$ are invertible by (2). Theorem 2 is now a consequence of Theorem 3.1 and the following Lemma (which we prove in arbitrary dimension).

Lemma 4.1. *If (X,f) of dimension $n \geq 4$ is spin cobordant over a circle to (X',f') such that $f'_* : \pi_1 X' \to \mathbb{Z}$ is onto then (X,f) is spin cobordant over a circle to (X',f') via a cobordism without n–handles.*

Proof. Any spin cobordism over a circle from (X,f) to (X',f') can be split into a union $U \cup V$, where U consists of k–handles, $1 \leq k \leq n-1$, and V of n–handles. Both U and V are spin cobordisms over a circle glued along (X'',f''), where $X'' = X' \,\#\, \ell\,(S^1 \times S^{n-1})$ and $f'' : X'' \to S^1$ coincides on the punctured copy of X' with $f' : X' \to S^1$. This can be seen by flipping the cobordism, so that V is obtained by attaching 1–handles to $[0,1]\times X'$. With an extra effort, one can also ensure that the restriction of $f'' : X'' \to S^1$ to each copy of $S^1 \times S^{n-1}$ is homotopic to zero: just slide one foot of a 1–handle if necessary around X' and use the fact that $f'_* : \pi_1 X' \to \mathbb{Z}$ is onto.

Next, construct a cobordism V' from X'' to X' by attaching 2–handles to $[0,1] \times X''$, one for each copy of $S^1 \times S^{n-1}$, along embedded circles $S^1 \subset S^1 \times S^{n-1}$ generating $H_1(S^1 \times S^{n-1};\mathbb{Z})$ and having bounding spin structure (to ensure that the resulting cobordism is spin). Since the restriction of f'' to each of the $S^1 \times S^{n-1}$ in X'' is homotopic to zero, the cobordism V' is automatically over a circle. The union $U \cup V'$ is then the desired cobordism. □

4.4. Vanishing of $\alpha(X)$ is sufficient in dimensions $n \geq 5$

Let X be a spin manifold of dimension $n \geq 5$ and $f : X \to S^1$ be a smooth map such that $f_* : \pi_1 X \to \mathbb{Z}$ is onto. One can add 2–handles to $[0,1] \times X$ to get a spin cobordism W to a spin manifold X' over S^1 so that $f'_* : \pi_1 X' \to \mathbb{Z}$ is an isomorphism. Since the 2–handles are attached along circles mapped to zero by f_*, the cobordism W is over a circle. The cobordism W can be flipped so that X is obtained from X' by adding $n-1$ handles.

The existence of W has two important implications. One is that $\alpha(X') = \alpha(X) = 0$, and the other is that $D^c(X,g)$ are invertible for a generic metric g on X if and only if $D^c(X',g')$ are invertible for a generic metric g' on X', see Theorem 3.1.

Now, according to Joachim and Schick [16], the vanishing of $\alpha(X')$ implies existence of a metric g' of positive scalar curvature on X' (this is known as the Gromov–Lawson–Rosenberg conjecture; it has been proved in particular for manifolds whose fundamental group is free abelian). Therefore, all operators $D^c(X',g')$ are invertible by (2).

Remark 4.2. Note that the above construction does not necessarily result in a metric of positive scalar curvature on X itself, because X is obtained from X' using handles of codimension two. While the invertibility of D^c can be transported across such a handle, transporting the positive scalar curvature condition would require codimension at least three (cf. [14] as amended in [24] and [30]).

5. An integral lift of the Rohlin invariant

Let X be a $\mathbb{Z}[\mathbb{Z}]$–homology $S^1 \times S^3$, that is, a smooth oriented 4-manifold such that $H_*(X;\mathbb{Z}) = H_*(S^1 \times S^3;\mathbb{Z})$ and $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, where $\tilde{X}$ is the universal abelian cover of X. Choose an embedded connected 3–manifold $Y \subset X$ whose fundamental class generates $H_3(X;\mathbb{Z}) = \mathbb{Z}$ (note that Y need not be a homology sphere). Define the Rohlin invariant of X by the formula

$$\rho(X) = \rho(Y,\sigma) \pmod 2.$$

Here, σ is a spin structure on Y induced by either of the two spin structures on X (the two choices differ by a cohomology class that vanishes on Y hence induce the same spin structure on Y), and $\rho(Y,\sigma) = \operatorname{sign}(W)/8 \pmod 2$ is the usual Rohlin invariant, where

W is any smooth compact spin 4-manifold with spin boundary (Y, σ). The invariant $\rho(X)$ is well–defined, see [25].

Let $f : X \to S^1$ be a smooth map such that $f_* : \pi_1 X \to \mathbb{Z}$ is onto. Let $Y \subset X$ be the preimage of a regular value of f. Without loss of generality, one may assume that Y is connected so that the fundamental class of Y generates $H_3(X;\mathbb{Z})$ and $\rho(X) = \rho(Y, \sigma)$. Cut X open along Y to obtain a spin cobordism U from Y to itself. The hypothesis that X is a $\mathbb{Z}[\mathbb{Z}]$–homology $S^1 \times S^3$ implies that U is a homology cobordism, i.e. that it has the homology of $Y \times I$. In particular, we have

$$\tilde{X} = \ldots \cup U \cup U \cup \ldots$$

Given a smooth compact spin 4-manifold W with boundary $\partial W = Y$, consider

$$W_U = W \cup U \cup U \cup \ldots,$$

which is a 4-manifold with periodic end modeled on $\tilde{X}$. Any choice of metric g on X naturally induces metrics on $\tilde{X}$ and on the end of W_U. The latter metric can be completed to a metric on W_U by choosing an appropriate metric on W. The metrics on $\tilde{X}$ and W_U will be called g again.

By the excision principle, the Dirac operator $D(W_U, g)$ is Fredholm if and only if the operator $D(\tilde{X}, g)$ is Fredholm. According to Theorem 1, for a generic metric g on X, the Dirac operator

$$D(W_U, g) : L^2_1\,(W_U, S) \to L^2\,(W_U, S)$$

and its chiral counterparts

$$D^{\pm}(W_U, g) : L^2_1\,(W_U, S^{\pm}) \to L^2\,(W_U, S^{\mp})$$

are Fredholm. With respect to this choice of metric, define the metric dependent invariant

$$w(X, g) = \operatorname{ind}_{\mathbb{C}} D^+(W_U, g) + \frac{1}{8}\,\operatorname{sign}(W).$$

Observe that, in general, $w(X, g)$ is a rational number. It is an integer if $\partial W = Y$ is an integral homology sphere.

Theorem 5.1. *The invariant $w(X, g)$ is well–defined (i.e. is independent of W) and* $w(X, g) = \rho(X) \pmod 2$.

Proof. Let W and W' be two choices of a smooth compact spin 4-manifold with boundary Y. Using the excision principle for the Dirac operator, see Bunke [8] and Charbonneau [11], and the index theorem, we obtain

$$\begin{aligned} \operatorname{ind}_{\mathbb{C}} D(W'_U) - \operatorname{ind}_{\mathbb{C}} D(W_U) &= \operatorname{ind}_{\mathbb{C}} D(-W \cup W') \\ &= -\frac{1}{24}\int_{-W\cup W'} p_1 = -\frac{1}{8}\,\operatorname{sign}(-W \cup W') = \frac{1}{8}\,\operatorname{sign} W - \frac{1}{8}\,\operatorname{sign} W', \end{aligned}$$

which proves that $w(X, g)$ is independent of the choice of W.

Let Y and Y' be two choices for cutting X open, and choose lifts of each to the infinite cyclic cover $\tilde{X}$ with covering translation $\tau : \tilde{X} \to \tilde{X}$. Since Y and Y' are both compact, the translate $\tau^p(Y')$ is disjoint from Y for sufficiently large $p \in \mathbb{Z}$. Both Y and $\tau^p(Y')$ separate $\tilde{X}$, which implies that they become the boundary components of a smooth spin manifold $V \subset \tilde{X}$. Because of the condition $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, the manifold V is a homology cobordism, i.e. it has the homology of the product $Y \times I$. The independence of $w(X,g)$ of the choice of Y now follows by the argument in the previous paragraph applied to $W' = W \cup V$.

To verify the second claim observe that $\operatorname{ind}_{\mathbb{C}} D^+(W_U,g)$ is always even because the operator $D^+(W_U,g)$ is quaternionic linear. Therefore,

$$w(X,g) = \operatorname{ind}_{\mathbb{C}} D^+(W_U,g) + \frac{1}{8}\operatorname{sign} W = \frac{1}{8}\operatorname{sign} W = \rho(X) \pmod 2.$$

□

6. Metrics of positive scalar curvature

In this section we will study the β–invariant of Cappell and Shaneson defined in [10] for certain non-orientable 4-manifolds (they called their invariant α but we will call it β to avoid confusion with the α–invariant of Section 4.1). While the original definition used the language of surgery theory and normal maps, we will present it in perhaps simpler terms. After that, we will use end periodic techniques to show that some manifolds with non–trivial β–invariant do not admit metrics of positive scalar curvature.

6.1. The Cappell–Shaneson invariant

Let X' be a non-orientable manifold with $H_1(X';\mathbb{Z}) = \mathbb{Z}$, with a chosen generator of $H^1(X';\mathbb{Z})$. This generator comes from a map $f : X' \to S^1$, and reduces mod 2 to $w_1(X')$. The orientation double cover $X \to X'$ has a canonical orientation coming from the standard construction of X (cf. [12, pp. 162–163]) and we assume in addition that $H_1(X;\mathbb{Z}) = \mathbb{Z}$ and $w_2(X) = 0$. Let $t : X \to X$ be the covering translation; it is of course orientation reversing.

Choose a regular value for f; its preimage in X' is an embedded 3-manifold Y, which is transversally oriented. Assume that Y is connected, and cut X' open along Y to obtain a manifold V. Note that both V and Y are orientable, although not necessarily oriented. Choose a lift of Y to X, and note that this lift is oriented because it is transversally oriented, and X itself is canonically oriented. This choice determines a particular lift of V to X as well: choose the lift of V such that the normal vector to the given lift of Y (determined, as noted before, by the map to the circle) points out of V. Note that our lift of Y acquires a spin structure σ from that of X (σ is independent of choice of spin structure on X since the two choices differ by a cohomology class that vanishes on Y). Following [10], let

$$\beta(X',f) = \rho(Y,\sigma) - \frac{1}{16}\operatorname{sign}(V) \pmod 2,$$

where we are implicitly using the orientation on the lift of V specified above.

Lemma 6.1. *The invariant $\beta(X', f)$ is independent of the choice of f but may change sign for a different choice of the lift of Y to X.*

Proof. Let Y_0 and Y_1 be two choices. Note that we can choose homotopic functions $f_k : X' \to S^1$, $k = 0, 1$, with $1 \in S^1$ a regular value for both, and $Y_k = f_k^{-1}(1)$. Choose a homotopy $F : X' \times I \to S^1$ between f_0 and f_1 such that $1 \in S^1$ is a regular value for F. Then $W = F^{-1}(1)$ is a spin cobordism between Y_0 and Y_1 which is transversally oriented. Note that the choice of lift of Y_0 gives a particular lift of W to $X \times I$, and therefore a particular lift of Y_1 as well.

Observe that the union of W and $t(W)$ separates $X \times I$ into two pieces, one of which has oriented boundary $W \cup -V_1 \cup t(W) \cup V_0$. Therefore,

$$\operatorname{sign}(V_1) - \operatorname{sign}(V_0) = 2 \operatorname{sign}(W)$$

but also

$$\rho(Y_1) - \rho(Y_0) = \frac{1}{8} \operatorname{sign}(W).$$

Plugging these two formulas into the definition of β completes the proof. □

The following example shows why the Rohlin invariant of Y is, by itself, not an invariant of X'.

Example 6.2. Let $Y = \Sigma(2, 3, 5)$ be the Poincaré homology sphere then Y bounds a smooth spin manifold W with intersection form E_8 built with only 2–handles. It is easy to see (this is Kirby calculus folklore, cf. [26]) that the double of W along Y is diffeomorphic to $\#_8 S^2 \times S^2$. Let X' be the manifold obtained from the double of W by puncturing each copy of W and gluing the resulting 3–sphere boundary components to get a non-orientable manifold; this is of course the same as the connected sum of the double of W with $S^1 \tilde{\times} S^3$.

We can compute $\beta(X')$ by spllitting either along Y, or along the 3–sphere. Note that $\rho(Y) = 1 \pmod 2$ while $\rho(S^3) = 0 \pmod 2$. Computing with the splitting of X' along Y, we get that $\beta(X', f) = \rho(Y) - \operatorname{sign}(W \# W)/16 = 1 - (-1) = 0 \pmod 2$. Splitting X' along the 3-sphere gives again $\beta(X', f) = \rho(S^3) - \operatorname{sign}(\#_8 S^2 \times S^2)/16 = 0 - 0 = 0$ (mod 2). Note that X' has a metric of positive scalar curvature, which is consistent with Theorem 6.3 below.

6.2. An obstruction to positive scalar curvature

Let X' be a non–orientable 4-manifold as described in the first paragraph of Section 6.1.

Theorem 6.3. *If X' admits a metric of positive scalar curvature, then $\beta(X', f) = 0$ for any choice of f.*

Proof. Suppose that X' has a metric g of positive scalar curvature and define

$$w_{\mathrm{cs}}(X', f, g) = \operatorname{ind}_{\mathbb{C}} D^+(W \cup -V \cup V \cup -V \cup \ldots, g) + \operatorname{sign} W/8 - \operatorname{sign} V/16,$$

where W is any smooth compact spin manifold with boundary Y. That the Dirac operator in this formula is Fredholm follows from Theorem 2.1 after we observe that all the operators $D^c(X,g)$ are invertible by (2). This can also be deduced directly from Gromov–Lawson [13].

Observe that $w_{\mathrm{cs}}(X',f,g)$ reduces mod 2 to the Cappell–Shaneson invariant $\beta(X',f)$. On the other hand, we have

$$\begin{aligned}
&w_{\mathrm{cs}}(X',f,g) = \\
&\operatorname{ind} D^+(W\cup(-V\cup V)\cup(-V\cup V)\cup\ldots)+\operatorname{sign}W/8-\operatorname{sign}V/16 = \\
&\operatorname{ind} D^+((W\cup -V)\cup(V\cup -V)\cup\ldots)+\operatorname{sign}(W\cup -V)/8+\operatorname{sign}V/16 = \\
&\operatorname{ind} D^+(-W\cup(V\cup -V)\cup\ldots)+\operatorname{sign}(-W)/8+\operatorname{sign}V/16 = \\
&\operatorname{ind} D^-(W\cup(-V\cup V)\cup\ldots)+\operatorname{sign}(-W)/8+\operatorname{sign}V/16 = \\
&-\operatorname{ind} D^+(W\cup(-V\cup V)\cup\ldots)-\operatorname{sign}W/8+\operatorname{sign}V/16 = -w_{\mathrm{cs}}(X',f,g)
\end{aligned}$$

(the third line above is obtained by rearranging, and the fourth by replacing $W\cup -V$ with $-W$ and using the excision principle for the index of the Dirac operator as in the proof of Theorem 5.1). Therefore, $w_{\mathrm{cs}}(X',f,g)=0$ and $\beta(X',f)=0 \pmod 2$. □

Examples of manifolds to which Theorem 6.3 applies are not difficult to come by, and indeed are among the first known examples of exotic 4-manifolds. Let $Z' = S^1\tilde{\times}S^3$ be the non-orientable S^3–bundle over S^1, and consider the manifold $Z'_k = Z' \#_k (S^2\times S^2)$. Akbulut has constructed [1] a manifold X'_1 homotopy equivalent (in fact, using Freedman's work, homeomorphic) to Z'_1 but not diffeomorphic to it, which is detected by the invariant β. Using the fact that β is not changed by connected sum with $S^2\times S^2$, it follows that there are such manifolds X'_k for all $k\geq 1$. According to Theorem 6.3 none of these manifolds admits a metric of positive scalar curvature. The question of the existence of an exotic $S^1\tilde{\times}S^3$ is still open.

Akbulut's construction is fairly strenuous, because he is trying to keep k small; there are easier constructions that produce exotic X'_k for somewhat larger k.

Example 6.4. The manifold $X'_{11} = Z'\# K3$ is homeomorphic to Z'_{11}. To see this, one can use Freedman's classification of simply connected topological 4-manifolds to deduce that $K3$ is homeomorphic to $E\# E\#_3 S^2\times S^2$, where E has the E_8 intersection form. Then

$$Z'\# K3 = Z'\# E\# E\#_3 S^2\times S^2 = Z'\#(-E)\# E\#_3 S^2\times S^2,$$

where the last homeomorphism is obtained by sending E around an orientation reversing loop. But $(-E)\# E\#_3 S^2\times S^2$ is just a connected sum $\#_{11} S^2\times S^2$.

The Cappell–Shaneson invariant of X'_{11} equals $\rho(S^3,\sigma)-\operatorname{sign}(K3)/16 = 1 \pmod 2$, therefore, X'_{11} has no metric of positive scalar curvature.

On the other hand, the double cover of X'_{11} is $(S^1\times S^3)\# K3\#(-K3)$ which is diffeomorphic to $(S^1\times S^3)\#_{22}(S^2\times S^2)$, as in Example 6.2. So the double cover of X'_{11} has a metric of positive scalar curvature even though X'_{11} itself does not. Such a phenomenon has already been observed in higher dimensions by Bérard–Bergery [5] and Rosenberg [22], and in dimension 4 by LeBrun [18] and Hanke, Kotschick and Wehrheim [15].

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