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A generalised Joyce construction for a family of nonlinear partial differential equations

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Abstract. We show that a certain family of nonlinear fourth order partial differential equations in two variables can be reduced to linear equations. This extends a construction of Joyce and also the description by Chern and Terng of affine maximal surfaces.

The title refers to a construction of Dominic Joyce of self-dual Riemannian metrics with two Killing fields [5]. In this note we show that the equations in this construction (in the form described by Calderbank and Pedersen in [2]) can be extended to a family of nonlinear fourth order PDEs in two dimensions, essentially reducing them to linear equations. One equation in this family is the *affine maximal equation*, and we show that the construction in this case is almost identical to one due to Chern and Terng.

1. The main result

We consider convex functions $u(x_i)$ defined on a domain in $\mathbf{R}^n$ and write $J = \det(u_{ij})$, where (u_{ij}) is the Hessian $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$. Let ψ be any smooth, strictly convex function on the half-line $(0, \infty)$ and consider the functional

$$\mathcal{F}(u) = \int \psi(J(u))\, dx_1 \dots dx_n. \tag{1}$$

The corresponding Euler-Lagrange equations $\delta\mathcal{F} = 0$ are

$$\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(J\psi'(J) u^{ij}\right) = 0, \tag{2}$$

where (u^{ij}) is the matrix inverse of (u_{ij}). This a nonlinear fourth order PDE for the function u. In the case when $\psi(J) = -J^\alpha$ for some $\alpha \in (0,1)$ these equations have been studied, from the analytical point of view (estimates, regularity etc.) by Trudinger and Wang [6], [7]. The case covered by Joyce's original construction is when $n = 2$ and $\psi(J) = -\log J$, as we will discuss further in Section 3. (Of course we only consider the functional $\mathcal{F}$ as a motivation for writing down the partial differential equations (2), and the actual convergence of the integral (1) is irrelevant.) Let us say that a point (x_1, x_2) is an "ordinary point" if the derivative ∇J does not vanish there.

To state our result, let f be a solution of the equation

$$f'(t) = t^{1/2}\psi''(t).$$

Thus $f' > 0$ and f is a $1-1$ map from $(0,\infty)$ to some finite or infinite interval I. Let p be the inverse map raised to the power $-1/2$, so $J^{-1/2} = p(r)$ when $r = f(J)$. Now consider the linear second order PDE for a function $\xi(H,r)$ defined on some domain in $\mathbf{R}\times I$:

$$\frac{\partial^2\xi}{\partial H^2} + \frac{1}{p(r)}\frac{\partial}{\partial r}\left(p(r)\frac{\partial\xi}{\partial r}\right) = 0. \tag{3}$$

Theorem 1.1. *Suppose ξ_1,ξ_2 are two solutions of equation (3) and $\det\left(\frac{\partial(\xi_1,\xi_2)}{\partial(H,r)}\right)$ is positive at some point (H_0,r_0). Define three 1-forms $\epsilon_1,\epsilon_2,\epsilon$ by*

$$\epsilon_1 = p(r)\left(\frac{\partial\xi_2}{\partial r}dH - \frac{\partial\xi_2}{\partial H}dr\right) \qquad \epsilon_2 = p(r)\left(-\frac{\partial\xi_1}{\partial r}dH + \frac{\partial\xi_1}{\partial H}dr\right)$$

$$\epsilon = \xi_1\epsilon_1 + \xi_2\epsilon_2.$$

Then $\epsilon,\epsilon_1,\epsilon_2$ are closed 1-forms and $\epsilon_1\wedge\epsilon_2$ is non-vanishing near (H_0,r_0). Thus we can find functions u,x_1,x_2 with $du=\epsilon, dx_1=\epsilon_1, dx_2=\epsilon_2$ and x_1,x_2 give local co-ordinates around (H_0,r_0). If we regard u as a function of (x_1,x_2), it is a solution of the equation (2) on a suitable domain, and all points in this domain are ordinary points.

Conversely, any solution of (2) with $n=2$, in the neighbourhood of an ordinary point, is obtained in this way, with solutions ξ_1,ξ_2 of (3) which are unique up to translation in the H-variable and the addition of constants.

In sum, the local study of the nonlinear equation (2) in two dimensions is essentially equivalent to that of the linear equation (3). The construction has some similarities with other well-known "explicit" solutions of nonlinear PDE in low dimensions, such as the Weierstrasse representation of minimal surfaces and the correspondence, also going back to the nineteenth century, between solutions of the Monge-Ampère equation $\det(u_{ij})=1$ and harmonic functions, in the two dimensional case.

2. The proof

This is entirely elementary. From now on we always suppose the dimension n is 2, and we work locally so we will not specify the precise domains of definition of the various functions. Given a convex function u we consider the Riemannian metric $g=\sum u_{ij}\,dx_i dx_j$, and in particular the *conformal structure* which this defines. Write ξ_i for the Legendre transform co-ordinates $\xi_i = \frac{\partial u}{\partial x_i}$. Suppose now that we have some other local co-ordinates λ_1,λ_2, inducing the same orientation as x_1,x_2 (i.e. $\det\left(\frac{\partial x_i}{\partial\lambda_a}\right) > 0$). We can write the metric $g=\sum g_{ab}\,d\lambda_a d\lambda_b$ in these co-ordinates. Recall that the co-ordinates are called *isothermal* if the matrix (g_{ab}) is a multiple of the identity matrix at each point, or in other words

$$g = V\left(d\lambda_1^2 + d\lambda_2^2\right)$$

for a positive function $V(\lambda_1, \lambda_2)$. Let ϵ_{ij} denote the alternating tensor $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$.

Observation 1. The co-ordinates λ_1, λ_2 are isothermal if and only if the partial derivatives $\frac{\partial x_i}{\partial \lambda_a}, \frac{\partial \xi_j}{\partial \lambda_b}$ are related by the four equations

$$\frac{\partial \xi_i}{\partial \lambda_a} = \epsilon_{ab}\epsilon_{ij}\sqrt{J}\ \frac{\partial x_j}{\partial \lambda_b}, \tag{4}$$

where J is now considered a function of λ_1, λ_2.

To see this we write

$$g = \sum_{ij} u_{ij}\, dx_i dx_j = \sum_j d\xi_j dx_j = \sum_{j,a,b} \frac{\partial \xi_j}{\partial \lambda_a}\frac{\partial x_j}{\partial \lambda_b}\ d\lambda_a d\lambda_b.$$

So the isothermal condition is

$$\sum_j \frac{\partial \xi_j}{\partial \lambda_a}\frac{\partial x_j}{\partial \lambda_b} = V\delta_{ab}.$$

In matrix notation, if $A = \left(\frac{\partial x_i}{\partial \lambda_a}\right), B = \left(\frac{\partial \xi_i}{\partial \lambda_a}\right)$, this is

$$A^T B = V \cdot I,$$

or in other words

$$B^T = VA^{-1}. \tag{5}$$

Taking determinants we have $\det B \det A = V^2$. On the other hand, the matrix (u_{ij}) is BA^{-1}, so $J = \det B \det A^{-1}$. Thus $V = \sqrt{J}\det A$ and (5) is

$$B^T = \sqrt{J}(\det A)\ A^{-1},$$

which is the same as (4) by the formula for the inverse of a 2×2 matrix.

Now consider any positive function $P(\lambda_1, \lambda_2)$ and the pair of second order linear PDEs

$$\frac{\partial}{\partial \lambda_1}\left(P\frac{\partial \xi}{\partial \lambda_1}\right) + \frac{\partial}{\partial \lambda_2}\left(P\frac{\partial \xi}{\partial \lambda_2}\right) = 0, \tag{6}$$

$$\frac{\partial}{\partial \lambda_1}\left(P^{-1}\frac{\partial x}{\partial \lambda_1}\right) + \frac{\partial}{\partial \lambda_2}\left(P^{-1}\frac{\partial x}{\partial \lambda_2}\right) = 0. \tag{7}$$

Observation 2. If $\xi(\lambda_1, \lambda_2)$ is a solution of (6) then there is a solution $x(\lambda_1, \lambda_2)$ of the first order system

$$\frac{\partial x}{\partial \lambda_a} = P\epsilon_{ab}\frac{\partial \xi}{\partial \lambda_b}, \tag{8}$$

unique up to the addition of a constant, and $x(\lambda_1, \lambda_2)$ satisfies (7).

This is because the consistency condition for the first order system is

$$\frac{\partial}{\partial\lambda_2}\left(P\frac{\partial\xi}{\partial\lambda_2}\right)=\frac{\partial}{\partial\lambda_1}\left(-P\frac{\partial\xi}{\partial\lambda_1}\right),$$

which is the equation (6). In a different language, we are saying that the 1-form

$$\epsilon = P\frac{\partial\xi}{\partial\lambda_2}d\lambda_1 - P\frac{\partial\xi}{\partial\lambda_1}d\lambda_2$$

is closed, so can be written as dx for a function x. Note that, changing P to P^{-1}, there is a complete symmetry between x and ξ so we can also start with a solution of (7) and construct a solution of (6).

Observation 3. Suppose ξ_1,ξ_2 are two solutions of (6), with $\det\left(\frac{\partial\xi_i}{\partial\lambda_a}\right)>0$. Let x_2 be the solution of (7) corresponding to ξ_1 by (8), and x_1 be the solution corresponding to $-\xi_2$. Then $\det\left(\frac{\partial x_i}{\partial\lambda_a}\right)>0$, so x_1,x_2 give local co-ordinates. Write

$$\epsilon=\xi_1dx_1+\xi_2dx_2.$$

Then ϵ is a closed 1-form and so $\epsilon=du$ for a function u. If we express u as a function of x_1,x_2 then $\frac{\partial u}{\partial x_i}=\xi_i$.

This is straightforward. The conditions (8) imply that

$$\det\left(\frac{\partial x_i}{\partial\lambda_a}\right)=P^2\det\left(\frac{\partial\xi_i}{\partial\lambda_a}\right)>0.$$

We have $d\epsilon=d\xi_1dx_1+d\xi_2dx_2$ and, writing $\xi_{i,a}=\frac{\partial\xi_i}{\partial\lambda_a}$,

$$d\epsilon=(\xi_{1,1}d\lambda_1+\xi_{1,2}d\lambda_2)(-\xi_{2,2}d\lambda_1+\xi_{2,1}d\lambda_2)+(\xi_{2,1}d\lambda_1+\xi_{2,2}d\lambda_2)(\xi_{1,2}d\lambda_1-\xi_{1,1}d\lambda_2)=0.$$

Now return to our functions $\psi(J),f(J)$ and the Euler-Lagrange equation (2).

Observation 4. A convex function u satisfies equation (2) if and only if $f(J)$ is harmonic with respect to the metric $g=\sum u_{ij}dx_idx_j$.

This is true in any dimension. The formula for the derivative of an inverse matrix is

$$\frac{\partial}{\partial x_k}u^{ij}=-\sum_{pq}u^{ip}u_{pqk}u^{qj},$$

whereas the formula for the derivative of the determinant is

$$\frac{\partial J}{\partial x_i}=J\sum_{pq}u^{pq}u_{pqi}.$$

These yield the identity

$$\sum_j \frac{\partial}{\partial x_j} u^{ij} = -\sum_{pqj} u^{ip} u_{jpq} u^{qj} = -\sum_p u^{ip} J^{-1} \frac{\partial J}{\partial x_p}. \tag{9}$$

Our Euler-Lagrange equation (2) is

$$\sum_i \frac{\partial v_i}{\partial x_i} = 0,$$

where v is the vector field in the (x_1, x_2) plane with components

$$\begin{aligned} v_i &= \sum_j \frac{\partial}{\partial x_j} (J\psi'(J) u^{ij}) \\ &= \sum_j \frac{\partial (J\psi'(J))}{\partial x_j} u^{ij} + J\psi'(J) \frac{\partial u^{ij}}{\partial x_j}. \end{aligned} \tag{10}$$

By the definition of the function f we have

$$\frac{\partial}{\partial x_j}(J\psi'(J)) = \sqrt{J}\ \frac{\partial f(J)}{\partial x_j} + \psi'(J) \frac{\partial J}{\partial x_j}.$$

Using (9) we obtain

$$v_i = \sqrt{J}\ \frac{\partial f(J)}{\partial x_j} u^{ij}.$$

Thus the equation (10) is the Laplace equation in the metric g:

$$\sum_i \frac{\partial}{\partial x_i} \left(\sqrt{J}\ u^{ij} \frac{\partial f(J)}{\partial x_j} \right) = 0.$$

With these four observations the main result, Theorem 1, is almost obvious. Suppose we start with an ordinary point of a solution u to (2). Then $r = f(J)$ is harmonic by Observation 4, and we can suppose that its derivative does not vanish in the region considered. There is then a conjugate harmonic function H, which by definition is one such that the local co-ordinates (H, r) are isothermal. By Observations 1 and 2 the functions x_i, ξ_j satisfy the equations (6),(7) respectively with $\lambda_1 = H, \lambda_2 = r$ and $P = p(r)$. Since P does not depend on H, equation (6) can be written in the form (3). By Observations 2 and 3 we can recover the original function u from the two solutions ξ_1, ξ_2 of the linear PDE (or, equally well, the two solutions x_1, x_2). It is also clear that, conversely, starting with any two solutions ξ_1, ξ_2 to the linear equation we construct a solution to (2) by this method.

Note that the conjugate function H, in this situation, can be defined simply as the solution of the system

$$\frac{\partial H}{\partial x_i} = \epsilon_{ij} v_j, \tag{11}$$

so we can also think of H as the Hamiltonian generating the area-preserving vector field v defined by (10). We leave this as an exercise for the reader.

3. Examples and discussion

(1) The functional $\mathcal{F}$ and hence the differential equation (2) depends on the choice of a function ψ. Such functions come in pairs which give rise to essentiallty equivalent equations. To see this, let $u^*(\xi_1, \xi_2)$ be the Legendre transform of $u(x_1, x_2)$. The Hessian $(\frac{\partial^2 u^*}{\partial\xi_i\partial\xi_j})$ at the point $\xi_i = \frac{\partial u}{\partial x_i}$ is the inverse of the Hessian of u at x_i. Thus we can write

$$\mathcal{F} = \int \psi(J_*^{-1}) J_* \ d\xi_1 d\xi_2,$$

where $J_* = \det(\frac{\partial^2 u^*}{\partial\xi_i\partial\xi_j})$. This implies that the Legendre transform takes a solution u of the equation (2) associated with ψ to a solution u^* of the equation associated to the function

$$\psi^*(t) = t\psi(t^{-1}).$$

In our construction this just means replacing the function $p(r)$ by $p(-r)^{-1}$ and interchanging the roles of the co-ordinates x_i, ξ_i.

(2) If $\psi(t) = -\log t$ then equation (2) is $\sum \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0$. This is the equation defining a zero scalar curvature Kahler metric in "symplectic" co-ordinates, see [1], [4]. Explicitly, we introduce two further co-ordinates θ_1, θ_2 and consider the Riemannian metric, in four dimensions,

$$\sum u_{ij} dx_i dx_j + \sum u^{ij} d\theta_i d\theta_j.$$

The well-known fact that a Kahler metric on a complex surface is self-dual if and only if the scalar curvature vanishes gives the link with Joyce's original formulation of his construction. Under the Legendre transform we get another description corresponding to the function $\psi^*(t) = t\log t$. This leads to the equations describing zero scalar curvature metrics in "complex" co-ordinates. When $\psi(t) = -\log t$ we get $p(r) = r$ and equation (3) is the equation defining axi-symmetric harmonic functions on $\mathbf{R}^3$.

(3) For any function $p(r)$ there is an obvious solution $\xi = H$ to (3). Thus we get a special family of solutions to (2) with $\xi_1 = H$ and ξ_2 some other solution of (3). There is another special family, which corresponds to this under the Legendre transform, when ξ_1 is a function of r only, so $x_2 = H$. These correspond to second order equations of Monge-Ampère type which give special solutions of (2). For example, in the zero scalar curvature case above we have the special solutions where $\xi_1 = \log r$. The function u satisfies the equation $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = e^{-\xi_1/2}$ and the corresponding four dimensional Riemannian metric is Ricci-flat.

(4) For the Trudinger-Wang equations, when $\psi(t) = -t^\alpha$ with $0 < \alpha < 1/2$, we get $p(r) = r^{1/(1-2\alpha)}$ (up to factor which is irrelevant because it drops out of the equation (3)). If $\alpha = 1/2$ we get $p(r) = e^{r/2}$.

(5) Consider the graph of a function $u(x_1, x_2)$ as a surface in $\mathbf{R}^3$. The Gauss curvature is

$$K = \frac{\det(u_{ij})}{(1 + |\nabla u|^2)^2} = \frac{J}{(1 + |\nabla u|^2)^2},$$

and the induced area form is

$$dA = (1 + |\nabla u|^2)^{1/2} dx_1 dx_2.$$

Thus

$$K^{1/4} dA = J^{1/4} dx_1 dx_2.$$

The left hand side is invariant under Euclidean transformations of $\mathbf{R}^3$ while the right hand side is invariant under unimodular affine transformations of $\mathbf{R}^2$. Since these two groups generate all unimodular affine transformations of $\mathbf{R}^3$ we see that this 2-form is an *affine* invariant of the surface. Hence the integral (1), when we take $\psi(t) = t^{1/4}$, is an affine invariant—the *affine area* of the surface. The graphs of the solutions of equation (2), when $\psi(t) = t^{1/4}$, are "affine maximal" surfaces in $\mathbf{R}^3$. According to Chern and Terng [3] these surfaces can be described locally as follows. Let F_1, F_2, F_3 be harmonic functions of variables H, r and write $\vec{F} = \vec{F}(H, r)$ for the corresponding vector-valued function. Then the condition $\frac{\partial^2 \vec{F}}{\partial r^2} + \frac{\partial^2 \vec{F}}{\partial H^2} = 0$ implies the consistency of the first order system

$$\frac{\partial \vec{Z}}{\partial H} = \vec{F} \times \frac{\partial \vec{F}}{\partial r} \qquad \frac{\partial \vec{Z}}{\partial r} = -\vec{F} \times \frac{\partial \vec{F}}{\partial H} \tag{12}$$

so there is a solution $\vec{Z}$ which, under appropriate non-degeneracy conditions, parametrises a surface in $\mathbf{R}^3$. Chern and Terng show that these surfaces are precisely the affine maximal surfaces. We want to relate this description to ours. Notice first that our equations can be written in a similar form. Given a pair of solutions ξ_1, ξ_2 of (3) we define a vector valued function Ξ with components $\xi_1, \xi_2, 1$. Then if $\vec{Z} = (x_1, x_2, u)$ our description is the first order system

$$\frac{\partial \vec{Z}}{\partial H} = p(r)\, \Xi \times \frac{\partial \Xi}{\partial r} \qquad \frac{\partial \vec{Z}}{\partial r} = -p(r)\, \Xi \times \frac{\partial \Xi}{\partial H}. \tag{13}$$

In the case when $\psi(t) = t^{1/4}$ we get $p(r) = r^2$ so the equation (3) is not the ordinary Laplace equation in the variables (H, r). However, it is easy to check that a function $F(H, r)$ satisfies the Laplace equation if and only if $\xi = r^{-1} F$ satisfies the equation (3), with $p(r) = r^2$. Now take two harmonic functions F_1, F_2 and set $F_3(H, r) = r$. Then a few lines of calculation show that the system (12) is identical to the system (13), when in the latter we use the functions $\xi_i = r^{-1} F_i$.

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