GGT Journal of Gökova Geometry Topology
Volume 4 (2010) 67 – 81

Criteria for cuspidal S_k singularities and their applications

Kentaro Saji

Abstract. We give useful criteria for $S_1^{\pm}$ singularities in the Mond classification table, and cuspidal $S_k^{\pm}$ singularities. As applications, we give a simple proof of a result given by Mond and a characterization of cuspidal $S_k^{\pm}$ singularities for the composition of a cuspidal edge and a fold map indicated by Arnol'd for the case $k = 1$.

1. Introduction

Singularities of smooth map-germs have long been studied, up to the equivalence under coordinate changes in both source and target. There are two separate problems: the classification and the recognition. The classification is well understood with many good references in the literature. Which germ on the classification table is a given germ equivalent to? Describing simple criteria for this question is the recognition and we will do it in this paper. In the previous method used for the recognition a given map-germ is first normalized and then its jet is studied. The criteria of the recognition without using normalization are not only more convenient but also indispensable in some cases. We call criteria without normalizing *general criteria* for a while. In fact, in the case of wave front surfaces in 3-space, general criteria for the cuspidal edge and the swallowtail were given in [11] where we studied the local and global behavior of flat fronts in hyperbolic 3-space using them. Moreover the singular curvature on the cuspidal edge was introduced and its properties were investigated in [17]. Furthermore, a general criterion for the cuspidal cross cap was given in [4], where we studied maximal surfaces and constant mean curvature one surfaces in the Lorentz-Minkowski 3-space and described a certain duality between the swallowtails and the cuspidal cross caps. The cuspidal cross cap is also called the *cuspidal S_0 singularity*. In [8], general criteria for the cuspidal lips and the cuspidal beaks were given and the horo-flat surfaces in hyperbolic space were investigated. Recently, several applications of these criteria were considered in various situations [6, 7, 8, 9, 12, 18]. Criteria for higher dimensional A-type singularities of wave fronts and their applications were considered in [16].

2000 Mathematics Subject classification. 57R45, 53A05
Key words and phrases. singularities, criteria, Mond singularity, wave front.

In this paper, we first give general criteria for the *Chen Matumoto Mond* $\pm$ *singularities* $S_1^{\pm}$ which are map-germs defined by

$$S_1^{\pm} : (x, y) \mapsto \big(x, y^2, y(x^2 \pm y^2)\big) \tag{1}$$

at the origin (See Figure 1).

FIGURE 1. The Chen Matumoto Mond $+$ singularity (left) and $-$ singularity (right).

X. Y. Chen and T. Matumoto showed these singularities and their suspensions are the generic singularities of one-parameter families of n-dimensional manifolds in $\boldsymbol{R}^{2n+1}$ ([3]). In [14], D. Mond classified simple singularities $\boldsymbol{R}^2 \to \boldsymbol{R}^3$ with respect to the $\mathcal{A}$-equivalence, giving a criterion for map-germs of the forms $(x, y) \mapsto (x, y^2, f(x, y))$ [14, Theorem 4.1.1]. The Chen Matumoto Mond $\pm$ singularities appear as $S_1^{\pm}$ singularities in his classification table [14]. In this paper, we also give criteria for the *cuspidal* $S_k^{\pm}$ *singularities*, which are map-germs defined by

$$cS_k^{\pm} : (x, y) \mapsto \big(x, y^2, y^3(x^{k+1} \pm y^2)\big), \qquad (k = 0, 1, \ldots)$$

at the origin (See Figure 2). These are kinds of "cusped" $S_k^{\pm}$ singularities. If k is even,

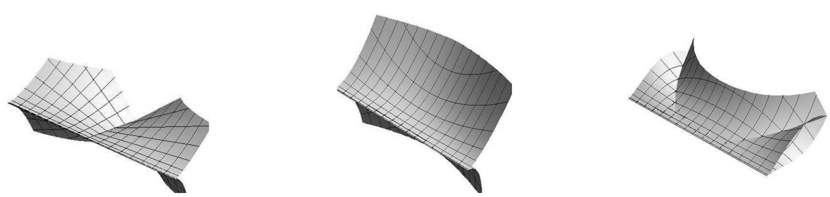

FIGURE 2. cS_0 singularity (left), cS_1^{+} singularity (center) and cS_1^{-} singularity (right).

the cuspidal S_k^{+} singularity and the cuspidal S_k^{-} singularity are $\mathcal{A}$-equivalent. If $k = 0$, this is the *cuspidal cross cap*. We state criteria for the cuspidal $S_k^{\pm}$ singularities as a generalization of the criterion for the cuspidal cross cap given in [4]. It is known that the cuspidal $S_k^{\pm}$ singularities appear as singularities of frontal surfaces (for the definition of frontal surfaces, see §3). As applications, we give a simple proof of the properties on singularities of tangent developable surfaces given by Mond [13] and an interpretation of the degree of contactness about V. I. Arnol'd's observation [1] in §4. All maps considered here are of class C^∞.

The author would like to express his sincere gratitude to Professor Takashi Nishimura for his invaluable advice and comments. In particular, he had suggested a proof of Theorem 3.2 using Thom's splitting lemma simpler than the preliminary version. This work was partially done during the author's stay at the University of Sydney, as a participant of the JSPS joint research program with Australia 2008. The author would like to thank Professors Laurentiu Paunescu and Satoshi Koike for fruitful discussions and kind hospitality. The author is grateful to the referee for careful reading and helpful comments.

2. Criteria for the Chen Matumoto Mond $\pm$ singularities

In this section, we show criteria for the Chen Matumoto Mond singularities of surfaces. If a map-germ $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ satisfies $\operatorname{rank} df_{\mathbf{0}} = 1$, the singular point $\mathbf{0}$ is called *corank one*. If $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ has a corank one singular point at $\mathbf{0}$, then there exist vector fields (ξ, η) near the origin such that $df_{\mathbf{0}}(\eta_{\mathbf{0}}) = \mathbf{0}$ and $\xi_{\mathbf{0}}, \eta_{\mathbf{0}} \in T_{\mathbf{0}}\boldsymbol{R}^2$ are linearly independent. We define a function $\varphi : (\boldsymbol{R}^2, \mathbf{0}) \to \boldsymbol{R}$ by

$$\varphi = \det(\xi f,\ \eta f,\ \eta\eta f), \tag{2}$$

where $\zeta g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is the directional derivative of a vector valued function g by a vector field ζ. We call $\eta_{\mathbf{0}}$ the *null direction* (cf. [11]).

Definition 2.1. Two map-germs $f_i : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ $(i = 1, 2)$ are *$\mathcal{A}$-equivalent* if there exist germs of C^∞-diffeomorphisms d_s and d_t such that $d_t \circ f_1 = f_2 \circ d_s$ holds, where $d_s : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^2, \mathbf{0})$ and $d_t : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$.

Theorem 2.2. *Let $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a map-germ and $\mathbf{0}$ a corank one singular point. Then f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the Chen Matumoto Mond $-$ singularity if and only if φ has a critical point at $\mathbf{0}$ and $\det \operatorname{Hess} \varphi(\mathbf{0}) > 0$. On the other hand f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the Chen Matumoto Mond $+$ singularity if and only if φ has a critical point at $\mathbf{0}$, $\det \operatorname{Hess} \varphi(\mathbf{0}) < 0$ and two vectors $\xi f(\mathbf{0})$ and $\eta\eta f(\mathbf{0})$ are linearly independent.*

Remark 2.3.

- The additional condition in the case $\det \operatorname{Hess} \varphi < 0$ cannot be removed. For example, $(x, xy + y^3, xy + 2y^3)$ satisfies $\det \operatorname{Hess} \varphi(\mathbf{0}) < 0$ but $\xi f(\mathbf{0})$ and $\eta\eta f(\mathbf{0})$ are linearly dependent. It is known that this map-germ is not $\mathcal{A}$-equivalent to the Chen Matumoto Mond singularity. If $\det \operatorname{Hess} \varphi > 0$, then $\xi f(\mathbf{0})$ and $\eta\eta f(\mathbf{0})$ are automatically linearly independent.
- Using the above function φ, we can write the recognition criterion for Whitney umbrella by $\xi\varphi \neq 0$, that is, $d\varphi \neq \mathbf{0}$.
- Since $\eta f(\mathbf{0}) = \mathbf{0}$, Theorem 2.2 implies that the Chen Matumoto Mond singularity is three determined, namely, if the 3-jet of a map-germ $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is $\mathcal{A}$-equivalent to the Chen Matumoto Mond singularity, then f is $\mathcal{A}$-equivalent to the Chen Matumoto Mond singularity.

To prove Theorem 2.2, the following lemmas play the key role.

Lemma 2.4. *The conditions in Theorem* 2.2 *are independent of the choice of vector fields* (ξ,η).

Lemma 2.5. *The conditions in Theorem* 2.2 *are independent of the choice of coordinates on the target.*

Proof of Lemma 2.4. Let us put

$$\begin{cases} \overline{\xi} &= a_{11}\xi + a_{12}\eta \\ \overline{\eta} &= a_{21}\xi + a_{22}\eta \end{cases}, \quad \big((a_{ij}) : \boldsymbol{R}^2 \to GL(2,\boldsymbol{R}),\ a_{21}(\mathbf{0}) = 0\big),$$

and

$$\overline{\varphi} = \det(\overline{\xi}f,\ \overline{\eta}f,\ \overline{\eta}\,\overline{\eta}f).$$

Then by a straight calculation, we have

$$\begin{aligned} \overline{\xi}f &= a_{11}\xi f + a_{12}\eta f, \\ \overline{\eta}f &= a_{21}\xi f + a_{22}\eta f \quad \text{and} \\ \overline{\eta}\,\overline{\eta}f &= *\xi f + *\eta f + a_{21}(a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f) + a_{22}^2\eta\eta f. \end{aligned}$$

Thus it follows that the linear independence of two vectors $\xi f(\mathbf{0})$ and $\eta\eta f(\mathbf{0})$ does not depend on the choice of vector fields. Hence, we have

$$\overline{\varphi} = (a_{11}a_{22} - a_{12}a_{21})\Big(a_{21}\det(\xi f,\ \eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f) + a_{22}^2\det(\xi f,\ \eta f,\ \eta\eta f)\Big).$$

Now it is sufficient to prove that

$$\xi m(\mathbf{0}) = \eta m(\mathbf{0}) = 0 \text{ and } \operatorname{Hess} m(\mathbf{0}) = O, \qquad \text{where } m := a_{21}\det(\xi f,\ \eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f).$$

Since m contains the terms a_{21} and ηf, which vanish at the origin, it holds that $\xi m(\mathbf{0}) = \eta m(\mathbf{0}) = 0$. Next, we have assumed that φ has a critical point at $\mathbf{0}$, namely,

$$\xi\varphi(\mathbf{0}) = \det(\xi f,\ \xi\eta f,\ \eta\eta f)(\mathbf{0}) = 0. \tag{3}$$

Since $\xi\eta - \eta\xi$ is a vector field, and ξ and η are linearly independent, $\xi\eta - \eta\xi$ is a linear combination of ξ and η at each point. Moreover, $\xi\eta f(\mathbf{0}) - \eta\xi f(\mathbf{0})$ is parallel to $\xi f(\mathbf{0})$, since η is the null vector at $\mathbf{0}$. Thus we see that $-(\xi\eta + \eta\xi)f = -2\xi\eta f + (\xi\eta - \eta\xi)f$ is a linear combination of $\xi\eta f$ and ξf at $\mathbf{0}$. Thus,

$$\xi\xi m(\mathbf{0}) = 2\xi a_{21}(\mathbf{0})\det(\xi f,\ \xi\eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f)(\mathbf{0}) = 0$$

holds, since $a_{21}(\mathbf{0}) = 0$. By the same reason and (3), we also have

$$\begin{aligned} \xi\eta m(\mathbf{0}) = \xi a_{21}(\mathbf{0})&\det(\xi f,\ \eta\eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f)(\mathbf{0}) \\ &+ \eta a_{21}(\mathbf{0})\det(\xi f,\ \xi\eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f)(\mathbf{0}) = 0. \end{aligned}$$

Furthermore, $\eta\eta\, m(\mathbf{0}) = 2\eta a_{21}(\mathbf{0})\det(\xi f,\ \eta\eta f,\ a_{21}\xi\xi f + a_{22}\xi\eta f + a_{22}\eta\xi f)(\mathbf{0}) = 0$ as well. Hence $\operatorname{Hess} m(\mathbf{0}) = O$ holds. $\square$

Proof of Lemma 2.5. Take a diffeomorphism $\Phi(X) = \big(\Phi_1(X), \Phi_2(X), \Phi_3(X)\big)$ of $\boldsymbol{R}^3$, where $X = (X_1, X_2, X_3)$. Put $f = (f_1, f_2, f_3)$ and

$$\tilde{\varphi} = \det\Big(\xi(\Phi \circ f),\ \eta(\Phi \circ f),\ \eta\eta(\Phi \circ f)\Big).$$

Then the first component of the vector $\eta\eta(\Phi \circ f) = \eta\big(d\Phi(\eta f)\big)$ is calculated as

$$\begin{aligned}\eta\left(\sum_{i=1}^{3} \frac{\partial \Phi_1}{\partial X_i}\eta f_i\right) &= \sum_{i=1}^{3}\left(\left(\sum_{j=1}^{3} \frac{\partial^2 \Phi_1}{\partial X_i \partial X_j}\eta f_j\right)\eta f_i + \frac{\partial \Phi_1}{\partial X_i}\eta\eta f_i\right) \\ &= \operatorname{Hess}\Phi_1(\eta f, \eta f) + d\Phi_1(\eta\eta f).\end{aligned} \tag{4}$$

Hence the linear independence of $\xi f(\mathbf{0})$ and $\eta\eta f(\mathbf{0})$ does not depend on the choice of the coordinates of the target. By (4) again, it holds that

$$\tilde{\varphi} \quad = \quad \det\left(d\Phi(\xi f),\ d\Phi(\eta f),\ \begin{pmatrix}\operatorname{Hess}\Phi_1(\eta f, \eta f) \\ \operatorname{Hess}\Phi_2(\eta f, \eta f) \\ \operatorname{Hess}\Phi_3(\eta f, \eta f)\end{pmatrix}\right) + (\det d\Phi)\,\varphi.$$

Thus by the same argument as above, it is sufficient to prove that $\operatorname{Hess} M(\mathbf{0}) = O$, where

$$M := \det\Big(d\Phi(\xi f),\ d\Phi(\eta f),\ \operatorname{Hess}\Phi_i(\eta f, \eta f)_{i=1,2,3}\Big).$$

Since ηf vanishes at the origin, $\operatorname{Hess} M(\mathbf{0}) = O$ holds. $\square$

Using these Lemmas, we prove Theorem 2.2.

Proof of Theorem 2.2. The necessity of the conditions is immediate from the calculation for the formula (1) and Lemmas 2.4 and 2.5. We prove that the conditions are sufficient. Let us assume the conditions in Theorem 2.2. By Lemmas 2.4 and 2.5, we can change vector fields (ξ, η) and coordinates on the target. Moreover, since the conditions do not depend on the coordinates on the source, we may change coordinates on the source. Since f is corank one at $\mathbf{0}$, by the implicit function theorem, f is $\mathcal{A}$-equivalent to the map-germ defined by $(x, y) \mapsto (x, f_2(x, y), f_3(x, y))$ at the origin. By the target coordinate change, f is $\mathcal{A}$-equivalent to the map-germ $(x, yg(x, y), yh(x, y))$. Since f has a singularity at the origin, there is no constant term in g and h. Moreover, we have the following lemma.

Lemma 2.6. *At the origin, g_y or h_y does not vanish, where $g_y = \partial g/\partial y$ and $h_y = \partial h/\partial y$.*

Proof. Since $(\partial/\partial y)\big(x, yg(x, y), yh(x, y)\big)(\mathbf{0}) = \mathbf{0}$ holds, we may choose $\xi = \partial/\partial x$, and $\eta = \partial/\partial y$. Then it holds that

$$\varphi = \det\begin{pmatrix} 1 & 0 & 0 \\ * & g + yg_y & 2g_y + yg_{yy} \\ * & h + yh_y & 2h_y + yh_{yy}\end{pmatrix}.$$

Since $g(\mathbf{0}) = h(\mathbf{0}) = 0$, we have $\varphi_{yy}(\mathbf{0}) = 6(g_y h_{yy} - g_{yy} h_y)(\mathbf{0})$.

In the case of $\det \mathrm{Hess}\,\varphi(\mathbf{0}) > 0$, if we assume that $g_y(\mathbf{0}) = h_y(\mathbf{0}) = 0$, then $\varphi_{yy}(\mathbf{0}) = 0$ holds and hence $\det \mathrm{Hess}\,\varphi(\mathbf{0}) = -(\varphi_{xy})^2(\mathbf{0}) \leq 0$ at the origin, which contradicts the assumption. Therefore, in this case $g_y(\mathbf{0}) \neq 0$ or $h_y(\mathbf{0}) \neq 0$ holds.

On the other hand, in the case of $\det \mathrm{Hess}\,\varphi < 0$, we have the additional condition which implies that $\eta\eta f(\mathbf{0}) \neq \mathbf{0}$. Thus we have $g_y(\mathbf{0}) \neq 0$ or $h_y(\mathbf{0}) \neq 0$. □

Let us continue the proof of Theorem 2.2. By Lemma 2.6, we may assume $g_y(\mathbf{0}) \neq 0$. Since by the implicit function theorem, the set $\{g(x,y) = 0\}$ is a regular curve, we take a new coordinate system $(x, \tilde{y})$ satisfying $g(x, \tilde{y}) = 0$ on $\tilde{y} = 0$. Then we may assume that $f(x, \tilde{y}) = \big(x, \tilde{y}^2 g(x, \tilde{y}), \tilde{y} h(x, \tilde{y})\big)$, $g(0,0) \neq 0$. Furthermore, considering a coordinate change $(\overline{x}, \overline{y}) = \big(x, \tilde{y}\sqrt{|g(x,\tilde{y})|}\big)$ and rewriting $(\overline{x}, \overline{y})$ by (x, y), we may assume $f(x,y) = (x, y^2, y\overline{h}(x,y))$. Now we set

$$\overline{h}_1(x,y) = \frac{\overline{h}(x,y) + \overline{h}(x,-y)}{2}, \qquad \overline{h}_2(x,y) = \frac{\overline{h}(x,y) - \overline{h}(x,-y)}{2}.$$

Then $\overline{h}(x,y) = \overline{h}_1(x,y) + \overline{h}_2(x,y)$ holds and $\overline{h}_1(x,y)$ (resp. $\overline{h}_2(x,y)$) is an even (resp. odd) function with respect to y. Then there exist functions $\tilde{h}_1(x,y)$ and $\tilde{h}_2(x,y)$ such that

$$\overline{h}_1(x,y) = \tilde{h}_1(x,y^2), \qquad \overline{h}_2(x,y) = y\tilde{h}_2(x,y^2).$$

Then we have $f(x,y) = \big(x, y^2, y\tilde{h}_1(x,y^2) + y^2\tilde{h}_2(x,y^2)\big)$. Considering a coordinate change $\tilde{\Theta}(X,Y,Z) = (X, Y, Z - Y\tilde{h}_2(X,Y))$ and replacing f by $\tilde{\Theta} \circ f$, we may set

$$f(x,y) = \big(x, y^2, y\tilde{h}_1(x,y^2)\big).$$

Since the function φ defined by (2) for this map has the form $-2\tilde{h}_1(x,y^2) + y\,*$, it holds that $(\partial/\partial x)\tilde{h}_1(\mathbf{0}) = 0$. Here $*$ means a function. Thus there exists a function $\tilde{f}$ such that

$$f(x,y) = \left(x, y^2, y\Big[\alpha x^2 + \beta y^2\big(1 + \tilde{f}(x,y^2)\big)\Big]\right), \quad \tilde{f}(0,0) = 0.$$

Note that the function φ for this map has the form

$$-2\alpha x^2 + 6\beta y^2 + (\text{higher order term}).$$

Considering a diffeomorphism θ defined by

$$(u,v) = \theta(x,y) = (x, \theta_2(x,y)) = \left(x, y\sqrt{1 + \tilde{f}(x,y^2)}\right)$$

and the inverse map $\theta^{-1}(u,v) = (u, \vartheta_2(u,v))$, we have the following lemma.

Lemma 2.7. *There exists a function f_2 satisfying $f_2(\mathbf{0}) \neq 0$ and $\vartheta_2(u,v) = v f_2(u,v^2)$.*

Proof. Substituting $v = 0$ in the identity

$$\theta \circ \theta^{-1}(u,v) = \left(u, \vartheta_2(u,v)\sqrt{1 + \tilde{f}(u, \vartheta_2(u,v)^2)}\right) = (u,v),$$

we have $\vartheta_2(u,0)=0$. Next, we take $(x,y)=\theta^{-1}(u,-v)$. Then we have $v=-\theta_2(x,y) = \theta_2(x,-y)$. Since $(u,v)=(x,\theta_2(x,-y))$, it holds that $\theta^{-1}(u,v)=(x,-y)$. Thus $\vartheta_2(u,-v)=-\vartheta_2(u,v)$. Hence ϑ_2 satisfies that $\vartheta_2(u,0)=0$ and $\vartheta_2(u,-v)=-\vartheta_2(u,v)$ for any (u,v) near $\mathbf{0}$. Then by the same argument as the construction of $\bar{h}_2(x,y)$ in the proof of Theorem 2.2, we have the lemma. □

By Lemma 2.7, the composition $f\circ\theta^{-1}$ has the expression

$$\big(u, v^2 f_2(u,v^2)^2, v f_2(u,v^2)(\alpha u^2+\beta v^2)\big).$$

Considering a diffeomorphism $\Theta(X,Y,Z)=(X, Y f_2(X,Y)^2, Z f_2(X,Y))$, we see that $\Theta^{-1}\circ f\circ\theta^{-1}$ has the form $(u,v^2,v(\alpha u^2+\beta v^2))$. This is $\mathcal{A}$-equivalent to the desired map-germ because we see that

$$-\operatorname{sgn}(\alpha\beta)=\operatorname{sgn}(\det \operatorname{Hess}\varphi(\mathbf{0})).$$

□

3. Criteria for cuspidal $S_k^{\pm}$ singularities of frontals

In this section, we shall introduce the notion of frontal surfaces and give criteria for the cuspidal $S_k^{\pm}$ singularities of frontals.

3.1. Preliminaries on the frontals

The projective cotangent bundle $PT^*\boldsymbol{R}^3$ of $\boldsymbol{R}^3$ has the canonical contact structure and can be identified with the projective tangent bundle $PT\boldsymbol{R}^3$. A smooth map-germ $f:(\boldsymbol{R}^2,\mathbf{0})\to(\boldsymbol{R}^3,\mathbf{0})$ is called a *frontal* if there exists a never-vanishing vector field ν of $\boldsymbol{R}^3$ along f such that $L:=(f,[\nu]):(\boldsymbol{R}^2,\mathbf{0})\to(\boldsymbol{R}^3\times P^2, L(\mathbf{0}))=(PT\boldsymbol{R}^3, L(\mathbf{0}))$ is an isotropic map, that is, the pull-back of the canonical contact form of $PT\boldsymbol{R}^3$ vanishes on $\boldsymbol{R}^2$, where P^2 means the projective space and $[\nu]$ means the projective class of ν. This condition is equivalent to the following orthogonality condition:

$$\langle df(X_p),\nu(p)\rangle = 0 \qquad (\forall p\in\boldsymbol{R}^2,\quad \forall X_p\in T_p\boldsymbol{R}^2),$$

where $\langle\ ,\ \rangle$ is the canonical inner product on $\boldsymbol{R}^3$. The vector field ν is called the *normal vector* of the frontal f. The plane perpendicular to $\nu(p)$ is called the *limiting tangent plane* at p. A frontal f is called a *front* if $L=(f,[\nu])$ is an immersion (cf. [1], see also [11]). The function

$$\lambda(u,v):=\det(f_u,f_v,\nu)$$

is called the *signed area density function*, where (u,v) is the coordinate system on $\boldsymbol{R}^2$.

Let $\mathbf{0}$ be a singular point of a frontal $f:(\boldsymbol{R}^2,\mathbf{0})\to(\boldsymbol{R}^3,\mathbf{0})$. Then the set of singular points $S(f)$ of f coincides with the zeros of λ near $\mathbf{0}$. If $d\lambda(\mathbf{0})\neq 0$, then $\mathbf{0}$ is called a *non-degenerate singular point*. Assume now that $\mathbf{0}$ is a non-degenerate singular point. Then there exists a regular curve $\gamma(t):((-\varepsilon,\varepsilon),0)\to(\boldsymbol{R}^2,\mathbf{0})$ $(\varepsilon>0)$ such that the image of γ is $S(f)$. Also the dimension of the kernel $\ker(df_{\gamma(t)})$ is equal to one and there is a never-vanishing vector $\eta(t)$ such that $\eta(t)$ spans $\ker(df_{\gamma(t)})$. We call η the *null vector field*. We define a function ψ on $S(f)$ by

$$\psi(t) = \det\left(\frac{df\circ\gamma}{dt}(t), \nu\circ\gamma(t), d\nu_{\gamma(t)}\big(\eta(t)\big)\right) \quad \text{for} \quad t\in(-\varepsilon,\varepsilon). \tag{5}$$

This function was originally defined in [4]. The signed area density function λ, the non-degeneracy and the null vector field η were introduced in [11].

3.2. Criterion for the $(2,5)$-cusp

If we substitute $u=0$ in the normal form of the cuspidal S_k^ϵ singularity given by $\big(u, v^2, v^3(u^{k+1}+\epsilon v^2)\big)$, $\epsilon=\pm1$, it reduces to a $(2,5)$-cusp curve through $\mathbf{0}$. In this subsection, we state a criterion for the $(2,5)$-cusp, namely, the map-germ given by $t\mapsto(t^2,t^5,0)$ at $t=0$.

Lemma 3.1. *Let $c(t): I\to \boldsymbol{R}^3$ be a curve and $0\in I$.*

(i) *Assume that c satisfies $c_0'=\mathbf{0}$, $c_0''\neq\mathbf{0}$,$c_0'''=c_0^{(4)}=\mathbf{0}$, and c_0'' and $c_0^{(5)}$ are linearly independent, then c at $t=0$ is $\mathcal{A}$-equivalent to the $(2,5)$-cusp.*

(ii) *Assume that c satisfies $c_0'=\mathbf{0}$, $c_0''\neq0$ and two vectors c_0'' and c_0''' are linearly dependent, that is, there exists $\ell\in\boldsymbol{R}$ such that $c_0'''=\ell c_0''$. If two vectors c_0'' and $3c_0^{(5)}-10\ell c_0^{(4)}$ are linearly independent in addition, then c at $t=0$ is $\mathcal{A}$-equivalent to the $(2,5)$-cusp.*

Here, $c'=c^{(1)}=dc/dt$, $c^{(j)}=dc^{(j-1)}/dt$, and $c_0^{(j)}=c^{(j)}(0)$ $(j=1,\dots,5)$.

Proof. One can prove (i) by a fundamental argument. So, we omit its proof. We shall prove (ii). Suppose that c satisfies the assumptions of (ii) except the last condition. Then c is written as

$$\begin{aligned} c(t) = \big(&a_2t^2+a_3t^3+a_4t^4+a_5t^5+o(t^5), ka_2t^2+ka_3t^3+b_4t^4+b_5t^5+o(t^5),\\ &ka_2t^2+ka_3t^3+c_4t^4+c_5t^5+o(t^5)\big),\ a_2,a_3,a_4,a_5,b_4,b_5,c_4,c_5,k\in\boldsymbol{R} \text{ and } a_2\neq0,\end{aligned}$$

where $o(t^5)$ is a Landau notation. Considering a coordinate change on the target given by $(X,Y,Z)\mapsto(X,Y-kX,Z-kX)$, we see that c is $\mathcal{A}$-equivalent to

$$\begin{aligned}\big(a_2t^2+a_3t^3+a_4t^4+a_5t^5+o(t^5), (b_4-ka_4)t^4+(b_5-ka_5)t^5+o(t^5),\\ (c_4-ka_4)t^4+(c_5-ka_5)t^5+o(t^5)\big).\end{aligned}$$

Next, considering a parameter change $t\mapsto t-(a_3/2a_2)t^2$, we get

$$\begin{aligned}\Bigg(a_2t^2+\left(-\frac{5a_3^2}{4a_2}+a_4\right)t^4+\left(\frac{3a_3^3}{4a_2^2}-\frac{2a_3a_4}{a_2}+a_5\right)t^5+o(t^5),\\ (b_4-ka_4)t^4+\left(-\frac{2a_3}{a_2}(b_4-ka_4)+b_5-ka_5\right)t^5+o(t^5),\\ (c_4-ka_4)t^4+\left(-\frac{2a_3}{a_2}(c_4-ka_4)+c_5-ka_5\right)t^5+o(t^5)\Bigg).\end{aligned}$$

Lastly, considering the coordinate change

$(X, Y, Z) \mapsto (X - (-5a_3^2 + 4a_2a_4)X^2/(4a_2^2), Y - (b_4 - ka_4)X^2/a_2^2, Z - (c_4 - ka_4)X^2/a_2^2),$

we get

$$\left(a_2t^2 + \left(\frac{3a_3^3}{4a_2^2} - \frac{2a_3a_4}{a_2} + a_5\right)t^5 + o(t^5), \left(-\frac{2a_3}{a_2}(b_4 - ka_4) + b_5 - ka_5\right)t^5 + o(t^5),\right.$$
$$\left.\left(-\frac{2a_3}{a_2}(c_4 - ka_4) + c_5 - ka_5\right)t^5 + o(t^5)\right).$$

By a direct calculation we see the last condition of (ii) is equivalent to the condition

$$a_2(b_5 - ka_5) - 2a_3(b_4 - ka_4) \neq 0 \text{ or } a_2(c_5 - ka_5) - 2a_3(c_4 - ka_4) \neq 0 \quad \text{at} \quad t = 0.$$

This is also the assumption of (i) for the curve with respect to the last coordinate change and we complete the proof. ☐

3.3. Criteria for cuspidal $S_k^\pm$ singularities

Criteria for the cuspidal $S_k^\pm$ singularities are stated as follows:

Theorem 3.2. *Let $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a frontal and fix a representative ν of the normal vector of f. The map-germ f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal $S_{k-1}^\pm$ singularity $(k \geq 2)$ if and only if the following* (a)-(d) *hold*:

(a) *$\mathbf{0}$ is a non-degenerate singular point and the null vector is transverse to $S(f)$ at $\mathbf{0}$.*
(b) *There exists a curve $c : ((-\varepsilon, \varepsilon), 0) \to (\boldsymbol{R}^2, \mathbf{0})$ such that $c'(0)$ is parallel to $\eta(\mathbf{0})$, $\hat{c}_0' = 0, \hat{c}_0'' \neq 0$ and there exists ℓ satisfying $\hat{c}_0''' = \ell\hat{c}_0''$ and the determinant $a := \det(\hat{\gamma}', \hat{c}'', 3\hat{c}^{(5)} - 10\ell\hat{c}^{(4)})(0) \neq 0$, where $\hat{c} = f \circ c$ and $\hat{\gamma} = f \circ \gamma$.*
(c) *$\psi(0) = \psi'(0) = \cdots = \psi^{(k-1)}(0) = 0$ and $b := \psi^{(k)}(0) \neq 0$, where ψ is the function defined by* (5).
(d) *If k is even, sign $\pm$ of the cuspidal $S_{k-1}^\pm$ singularity coincides with the sign of the product $ab = \det(\hat{\gamma}', \hat{c}'', 3\hat{c}^{(5)} - 10\ell\hat{c}^{(4)})(0) \cdot \psi^{(k)}(0)$. Here, we choose η and t so that $c'(0)$ points the same direction as the null vector $\eta(0)$ and that $(\gamma', \eta)(0)$ is positively oriented.*

To prove Theorem 3.2, we show at first the following lemma.

Lemma 3.3. *The conditions in Theorem* 3.2 *do not depend either choice of coordinates on the source, the parameter of c, the parameter of γ, the choice of representative ν, the choice of η or on the choice of coordinates on the target.*

It is easy to check that the condition (a) does not depend on all the choices by Lemma 3.1. Since linear independence is not changed by a diffeomorphism, the condition (b) does not depend on all the choices. We shall prove that either of the conditions (c) and (d) does not depend on all the choices.

Proof for the condition (c). Note that the condition (c) is not changed on the non-zero functional multiple of ψ on $S(f)$. Thus it does not depend on the choices of ν, η and the parameter of γ. Hence it is sufficient to prove that the condition (c) does not depend on the choice of the coordinates on the target.

Let $\Phi : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a diffeomorphism-germ and $d\Phi$ its derivative. The map $d\Phi$ can be considered as a $GL(3, \boldsymbol{R})$-valued function $q \mapsto d\Phi_q$. Since $Au \times Av$ is $(\det A)\,{}^tA^{-1}u \times v$ for any vectors u and v in the 3-space and any non-singular matrix A, we can take $\tilde{\nu} = {}^t(d\Phi)^{-1}\nu$ as a normal vector field of $\tilde{f} = \Phi \circ f$. So, we shall prove

$$\tilde{\psi}(t) = \det\left((\tilde{f} \circ \gamma)'(t),\ \tilde{\nu} \circ \gamma(t),\ d\tilde{\nu}_{\gamma(t)}(\eta(t))\right)$$

is a non-zero functional multiple of $\psi(t)$.

Since the condition does not depend on the choices of coordinates on the source, choice of η and choice of ν, we may assume that $S(f) = \{v = 0\}$, $\eta = \partial/\partial v$ on $\gamma(t)$, ν is the unit normal vector and $f(u, 0)$ is the arc-length parameter. Under this assumption, f_u, ν, ν_v are orthogonal each other, since we see $\langle \nu, \nu_v \rangle = 0$ from $\langle \nu, \nu \rangle = 1$ and $\langle f_u, \nu_v \rangle = -\langle f_{uv}, \nu \rangle = \langle f_v, \nu_u \rangle = 0$ on $S(f)$ from $f_v = \mathbf{0}$ on $S(f)$. Hence $\nu \times \nu_v$ is parallel to f_u. Thus we have $\psi = \det(f_u, \nu, \nu_v) = \langle f_u, \nu \times \nu_v \rangle$, and it holds that $\nu \times \nu_v = \psi f_u$. Then, it follows that

$$\begin{aligned}
\tilde{\psi}(t) &= \det\Big(d\Phi_{f(\gamma(t))} f_u(\gamma(t)),\ {}^t(d\Phi_{f(\gamma(t))})^{-1}\nu(\gamma(t)),\ \big({}^t(d\Phi_{f(\gamma(t))})^{-1}\nu(\gamma(t))\big)_v\Big) \\
&= \det\left(d\Phi_{f(\gamma(t))} f_u(\gamma(t)),\ {}^t(d\Phi_{f(\gamma(t))})^{-1}\nu(\gamma(t)),\ {}^t(d\Phi_{f(\gamma(t))})^{-1}\nu_v(\gamma(t))\right).
\end{aligned}$$

Here note that $({}^t(d\Phi_{f(\gamma(t))})^{-1})_v = 0$ on $S(f)$ because $f_v(\gamma(t)) = 0$ and $\det(d\Phi_{f(\gamma(t))}) \neq 0$. Omitting (t), $\gamma(t)$ and $f(\gamma(t))$, we can modify

$$\begin{aligned}
\tilde{\psi} &= \det\left(d\Phi f_u,\ {}^t(d\Phi)^{-1}\nu,\ {}^t(d\Phi)^{-1}\nu_v\right) = \left\langle d\Phi f_u,\ {}^t(d\Phi)^{-1}\nu \times {}^t(d\Phi)^{-1}\nu_v \right\rangle \\
&= \left\langle d\Phi f_u,\ \det({}^t(d\Phi)^{-1}) d\Phi(\nu \times \nu_v) \right\rangle = \det({}^t(d\Phi)^{-1}) \langle d\Phi f_u, d\Phi(\psi f_u) \rangle \\
&= \det((d\Phi)^{-1}) \langle d\Phi f_u, d\Phi f_u \rangle \psi.
\end{aligned}$$

Since $\det((d\Phi)^{-1}) \langle d\Phi f_u, d\Phi f_u \rangle$ is a function which never vanishes on $S(f)$, the condition (c) does not depend on the choice of the coordinate system on the target. $\square$

Proof for the condition (d). When the direction of the representative ν of $[\nu]$ is changed to opposite direction, the signs of both a and b are not changed. When the parameter of γ reverses, the sign of a is unchanged, and if k is even then the sign of b is unchanged because of the positivity of the basis (γ', η). If the orientation of the target is changed, then signs of both a and b are changed. Hence in all the cases, $\operatorname{sgn}(ab)$ is not changed. $\square$

Proof of Theorem 3.2. Assume that a map-germ f satisfies the conditions in Theorem 3.2. By the same argument as in the proof of Theorem 2.2, we may assume that

$$f(u, v) = \big(u, vg(u, v), vh(u, v)\big).$$

Consider the new coordinate system $(u, \tilde{v})$ satisfying $S(f) = \{\tilde{v} = 0\}$ and rewrite $\tilde{v}$ by v. Then,we get $g = h = 0$ on $v = 0$. Thus, there exist functions $\tilde{g}(u, v)$ and $\tilde{h}(u, v)$ such that

$f(u,v) = \big(u, v^2\tilde{g}(u,v), v^2\tilde{h}(u,v)\big)$. By the same argument as in the proof of Theorem 2.2 again, we may assume that

$$f(u,v) = (u, v^2, v^3\overline{h}(u,v^2)).$$

The normal vector $[\nu]$ of f and $d\nu(\eta)$ are given by

$$\nu = \left(*, -3v\bar{h}(u,v^2) - 2v^3\frac{\partial\bar{h}}{\partial v}(u,v^2), 2\right) \text{ and } d\nu(\eta) = d\nu(\partial/\partial v) = \big(*, -3\bar{h}(u,v^2) + v*, 0\big).$$

Thus the function ψ of this map is $6\overline{h}(t,0)$. Thus the condition (c) is written as

$$\overline{h} = (\partial/\partial u)\overline{h} = \ldots = (\partial^{k-1}/\partial u^{k-1})\overline{h} = 0 \quad \text{and} \quad (\partial^k/\partial u^k)\overline{h} \neq 0 \quad \text{at} \quad \mathbf{0}.$$

Consider a curve $(o(v), v)$, where $o(v)$ is a Landau notation again. Then this curve is tangent to η at $\mathbf{0}$ and all curves passing through $\mathbf{0}$ tangent to $\eta = \partial/\partial v$ at $\mathbf{0}$ are written by this form. Since $(\partial/\partial u)\overline{h}(\mathbf{0}) = 0$, $v^3\overline{h}(o(v),v)$ has no terms of v^4. Thus the condition (b) is equivalent to $(\partial^5/\partial v^5)\big(v^3\overline{h}(o(v),v)\big)(0) \neq 0$. Hence, the coefficient of v^2 in $\overline{h}(u,v^2)$ is not zero. Thus it follows that there exist functions h, $\overline{g}$ and non-zero real numbers α, β such that

$$\overline{h}(u,v^2) = \alpha v^2 + \beta u^k h(u) + v^2\alpha\overline{g}(u,v^2), \quad h(0) = 1.$$

By the coordinate system change

$$\begin{aligned} U &= u\sqrt[k]{h(u)} \\ V &= \sqrt{|\alpha|}v\sqrt{1+\overline{g}(u,v^2)}, \end{aligned} \tag{6}$$

$\overline{h}$ becomes $\operatorname{sgn}(\alpha)V^2 + \beta U^k$. One can easily see that the inverse map of (6) is given by

$$\begin{aligned} u &= UH(U) \\ v &= VG(U,V^2), \end{aligned}$$

using functions G, H whose constant terms are not zero. Hence f is $\mathcal{A}$-equivalent to

$$f(U,V) = \big(UH(U), V^2G(U,V^2)^2, V^3G(U,V^2)^3(\operatorname{sgn}(\alpha)V^2 + \beta U^k)\big).$$

Now we consider a map-germ $\big(u, v^2, v^3(\operatorname{sgn}(\alpha)v^2 + \beta u^k)\big)$ and a diffeomorphism

$$\Psi(X,Y,Z) = \big(XH(X), YG(X,Y)^2, ZG(X,Y)^3\big).$$

Then it follows that f is $\mathcal{A}$-equivalent to $\big(u, v^2, v^3(\operatorname{sgn}(\alpha)v^2 + \beta u^k)\big)$.

Here, we have $ab = 6(6!k!)\operatorname{sgn}(\alpha)\beta$. By a suitable scale change, if k is odd or k is even and $\operatorname{sgn}(\alpha)\beta > 0$, then f is $\mathcal{A}$-equivalent to $(u, v^2, v^3(v^2 + u^k))$. If k is even and $\operatorname{sgn}(\alpha)\beta < 0$, then f is $\mathcal{A}$-equivalent to $(u, v^2, v^3(v^2 - u^k))$. □

4. Applications

In this section, we give two applications of our criteria.

Let $\boldsymbol{s} : ((-\varepsilon, \varepsilon), 0) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a space curve such that its curvature never vanishes, with the arclength parameter. Let $\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}$ be its Frenet frame and κ, τ its curvature and torsion respectively. A map $(t, u) \mapsto \boldsymbol{s}(t) + u\boldsymbol{e}(t)$ is called the *tangent developable surface* of $\boldsymbol{s}$. In [15], Mond proved the following theorem.

Theorem 4.1 (Mond [15])**.** *The germ of the tangent developable surface of* $\boldsymbol{s}$ *at* $(0, 0)$ *is* $\mathcal{A}$*-equivalent to the cuspidal* S_1^+ *singularity if* $\tau = \tau' = 0$ *and* $\tau'' \neq 0$ *at* 0.

Remark 4.2. Mond also classified the case that $\tau = \tau' = \cdots = \tau^{(k-1)} = 0$, $\tau^{(k)} \neq 0$ for $k = 3$ and 4. By Ishikawa's theorem [5], the developable surfaces do not have any cuspidal S_{k-1} singularities for $k > 2$.

We shall prove Theorem 4.1 using our criteria as an application.

Proof. Let $\boldsymbol{s}$ be a space curve and $f(t, u) = \boldsymbol{s}(t) + u\boldsymbol{e}(t)$ the tangent developable surface of $\boldsymbol{s}$. Then $S(f) = \{u = 0\}$ and $\eta = -\partial/\partial t + \partial/\partial u$. Since $\lambda = \det(\boldsymbol{e} + u\kappa\boldsymbol{n}, \boldsymbol{e}, \boldsymbol{b}) = -\kappa u$, we see that $d\lambda \neq 0$ and the singularities are non-degenerate. Let us consider a curve

$$c : t \mapsto \left(-t, -\frac{\boldsymbol{s}(-t) \cdot \boldsymbol{e}(0)}{\boldsymbol{e}(-t) \cdot \boldsymbol{e}(0)} \right),$$

in the (t, u)-space and put $\hat{c} = f \circ c$. Then, we see that c satisfies the condition (b) of Theorem 3.2. In fact, by a direct calculation we have

$$\begin{aligned}
&\hat{c}'(0) = \boldsymbol{0}, \\
&\hat{c}''(0) = -\kappa(0)\boldsymbol{n}(0), \\
&\hat{c}'''(0) = 2\kappa'(0)\boldsymbol{n}(0), \\
&\hat{c}^{(4)}(0) = *\boldsymbol{e}(0) + *\boldsymbol{n}(0), \quad \text{and} \\
&\hat{c}^{(5)}(0) = *\boldsymbol{e}(0) + *\boldsymbol{n}(0) + 4\kappa(0)\tau''(0)\boldsymbol{b}(0).
\end{aligned}$$

Hence $a = -12\det(\boldsymbol{e}, \kappa\boldsymbol{n}, \kappa\tau''\boldsymbol{b})(0) \neq 0$ holds. Moreover, since we can take $\nu = \boldsymbol{b}$, we have $d\nu(\eta) = -(\partial\boldsymbol{b}/\partial t) = \tau\boldsymbol{n}$. So,

$$\psi(t) = \det\big(\hat{\gamma}'(t),\ \nu(\gamma(t)),\ \eta(t)\nu(\gamma(t))\big) = \det(\boldsymbol{e}(t), \boldsymbol{b}(t), \tau\boldsymbol{n}(t)) = -\tau(t)$$

and hence $b = -\tau''(0)$ by the assumption. Since $ab > 0$, f at $(0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity by Theorem 3.2. $\square$

Now we consider another property of the cuspidal S_k singularity. The following observation about the cuspidal cross cap was given by Arnol'd [1, p.120 Example 3]. Let $f : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a cuspidal edge and $F : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ a "generic" fold. Then the map-germ $F \circ f$ at $\boldsymbol{0}$ is a cuspidal cross cap, where the *cuspidal edge* is a map-germ defined by $(u, v) \mapsto (u, v^2, v^3)$ at the origin and the *fold* is a map-germ defined by $(x, y, z) \mapsto (x, y, z^2)$ at the origin. Here, we generalize this observation and clarify the meaning of *genericness*.

Theorem 4.3. *Let $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a map-germ $\mathcal{A}$-equivalent to the cuspidal edge and $F : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a map-germ $\mathcal{A}$-equivalent to the fold. Assume that the following three conditions hold:*

(A) *the limiting tangent plane LT of f at $\mathbf{0}$ does not contain the kernel* $\ker dF_{\mathbf{0}}$*,*
(B) *LT is transverse to $S(F)$ at $\mathbf{0}$,*
(C) *the singular curve $\hat{\gamma} = f(S(f))$ has a k-point contact with $S(F)$ at $\mathbf{0}$.*

Then the composition $F \circ f$ at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal $S^{\pm}_{k-1}$ singularity.

In the case of even k, the sign $\pm$ is determined by the following rule. Since k is the order of contact between $\hat{\gamma}$ and $S(F)$, it holds that $\hat{\gamma}$ is locally located on the half space bounded by $S(F) \subset \boldsymbol{R}^3$. If $\mathrm{Im}(f)$ *is also locally located only on the same side as $\hat{\gamma}$, then $F \circ f$ is $\mathcal{A}$-equivalent to the cuspidal S^{+}_{k-1} singularity. Otherwise, $F \circ f$ is $\mathcal{A}$-equivalent to the cuspidal S^{-}_{k-1} singularity* (*See Figure* 3).

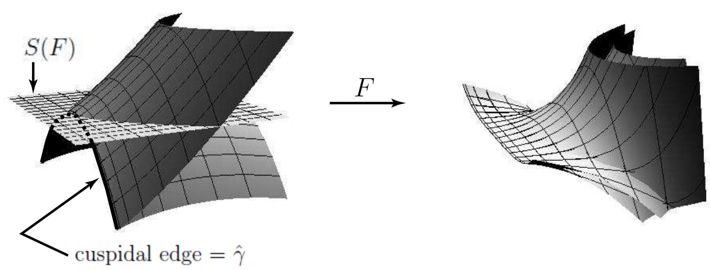

FIGURE 3. Left : 2-point contact of $\hat{\gamma}$ and $S(F)$ Right : cuspidal S^{-}_{1} singularity of $F \circ f$

Since the condition (A) means that the normal vector $\nu(\mathbf{0})$ of f is not perpendicular to $\ker dF_{\mathbf{0}}$ and the condition (B) means that the normal vector $\nu(\mathbf{0})$ of f is not perpendicular to the tangent plane of $S(F)$, the conditions (A) and (B) are generic conditions. It should be remarked that folding maps for smooth surfaces are considered in [2, 10].

Proof. Let f and F be map-germs satisfying the conditions in Theorem 4.3. In the following diagram:

$$(\boldsymbol{R}^2; (u,v), \mathbf{0}) \xrightarrow{f} (\boldsymbol{R}^3; (x,y,z), \mathbf{0}) \xrightarrow{F} (\boldsymbol{R}^3; (X,Y,Z), \mathbf{0}),$$

the conditions and assertions of the theorem do not depend on the choice of coordinate systems on each space. So, we take the coordinate systems (x,y,z) and (X,Y,Z) which satisfy $F(x,y,z) = (x,y,z^2)$. Moreover, we can take the coordinate system (u,v) so that $S(f) = \{v = 0\}$ and $\eta = \partial/\partial v$ there. Denote $f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$. Then by the transversality condition, either $(\partial/\partial u)f_1(0,0) \neq 0$ or $(\partial/\partial u)f_2(0,0) \neq 0$

holds. By the implicit function theorem, we may assume $f(u,v) = (u, f_2(u,v), f_3(u,v))$. Then by the conditions $S(f) = \{v = 0\}$ and $\eta = \partial/\partial v$, f has the following form: $f(u,v) = (u, a_2(u)+v^2 b_2(u,v), a_3(u)+v^2 b_3(u,v))$. By the condition (A) that the limiting tangent plane does not contain the Z-axis, it holds that $b_2(0,0) \neq 0$. By the coordinate change $\tilde{u} = u$, $\tilde{v} = v\sqrt{b_2(u,v)}$, we may assume $f(u,v) = (u, a_2(u)+v^2, a_3(u)+v^2 b_3(u,v))$. Since $\alpha(x,y,z) = (x, y-a_2(x), z)$ and $\beta(X,Y,Z) = (X, Y-a_2(X), Z)$ are both diffeomorphisms, considering $\alpha \circ f$ and $\beta \circ F \circ \alpha^{-1}$, and rewriting them by f and F, we may assume that $f(u,v) = (u, v^2, a_3(u) + v^2 b_3(u,v))$ and $F(x,y,z) = (x,y,z^2)$ again. So,

$$F \circ f(u,v) = \big(u, v^2, a_3(u)^2 + 2v^2 a_3(u) b_3(u,v) + v^4 b_3(u,v)^2\big).$$

Then $\partial(F\circ f)/\partial u = (1, 0, 2a_3 a_3' + 2v^2 *)$ and $\partial(F\circ f)/\partial v = 2v(0, 1, 2a_3 b_3 + v a_3 (b_3)_v + v^2 *)$, where $*$ means a function, $a_3' = da_3/du$ and $(b_3)_v = \partial b_3/\partial v$. We can take ν to be $(2a_3 a_3' + 2v^2 *, 2a_3 b_3 + v a_3 (b_3)_v + v^2 *, -1)$ as a normal vector. Then we see that the signed area density function λ is a non-zero functional multiple of v. Thus the non-degeneracy of all singularities of $F \circ f$ follows. Since $f|_{S(f)} = (u, 0, a_3(u))$, the condition (C) implies $a_3' = \cdots = a_3^{(k-1)} = 0$ and $a_3^{(k)} \neq 0$ at 0.

Since f is $\mathcal{A}$-equivalent to the cuspidal edge at the origin, $(b_3)_v(0,0) \neq 0$ holds. The function ψ of $F \circ f$ defined by (5) is given by $3a_3(t)(b_3)_v(t,0)(1 + 4a_3(t)^2 a_3'(t)^2)$ because of $d\nu(\eta)(t,0) = \nu_v(t,0) = \big(0, 3a_3(t)(b_3)_v(t,0), 0\big)$.

If $k = 1$, then we have the conclusion by Corollary 1.5 of [4]. If $k \geq 2$, we have $b_3(0,0) \neq 0$ by the transversality condition (B). Now, we consider a curve $c(t) = (0,t)$ and $\hat{c}(t) = F \circ f(c(t)) = \big(0, t^2, t^4 b_3(0,t)^2\big)$. Since $b_3(0,0) \neq 0$ and $(b_3)_v(0,0) \neq 0$, we see that the conditions (b) of Theorem 3.2 are satisfied. If k is odd, by Theorem 3.2 $F \circ f$ is $\mathcal{A}$-equivalent to the cuspidal S_{k-1} singularity. If k is even, $F \circ f$ is equivalent to the cuspidal S^+_{k-1} singularity (resp. the cuspidal S^-_{k-1} singularity) if and only if $a_3^{(k)}(0) b_3(0,0) > 0$ (resp. $a_3^{(k)}(0) b_3(0,0) < 0$). Since $S(F) = \{(x,y,z) \,|\, z = 0\}$ and $f(u,v) = (u, v^2, a_3(u)+v^2 b_3(u,v))$, one can easily see that the condition $a_3^{(k)}(0) b_3(0,0) > 0$ is equivalent to the condition that $\mathrm{Im}(f)$ is locally located on the half space bounded by the xy-plane such that $f(u,0)$ lies in. This completes the proof. $\square$

Mond's criteria [14, Theorem 4.1.1] is useful for normalized germs. But in general, like for the examples of this section, our criteria seem more useful.

References

[1] V. I. Arnol′d, *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[2] J. W. Bruce and T. C. Wilkinson, Folding maps and focal sets, In *Singularity theory and its applications, Part I (Coventry, 1988/1989)*, volume 1462 of *Lecture Notes in Math.*, pages 63–72. Springer, Berlin, 1991.

[3] X.-Y. Chen and T. Matumoto, On generic 1-parameter families of C^∞-maps of an n-manifold into a $(2n-1)$-manifold, *Hiroshima Math. J.*, **14**(1985), no. 3, 547–550.

[4] S. Fujimori, K. Saji, M. Umehara and K. Yamada, Singularities of maximal surfaces, *Math. Z.*, **259**(2008), no. 4, 827–848.

[5] G. Ishikawa, Determinacy of the envelope of the osculating hyperplanes to a curve, *Bull. London Math. Soc.*, **25**(1993), no. 6, 603–610.

[6] G. Ishikawa and Y. Machida, Singularities of improper affine spheres and surfaces of constant Gaussian curvature, *Internat. J. Math.*, **17**(2006), no. 3, 269–293.

[7] S. Izumiya and K. Saji, The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces, *J. Singul.*, **2**(2010), 97–127.

[8] S. Izumiya, K. Saji and M. Takahashi, Horospherical flat surfaces in hyperbolic 3-space, *J. Math. Soc. Japan*, **62**(2010), 789–849.

[9] S. Izumiya, K. Saji and N. Takeuchi, Circular surfaces, *Adv. Geom.*, **7**(2007), no. 2, 295–313.

[10] S. Izumiya, M. Takahashi and F. Tari, Folding maps on spacelike and timelike surfaces and duality, *Osaka J. Math.*, **47**(2010), 839–862.

[11] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, Singularities of flat fronts in hyperbolic space, *Pacific J. Math.*, **221**(2005), no. 2, 303–351.

[12] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space and their caustics, *J. Math. Soc. Japan*, **59**(2007), no. 1, 265–299.

[13] D. Mond, On the tangent developable of a space curve, *Math. Proc. Cambridge Philos. Soc.*, **91**(1982), no. 3, 351–355.

[14] D. Mond, On the classification of germs of maps from $\mathbf{R}^2$ to $\mathbf{R}^3$, *Proc. London Math. Soc. (3)*, **50**(1985), no. 2, 333–369.

[15] D. Mond, Singularities of the tangent developable surface of a space curve, *Quart. J. Math. Oxford Ser. (2)*, **40**(1989), no. 157, 79–91.

[16] K. Saji, M. Umehara and K. Yamada, A_k singularities of wave fronts, *Math. Proc. Cambridge Philos. Soc.*, **146**(2009), no. 3, 731–746.

[17] K. Saji, M. Umehara and K. Yamada, The geometry of fronts, *Ann. of Math. (2)*, **169**(2009), no. 2, 491–529.

[18] T. Sasaki, K. Yamada and M. Yoshida, The hyperbolic Schwarz map for the hypergeometric differential equation, *Experiment. Math.*, **17**(2008), no. 3, 269–282.

Department of Mathematics, Faculty of Education, Gifu University, Yanagido 1-1, Gifu, 501-1193, Japan.

E-mail address: `ksaji@gifu-u.ac.jp`