Isotropy subgroup of the Hurwitz action of the 4-braid group on braid systems

Yoshiro Yaguchi

Abstract. We study the Hurwitz action of the 4-braid group $B_4$ on the 4-fold direct product $B_4^5$ of the 5-braid group $B_5$ and determine explicitly a system of generators of the isotropy subgroup of the Hurwitz action of $B_4$ at the standard generators of $B_5$. As a corollary 125 representatives of its coset space are also presented.

1. Introduction

Let $B_n$ denote the $n$-braid group, which has the following presentation [1, 2]:

$$
\left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| = 1, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \right\rangle,
$$

where $\sigma_i$ is the $i$-th standard generator.

Definition 1.1. The Hurwitz action of the $n$-braid group $B_n$ on the $n$-fold Cartesian product $G^n$ of a group $G$ is the right action defined by

$$(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, b_{i+2}, \ldots, b_n) \cdot \sigma_i = (b_1, \ldots, b_{i-1}, b_{i+1}, b_i b_{i+1} b_{i+2}, b_{i+3}, \ldots, b_n),$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of $B_n$.

The Hurwitz action by the standard generators $\sigma_i$ and their inverses $\sigma_i^{-1}$ are called elementary transformations.

When $G$ is the mapping class group of a Riemann surface $\Sigma_g$ of genus $g$ and when each $b_j$ ($j = 1, \ldots, n$) is a positive Dehn twist along a simple closed curve on $\Sigma_g$, the orbit of an $n$-tuple $(b_1, \ldots, b_n)$ by the Hurwitz action of $B_n$ corresponds to an equivalence class of a genus $g$ Lefschetz fibration $f : W \to D^2$ over a 2-disk with $n$ singular fibers (cf. [3]). When $G$ is the $m$-braid group $B_m$ and when each $b_j$ ($j = 1, \ldots, n$) is a conjugate of $\sigma_1$, the orbit of an $n$-tuple $(b_1, \ldots, b_n)$ corresponds to an equivalence class of an “algebraic” braided surface in a bi-disk $D^2 \times D^2$ with $n$ branch points [4, 6, 7]. Here we say that two braided surfaces $S$ and $S'$ are equivalent if there is a fiber-preserving diffeomorphism $f : D^2 \times D^2 \to D^2 \times D^2$ (that is, $f(D^2 \times \{x\}) = D^2 \times \{g(x)\}$ for some diffeomorphism $g : D^2 \to D^2$) carrying $S$ to $S'$ rel $D^2 \times \partial D^2$. Moreover if the fiber-preserving diffeomorphism $f : D^2 \times D^2 \to D^2 \times D^2$ keeps every fiber $D^2 \times \{x\}$ setwise for

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isomorphic if and only if their monodromies determine the same subgroup of the Hurwitz action of $S$ and of length $S$ quite easy to solve. Two braided surfaces obtained by the Hurwitz action of $S$ equal to the cardinal number of the orbit of an element of $B_4$ and its coset space which consists of 16 elements. Therefore it is important to study the orbit of an $n$-tuple $(b_1, \ldots, b_n) \in B_m^n$ under the Hurwitz action. We shall consider a special case of $(\sigma_1, \ldots, \sigma_n)$ in $B_m^n$ for $m = 5$ since it corresponds to the “standard” braided surface of degree $n + 1$ with $n$ branch points, and it is related to an unknotted surface braid (cf. [4, 7]).

An element of the $n$-fold direct product $B_m^n$ of $B_m$ is called a braid system of degree $m$ and of length $n$ (cf. [4]).

Let $(b_1, \ldots, b_n)$ be a braid system of degree $m$ and of length $n$. We denote the isotropy subgroup of the Hurwitz action of $B_n$ at the braid system $(b_1, \ldots, b_n)$ by $H(b_1, \ldots, b_n)$. Namely, $H(b_1, \ldots, b_n) = \{ \beta \in B_n \mid (b_1, \ldots, b_n) \cdot \beta = (b_1, \ldots, b_n) \}$. The elements of the orbit of $(b_1, \ldots, b_n)$ under the Hurwitz action correspond to the cosets of $B_n$ by the isotropy subgroup $H(b_1, \ldots, b_n)$. In particular, the cardinal number of the orbit is that of the coset space $H(b_1, \ldots, b_n) \backslash B_n$.

In [8] we described the isotropy subgroup of Hurwitz action of $B_3$ at the standard generators of $B_4$ and its coset space which consists of 16 elements.

In this paper, we study the isotropy subgroup of Hurwitz action of $B_4$ at those of $B_5$ and its cosets. To state the main theorem, we need some preparations.

For integers $i$ ($0 \leq i \leq 3$) and $s$ ($0 \leq s \leq 4$), let $a_i^s$, $b_i^s$, $c_i^s$, $d_i^s$, $e_i^s$, $f_i^s$, $g_i^s$, $h_i^s$, $j_i^s$, $k_i^s$, $l_i^s$, $m_i^s$, $n_i^s$, $o_i^s$, $p_i^s$, $q_i^s$, $r_i^s$, $s_i^s$, $t_i^s$, $u_i^s$, $v_i^s$, $w_i^s$, $x_i^s$, $y_i^s$, $z_i^s$, be elements of $B_4$ defined by

$$a_i^s = (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i, \quad b_i^s = (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i\sigma_2,$$
$$c_i^s = (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i\sigma_3, \quad d_i^s = (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i\sigma_2^2,$$
$$e_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2, \quad f_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_2^2,$$
$$g_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2\sigma_3, \quad h_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_2^2\sigma_3,$$
$$i_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2(\sigma_2\sigma_3)^2, \quad j_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2(\sigma_2\sigma_3)^2\sigma_2,$$
$$k_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2(\sigma_2\sigma_3)^3, \quad l_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2(\sigma_2\sigma_3)^3\sigma_2,$$
$$m_i^s = (\sigma_1\sigma_2\sigma_3)^s\sigma_1^2(\sigma_2\sigma_3)^4.$$

Now we define two subsets of $B_4$ denoted by $X$ and $S$.

Let $X = \left( \bigcup_{i=0}^{3} A_i \right) \cup \left( \bigcup_{i=0}^{3} B_i \right) \cup \left( \bigcup_{i=0}^{3} C_i \right) \cup \left( \bigcup_{i=0}^{3} D_i \right) \cup \left( \bigcup_{i=0}^{8} E_i \right)$ be the subset of $B_4$ defined by
\[ \mathcal{A}_i = \{ \sigma_i^s \mid 0 \leq s \leq 4 \} \quad (i = 0, \ldots, 3), \]
\[ \mathcal{B}_i = \{ b_i^s \mid 0 \leq s \leq 4 \} \quad (i = 0, \ldots, 3), \]
\[ \mathcal{C}_i = \{ c_i^s \mid 0 \leq s \leq 4 \} \quad (i = 0, \ldots, 3), \]
\[ \mathcal{D}_i = \{ d_i^s \mid 0 \leq s \leq 4 \} \quad (i = 0, \ldots, 3), \]
\[ \mathcal{E}_i = \{ e_i^s \mid 0 \leq s \leq 4 \} \quad (i = 0, \ldots, 8). \]

Moreover, let \( \mathcal{Y}_j \) \((j = 1, 2, 3)\) be the subset of \( \mathcal{X} \) defined by
\[ \mathcal{Y}_1 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_3 \cup \mathcal{E}_3 \cup \mathcal{E}_4, \]
\[ \mathcal{Y}_2 = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_4 \cup \mathcal{E}_5 \cup \mathcal{E}_6 \cup \mathcal{E}_7, \]
\[ \mathcal{Y}_3 = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_7 \cup \mathcal{E}_8, \]
and let \( \mathcal{Z}_j \) be the complement of \( \mathcal{Y}_j \) in \( \mathcal{X} \) for \( j = 1, 2, 3 \).

Finally a subset \( \mathcal{S} \) of \( \mathcal{B}_4 \) is defined by
\[ \mathcal{S} = \{ \beta \sigma_j^2 \beta^{-1} \mid \beta \in \mathcal{Y}_j, \ j = 1, 2, 3 \} \cup \{ \beta \sigma_j^2 \beta^{-1} \mid \beta \in \mathcal{Z}_j, \ j = 1, 2, 3 \}. \]

The following theorem is the main result of this paper.

**Theorem 1.2.** Let \( s_1, s_2, s_3, s_4 \) be the standard generators of \( \mathcal{B}_5 \). Then, we have

1. \( H(s_1, s_2, s_3, s_4) \) is generated by \( \mathcal{S} \).
2. \( \#(H(s_1, s_2, s_3, s_4)\backslash \mathcal{B}_n) = 125 \). Moreover, all the representatives are given by \( \mathcal{X} \).

**Remark 1.3.** For any integer \( n \geq 2 \), the equality \( \#(H(s_1, \ldots, s_n)\backslash \mathcal{B}_n) = (n + 1)^{n-1} \) has been proven in [5]. In particular, the first half of Theorem 1.2(2) was known. So, the purpose of this paper is to give explicitly a system of generators \( \mathcal{S} \) of the isotropy subgroup and 125 representatives of its cosets. In fact, we give a direct proof of Theorem 1.2 without using the above mentioned result of [5].

**Remark 1.4.** In [9], for any integer \( n \geq 2 \) and any element \( \varphi \) of the symmetric group \( S_n \), the author shows the equality \( \#(H(s_{\varphi(1)}, \ldots, s_{\varphi(n)})\backslash \mathcal{B}_n) = (n + 1)^{n-1} \).

## 2. Proof of Theorem 1.2

The notation \( \beta_1 \equiv \beta_2 \pmod{H} \) stands for \( H \beta_1 = H \beta_2 \) (or equivalently \( \beta_1 \beta_2^{-1} \in H \)), where \( H \) is a subgroup of \( \mathcal{B}_4 \) and \( \beta_1, \beta_2 \in \mathcal{B}_4 \). In the middle column of Table 2.1, \( b_1 \ast b_2 \) stands for \( b_2^{-1} b_1 b_2 \) for \( b_1, b_2 \in \mathcal{B}_5 \) and “1, 2, 3, 4” are abbreviations of “\( s_1, s_2, s_3, s_4 \)” of \( \mathcal{B}_5 \), respectively. In the right column of Table 2.1, \((ij)\) is the transposition of \( i \) and \( j \) for \( 1 \leq i \neq j \leq 5 \), which is an element of the symmetric group \( S_5 \). For each \( \beta \in \mathcal{X} \), we write the result of \( (1, 2, 3, 4) \cdot \beta \) in the middle column of Table 2.1 and that of \( (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta \) in the right column of Table 2.1, where \( \pi \) is the natural homomorphism from \( \mathcal{B}_5 \) to \( S_5 \) which sends \( i (= s_i) \) to the transposition \((i \ i + 1)\). For any distinct elements \( \beta_1 \) and \( \beta_2 \) of \( \mathcal{X} \), we see that \( (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta_1 \neq (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta_2 \). Thus, \( (s_1, s_2, s_3, s_4) \cdot \beta_1 \neq (s_1, s_2, s_3, s_4) \cdot \beta_2 \), and we obtain the following lemma.

**Lemma 2.1.** For any distinct elements \( \beta_1 \) and \( \beta_2 \) of \( \mathcal{X} \), \( \beta_1 \neq \beta_2 \pmod{H(s_1, s_2, s_3, s_4)} \).

The following proposition is given in [8] and it is useful to prove Lemma 2.3.
Proposition 2.2. Let \((b_1, b_2)\) be a braid system of degree \(m\) and of length 2.

1. \(\sigma_1^2 \in H(b_1, b_2)\) if and only if \(b_1b_2 = b_2b_1\).
2. \(\sigma_1^3 \in H(b_1, b_2)\) if and only if \(b_1b_2b_1 = b_2b_1b_2\).

Lemma 2.3.

1. If \(\beta \in \mathcal{Y}_j\) for \(j = 1, 2\) or 3, then \(\beta\sigma_1^2 \equiv \beta \pmod{H(s_1, s_2, s_3, s_4)}\).
2. If \(\beta \in \mathcal{Z}_j\) for \(j = 1, 2\) or 3, then \(\beta\sigma_1^3 \equiv \beta \pmod{H(s_1, s_2, s_3, s_4)}\).

\[
\begin{array}{|c|c|}
\hline
\beta & (1, 2, 3, 4) \cdot \beta \\
\hline
\mathcal{A}_0 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 2, 3, 4) \\
2 \quad (2, 3, 4, 1 * (234)) \\
3 \quad (3, 4, 1 * (234), 1) \\
4 \quad (4, 1 * (234), 1, 2) \\
5 \quad (1 * (234), 1, 2, 3)
\end{array}
\end{array} \\
\mathcal{A}_1 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 3, 4, 2 * (34)) \\
2 \quad (2, 4, 1 * (234), 1 * 2) \\
3 \quad (3, 1 * (234), 1, 2 * 3) \\
4 \quad (4, 1, 2, 3 * 4) \\
5 \quad (1 * (234), 2, 3, 1 * (23))
\end{array}
\end{array} \\
\mathcal{A}_2 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 4, 2 * (34), 2) \\
2 \quad (2, 1 * (234), 1 * 2, 3) \\
3 \quad (3, 1, 2 * 3, 4) \\
4 \quad (4, 2, 3 * 4, 1 * (234)) \\
5 \quad (1 * (234), 3, 1 * (23), 1)
\end{array}
\end{array} \\
\mathcal{A}_3 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 2 * (34), 2, 3) \\
2 \quad (2, 1 * 2, 3, 4) \\
3 \quad (3, 2 * 3, 4, 1 * (234)) \\
4 \quad (4, 3, 4, 1 * (234), 1) \\
5 \quad (1 * (234), 1, 2 * (23), 1, 2)
\end{array}
\end{array} \\
\mathcal{B}_0 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 3, 2 * 3, 4) \\
2 \quad (2, 4, 3 * 4, 1 * (234)) \\
3 \quad (3, 1 * (234), 1 * (23), 1) \\
4 \quad (4, 1, 2 * (34), 2) \\
5 \quad (1 * (234), 2, 1 * 2, 3)
\end{array}
\end{array} \\
\mathcal{B}_1 & \begin{array}{c}
\begin{array}{c}
1 \quad (1, 4, 3 * 4, 2 * (34)) \\
2 \quad (2, 1 * (234), 1 * (23), 1 * 2) \\
3 \quad (3, 1, 2 * (34), 2 * 3) \\
4 \quad (4, 2, 1 * 2, 3 * 4) \\
5 \quad (1 * (234), 3, 2 * 3, 1 * (23))
\end{array}
\end{array} \\
\hline
\end{array}
\]

Table 2.1(a)
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<td>$(15, (24), (23), (14))$</td>
</tr>
</tbody>
</table>

Table 2.1(b)
### Isotropy subgroup of the Hurwitz action of the 4-braid group

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( (1, 2, 3, 4) \cdot \beta )</th>
<th>( (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_2 )</td>
<td>( d_1^0 ) (1, 2 * 3, 4, 2)</td>
<td>( (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta )</td>
</tr>
<tr>
<td></td>
<td>( d_1^1 ) (2, 3 * 4, 1 * (234), 3)</td>
<td>((12), (24), (45), (23))</td>
</tr>
<tr>
<td></td>
<td>( d_2^2 ) (3, 1 * (23), 1, 4)</td>
<td>((23), (35), (15), (34))</td>
</tr>
<tr>
<td></td>
<td>( d_2^3 ) (4, 2 * (34), 2, 1 * (234))</td>
<td>((34), (14), (12), (45))</td>
</tr>
<tr>
<td></td>
<td>( d_2^4 ) (1 * (234), 1 * 2, 3, 1)</td>
<td>((45), (25), (23), (15))</td>
</tr>
<tr>
<td>( D_3 )</td>
<td>( d_3^0 ) (1, 3 * 4, 2 * (34), 3)</td>
<td>((15), (13), (34), (12))</td>
</tr>
<tr>
<td></td>
<td>( d_3^1 ) (2, 1 * (23), 1 * 2, 4)</td>
<td>((23), (14), (13), (45))</td>
</tr>
<tr>
<td></td>
<td>( d_3^2 ) (3, 2 * (34), 2 * 3, 1 * (234))</td>
<td>((23), (14), (13), (45))</td>
</tr>
<tr>
<td></td>
<td>( d_3^3 ) (4, 1 * 2, 3 * 4, 1)</td>
<td>((34), (25), (24), (15))</td>
</tr>
<tr>
<td></td>
<td>( d_3^4 ) (1 * (234), 2 * 3, 1 * (23), 2)</td>
<td>((15), (24), (14), (23))</td>
</tr>
<tr>
<td>( \mathcal{E}_0 )</td>
<td>( e_0^0 ) (1 * 2, 1, 3, 4)</td>
<td>((13), (12), (34), (45))</td>
</tr>
<tr>
<td></td>
<td>( e_0^1 ) (2 * 3, 2, 4, 1 * (234))</td>
<td>((24), (23), (45), (15))</td>
</tr>
<tr>
<td></td>
<td>( e_0^2 ) (3 * 4, 3, 1 * (234), 1)</td>
<td>((35), (34), (15), (12))</td>
</tr>
<tr>
<td></td>
<td>( e_0^3 ) (1 * (23), 4, 1, 2)</td>
<td>((14), (45), (12), (23))</td>
</tr>
<tr>
<td></td>
<td>( e_0^4 ) (2 * (34), 1 * (234), 2, 3)</td>
<td>((25), (15), (23), (34))</td>
</tr>
<tr>
<td>( \mathcal{E}_1 )</td>
<td>( e_1^0 ) (1 * 2, 3, 1, 4)</td>
<td>((13), (34), (12), (45))</td>
</tr>
<tr>
<td></td>
<td>( e_1^1 ) (2 * 3, 4, 2, 1 * (234))</td>
<td>((24), (45), (23), (15))</td>
</tr>
<tr>
<td></td>
<td>( e_1^2 ) (3 * 4, 1 * (234), 3, 1)</td>
<td>((35), (34), (15), (12))</td>
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<tr>
<td></td>
<td>( e_1^3 ) (1 * (23), 1, 4, 2)</td>
<td>((14), (12), (45), (23))</td>
</tr>
<tr>
<td></td>
<td>( e_1^4 ) (2 * (34), 2, 1 * (234), 3)</td>
<td>((25), (23), (15), (34))</td>
</tr>
<tr>
<td>( \mathcal{E}_2 )</td>
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<td>((24), (45), (15), (23))</td>
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<td>( e_2^2 ) (3 * 4, 1 * (234), 1, 3)</td>
<td>((35), (15), (12), (34))</td>
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<td>((14), (12), (23), (45))</td>
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<td></td>
<td>( e_2^4 ) (2 * (34), 2, 3, 1 * (234))</td>
<td>((25), (23), (34), (15))</td>
</tr>
<tr>
<td>( \mathcal{E}_3 )</td>
<td>( e_3^0 ) (1 * 2, 4, 3 * 4, 1)</td>
<td>((13), (45), (35), (12))</td>
</tr>
<tr>
<td></td>
<td>( e_3^1 ) (2 * 3, 1 * (234), 1 * (23), 2)</td>
<td>((24), (15), (14), (23))</td>
</tr>
<tr>
<td></td>
<td>( e_3^2 ) (3 * 4, 1, 2 * (34), 3)</td>
<td>((35), (12), (25), (34))</td>
</tr>
<tr>
<td></td>
<td>( e_3^3 ) (1 * (23), 2, 1 * 2, 4)</td>
<td>((14), (23), (13), (45))</td>
</tr>
<tr>
<td></td>
<td>( e_3^4 ) (2 * (34), 3, 2 * 3, 1 * (234))</td>
<td>((25), (34), (24), (15))</td>
</tr>
<tr>
<td>( \mathcal{E}_4 )</td>
<td>( e_4^0 ) (1 * 2, 4, 1, 3 * 4)</td>
<td>((13), (45), (12), (35))</td>
</tr>
<tr>
<td></td>
<td>( e_4^1 ) (2 * 3, 1 * (234), 2, 1 * (23))</td>
<td>((24), (15), (23), (14))</td>
</tr>
<tr>
<td></td>
<td>( e_4^2 ) (3 * 4, 1, 3 * (34))</td>
<td>((35), (12), (34), (25))</td>
</tr>
<tr>
<td></td>
<td>( e_4^3 ) (1 * (23), 2, 4, 1 * 2)</td>
<td>((14), (23), (45), (13))</td>
</tr>
<tr>
<td></td>
<td>( e_4^4 ) (2 * (34), 3, 1 * (234), 2 * 3)</td>
<td>((25), (34), (15), (24))</td>
</tr>
<tr>
<td>( \mathcal{E}_5 )</td>
<td>( e_5^0 ) (1 * 2, 1, 4, 3 * 4)</td>
<td>((13), (12), (45), (35))</td>
</tr>
<tr>
<td></td>
<td>( e_5^1 ) (2 * 3, 2, 1 * (234), 1 * (23))</td>
<td>((24), (23), (15), (14))</td>
</tr>
<tr>
<td></td>
<td>( e_5^2 ) (3 * 4, 3, 1, 2 * (34))</td>
<td>((35), (34), (12), (25))</td>
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<tr>
<td></td>
<td>( e_5^3 ) (1 * (23), 4, 2, 1 * 2)</td>
<td>((14), (45), (23), (13))</td>
</tr>
<tr>
<td></td>
<td>( e_5^4 ) (2 * (34), 1 * (234), 3, 2 * 3)</td>
<td>((25), (15), (34), (24))</td>
</tr>
</tbody>
</table>
Proof of Lemma 2.3. The result of \((s_1, s_2, s_3, s_4) \cdot \beta\) \(= (1, 2, 3, 4) \cdot \beta\) for each \(\beta \in X\) is written in the middle column of the table above. In this proof, we denote it by \((b_1, b_2, b_3, b_4)\) for each \(\beta \in X\).

1. If \(\beta \in Y_j\) for \(j = 1, 2, 3\), we see that \(b_j b_{j+1} = b_{j+1} b_j\) by direct calculations. So, by Proposition 2.2(1), we have \((s_1, s_2, s_3, s_4) \cdot \beta \sigma_j^2 = (s_1, s_2, s_3, s_4) \cdot \beta\).

2. If \(\beta \in Z_j\) for \(j = 1, 2, 3\), we see that \(b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}\) by direct calculations. So, by Proposition 2.2(2), we have \((s_1, s_2, s_3, s_4) \cdot \beta \sigma_j^3 = (s_1, s_2, s_3, s_4) \cdot \beta\). \(\square\)

By Lemma 2.3 \(\beta \sigma_j^2 \beta^{-1} \in H(s_1, s_2, s_3, s_4)\) if \(\beta \in Y_j\) and \(\beta \sigma_j^3 \beta^{-1} \in H(s_1, s_2, s_3, s_4)\) if \(\beta \in Z_j\) hold for \(j = 1, 2, 3\). Thus, we see that any element of \(S\) belongs to \(H(s_1, s_2, s_3, s_4)\) and hence \((S) \subset H(s_1, s_2, s_3, s_4)\).

Now we consider the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathcal{S} \\
\downarrow i' & & \downarrow i \\
\langle S \rangle \setminus B_4 & \xrightarrow{j} & H(s_1, s_2, s_3, s_4) \setminus B_4
\end{array}
\]

where \(i\) and \(i'\) are the restrictions \(p|_X\) and \(p'|_X\) of the natural projections
Isotropy subgroup of the Hurwitz action of the 4-braid group

\( p : B_4 \rightarrow H(s_1, s_2, s_3, s_4) \backslash B_4 \) and \( p' : B_4 \rightarrow \langle S \rangle \backslash B_4 \), and \( j \) is the natural surjection. By Lemma 2.1 we see that \( i \) is injective and hence \( i' \) is also injective.

We will show that \( i' \) is surjective in Lemma 2.9 by preparing Lemmas 2.4, 2.5, 2.6, 2.7 and 2.8. Hereafter we will use the abbreviation \( \alpha \equiv \beta \) instead of \( \alpha \equiv \beta \pmod{\langle S \rangle} \) unless otherwise stated.

**Lemma 2.4.** For \( 1 \leq j \leq 3 \), we have the following formulas.

1. If \( \beta \in \mathcal{Y}_j \), then \( \beta \sigma_j^2 \equiv \beta \).
2. If \( \beta \in \mathcal{Z}_j \), then \( \beta \sigma_j^3 \equiv \beta \).

**Proof.** This is obtained immediately from the definition of \( \mathcal{S} \). \( \square \)

**Lemma 2.5.** For \( 0 \leq s \leq 4 \), we have the following formulas.

1. \( a_0^s(\sigma_2 \sigma_3)^4 \equiv a_0^s \)
2. \( e_2^s(\sigma_2 \sigma_3)^3 \sigma_2 \equiv e_2^s \) and
3. \( e_0^s(\sigma_2 \sigma_3)^5 \sigma_2 \equiv e_0^s \).

**Proof.** (1) In the braid group \( B_4 \), we have \( a_0^s(\sigma_2 \sigma_3)^4 = a_0^s \sigma_2^2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( a_0^s \) is in \( A_0 \subset \mathcal{Z}_2 \), by Lemma 2.4(2) we have \( a_0^s \sigma_2^2 \sigma_3 \sigma_2 \sigma_3^2 \equiv a_0^s \sigma_2^2 \sigma_3 \sigma_2 \sigma_3^2 \). In \( B_4 \), we have \( a_0^s \sigma_2^2 \sigma_3 \sigma_2 \sigma_3^2 = d_1 \sigma_j \sigma_2 \sigma_3 \). Since \( d_1 \sigma_j \in D_1 \subset \mathcal{Y}_3 \), by Lemma 2.4(1) we have \( d_1 \sigma_j \sigma_2 \sigma_3 = d_1 \sigma_j \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( a_0^s \) is in \( \mathcal{Z}_2 \), by Lemma 2.4(2) we have \( a_0^s \sigma_2 \sigma_3 \sigma_2 = a_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Thus, we obtain \( a_0^s(\sigma_2 \sigma_3)^4 \equiv a_0^s \).

(2) Since \( e_2^s(\sigma_2 \sigma_3)^3 \sigma_2 = e_2^s \sigma_2 \sigma_3 \sigma_2 \) and \( e_2^s \in \mathcal{Z}_3 \), using Lemma 2.4(2) we have \( e_2^s \sigma_2 \sigma_3 \sigma_2 \equiv e_2^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_2^s \) is in \( \mathcal{Y}_2 \), using Lemma 2.4(1) we have \( e_2^s \sigma_2 \sigma_3 \sigma_2 \equiv e_2^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_2^s \) is in \( \mathcal{Y}_3 \), using Lemma 2.4(1) we have \( e_2^s \sigma_2 \sigma_3 \sigma_2 \equiv e_2^s \).

(3) Since \( e_0^s(\sigma_2 \sigma_3)^5 \sigma_2 = e_0^s \sigma_2 \sigma_3 \sigma_2 \) and \( e_0^s \in \mathcal{Y}_3 \), by Lemma 2.4(1) we have \( e_0^s \sigma_2 \sigma_3 \equiv e_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_0^s \) is in \( \mathcal{Y}_2 \), by Lemma 2.4(1) we have \( e_0^s \sigma_2 \sigma_3 \equiv e_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_0^s \) is in \( \mathcal{Z}_3 \), by Lemma 2.4(2) we have \( e_0^s \sigma_2 \sigma_3 \equiv e_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_0^s \) is in \( \mathcal{Y}_4 \), by Lemma 2.4(1) we have \( e_0^s \sigma_2 \sigma_3 \sigma_2 \equiv e_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). Since \( e_0^s \) is in \( \mathcal{Y}_2 \), by Lemma 2.4(1) we have \( e_0^s \sigma_2 \sigma_3 \sigma_2 \sigma_3^2 \). \( \square \)

**Lemma 2.6.** \( (\sigma_1 \sigma_2 \sigma_3)^5 \equiv 1 \).

**Proof.** In \( B_4 \), we have \( (\sigma_1 \sigma_2 \sigma_3)^5 = e_0^s \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \). Since \( e_0^s \in \mathcal{Y}_2 \), by Lemma 2.4(1) \( e_0^s \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \equiv e_0^s \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \). Since \( e_0^s \) is in \( \mathcal{Z}_3 \), by Lemma 2.4(2) we have \( a_0^s \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \equiv a_0^s \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \). In \( B_4 \), we have \( a_0^s \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_3 = b_0^s \sigma_3 \sigma_2 \sigma_3 \). Since \( b_0^s \) is in \( \mathcal{Y}_1 \), by Lemma 2.4(1) we have \( b_0^s \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \). Since \( b_0^s \) is in \( \mathcal{Y}_2 \), by Lemma 2.4(1) we have \( b_0^s \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 = (\sigma_2 \sigma_3)^4 \). Using Lemma 2.5(1), we have \( (\sigma_2 \sigma_3)^4 = a_0^s(\sigma_2 \sigma_3)^4 \equiv a_0^s = 1 \). \( \square \)
Lemma 2.7. (1) The following equations hold.

(i) \( a_0^s \sigma_1^2 \equiv a_0^s \sigma_1 \equiv e_0^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 1 \pmod{5} \),

(ii) \( a_1^s \sigma_2 \equiv c_1^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 2 \pmod{5} \),

(iii) \( a_2^s \sigma_2 \equiv b_0^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 3 \pmod{5} \),

(iv) \( b_1^s \sigma_1 \equiv c_1^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 3 \pmod{5} \),

(v) \( b_2^s \sigma_2^3 \equiv e_0^s \sigma_1 \equiv b_3^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 4 \pmod{5} \),

(vi) \( c_0^s \sigma_3^s \equiv c_0^s \sigma_1 \equiv e_5^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 1 \pmod{5} \),

(vii) \( d_0^s \sigma_2^2 \equiv e_2^s \sigma_1 \equiv e_2^u \) for \( s, t, u \in \{0, 1, 2, 3, 4\} \) with \( u \equiv t + 2 \equiv s + 3 \pmod{5} \),

(viii) \( d_1^s \sigma_1 \equiv e_4^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 2 \pmod{5} \),

(ix) \( d_2^s \sigma_1^3 \equiv e_0^u \sigma_1 \equiv e_1^u \) for \( s, t, u \in \{0, 1, 2, 3, 4\} \) with \( u \equiv t + 2 \equiv s + 3 \pmod{5} \),

(x) \( d_3^s \sigma_1 \equiv e_3^s \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t \equiv s + 2 \pmod{5} \).

(2) For each \( s \in \{0, 1, 2, 3, 4\} \), the following equations hold.

(i) \( a_0^s \sigma_2^2 \equiv b_0^s \sigma_2 \equiv d_0^s \),

(ii) \( a_1^s \sigma_2^3 \equiv b_1^s \sigma_2 \equiv d_1^s \),

(iii) \( a_2^s \sigma_2^3 \equiv b_2^s \sigma_2 \equiv d_2^s \),

(iv) \( a_3^s \sigma_2^3 \equiv b_3^s \sigma_2 \equiv d_3^s \),

(v) \( c_0^s \sigma_2 \equiv c_3^s \),

(vi) \( c_1^s \sigma_2 \equiv c_3^s \),

(vii) \( e_0^s \sigma_2 \equiv e_1^s \),

(viii) \( e_2^s \sigma_2 \equiv e_3^s \),

(ix) \( e_4^s \sigma_2 \equiv e_6^s \),

(x) \( e_5^s \sigma_2 \equiv e_7^s \).

(3) For each \( s \in \{0, 1, 2, 3, 4\} \), the following equations hold.

(i) \( a_0^s \sigma_3^2 \equiv c_0^s \sigma_3 \equiv b_3^s \),

(ii) \( a_1^s \sigma_3^2 \equiv c_1^s \sigma_3 \equiv b_0^s \),

(iii) \( a_2^s \sigma_3^2 \equiv c_2^s \sigma_3 \equiv b_1^s \),

(iv) \( a_3^s \sigma_3^2 \equiv c_3^s \sigma_3 \equiv b_2^s \),

(v) \( d_0^s \sigma_3 \equiv d_2^s \),

(vi) \( d_1^s \sigma_3 \equiv d_3^s \),

(vii) \( e_0^s \sigma_2 \equiv e_2^s \sigma_3 \equiv e_6^s \),

(viii) \( e_1^s \sigma_3 \equiv e_3^s \),

(ix) \( e_3^s \sigma_3 \equiv e_1^s \),

(x) \( e_5^s \sigma_3 \equiv e_8^s \).

Proof. First, we prove (2).

Since \( d_i^s = b_i^s \sigma_2 = a_i^s \sigma_2^s \) for \( 0 \leq i \leq 3 \), we have (i), (ii), (iii) and (iv).

Let \( i \) be 0 or 1. Then, we have \( c_i^s \sigma_2 = b_{i+1}^s \sigma_3^{-1} \) in \( B_4 \). Since \( b_{i+1}^s \in Z_3 \), by Lemma 2.4(2) we have \( b_{i+1}^s \sigma_3^{-1} = b_{i+1}^s \sigma_3^2 = c_{i+2}^s \). Hence, (v) and (vi) hold.
Let \( i \in \{0, 2, 4, 6\} \). Then, \( e^*_i \sigma_2 = e^*_i + 1 \). Hence, (vii), (ix), (x) and \( e^*_2 \sigma_2^2 \equiv e^*_3 \sigma_2 \) of (viii) hold.

We show \( e^*_3 \sigma_2 \equiv e^*_5 \) of (viii). Note that \( e^*_3 \sigma_2 = e^*_2 \sigma_2^2 \). By Lemma 2.5(2), we have \( e^*_3 \sigma_2 \equiv e^*_2 (\sigma_2 \sigma_3)^3 \sigma_2^3 = e^*_5 \sigma_2^3 \). Since \( e^*_5 \in \mathbb{Z}_2 \), by Lemma 2.4(2) we have \( e^*_3 \sigma_2 \equiv e^*_5 \).

Next, we prove (3).

Let \( i \in \{0, 1, 2, 3\} \). By Lemma 2.5(1), \( a^*_i = a^*_0 (\sigma_2 \sigma_3)^i \equiv a^*_0 (\sigma_2 \sigma_3)^{i + 1} \). Then, \( a^*_i \sigma_2 \equiv a^*_0 (\sigma_2 \sigma_3)^{i + 1} \sigma_2 \sigma_3 \). When \( i = 0 \), we see that \( a^*_0 (\sigma_2 \sigma_3)^{i + 1} \sigma_2 \sigma_3^3 = b^*_3 \sigma_2^3 \). Since \( b^*_3 \in \mathbb{Z}_3 \), by Lemma 2.4(2) we have \( b^*_3 \sigma_2^3 \equiv b^*_3 \). When \( i \in \{1, 2, 3\} \), by Lemma 2.5(1) we have \( a^*_i (\sigma_2 \sigma_3)^{i + 1} \sigma_2 \sigma_3 \equiv a^*_0 (\sigma_2 \sigma_3)^{i - 1} \sigma_2 \sigma_3 = b^*_i \sigma_3^3 \). Since \( b^*_i \in \mathbb{Z}_3 \), by Lemma 2.4(2) we have \( b^*_i \sigma_3^3 \equiv b^*_i \). Since \( c^*_i = a^*_i \sigma_2 \), we have (i), (ii), (iii) and (iv).

Let \( i \) be 0 or 1. Then, \( d^*_i \sigma_3 = a^*_i + 2 \sigma_3^{-1} \) in \( B_4 \). Since \( a^*_i \in \mathbb{Z}_2 \), by Lemma 2.4(2) we have \( a^*_i + 2 \sigma_3^{-1} \equiv a^*_i + 2 \). Thus, we have (v) and (vi).

Let \( i \in \{1, 3, 5, 7\} \). Then, \( e^*_i \sigma_3 = e^*_i + 1 \). Hence, (vii), (ix), (x) and \( e^*_3 \sigma_3 \equiv e^*_6 \) of (vii) hold.

We show \( e^*_0 \sigma_3 \equiv e^*_5 \) of (vii). Since \( e^*_0 \sigma_3 = e^*_5 \sigma_3^{-1} \sigma_2^{-1} \) in \( B_4 \) and \( e^*_5 \in \mathbb{Y}_3 \), by Lemma 2.4(1) we have \( e^*_5 \sigma_3^{-1} \sigma_2^{-1} \equiv e^*_5 \sigma_3^{-1} \). Since \( e^*_5 \in \mathbb{Y}_2 \), by Lemma 2.4(1) we have \( e^*_5 \sigma_2^{-1} \equiv e^*_5 \sigma_2^{-1} \).

Finally, we prove (1).

We show (i). It is obvious that \( a^*_0 \sigma_1^3 = e^*_0 \). We show \( a^*_0 \sigma_1 = a^*_3 \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 1 \) (mod 5). Note that \( a^*_0 \sigma_1 = (\sigma_1 \sigma_2 \sigma_3)^{s + 1} (\sigma_2 \sigma_3)^{-1} \). When \( s \neq 4 \), \( (\sigma_1 \sigma_2 \sigma_3)^{s + 1} (\sigma_2 \sigma_3)^{-1} = a^*_0 (\sigma_2 \sigma_3)^{-1} \). By Lemma 2.5(1), we have \( a^*_0 (\sigma_2 \sigma_3)^{-1} \equiv a^*_0 (\sigma_2 \sigma_3)^{-3} \sigma_2^{-1} \). Since \( a^*_0 = 1 \), by Lemma 2.6(1), \( a^*_0 (\sigma_2 \sigma_3)^{-3} \sigma_2^{-1} = a^*_0 (\sigma_2 \sigma_3)^{-3} = a^*_0 \).

We show (iii) before (ii). Since \( a^*_5 \in \mathbb{Y}_1 \), it suffices to prove \( a^*_5 (\sigma_2 \sigma_3)^{-1} = b^*_0 \) with \( t = s + 3 \) (mod 5). Since \( b^*_0 \sigma_1 (\sigma_1 \sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} \), we will prove \( (\sigma_1 \sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} = a^*_0 \), where \( t = s + 3 \) (mod 5). When \( t = 0, 1, 2 \), we have \((\sigma_1 \sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} = a^*_0 (\sigma_2 \sigma_3)^{-2} \). By Lemma 2.5(1), we have \( a^*_0 (\sigma_2 \sigma_3)^{-2} \equiv a^*_0 (\sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} = a^*_0 + 2 (\sigma_2 \sigma_3)^{t + 2} = a^*_0 \). When \( t = 3, 4 \), by Lemma 2.6(1), \( (\sigma_1 \sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} = a^*_0 (\sigma_2 \sigma_3)^{-2} \). By Lemma 2.5(1), \( a^*_0 (\sigma_2 \sigma_3)^{-2} \equiv a^*_0 (\sigma_2 \sigma_3)^{t + 2} (\sigma_2 \sigma_3)^{-2} = a^*_0 + 2 (\sigma_2 \sigma_3)^{t + 2} = a^*_0 \).

We show (ii). Note that \( a^*_0 + 1 = b^*_0 \). Since \( b^*_0 \in \mathbb{Y}_1 \), by Lemma 2.4(1) we have \( b^*_0 \sigma_1 \sigma_3 = b^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3 \). By (1)(iii), \( b^*_0 \sigma_1^{-1} \equiv a^*_2 \) for \( t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 2 \) (mod 5). Thus, we have \( b^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3 \equiv a^*_2 \).

We show (iv). In \( B_4 \), \( b^*_i (\sigma_1 \sigma_2 \sigma_3)^{s + 3} (\sigma_2 \sigma_3)^{-2} \sigma_3^{-2} \). Since \( a^*_0 = 3 \), \( (\sigma_1 \sigma_2 \sigma_3)^{s + 3} (\sigma_2 \sigma_3)^{-2} \sigma_3^{-1} \)(mod 5). When \( s = 0 \) or 1, it is \( a^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3^{-2} \). Since \( a^*_0 = 3 \), \( (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \)(mod 5). When \( s = 0 \) or 1, it is \( a^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \). Since \( a^*_0 = 3 \), \( (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \)(mod 5). We show (iv). In \( B_4 \), \( b^*_i (\sigma_1 \sigma_2 \sigma_3)^{s + 3} (\sigma_2 \sigma_3)^{-2} \sigma_3^{-1} \sigma_3^{-1} \)(mod 5). When \( s = 0 \) or 1, it is \( a^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \). Since \( a^*_0 = 3 \), \( (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \)(mod 5). When \( s = 0 \) or 1, it is \( a^*_0 (\sigma_2 \sigma_3)^{-1} \sigma_3^{-1} \).
$b_1^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_3^{-1} = a_0^{-2}\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_3^{-1} \equiv c_1^{s-2} = c_1$.

We show (v). First, we show $b_2\sigma_1 \equiv e_0^t$ with $t \equiv s + 4 \pmod{5}$. Note that $b_2^s\sigma_1 = b_2^s\sigma_3^{-1}\sigma_1 = a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}$. By Lemma 2.5(1), we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} \equiv a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}$. When $s \neq 0$, we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} = e_2^{-1}\sigma_2^{-1}\sigma_3^{-1}$.

Since $e_2^{-1} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $e_2^{-1}\sigma_2^{-1}\sigma_3^{-1} = e_2^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv e_3^{-1}\sigma_3^{-1}$. Since $e_3^{-1} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_3^{-1}\sigma_3^{-1} \equiv e_3^{-1}\sigma_3^{-1} \equiv e_4^{-1}\sigma_3^{-1}$. Since $e_4^{-1} \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_4^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv e_4^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv e_6^{-1}$. When $s = 0$, by Lemma 2.6 we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} \equiv (\sigma_2\sigma_3)^{s}\sigma_3^{-1} = e_0^t \equiv e_0^t$.

By the same argument as in the case where $s \neq 0$, we see that $a_0^t\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv e_0^t = e_0^t$.

Next we show $b_2^s\sigma_1^2 \equiv b_3$ with $t \equiv s + 4 \pmod{5}$. Note that $b_2^s\sigma_1^2 = b_2^s\sigma_3^{-1}\sigma_1^2 = a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2$. By Lemma 2.5(1), we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2 = a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2$. When $s \neq 0$, we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2 = a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2 = a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2$. Since $a_0^t \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} \equiv a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}$. By Lemma 2.5(1), we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} \equiv a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1} \equiv b_3^{-1} = b_3$. When $s = 0$, by Lemma 2.6, we have $a_0^t(\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2 \equiv (\sigma_2\sigma_3)^{s}\sigma_3^{-1}\sigma_1^2 \equiv a_0^t\sigma_3^{-1}\sigma_1^2$. By the same argument as in the case where $s \neq 0$, we see that $a_0^t\sigma_3^{-1}\sigma_1^2 \equiv b_3^{-1} = b_3$.

We show (vi). Note that $c_0^t\sigma_1^2 = a_0^t\sigma_3^{-1}\sigma_1^2 = a_0^t\sigma_3^{-1}\sigma_1^2$. By (1)(i), we have $a_0^t\sigma_3^{-1}\sigma_1^2 \equiv a_0^t\sigma_3^{-1}\sigma_1^2 \equiv a_0^t\sigma_3^{-1}\sigma_1^2$ with $t \equiv s + 1 \pmod{5}$. By (3)(iv), we have $a_0^t\sigma_3^{-1}\sigma_1^2 \equiv c_0^t\sigma_1^2$. Thus, we have $c_0^t\sigma_1^2 \equiv c_0^t\sigma_1^2$ with $t \equiv s + 1 \pmod{5}$. On the other hand, by (1)(i) we have $a_0^t\sigma_3^{-1}\sigma_1^2 \equiv e_0^t\sigma_3^{-1}\sigma_1^2$. By (3)(vii), we see that $e_0^t\sigma_3^{-1}\sigma_1^2 \equiv e_0^t\sigma_3^{-1}\sigma_1^2$.

We show (vii). Since $d_0^t \in \mathcal{Z}_1$, by Lemma 2.4(2) we have $d_0^t\sigma_1^2 \equiv d_0^s\sigma_1^2$. Note that $d_0^t\sigma_1 \equiv d_0^t\sigma_1 \equiv a_0^t\sigma_2^{-1}\sigma_3^{-1}$. Since $d_0^t \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^t\sigma_2^{-1}\sigma_3^{-1} \equiv e_0^t\sigma_2^{-1}\sigma_3^{-1}$. In $B_4$, we have $a_0^t\sigma_2^{-1}\sigma_3^{-1} \equiv (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1}$. When $s = 2, 3, 4$, we see that $(\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv e_4^{-2}$. When $s = 0, 1$, by Lemma 2.6 we have $(\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv e_4^{-2}$. Thus, we have $d_0^t\sigma_1^2 \equiv e_0^t$ with $u \equiv s + 3 \pmod{5}$. We show that $d_0^t\sigma_1 \equiv e_0^t$ for $t \in \{0, 1, 2, 3, 4\}$ with $t \equiv s + 3 \pmod{5}$. In $B_4$, $d_0^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1}$. When $s = 0, 2$, we have $d_0^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv e_0^t\sigma_2^{-1}\sigma_3^{-1}$. Since $e_0^t \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_0^t\sigma_3^{-1} \equiv e_0^t\sigma_3^{-1} \equiv c_0^t\sigma_3^{-1} \equiv e_0^t\sigma_3^{-1}$. When $s = 0, 2$, by Lemma 2.6 we have $(\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv e_0^t\sigma_3^{-1}\sigma_2^{-1}$.

By the same argument as in the case where $s \neq 4$, we see that $e_0^t\sigma_3^{-1}\sigma_2^{-1} \equiv e_0^t\sigma_3^{-1}\sigma_2^{-1}$. We show (viii). In $B_4$, $d_0^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1}$. When $s \in \{0, 1, 2\}$, we have $(\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2^{-1}\sigma_3^{-1} \equiv a_0^t\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}$. Since $a_0^t \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^t\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv a_0^t\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}$. Since $a_0^t \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^t\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} \equiv a_0^t\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}$.
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\((\sigma_1 \sigma_2 \sigma_3)^{s-4} \sigma_2^2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \). By the same argument as in the case where \( s \in \{0, 1, 2\} \), we see that \((\sigma_1 \sigma_2 \sigma_3)^{s-4} \sigma_2^2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \equiv e_4^{-3} = e_4^t \).

We show (ix). First, we show \( d^2 \sigma_1 = e_4^t \) with \( t + 2 \equiv s + 3 \) (mod 5). Note that \( d^2 \sigma_1 = a_3^2 \sigma_4 \). Since \( a_3^2 \in Z_2 \) and \( a_6^0 \in Z_2 \), by Lemma 2.4(2) we have \( a_3^2 \sigma_4^2 \sigma_1 = a_3^2 \sigma_4^{-1} \sigma_1 = a_6^0 \sigma_3 \sigma_4 \). In \( B_4 \), we have \( a_6^0 \sigma_3 \sigma_4 \sigma_1 = (\sigma_1 \sigma_2 \sigma_3)^{s+1} \sigma_2^{-1} \sigma_3^{-1} \sigma_2^{-1} \). When \( s \neq 4 \), we see that \((\sigma_1 \sigma_2 \sigma_3)^{s+1} \sigma_2^{-1} \sigma_3^{-1} \sigma_2^{-1} = a_6^0 \sigma_3 \sigma_4 \sigma_1 \). Since \( a_6^0 \sigma_3 \sigma_4 \sigma_1 \in Z_2 \), by Lemma 2.4(2) we have \( a_6^0 \sigma_3 \sigma_4 \sigma_1 \equiv a_6^0 \sigma_3 \sigma_4 \sigma_1 \sigma_2^{-1} = e_4^{-1} + \sigma_2^{-1}. \) Since \( e_4^{-1} \in Y_3 \), by Lemma 2.4(1) we have \( e_4^{-1} + \sigma_2^{-1} = e_4^{-1} + \sigma_2^{-1} = e_4^{-1} + \sigma_2^{-1} \). Since \( e_4^{-1} \in Z_2 \), by Lemma 2.4(2) we have \( e_4^{-1} + \sigma_2^{-1} = e_4^{-1} + \sigma_2^{-1} \). By Lemma 2.5(2), we have \( e_4^{-1} + \sigma_2^{-1} = e_4^{-1} + \sigma_2^{-1} \). By the same argument as in the case where \( s \neq 4 \), we see \( a_6^0 \sigma_3 \sigma_4 \sigma_1 \equiv e_4^{-1} = e_4^t \).

Next, we show \( e_4^t \sigma_1 = e_4^t \) with \( u \equiv t + 2 \) (mod 5). Note that \( e_4^t \sigma_1 = (\sigma_1 \sigma_2 \sigma_3)^{t+2} \sigma_2^2 \sigma_1 \sigma_3 \). When \( t \in \{0, 1, 2\} \), it is \( e_4^{t+2} \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \). Since \( e_4^{t+2} \in Y_2 \), by Lemma 2.4(1) we have \( e_4^{t+2} \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \). Since \( e_4^{t+2} \in Y_2 \), by Lemma 2.4(1) we have that \( e_4^{t+2} \sigma_2 \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \). Since \( e_4^{t+2} \in Y_2 \), by Lemma 2.4(1) we have that \( e_4^{t+2} \sigma_2 \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \). By the same argument as in the case where \( t \in \{0, 1, 2\} \), we see that \( e_4^{t+2} \sigma_2 \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \).

We show (x). In \( B_4 \), \( d^3 \sigma_1 = a_3^2 \sigma_4 \sigma_1 \). Since \( a_3^2 \in Z_2 \), by Lemma 2.4(2) we have \( a_3^2 \sigma_4 \sigma_1 = a_3^2 \sigma_4 \sigma_1 = a_6^0 \sigma_3 \sigma_4 \sigma_1 \). By Lemma 2.5(1), we have \( a_6^0 \sigma_3 \sigma_4 \sigma_1 \equiv a_6^0 \sigma_3 \sigma_4 \sigma_1 \). By Lemma 2.5(1), we have \( a_6^0 \sigma_3 \sigma_4 \sigma_1 \equiv a_6^0 \sigma_3 \sigma_4 \sigma_1 \). By the same argument as in the case where \( s \in \{3, 4\} \), we have \( e_4^{t+2} \sigma_2 \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \). By the same argument as in the case where \( s \in \{3, 4\} \), we have \( e_4^{t+2} \sigma_2 \sigma_3 \equiv e_4^{t+2} \sigma_2 \sigma_3 \).

By Lemmas 2.4 and 2.7, we shall obtain Lemma 2.8.

**Lemma 2.8.** For any element \( \alpha \in X \) and for any power \( \sigma_j^s \) of each standard generator \( \sigma_j \) (\( j = 1, 2, 3 \)) of \( B_4 \), there is an element \( \beta \) of \( X \) such that \( \alpha \sigma_j^s \equiv \beta \).

**Proof.** Since \( X = Y_3 \coprod Z_3 \), \( \alpha \) belongs to either \( Y_3 \) or \( Z_3 \). Suppose that \( \alpha \in Y_3 \). Then, by Lemma 2.4(1) we have \( \sigma_j^s \equiv \alpha \) for an even number \( l \) or \( \alpha \sigma_j^l \equiv \alpha \sigma_j^l \) for an odd number \( l \).

We will show that for \( \alpha \in Y_3 \) there is an element \( \beta \) of \( X \) such that \( \alpha \sigma_j^s \equiv \beta \). By Lemma 2.7(1)(ii) and Lemma 2.4(1), we have \( \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 2 \) (mod 5). Similarly, by Lemma 2.4(1) and Lemma 2.7(1)(iii), \( (1)(iv), (1)(vii), (1)(x), (2)(v), (2)(vi), (2)(vii), (2)(ix), (2)(x), (3)(v), (3)(vi), (3)(viii), (3)(ix) \) and \( (3)(x) \) we have \( \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \equiv \alpha \sigma_j^2 \).

By Lemmas 2.4 and 2.7, we shall obtain Lemma 2.8.
Lemma 2.4(2), we have

\[ d_t^a \text{ for } s, t \in \{0, 1, 2, 3, 4\} \text{ with } t \equiv s + 2 \pmod{5}, d_t^a \sigma_1^j \equiv e_s^j \sigma_1 \equiv d_t^a \text{ for } s, t \in \{0, 1, 2, 3, 4\} \]

with \( t \equiv s + 2 \pmod{5} \); \( c_0^a \sigma_2 \equiv c_2^a \), \( c_2^a \sigma_2 \equiv c_3^a \), \( c_1^a \sigma_2 \equiv c_3^a \), \( c_3^a \sigma_2 \equiv c_1^a \), \( c_0^a \sigma_2 \equiv c_1^a \), \( c_1^a \sigma_2 \equiv c_2^a \), \( c_2^a \sigma_2 \equiv c_3^a \), \( d_0^a \sigma_3 \equiv d_2^a \), \( d_2^a \sigma_3 \equiv d_3^a \), \( d_3^a \sigma_3 \equiv d_0^a \), \( d_1^a \sigma_3 \equiv d_2^a \), \( d_2^a \sigma_3 \equiv d_3^a \), \( d_3^a \sigma_3 \equiv d_1^a \), \( e_1^a \sigma_4 \equiv e_2^a \), \( e_2^a \sigma_3 \equiv e_1^a \), \( e_3^a \sigma_3 \equiv e_1^a \), \( e_4^a \sigma_3 \equiv e_3^a \), \( e_5^a \sigma_3 \equiv e_5^a \) and \( e_6^a \sigma_3 \equiv e_7^a \) for \( s \in \{0, 1, 2, 3, 4\} \).

Hence, we have the result when \( \alpha \in \mathcal{Y}_j \).

We consider the case where \( \alpha \in \mathcal{Z}_j \). By Lemma 2.4(2) we have \( \alpha \sigma_j^l = \alpha \) for \( l \in 3\mathbb{Z} \), \( \alpha \sigma_j^l = \alpha \sigma_j \) for \( l \in 3\mathbb{Z} + 1 \) or \( \alpha \sigma_j^l = \alpha \sigma_j^2 \) for \( l \in 3\mathbb{Z} + 2 \). We will show that for \( \alpha \in \mathcal{Z}_j \), there are elements \( \beta \) and \( \gamma \) of \( \mathcal{X} \) such that \( \alpha \sigma_j \equiv \beta \) and \( \alpha \sigma_j^2 \equiv \gamma \). By Lemma 2.7(1)(i) and Lemma 2.4(2), we have \( a_0^a \sigma_j^l \equiv a_3^a \sigma_j^l \equiv a_0^a \sigma_j \equiv a_0^a \) \((i.e., a_0^a \sigma_1 \equiv a_3^a \sigma_1 \equiv a_0^a \) and \( a_0^a \sigma_1 \equiv a_0^a \)) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 1 \pmod{5} \). Similarly, by Lemma 2.4(2) and Lemma 2.7(1)(v), \( (1)(vi), (1)(vii), (1)(ix); (2)(i), (2)(ii), (2)(iii), (2)(iv), (2)(viii); (3)(i), (3)(ii), (3)(iii), (3)(iv) \) and \( (3)(vii) \) we have \( b_2^a \sigma_1 \equiv e_0^a \), \( e_0^a \sigma_1 \equiv b_2^a \) and \( b_1^a \sigma_1 \equiv b_2^a \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 4 \pmod{5} \), \( c_0^a \sigma_1^l \equiv c_2^a \sigma_1 \equiv e_0^a \), \( c_2^a \sigma_1^l \equiv c_4^a \sigma_1 \equiv c_0^a \) and \( e_2^a \sigma_1^l \equiv c_0^a \sigma_1 \equiv c_4^a \) for \( s, t \in \{0, 1, 2, 3, 4\} \) with \( t = s + 1 \pmod{5} \), \( d_0^a \sigma_1 \equiv e_0^a \), \( e_0^a \sigma_1 \equiv e_0^a \) and \( e_4^a \sigma_1 \equiv d_0^a \) for \( s, t \), \( u \in \{0, 1, 2, 3, 4\} \) with \( u = t + 2 \equiv s + 3 \pmod{5} \), \( d_2^a \sigma_1 \equiv e_0^a \), \( e_0^a \sigma_1 \equiv e_0^a \) and \( e_4^a \sigma_1 \equiv d_2^a \) for \( s, t, u \in \{0, 1, 2, 3, 4\} \) with \( u = t + 2 \equiv s + 3 \pmod{5} \); \( a_0^a \sigma_2 \equiv b_0^a \), \( b_0^a \sigma_2 \equiv a_0^a \), \( a_0^a \sigma_2 \equiv b_1^a \), \( b_1^a \sigma_2 \equiv d_1^a \), \( d_1^a \sigma_2 \equiv a_1^a \), \( a_1^a \sigma_2 \equiv b_2^a \), \( b_2^a \sigma_2 \equiv d_2^a \), \( d_2^a \sigma_2 \equiv a_2^a \), \( a_2^a \sigma_2 \equiv b_3^a \), \( b_3^a \sigma_2 \equiv d_3^a \), \( d_3^a \sigma_2 \equiv a_3^a \), \( a_3^a \sigma_2 \equiv e_0^a \), \( e_0^a \sigma_2 \equiv e_0^a \), \( e_0^a \sigma_3 \equiv e_0^a \), \( e_0^a \sigma_3 \equiv e_0^a \) and \( e_0^a \sigma_3 \equiv e_0^a \) for \( s \in \{0, 1, 2, 3, 4\} \). Hence, we have the result when \( \alpha \in \mathcal{Z}_j \).

By Lemma 2.8, we have the following lemma;

Lemma 2.9. \( i' \) is surjective.

Proof. Let \( \omega \) be an element in \( B_4 \). Then we may assume \( \omega = \sigma_1^{i_1} \cdots \sigma_4^{i_4} \), where \( i_l \in \{1, 2, 3\} \) and \( j_l \) is a non-zero integer for \( l = 1, \ldots, k \) and with \( i_l \neq i_k \) for \( |l - k| = 1 \). Since \( id = a_0^a \in \mathcal{X} \), we see by Lemma 2.8 that there exists an element \( \beta_l \in \mathcal{X} \) such that \( \omega = id \cdot \sigma_1^{i_1} \cdots \sigma_4^{i_4} \equiv \beta_1 \sigma_2^{j_1} \cdots \sigma_4^{j_k} \). Applying Lemma 2.8 repeatedly, we can inductively find elements \( \beta_1, \beta_2, \ldots, \beta_l \) in \( \mathcal{X} \) such that \( \omega = \beta_l \sigma_1^{i_l+1} \cdots \sigma_4^{j_k} \) for \( l = 1, \ldots, k \). Hence, we get \( \omega = \beta_k \in \mathcal{X} \) at the end.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Recall that \( i \) and \( i' \) are injective and \( j \) is surjective. Since \( i' \) is surjective by Lemma 2.9, the maps \( i' \), \( i = j \circ i' \) and \( j \) are bijective. The bijectivity of \( i \) implies that \( \mathcal{X} \) is a complete system of representatives of \( H(s_1, s_2, s_3, s_4) \setminus B_4 \). Since \( \mathcal{X} \) consists of 125 elements, we have \( \#(H(s_1, s_2, s_3, s_4) \setminus B_4) = 125 \). This completes the proof of Theorem 1.2(2). Moreover, the bijectivity of \( j \) implies \( H(s_1, s_2, s_3, s_4) = \langle S \rangle \). This completes the proof of Theorem 1.2(1).

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References


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