

Seiberg-Witten invariants of 3-orbifolds and non-Kähler surfaces

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ABSTRACT. A formula is given which computes the Seiberg-Witten invariant of a 3-orbifold from the invariant of the underlying manifold. As an application, we derive a formula for the Seiberg-Witten invariant of a non-Kähler complex surface, which was originally due to O. Biquard [4] and S.R. Williams [25] independently.

1. Introduction

Let Y be a closed, oriented 3-orbifold with $b_1 \geq 1$ whose singular set ΣY consists of a union of embedded circles. Consider a component $l \subset \Sigma Y$ with associated multiplicity α , and let Y_0 be the oriented 3-orbifold obtained from Y by removing l from the singular set. Our main result in this paper gives a relationship between the Seiberg-Witten invariants of Y and Y_0 .

Theorem 1.1. *Let S_Y, S_{Y_0} denote the set of $Spin^c$ -structures on Y, Y_0 respectively. There exists a canonical map $\phi : S_Y \rightarrow S_{Y_0}$, which is surjective and α to 1, such that the Seiberg-Witten invariants of Y and Y_0 obey the following equations*

$$SW_Y(\xi) = SW_{Y_0}(\phi(\xi)), \quad \forall \xi \in S_Y.$$

Remark 1.2. (1) The Seiberg-Witten invariant is defined as in Meng-Taubes [16], Taubes [22]. See Baldridge [2] for the extension to 3-orbifolds. In particular, when $b_1 = 1$, the Seiberg-Witten invariants of Y, Y_0 depend on a choice of orientation of $H^1(|Y|; \mathbb{R})$, where $|Y|$ stands for the underlying 3-manifold. With this understood, we remark that the same orientation of $H^1(|Y|; \mathbb{R})$ is used for Y, Y_0 in Theorem 1.1. Also, the sign of the Seiberg-Witten invariants depends on a choice of homology orientation of $|Y|$; the same is used for Y, Y_0 .

(2) A more explicit description of the map $\phi : S_Y \rightarrow S_{Y_0}$ is given in Section 2, together with a formula relating the determinant line bundles of the $Spin^c$ -structures $\xi, \phi(\xi)$, $\forall \xi \in S_Y$, cf. Proposition 2.6.

(3) Theorem 1.1 holds true more generally where Y is allowed to have non-empty boundary ∂Y , where ∂Y consists of a union of tori (cf. [16, 22]), and $\Sigma Y \cap \partial Y = \emptyset$.

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(4) Theorem 1.1 (together with Proposition 2.6) shows that, in contrast to geometric structures, the Seiberg-Witten invariant of a 3-orbifold Y is much less sensitive to the embedding of the singular set ΣY in $|Y|$, i.e. $\Sigma Y \subset |Y|$ as a knot or link.

(5) The Seiberg-Witten invariant of a 3-manifold is equivalent to the Turaev torsion of the manifold [16, 22, 23, 24]. The equivalence of the two invariants may be extended to 3-orbifolds by establishing an analog of Theorem 1.1 for the Turaev torsion.

Theorem 1.1 has the following topological consequence. Recall that an orientable 3-orbifold is called *pseudo-good* if it does not contain any bad 2-suborbifold. The importance of this notion lies in the fact that the basic theory of 3-manifolds was extended only to this class of 3-orbifolds (see e.g. [6]). It is known that Thurston's Geometrization Conjecture for 3-orbifolds (which is now proven, cf. [5]) implies that a pseudo-good 3-orbifold must be very good, i.e., it admits a finite, regular manifold cover (cf. [15]).

Every pseudo-good 3-orbifold admits a spherical splitting, i.e., the 3-orbifold may be cut open along a system of spherical 2-suborbifolds such that after capping off the boundary each component becomes irreducible. We remark that, unlike the connected sum decomposition of a 3-manifold into prime factors, the spherical splitting of a 3-orbifold is not known to be unique in general, and the issue of non-separating spherical 2-suborbifolds has to be treated differently (cf. [6], page 41). However, when there are no non-separating spherical 2-suborbifolds, Petronio showed in [20] that the 3-orbifold admits a so-called efficient spherical splitting for which the uniqueness statement holds.

The following corollary of Theorem 1.1 follows easily from the fact that if a 3-manifold with $b_1 > 1$ contains a non-separating 2-sphere, or is a connected sum of two manifolds both of which have nonzero b_1 , then its Seiberg-Witten invariant vanishes.

Corollary 1.3. *Let Y be an oriented 3-orbifold with $b_1 > 1$ and $SW_Y \neq 0$, whose singular set ΣY consists of a union of embedded circles. Then Y is pseudo-good and every spherical 2-suborbifold of Y is separating. Moreover, in the efficient spherical splitting of Y exactly one irreducible component has $b_1 \neq 0$.*

Corollary 1.3 has application in the study of smooth, fixed-point free circle actions on 4-manifolds.

In this paper, we shall focus on another application of Theorem 1.1, where we derive a formula for the Seiberg-Witten invariant of a non-Kähler complex surface. After this paper was completed, we found that a similar result was already obtained by O. Biquard [4] and S.R. Williams [25] (independently) using a different (and analytical) method. The Seiberg-Witten invariants of Kähler surfaces were determined by Brussee [7] and Friedman-Morgan [13] independently.

In order to state the formula, recall that according to the Enriques-Kodaira classification (cf. [3]), a minimal complex surface X with $b_2^+ \geq 1$ is either a rational or ruled surface, or an elliptic surface, or a surface of general type. Moreover, if X is non-Kähler, it must be an elliptic surface with Euler number zero. According to [12], Theorem 7.7, X is obtained from the product $E \times C$, where $E = \mathbb{C}/\Lambda$ is an elliptic curve and C is a curve of genus g , by doing logarithmic transforms on lifts $x_1, \dots, x_n \in \mathbb{C}$ of m_i -torsion points

ξ_i modulo Λ , with $\sum_i x_i \neq 0$. Note that the assumption $b_2^+ \geq 1$ implies that $g \neq 0$, which in turn implies that $b_2^+ > 1$.

Fix a basis $\mathbf{e}_1, \mathbf{e}_2$ of Λ . There are integers a_i, b_i such that

$$x_i = \frac{a_i \mathbf{e}_1 + b_i \mathbf{e}_2}{m_i}, \quad i = 1, 2, \dots, n.$$

Let Σ be the 2-orbifold whose underlying space is C , with singular points t_i of multiplicities m_i , where $t_i \in C$ is the point over which the logarithmic transform on x_i is performed. (Here we abuse the notation in the sense that when $m_i = 1$, t_i is in fact a regular point.) With this understood, there are associated orbifold complex line bundles E_1, E_2 over Σ which are defined as follows. Let $a_i = q_i m_i + u_i$, $b_i = r_i m_i + v_i$, where $0 \leq u_i, v_i < m_i$. Then the Seifert invariants of E_1, E_2 are $(\sum_i q_i, (m_i, u_i))$ and $(\sum_i r_i, (m_i, v_i))$ respectively. We remark that E_1, E_2 depend on the choice of the basis $\mathbf{e}_1, \mathbf{e}_2$, but the subgroup of orbifold complex line bundles generated by E_1, E_2 depends only on X (see Section 6 for more details). We denote the subgroup by Γ_X .

For any orbifold complex line bundle D over Σ , we denote by (D) the orbit of D under the action of Γ_X , i.e., $(D) = (D')$ if and only if $D \equiv D' \pmod{\Gamma_X}$. Finally, for an orbifold complex line bundle D with Seifert invariant $(d, (m_i, s_i))$, we denote the ‘‘background’’ degree d by $|D|$.

With the preceding understood, we have

Theorem 1.4. *Let X be a minimal, non-Kähler, complex surface with $b_2^+ > 1$. With the notations introduced above, the set of Spin^c -structures on X which have nonzero Seiberg-Witten invariant may be identified with a subset of orbits (D) , such that under this identification, if a Spin^c -structure \mathcal{L} corresponds to (D) and D has Seifert invariant $(d, (m_i, s_i))$, then in terms of Poincaré duality,*

$$c_1(\det \mathcal{L}) = (2d - 2g + 2)F + \sum_i (2s_i + 1 - m_i)F_i,$$

where F stands for a regular fiber and F_i stands for the fiber at t_i of the elliptic fibration on X . Moreover, the Seiberg-Witten invariant of X is given by

$$SW_X(\mathcal{L}) \equiv SW_X((D)) = \sum_{D' \in (D)} (-1)^{|D'|} \binom{2g-2}{|D'|}.$$

In the above formula, $\binom{2g-2}{|D'|} = 0$ if $|D'|$ lies outside the interval $[0, 2g-2]$.

The organization of the remaining sections is as follows. In Section 2 we define the map $\phi : S_Y \rightarrow S_{Y_0}$ and discuss its relevant properties. Sections 3 and 4 are devoted to the proof of Theorem 1.1, which is based on the gluing theorems of Morgan-Mrowka-Szabó [18] and Taubes [22] respectively. In Section 5 we compute some examples to illustrate the theorem using Seifert 3-manifolds. The proof of Theorem 1.4 about non-Kähler complex surfaces is given in Section 6 (see Theorem 6.2). Part of the material presented in this paper has been circulated in a preprint [9].

2. The map $\phi : S_Y \rightarrow S_{Y_0}$

In order to define the map $\phi : S_Y \rightarrow S_{Y_0}$, we shall decompose Y as a union of Y_+ and Y_- along a 2-torus T , where Y_+ is the complement of a regular neighborhood of l in $|Y|$, and Y_- is a regular neighborhood of l in Y , which is a 3-orbifold modeled by $D^2 \times \mathbb{S}^1/\mathbb{Z}_\alpha$. In the above model, D^2 is the unit disc in \mathbb{C} with the standard \mathbb{Z}_α -action generated by a rotation of angle $2\pi/\alpha$, and the \mathbb{Z}_α -action on $D^2 \times \mathbb{S}^1$ is given by the product action which is trivial on the \mathbb{S}^1 -factor. As D^2 is canonically oriented as a subset of \mathbb{C} , the orientation of Y determines an orientation of $l = \{0\} \times \mathbb{S}^1/\mathbb{Z}_\alpha$. Reversing the orientation of l means reversing the orientation of D^2 . Likewise, we decompose Y_0 as a union of Y_+ and $Y_{0,-}$. Here $Y_{0,-}$ is a regular neighborhood of l in Y_0 , identified with $D^2 \times \mathbb{S}^1$, where $D^2 = |D^2/\mathbb{Z}_\alpha|$ is the underlying 2-disc of the orbifold D^2/\mathbb{Z}_α . The orientation of l inherited from the identification $Y_{0,-} = D^2 \times \mathbb{S}^1$ is the same as the one inherited from $Y_- = D^2 \times \mathbb{S}^1/\mathbb{Z}_\alpha$.

Fixing such a decomposition for Y (resp. Y_0), one can describe a $Spin^c$ -structure on Y (resp. Y_0) in terms of $Spin^c$ -structures on Y_+ and Y_- (resp. Y_+ and $Y_{0,-}$) and some gluing datum along the 2-torus $T = \partial Y_+ = \partial Y_- (= \partial Y_{0,-}$ respectively). To be more precise, let a normal direction along T be chosen. This is equivalent to fixing an orientation on T . Then any $Spin^c$ -structure ξ^+ on Y_+ (likewise any ξ^- on Y_- , or ξ on Y) as an isomorphism class of principal $Spin^c(3)$ -bundles lifting the $SO(3)$ -bundle of oriented orthonormal frames admits a reduction to a principal $Spin^c(2)$ -bundle over a neighborhood of T ; that principal $Spin^c(2)$ -bundle corresponds to a $Spin^c$ -structure on T , which we will denote by $\xi_+|_T$, called the restriction of ξ_+ to T . It follows easily that $\xi_+|_T$ is isomorphic to the trivial $Spin^c$ -structure on T , i.e., the unique $Spin^c$ -structure on T that has a trivial determinant line bundle, which admits a reduction to a $Spin$ -structure on T . The $Spin$ -structures on $T = \mathbb{S}^1 \times \mathbb{S}^1$ are classified by $H^1(T; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, and each $Spin$ -structure gives a specific trivialization of the determinant line bundle of the $Spin^c$ -structure on T . There is a special $Spin$ -structure on T , denoted by ξ_0 , which is the product of the non-trivial $Spin$ -structure of each \mathbb{S}^1 -factor of T . We remark that if ξ_0 is identified with $0 \in H^1(T; \mathbb{Z}_2)$, then for any other $Spin$ -structure η on T which is identified with $\hat{\eta} \in H^1(T; \mathbb{Z}_2)$, $\hat{\eta}$ is non-zero on an \mathbb{S}^1 -factor of T if and only if η can be extended to a $Spin$ -structure on a solid torus bounded by T in which the \mathbb{S}^1 -factor of T bounds a disc.

With the preceding understood, let S_{Y_+}, S_{Y_-} be the set of $Spin^c$ -structures on Y_+, Y_- respectively, and let $S_{Y_+}^0, S_{Y_-}^0$ be the set of pairs $(\xi^+, h_+), (\xi^-, h_-)$, where $\xi^+ \in S_{Y_+}$, $\xi^- \in S_{Y_-}$, and h_+, h_- are homotopy classes of isomorphisms $\xi^+|_T \rightarrow \xi_0, \xi^-|_T \rightarrow \xi_0$ respectively. Note that the automorphisms $\xi_0 \rightarrow \xi_0$ are given by elements of $C^\infty(T; \mathbb{S}^1)$, whose homotopy classes may be identified with $H^1(T; \mathbb{Z})$. Likewise, for any $\xi^+ \in S_{Y_+}$, and $\xi^- \in S_{Y_-}$, the automorphisms of ξ^+, ξ^- are given by elements of $C^\infty(Y_+; \mathbb{S}^1), C^\infty(Y_-; \mathbb{S}^1)$ respectively. The following lemma determines the set of homotopy classes of elements of $C^\infty(Y_+; \mathbb{S}^1)$ and $C^\infty(Y_-; \mathbb{S}^1)$.

Lemma 2.1. *Let W be a compact orbifold (with or without boundary) whose singular set consists of a disjoint union of manifolds of co-dimension 2. Let $|W|$ denote the underlying topological space and let $C^\infty(W; \mathbb{S}^1)$ denote the space of smooth circle-valued functions on the orbifold W . Then there is a natural identification between $\pi_0(C^\infty(W; \mathbb{S}^1))$ and $H^1(|W|; \mathbb{Z})$.*

Proof. Since the singular set of W is a disjoint union of manifolds of co-dimension 2, $|W|$ is naturally a smooth manifold. Now given any $\varphi \in C^\infty(W; \mathbb{S}^1)$, it induces a continuous circle-valued function $\tilde{\varphi}$ on $|W|$. We denote by $[\varphi]$ the element in $H^1(|W|; \mathbb{Z})$ determined by the homotopy class of $\tilde{\varphi}$. The correspondence $\varphi \mapsto [\varphi]$ clearly induces a mapping $\theta : \pi_0(C^\infty(W; \mathbb{S}^1)) \rightarrow H^1(|W|; \mathbb{Z})$. We will show that θ is both onto and one to one.

Let $x \in H^1(|W|; \mathbb{Z})$ be given. Since $|W|$ is a smooth manifold, there is a smooth \mathbb{S}^1 -valued function $\tilde{\varphi}$ on $|W|$ which represents x . The pull back of $\tilde{\varphi}$ to W , which we denote by φ , is smooth because locally the projection $W \rightarrow |W|$ is given by $(w, z) \mapsto (w, z^m)$ for some $m > 1$, where $w \in \mathbb{R}^k$ and $z \in \mathbb{C}$. It is clear that $[\varphi] = x$, hence θ is onto.

To see that θ is one to one, let $\varphi_1, \varphi_2 \in C^\infty(W; \mathbb{S}^1)$ such that $[\varphi_1] = [\varphi_2]$. The induced continuous functions $\tilde{\varphi}_1, \tilde{\varphi}_2$ on $|W|$ are homotopic. We perturb $\tilde{\varphi}_1, \tilde{\varphi}_2$ into smooth functions $\tilde{\varphi}_{1+\epsilon}, \tilde{\varphi}_{2-\epsilon}$ on $|W|$ through a family of functions $\tilde{\varphi}_t$, with $1 \leq t \leq 1 + \epsilon$ and $2 - \epsilon \leq t \leq 2$ respectively. The pull back of $\tilde{\varphi}_t$ to W are continuous, but can be perturbed to a family of smooth functions on the local uniformizing systems, which after averaging become equivariant with respect to the local group actions on the uniformizing systems. In other words, $\varphi_1, \varphi_{1+\epsilon}$ and $\varphi_2, \varphi_{2-\epsilon}$ are homotopic through smooth functions on W . Finally, connect $\tilde{\varphi}_{1+\epsilon}, \tilde{\varphi}_{2-\epsilon}$ through a family of smooth functions $\tilde{\varphi}_t$ on $|W|$ with $1 + \epsilon \leq t \leq 2 - \epsilon$. Their pull backs to W , φ_t , are smooth functions on W . This shows that φ_1, φ_2 have the same class in $\pi_0(C^\infty(W; \mathbb{S}^1))$, and therefore θ is one to one. \square

We consider natural actions of $H^1(T; \mathbb{Z})$ on $S_{Y_+}^0$ and $S_{Y_-}^0$ defined as follows. For any $e \in H^1(T; \mathbb{Z})$, $(\xi^+, h_+) \in S_{Y_+}^0$, define

$$e \cdot (\xi^+, h_+) = (\xi^+, h_+ + e),$$

where $h_+ + e$ denotes the homotopy class of isomorphisms $\xi^+|_T \rightarrow \xi_0$ which is given by h_+ followed by an automorphism of ξ_0 whose homotopy class is represented by e . Likewise, for any $e \in H^1(T; \mathbb{Z})$, $(\xi^-, h_-) \in S_{Y_-}^0$, define

$$e \cdot (\xi^-, h_-) = (\xi^-, h_- + e).$$

With this understood, $H^1(|Y_+|; \mathbb{Z})$ and $H^1(|Y_-|; \mathbb{Z})$ act on $S_{Y_+}^0$ and $S_{Y_-}^0$ via homomorphisms $j_+ : H^1(|Y_+|; \mathbb{Z}) \rightarrow H^1(T; \mathbb{Z})$ and $j_- : H^1(|Y_-|; \mathbb{Z}) \rightarrow H^1(T; \mathbb{Z})$, which are induced by the embedding of T in $|Y_+|, |Y_-|$. As a consequence of Lemma 2.1, we note that if $e \in j_+ H^1(|Y_+|; \mathbb{Z})$ (reps. $e \in j_- H^1(|Y_-|; \mathbb{Z})$), then for any $(\xi^+, h_+) \in S_{Y_+}^0$ (resp. $(\xi^-, h_-) \in S_{Y_-}^0$), there is an automorphism \tilde{e} of ξ^+ (resp. ξ^-) such that $h_+ \circ \tilde{e}|_{\xi^+|_T} = h_+ + e$ (resp. $h_- \circ \tilde{e}|_{\xi^-|_T} = h_- + e$). We denote by \underline{S}_{Y_+} , \underline{S}_{Y_-} the quotient set of $S_{Y_+}^0$ and $S_{Y_-}^0$ under the action of $H^1(|Y_+|; \mathbb{Z})$ and $H^1(|Y_-|; \mathbb{Z})$ respectively. Finally, define an action of

$H^1(T; \mathbb{Z})$ on $\underline{S}_{Y_+} \times \underline{S}_{Y_-}$ by

$$e \cdot ((\xi^+, h_+), [(\xi^-, h_-)]) = ((\xi^+, h_+ + e), [(\xi^-, h_- + e)]),$$

where $e \in H^1(T; \mathbb{Z})$, $[(\xi^+, h_+)] \in \underline{S}_{Y_+}$, $[(\xi^-, h_-)] \in \underline{S}_{Y_-}$.

The same discussions apply to Y_0 , and we obtain the set $\underline{S}_{Y_{0,-}}$ and a similarly defined action of $H^1(T; \mathbb{Z})$ on $\underline{S}_{Y_+} \times \underline{S}_{Y_{0,-}}$.

Lemma 2.2. *There are natural identifications $\psi : \underline{S}_{Y_+} \times \underline{S}_{Y_-} / H^1(T; \mathbb{Z}) \rightarrow S_Y$ and $\psi_0 : \underline{S}_{Y_+} \times \underline{S}_{Y_{0,-}} / H^1(T; \mathbb{Z}) \rightarrow S_{Y_0}$.*

Proof. The map ψ is defined as follows. For any $((\xi^+, h_+), (\xi^-, h_-)) \in S_{Y_+}^0 \times S_{Y_-}^0$, we construct a $Spin^c$ -structure on Y by gluing ξ^+ and ξ^- along T via the isomorphism $h_-^{-1} \circ h_+ : \xi^+|_T \rightarrow \xi^-|_T$, and assign this $Spin^c$ -structure to $((\xi^+, h_+), (\xi^-, h_-))$. Call this map $\tilde{\psi} : S_{Y_+}^0 \times S_{Y_-}^0 \rightarrow S_Y$. Clearly $\tilde{\psi}$ factors through $\underline{S}_{Y_+} \times \underline{S}_{Y_-} / H^1(T; \mathbb{Z})$.

First, we check that $\tilde{\psi}$ is surjective onto S_Y . For any $\xi \in S_Y$, set $\xi^\pm \equiv \xi|_{Y_\pm} \in S_{Y_\pm}$. Then $\xi^\pm|_T = \xi|_T$. We pick any $h_+ : \xi^+|_T \rightarrow \xi_0$, and let $h_- = h_+$ under the identification $\xi^-|_T = \xi^+|_T$. Then it follows easily that $((\xi^+, h_+), (\xi^-, h_-))$ is sent to ξ under $\tilde{\psi}$.

Next we check that $\tilde{\psi}$ is one to one after factoring through the quotient set. To this end, let $((\xi_i^+, h_{+,i}), (\xi_i^-, h_{-,i}))$, $i = 1, 2$, be sent to the same image under $\tilde{\psi}$. Then there are isomorphisms $f_\pm : \xi_1^\pm \rightarrow \xi_2^\pm$, such that

$$h_{-,2}^{-1} \circ h_{+,2} \circ f_+|_T = f_-|_T \circ h_{-,1}^{-1} \circ h_{+,1}.$$

Without loss of generality, we may assume $\xi_1^\pm = \xi_2^\pm$, and consequently, f_\pm are given by elements of $C^\infty(Y_\pm; \mathbb{S}^1)$. Then the class of $f_\pm|_T$ in $H^1(T; \mathbb{Z})$, $[f_\pm|_T]$, lies in the image $j_\pm : H^1(|Y_\pm; \mathbb{Z}) \rightarrow H^1(T; \mathbb{Z})$. Replacing $h_{+,2}$ by $h_{+,2} \circ [f_+|_T]$, and $h_{-,1}$ by $[f_-|_T] \circ h_{-,1}$, which give the same elements in \underline{S}_{Y_+} and \underline{S}_{Y_-} respectively, we may assume instead

$$h_{-,2}^{-1} \circ h_{+,2} = h_{-,1}^{-1} \circ h_{+,1}.$$

Let $e \in H^1(T; \mathbb{Z})$ such that $h_{+,2} = h_{+,1} + e$, then $h_{-,2} = h_{-,1} + e$ also holds. This finishes the proof that ψ is one to one and onto.

The definition of the map ψ_0 and the verification that ψ_0 is one to one and onto are completely analogous. \square

We remark that Lemma 2.2 is analogous to Lemma 2.6 in Taubes [22]. However, the formulation of Lemma 2.6 in [22] is ambiguous when there are 2-torsions in the second cohomology. We thank Cliff Taubes for clarifying this issue for us.

With Lemma 2.2, the promised map $\phi : S_Y \rightarrow S_{Y_0}$ will be defined through a map $\phi' : \underline{S}_{Y_-} \rightarrow \underline{S}_{Y_{0,-}}$ that is $H^1(T; \mathbb{Z})$ -equivariant. In order to define ϕ' , we first take a closer look at the set of $Spin^c$ -structures on Y_- and $Y_{0,-}$ respectively.

As for $Y_- = D^2 \times \mathbb{S}^1 / \mathbb{Z}_\alpha$, an element of S_{Y_-} may be regarded as a \mathbb{Z}_α -equivariant $Spin^c$ -structure on $D^2 \times \mathbb{S}^1$. This said, there are α elements of S_{Y_-} , labelled by ξ_β^- , where $\beta = 0, 1, \dots, \alpha - 1$. It is more convenient to describe ξ_β^- in terms of the associated

\mathbb{Z}_α -equivariant spinor bundle S_β^- on $D^2 \times \mathbb{S}^1$. With this understood, for $\beta = 0$, S_0^- is the \mathbb{Z}_α -equivariant bundle on $D^2 \times \mathbb{S}^1$ given by $\mathbb{I} \oplus K_{D^2}^{-1}$, where \mathbb{I} is the trivial \mathbb{Z}_α -equivariant complex line bundle and K_{D^2} is the canonical bundle of D^2 which is regarded as a \mathbb{Z}_α -equivariant complex line bundle on $D^2 \times \mathbb{S}^1$. Moreover, for each $\beta \neq 0$, S_β^- is the \mathbb{Z}_α -equivariant bundle $S_0^- \otimes \underline{\mathbb{C}}_\beta$, where $\underline{\mathbb{C}}_\beta$ is the \mathbb{Z}_α -equivariant complex line bundle on $D^2 \times \mathbb{S}^1$ given by $D^2 \times \mathbb{S}^1 \times \mathbb{C}$ with the \mathbb{Z}_α -action

$$\lambda \cdot (z, x, w) = (\lambda z, x, \lambda^\beta w), \quad \lambda = \exp(2\pi i/\alpha), \quad (z, x) \in D^2 \times \mathbb{S}^1, w \in \mathbb{C}.$$

The set $S_{Y_{0,-}}$, where $Y_{0,-} = D^2 \times \mathbb{S}^1$, consists of only one element, denoted by $\xi^{0,-}$.

Lemma 2.3. *For $(\xi^-, h_-) \in S_{Y_-}^0$, let $c(\xi^-, h_-)$ denote the first Chern class of the determinant line bundle $\det \xi^-$ of ξ^- equipped with the trivialization given by the map $\det h_- : \det \xi^-|_T \rightarrow \det \xi_0$. (Here $c(\xi^-, h_-)$ is defined via Chern-Weil theory and lives in the relative cohomology group $H^2(|Y_-|, T; \mathbb{R})$.) Likewise, let $c(\xi^{0,-}, h_{0,-})$ be the first Chern class for $(\xi^{0,-}, h_{0,-}) \in S_{Y_{0,-}}^0$. Then*

- $\int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-})$ takes values in the set $\{2k + 1 | k \in \mathbb{Z}\}$.
- For any $(\xi^-, h_-) \in S_{Y_-}^0$, $\int_{D^2 \times \{pt\}} c(\xi^-, h_-)$ takes values in the set

$$\left\{ 2k + \frac{2\beta + 1}{\alpha} \mid k \in \mathbb{Z}, \beta = 0, 1, \dots, \alpha - 1 \right\}.$$

Proof. Consider $\int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-})$, for all $(\xi^{0,-}, h_{0,-}) \in S_{Y_{0,-}}^0$, first. Observe that $\xi^{0,-}$ admits a reduction to a *Spin*-structure on $Y_{0,-} = D^2 \times \mathbb{S}^1$. We fix such a *Spin*-structure η , and denote by $h : \xi^{0,-}|_T \rightarrow \eta|_T$ the corresponding isomorphism on T . Then η gives rise to a trivialization of $\det \xi^{0,-}$ on $Y_{0,-}$, with the trivialization of $\det \xi^{-,0}|_T$ given by $\det h : \det \xi^{0,-}|_T \rightarrow \det \eta|_T$. With this understood, $\int_{D^2 \times \{pt\}} c(\xi^{0,-}, h) = 0$, and

$$\int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-}) = \int_{D^2 \times \{pt\}} c(\xi^{0,-}, h) + \det h_{0,-} \circ \det h^{-1}([\partial D^2]),$$

where $\det h_{0,-} \circ \det h^{-1}$ is regarded as an element of $H^1(T; \mathbb{Z})$ and ∂D^2 is oriented as boundary of D^2 . Since $\eta|_T$ extends to η over $D^2 \times \mathbb{S}^1$, the difference between the *Spin*-structures η_T and ξ_0 is given by a class $\hat{\eta} \in H^1(T; \mathbb{Z}_2)$ which evaluates nontrivially on ∂D^2 . Consequently, $\det h_{0,-} \circ \det h^{-1}([\partial D^2]) = \hat{\eta}([\partial D^2]) = 1 \pmod{2}$, and $\int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-}) = 2k + 1$ for some $k \in \mathbb{Z}$.

Next we compute $\int_{D^2 \times \{pt\}} c(\xi^-, h_-)$, for all $(\xi^-, h_-) \in S_{Y_-}^0$. To this end, we introduce an oriented 3-orbifold $\Sigma \times \mathbb{S}^1$, where Σ is the 2-orbifold whose underlying 2-manifold is $|\Sigma| = \mathbb{S}^2$ and Σ has exactly one singular point of multiplicity α . It is easily seen that $\Sigma \times \mathbb{S}^1$ admits a decomposition as a union of Y_+ and Y_- along the common boundary T with $Y_+ = D^2 \times \mathbb{S}^1$.

There is a canonical *Spin*^c-structure ξ^0 on $\Sigma \times \mathbb{S}^1$ whose associated spinor bundle S^0 is given by $\mathbb{I} \oplus K_\Sigma^{-1}$. With this canonical *Spin*^c-structure ξ^0 , any *Spin*^c-structure ξ on $\Sigma \times \mathbb{S}^1$ corresponds to an orbifold complex line bundle E (always a pull-back from Σ)

such that the associated spinor bundle S is given by $S^0 \otimes E$. Let $(b, (\alpha, \beta))$, where $b \in \mathbb{Z}$ and $0 \leq \beta < \alpha$, be the Seifert invariant of E . Then

$$c_1(\det \xi)(|\Sigma|) = 2b + \frac{2\beta}{\alpha} + 2 + \left(\frac{1}{\alpha} - 1\right) = 2b + 1 + \frac{2\beta + 1}{\alpha}.$$

Now we apply the identification ψ in Lemma 2.2 to $Y = \Sigma \times \mathbb{S}^1$, and via ψ we identify ξ to a pair of elements (ξ^+, h_+) , (ξ^-, h_-) (note that $\xi^- = \xi_\beta^-$ by construction). Then

$$2b + 1 + \frac{2\beta + 1}{\alpha} = c_1(\det \xi)(|\Sigma|) = \int_{D^2 \times \{pt\}} c(\xi^+, h_+) + \int_{D^2 \times \{pt\}} c(\xi^-, h_-).$$

By the previous calculation, $\int_{D^2 \times \{pt\}} c(\xi^+, h_+) = 2s + 1$, $s \in \mathbb{Z}$. Thus

$$\int_{D^2 \times \{pt\}} c(\xi^-, h_-) \equiv \int_{D^2 \times \{pt\}} c(\xi_\beta^-, h_-) = 2k + \frac{2\beta + 1}{\alpha}, \quad k \in \mathbb{Z}.$$

This completes the proof of the lemma. \square

Note that the correspondences

$$(\xi^{0,-}, h_{0,-}) \mapsto \int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-}), \quad (\xi^-, h_-) \mapsto \int_{D^2 \times \{pt\}} c(\xi^-, h_-)$$

factor through the quotient sets $\underline{S}_{Y_{0,-}}$ and \underline{S}_{Y_-} . This gives definitions of the values

$$c([\xi^{0,-}, h_{0,-}]) \equiv \int_{D^2 \times \{pt\}} c(\xi^{0,-}, h_{0,-}), \quad c([\xi^-, h_-]) \equiv \int_{D^2 \times \{pt\}} c(\xi^-, h_-),$$

which allow us to distinguish elements of $\underline{S}_{Y_{0,-}}$ and \underline{S}_{Y_-} , because the group of orbifold complex line bundles on Y_- with a fixed trivialization on the boundary is torsion-free.

With the preceding understood, the definitions of the maps ϕ' and ϕ are given as follows.

Definition 2.4. (1) Recall that the elements of S_{Y_-} are labelled by ξ_β^- , $0 \leq \beta < \alpha$. There are two characteristics of ξ_β^- , any one of which uniquely determines ξ_β^- : (i) the spinor bundle of ξ_β^- is given by the \mathbb{Z}_α -equivariant bundle $(\mathbb{I} \oplus K_{D^2}^{-1}) \otimes \underline{\mathbb{C}}_\beta$ on $D^2 \times \mathbb{S}^1$, and (ii) (cf. the proof of Lemma 2.3)

$$c([\xi_\beta^-, h_-]) = 2k + \frac{2\beta + 1}{\alpha}, \quad k \in \mathbb{Z}.$$

With this understood, we define $\phi' : \underline{S}_{Y_-} \rightarrow \underline{S}_{Y_{0,-}}$ which is uniquely determined by the following condition: for any $[(\xi_\beta^-, h_-)] \in \underline{S}_{Y_-}$, $\phi'([\xi_\beta^-, h_-]) \in \underline{S}_{Y_{0,-}}$ such that if $c([\xi_\beta^-, h_-]) = 2k + \frac{2\beta + 1}{\alpha}$ for some $k \in \mathbb{Z}$ uniquely picked out by h_- , we require $\phi'([\xi_\beta^-, h_-])$ to satisfy

$$c(\phi'([\xi_\beta^-, h_-])) = 2k + 1.$$

The map ϕ' is clearly $H^1(T; \mathbb{Z})$ -equivariant.

(2) With the identifications ψ, ψ_0 in Lemma 2.2, the map $\phi : S_Y \rightarrow S_{Y_0}$ is defined to be the one induced by the $H^1(T; \mathbb{Z})$ -equivariant map from $\underline{S}_{Y_+} \times \underline{S}_{Y_-}$ to $\underline{S}_{Y_+} \times \underline{S}_{Y_{0,-}}$ given by

$$((\xi_+, h_+), [(\xi_-, h_-)]) \mapsto ((\xi_+, h_+), \phi'([(\xi_-, h_-)])).$$

Remark 2.5. We remark that although the definition of ϕ involves fixing a number of auxiliary data, such as an orientation of l and an orientation of T , as the identifications $Y_- = D^2 \times \mathbb{S}^1 / \mathbb{Z}_\alpha$ and $Y_{0,-} = D^2 \times \mathbb{S}^1$, the reduction of a $Spin^c$ -structure in a neighborhood of T to a $Spin^c$ -structure on T , as well as the definitions of $c([(\xi_0^-, h_{0,-})])$ and $c([(\xi^-, h_-)])$ all require it, one can easily verify that reversing the orientation of l or T gives rise to the same map ϕ . Hence ϕ is intrinsically defined. However, take note that the naming of ξ_β^- depends on the choice of orientation of l .

For any $\xi \in S_Y$, the determinant line bundle of ξ can be nicely related to the determinant line bundle of $\phi(\xi)$. Such a relation between $\det \xi$ and $\det \phi(\xi)$ is useful in computations.

In order to state the said relation between $\det \xi$ and $\det \phi(\xi)$, we need to introduce some notations. For each $k \in \mathbb{Z}$, we denote by E_k the orbifold complex line bundle on Y defined as follows. One first defines E_k on $Y_- = D^2 \times \mathbb{S}^1 / \mathbb{Z}_\alpha$ by the \mathbb{Z}_α -equivariant complex line bundle $\underline{\mathbb{C}}_k$ on $D^2 \times \mathbb{S}^1$, i.e., the one given by $D^2 \times \mathbb{S}^1 \times \mathbb{C}$ with the \mathbb{Z}_α -action

$$\lambda \cdot (z, x, w) = (\lambda z, x, \lambda^k w), \quad \lambda = \exp(2\pi i / \alpha), (z, x) \in D^2 \times \mathbb{S}^1, w \in \mathbb{C}.$$

This orbifold complex line bundle has a canonical trivialization on $\partial Y_- = \partial D^2 \times \mathbb{S}^1$ given by the \mathbb{Z}_α -equivariant nonzero section $(z, x) \mapsto z^k$. The bundle E_k is defined over the rest of Y by extending this trivialization. Observe two useful facts about E_k : (1) E_k descends to a complex line bundle on $|Y|$ if and only if k is divisible by α , in which case the descendant of E_k to $|Y|$ has c_1 given by $\frac{k}{\alpha} \cdot PD(l)$ in $H^2(|Y|; \mathbb{Z})$, and (2) when l is non-torsion, the first Chern class of E_k as an element of $H^2(|Y|; \mathbb{R})$ is given by $\frac{k}{\alpha} \cdot PD(l)$ (cf. [8], Lemma 3.6).

With the preceding understood, we have

Proposition 2.6. *Given any $\xi \in S_Y$, suppose $\xi|_{Y_-}$ is given by ξ_β^- for some $\beta = \beta_\xi$, $0 \leq \beta < \alpha$. Then*

$$\det \phi(\xi) = \det \xi \otimes E_{\alpha - 2\beta_\xi - 1}.$$

Proof. We use ψ in Lemma 2.2 to identify ξ with the orbit of $([(\xi^+, h_+)], [(\xi_\beta^-, h_-)])$. Then $\det \xi$ is given by $([(\det \xi^+, \det h_+)], [(\det \xi_\beta^-, \det h_-)])$, where $\det h_+ : \det \xi^+|_T \rightarrow \det \xi_0$ and $\det h_- : \det \xi_\beta^-|_T \rightarrow \det \xi_0$. Moreover, we normalize our choice by requiring that

$$c([(\xi_\beta^-, h_-)]) = \frac{2\beta + 1}{\alpha}.$$

With the preceding understood, recall that the spinor bundle of ξ_β^- is given by $(\mathbb{I} \oplus K_{D^2}^{-1}) \otimes \underline{\mathbb{C}}_\beta$. Thus $\det \xi_\beta^- = K_{D^2}^{-1} \otimes \underline{\mathbb{C}}_{2\beta}$. Noticing that $K_{D^2}^{-1} = \underline{\mathbb{C}}_1$, so that $\det \xi_\beta^- = \underline{\mathbb{C}}_{2\beta+1}$, we find that the trivialization of $\det \xi_\beta^-$ determined by the \mathbb{Z}_α -equivariant

non-zero section $(z, x) \mapsto z^{2\beta+1}$ gives the same evaluation of the relative first Chern class of $\det \xi_\beta^-$ on $D^2 \times \{pt\}$ as that determined by $\det h_-$ (cf. Lemma 3.6 of [8]). Hence the trivialization of $\det h_-$ may be identified with the trivialization given by the non-zero section $(z, x) \mapsto z^{2\beta+1}$.

With this understood, note that $\det \xi \otimes E_{\alpha-2\beta-1}$ is given by the pair

$$((\det \xi^+, \det h_+)), ((\det \xi_\beta^- \otimes E_{\alpha-2\beta-1}|_{Y_-}, h)),$$

where, with $\det \xi_\beta^- \otimes E_{\alpha-2\beta-1}|_{Y_-} = \underline{\mathbb{C}}_\alpha$, $h : (\det \xi_\beta^- \otimes E_{\alpha-2\beta-1}|_{Y_-})|_T \rightarrow \det \xi_0$ is given by the \mathbb{Z}_α -equivariant non-zero section $(z, x) \mapsto z^\alpha$, $(z, x) \in \partial D^2 \times \mathbb{S}^1 = T$. It follows that

$$\int_{D^2 \times \{pt\}} c_1(\det \xi_\beta^- \otimes E_{\alpha-2\beta-1}|_{Y_-}, h) = 1 = c(\phi'((\xi_\beta^-, h_-))),$$

which implies that $\det \phi(\xi) = \det \xi \otimes E_{\alpha-2\beta_\epsilon-1}$ as claimed. Hence the proposition. \square

Now we shall proceed to prove Theorem 1.1. Our strategy is to consider the 4-orbifolds $M = \mathbb{S}^1 \times Y$ and $M_0 = \mathbb{S}^1 \times Y_0$. By a theorem of Donaldson [10], the Seiberg-Witten invariant of $M = \mathbb{S}^1 \times Y$ for a $Spin^c$ -structure \mathcal{L} is non-zero only if \mathcal{L} is the pull-back of a $Spin^c$ -structure ξ on Y , and in this case one has

$$SW_M(\mathcal{L}) = SW_Y(\xi).$$

The same thing is true for M_0 and Y_0 . Thus, we shall consider the subset S_M of the set of $Spin^c$ -structures on M which are pull-backs of a $Spin^c$ -structure on Y , and likewise, consider the subset S_{M_0} of $Spin^c$ -structures on M_0 that are pull-backs of a $Spin^c$ -structure on Y_0 . Note that S_M, S_{M_0} may be canonically identified with S_Y, S_{Y_0} , so that there is correspondingly a surjective, α -to-one map, also denoted by ϕ , from S_M to S_{M_0} . Theorem 1.1 follows if one shows that

$$SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L})), \quad \forall \mathcal{L} \in S_M.$$

By setting $M_+ = \mathbb{S}^1 \times Y_+$, $M_- = \mathbb{S}^1 \times Y_-$, and $M_{0,-} = \mathbb{S}^1 \times Y_{0,-}$, and $N = \mathbb{S}^1 \times T$ which is a 3-torus, one similarly obtains decompositions of M as a union of M_+ with M_- along N , and M_0 as a union of M_+ with $M_{0,-}$ along N .

There are similarly defined sets \underline{S}_{M_+} , \underline{S}_{M_-} and $\underline{S}_{M_{0,-}}$, equipped with group actions by $H^1(N; \mathbb{Z})$, which are canonically identified with the sets \underline{S}_{Y_+} , \underline{S}_{Y_-} and $\underline{S}_{Y_{0,-}}$. The actions by $H^1(N; \mathbb{Z})$ and $H^1(T; \mathbb{Z})$ are compatible with the understanding that $H^1(T; \mathbb{Z})$ embeds in $H^1(N; \mathbb{Z})$ as a subgroup which is induced by the projection $N = \mathbb{S}^1 \times T \rightarrow T$. Moreover, there are identifications, still denoted by ψ, ψ_0 ,

$$\psi : \underline{S}_{M_+} \times \underline{S}_{M_-} / H^1(N; \mathbb{Z}) \rightarrow S_M \quad \text{and} \quad \psi_0 : \underline{S}_{M_+} \times \underline{S}_{M_{0,-}} / H^1(N; \mathbb{Z}) \rightarrow S_{M_0},$$

and a $H^1(N; \mathbb{Z})$ -equivariant map $\phi' : \underline{S}_{M_-} \rightarrow \underline{S}_{M_{0,-}}$, such that $\phi : S_M \rightarrow S_{M_0}$ is given by $Id \times \phi' / H^1(N; \mathbb{Z})$. We continue to use the notations introduced in the 3-dimensional setting for the corresponding objects in the 4-dimensional setting.

With the preceding understood, the equations

$$SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L})), \quad \forall \mathcal{L} \in S_M,$$

will be derived from the gluing theorems of Seiberg-Witten invariants along T^3 , as developed in Morgan, Mrowka and Szabó [18], as well as in Taubes [22]. This said, we actually need the extensions of the gluing theorems to 4-orbifolds. Since the singular sets of the 4-orbifolds are all lying in the complement of the gluing region, there are no essential complications in the analysis involved.

We remark that each of the gluing theorems has its own limitations. For instance, Taubes' gluing theorem requires that the 3-torus N be essential (in the sense of [22]), and Morgan-Mrowka-Szabó's gluing theorem needs the assumption that $b_2^+(M_+) > 0$. Thus the proof of Theorem 1.1 is given separately in two cases.

- (i) $b_2^+(M_+) > 0$, which means that $H_1(|Y_+|, T; \mathbb{Z}) \neq 0$. In the case of $b_1(Y) = 1$, this condition is equivalent to l being torsion in $H_1(|Y|; \mathbb{Z})$.
- (ii) N is essential, which is equivalent to l being non-torsion in $H_1(|Y|; \mathbb{Z})$.

Note that Case (i) and Case (ii) have overlaps, which means that for these cases, the gluing theorems of Morgan-Mrowka-Szabó and Taubes give independent proofs for Theorem 1.1.

3. Proof of Theorem 1.1: case (i)

Let $\mathcal{L} \in S_M$ be given. Via ψ in Lemma 2.2 we identify \mathcal{L} with a pair of elements (ξ^+, h_+) , (ξ_β^-, h_-) , where $\xi^+ = \mathcal{L}|_{M_+}$, $\xi_\beta^- = \mathcal{L}|_{M_-}$ for some $0 \leq \beta < \alpha$, and the maps $h_+ : \xi^+|_N \rightarrow \xi_0$, $h_- : \xi_\beta^-|_N \rightarrow \xi_0$ are isomorphisms. Recall that this identification means that \mathcal{L} is obtained by gluing ξ^+ with ξ_β^- along N via $h_-^{-1} \circ h_+$. We shall fix h_+ throughout, and in doing so, h_- is uniquely determined up to the actions by $H^1(|M_+|; \mathbb{Z})$ and $H^1(|M_-|; \mathbb{Z})$. Let $\Lambda = \{h_{-,i}\}$ be a set of elements of such h_- 's each of which represents an orbit of the $H^1(|M_-|; \mathbb{Z})$ -action. Furthermore, we shall normalize our choice of h_+ by requiring that there is a $h_{-,0} \in \Lambda$ such that $c([\xi_\beta^-, h_{-,0}]) = (2\beta + 1)/\alpha$.

The set Λ has the following properties. Let e_1 be the vector in $H^1(N; \mathbb{Z})$ which is the Poincaré dual of $\{pt\} \times T^2$ in N . Then if l is torsion of order m , the image $j_+ H^1(|M_+|; \mathbb{Z}) \subset H^1(N; \mathbb{Z})$ is generated by me_1 . If l is non-torsion, then no non-zero multiples of e_1 is contained in $j_+ H^1(|M_+|; \mathbb{Z})$. With this understood, in the former case we have $c([\xi_\beta^-, h_-]) = (2\beta + 1)/\alpha \pmod{2m}$, $\forall h_- \in \Lambda$, and in the latter case, Λ consists of a single element h_- with $c([\xi_\beta^-, h_-]) = (2\beta + 1)/\alpha$.

Analogously, via ψ_0 in Lemma 2.2 we identify $\phi(\mathcal{L})$ with a pair of elements (ξ^+, h_+) , and $(\xi^{0,-}, h_{0,-})$, where (ξ^+, h_+) is the one chosen in the previous paragraph, and the map $h_{0,-} : \xi^{0,-}|_N \rightarrow \xi_0$ is an isomorphism chosen out of a set Λ_0 , which is given by the image of the set Λ under the map ϕ' . When l is torsion of order m , we see that $c([\xi^{0,-}, h_{0,-}]) = 1 \pmod{2m}$, $\forall h_{0,-} \in \Lambda_0$, and if l is non-torsion, Λ_0 consists of a single element $h_{0,-}$ with $c([\xi^{0,-}, h_{0,-}]) = 1$.

Finally, the Dirac operator D associated to the *Spin*-structure ξ_0 has $\ker D = \mathbb{C}^2$, which consists of covariantly constant sections. This gives rise to a trivialization of ξ_0 . Viewing differently, if we regard ξ_0 as a *Spin^c*-structure, then it follows that there is a

special flat $U(1)$ -connection θ_0 on $\det \xi_0$, such that the associated Dirac operator D_{θ_0} has $\ker D_{\theta_0} = \mathbb{C}^2$. This gives rise to an identification of the space of equivalence classes of flat $U(1)$ -connections on $\det \xi_0$ modulo homotopically trivial gauge transformations with the space $H^1(N; \mathbb{R})$ via harmonic 1-forms, under which θ_0 is identified with the origin $0 \in H^1(N; \mathbb{R})$. See [18] for more details.

With the preceding understood, we now review the gluing theorem of Seiberg-Witten invariant of Morgan, Mrowka and Szabó in [18] (adapted to the present orbifold context). The key component of this analysis is the structure of L^2 -moduli spaces of Seiberg-Witten equations on 4-manifolds (or more generally 4-orbifolds) with cylindrical ends $T^3 \times [0, \infty)$. We consider M_+ first, which is an oriented 4-orbifold with boundary $N = T^3$. We endow M_+ with a Riemannian metric g that is flat near N and attach $N \times [0, \infty)$ to it extending the metric g naturally. Call the resulting cylindrical-end Riemannian 4-orbifold (\hat{M}_+, \hat{g}) .

With $\mathcal{L} \in S_M$ given, $\xi^+ \in S_{M_+}$ being the restriction of \mathcal{L} to M_+ , we extend ξ^+ naturally to a $Spin^c$ -structure on \hat{M}_+ . With this understood, one considers the L^2 -moduli space of Seiberg-Witten equations

$$\begin{cases} D_A \psi &= 0 \\ P_+ F_A &= \tau(\psi \otimes \psi^*) - i\mu \end{cases}$$

where μ is a compactly supported real-valued smooth self-dual 2-form. The L^2 -moduli space, denoted by $\mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu)$, is the space of (A, ψ) modulo gauge transformations by the group $C^\infty(\hat{M}_+; \mathbb{S}^1)$, where (A, ψ) satisfies the finite energy condition

$$\int_{\hat{M}_+} |F_A|^2 dvol < \infty.$$

Let $\chi_0(N)$ be the space of flat $U(1)$ -connections on $\det \xi^+|_N$ modulo gauge transformations by those $\varphi \in C^\infty(N; \mathbb{S}^1)$ where φ is the restriction of an element of $C^\infty(\hat{M}_+; \mathbb{S}^1)$. There is a continuous map $\partial_\infty : \mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu) \rightarrow \chi_0(N)$ which sends the gauge equivalence class of (A, ψ) to the class of the asymptotic value of $A|_{N \times \{t\}}$ as $t \rightarrow \infty$. With the isomorphism $h_+ : \xi^+|_N \rightarrow \xi_0$ chosen, $\chi_0(N)$ is identified with $H^1(N; \mathbb{R})/j_+ 2H^1(|M_+|; \mathbb{Z})$ through $\det h_+ : \det \xi^+|_N \rightarrow \det \xi_0$, as the corresponding space of flat $U(1)$ -connections on $\det \xi_0$ is identified with $H^1(N; \mathbb{R})/j_+ 2H^1(|M_+|; \mathbb{Z})$ via harmonic 1-forms. As h_+ is being fixed, we shall regard ∂_∞ as a map from $\mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu)$ into $H^1(N; \mathbb{R})/j_+ 2H^1(|M_+|; \mathbb{Z})$.

With the preceding understood, the structure theorem of Morgan, Mrowka and Szabó (cf. [18], Theorem 2.8) asserts that for a generic choice of μ , the L^2 -moduli space $\mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu)$ is a compact 1-dimensional manifold with boundary, such that the image of $\partial \mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu)$ under the map ∂_∞ lies in the lattice of points of even integer coordinates in $H^1(N; \mathbb{R})/j_+ 2H^1(|M_+|; \mathbb{Z})$. We remark that this assertion requires the condition that $b_2^+(M_+) > 0$, which is assumed to be true. Furthermore, we used the fact that the dimension of the moduli space $\mathcal{M}_{\mathcal{L}}$ of Seiberg-Witten equations associated to the $Spin^c$ -structure \mathcal{L} is 0. (This follows easily from the fact that \mathcal{L} is the pull-back of

a $Spin^c$ -structure on Y .) Indeed, according to [18], the dimension of $\mathcal{M}_{\hat{M}_+}(\xi^+, \hat{g}, \mu)$ is given by $\dim \mathcal{M}_{\mathcal{L}} + 1$ (cf. Theorem 2.4 in [18]), which equals 1 since $\dim \mathcal{M}_{\mathcal{L}} = 0$.

Similarly, we consider the L^2 -moduli space on \hat{M}_- , $\mathcal{M}_{\hat{M}_-}(\xi_\beta^-, \hat{g}, \mu)$, where $\xi_\beta^- \in S_{M_-}$ is the restriction of \mathcal{L} on M_- . Since $M_- = D^2 \times T^2 / \mathbb{Z}_\alpha$, we can choose a metric \hat{g} which has non-negative, somewhere positive scalar curvature. Taking $\mu = 0$, it follows that $\mathcal{M}_{\hat{M}_-}(\xi_\beta^-, \hat{g}, \mu)$ consists entirely of reducible solutions, i.e., $(A, 0)$ where A is flat. For the purpose of gluing we shall in fact consider the based version $\mathcal{M}_{\hat{M}_-}^0(\xi_\beta^-, \hat{g}, \mu)$, which is the space of finite energy solutions modulo homotopically trivial gauge transformations. Thus $\mathcal{M}_{\hat{M}_-}^0(\xi_\beta^-, \hat{g}, \mu)$ is given by the set of $[A]$'s, where $[A]$ is the equivalence class of flat $U(1)$ -connections on $\det \xi_\beta^-$ modulo homotopically trivial gauge transformations. Now for each $h_- \in \Lambda$, the isomorphism $\det h_- : \det \xi_\beta^-|_N \rightarrow \det \xi_0$ embeds $\mathcal{M}_{\hat{M}_-}^0(\xi_\beta^-, \hat{g}, \mu)$ into $H^1(N; \mathbb{R})$ as follows: $[A] \in \mathcal{M}_{\hat{M}_-}^0(\xi_\beta^-, \hat{g}, \mu)$ is sent to the image of $[A]|_N$ in $H^1(N; \mathbb{R})$ via the map induced by $\det h_-$. For each $h_- \in \Lambda$, we denote by $\Omega_\beta(h_-)$ the corresponding image in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$.

The L^2 -moduli space on $\hat{M}_{0,-}$ has a similar structure. More precisely, we consider $\mathcal{M}_{\hat{M}_{0,-}}(\xi^{0,-}, \hat{g}, \mu)$, the L^2 -moduli space on $\hat{M}_{0,-}$. Again, since $M_{0,-} = D^2 \times T^2$, we can choose a metric \hat{g} which has non-negative, somewhere positive scalar curvature. Taking $\mu = 0$, it follows that $\mathcal{M}_{\hat{M}_{0,-}}(\xi^{0,-}, \hat{g}, \mu)$ consists entirely of reducible solutions. Let $\mathcal{M}_{\hat{M}_{0,-}}^0(\xi^{0,-}, \hat{g}, \mu)$ be the based version. Then for each $h_{0,-} \in \Lambda_0$, the isomorphism $\det h_{0,-}$ embeds $\mathcal{M}_{\hat{M}_{0,-}}^0(\xi^{0,-}, \hat{g}, \mu)$ into $H^1(N; \mathbb{R})$. Denote by $\Omega_{0,-}(h_{0,-})$ the corresponding image in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$, $\forall h_{0,-} \in \Lambda_0$.

Lemma 3.1. *Regard e_1 as a coordinate function on $H^1(N; \mathbb{Z})/j_+2H^1(|M_+|; \mathbb{Z})$, which takes values only in $\mathbb{R}/2m\mathbb{Z}$ when l is torsion of order m . Then for any $h_- \in \Lambda$, $h_{0,-} \in \Lambda_0$, the subsets $\Omega_\beta(h_-)$, $\Omega_{0,-}(h_{0,-})$ are 2-dimensional subspaces defined by $e_1 = -(2\beta + 1)/\alpha$ and $e_1 = -1$ respectively. Consequently, $\Omega_\beta(h_-) \equiv \Omega_\beta$, $\Omega_{0,-}(h_{0,-}) \equiv \Omega_{0,-}$ are independent of h_- , $h_{0,-}$ respectively, and both Ω_β and $\Omega_{0,-}$ miss the lattice of even integer coordinates in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$. Moreover, Ω_β is isotopic to $\Omega_{0,-}$ in the complement of the lattice of even integer coordinates in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$.*

Proof. First, consider the embedding of $\mathcal{M}_{\hat{M}_{0,-}}^0(\xi^{0,-}, \hat{g}, \mu)$ into $H^1(N; \mathbb{R})$ via $\det h_{0,-}$ for any given $h_{0,-} \in \Lambda_0$. Let $[A] \in \mathcal{M}_{\hat{M}_{0,-}}^0(\xi^{0,-}, \hat{g}, \mu)$ be any element. Note that the holonomy of $[A]|_N$ is zero around the loop ∂D^2 . By modifying A with an imaginary valued harmonic 1-form which vanishes on ∂D^2 , we can assume that $[A]|_N$ has integral holonomy, thus defining a trivialization h of $\det \xi^{0,-}|_N$. Note that this modification does not change the e_1 -coordinate of the image of $[A]$ in $H^1(N; \mathbb{R})$ since the harmonic 1-form used vanishes on ∂D^2 . With respect to the trivialization h , the relative first Chern class of $\det \xi^{0,-}$ is zero. It follows then that

$$c([\xi^{0,-}, h_{0,-}]) = -h \circ h_{0,-}^{-1}([\partial D^2]) = -e_1([A]),$$

where $e_1([A])$ stands for the e_1 -coordinate of the image of $[A]$ in $H^1(N; \mathbb{R})$. This shows that the e_1 -coordinate on $\Omega_{0,-}(h_{0,-})$, taking values in $\mathbb{R}/2m\mathbb{Z}$ when l is torsion of order m , is constant equal to -1 . Consequently, $\Omega_{0,-}(h_{0,-}) \equiv \Omega_{0,-}$ is independent of $h_{0,-} \in \Lambda_0$. It is the 2-dimensional subspace defined by $e_1 = -1$ because $\mathcal{M}_{M_{0,-}}^0(\xi^{0,-}, \hat{g}, \mu)$ is a 2-dimensional space. Clearly, $\Omega_{0,-}$ misses the lattice of even integer coordinates in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$ since $e_1 = -1$ on $\Omega_{0,-}$.

The case of $\Omega_\beta(h_-)$ is similar. For any $[A] \in \mathcal{M}_{M_-}(\xi_\beta^-, \hat{g}, \mu)$, we regard it as a flat $U(1)$ -connection on $D^2 \times T^2$, where $M_- = D^2 \times T^2/\mathbb{Z}_\alpha$. Let $\hat{N} = \partial(D^2 \times T^2)$ which covers N , and let \hat{e}_1 be the corresponding coordinate on $H^1(\hat{N}; \mathbb{R})$. Then $\hat{e}_1 = \alpha \cdot e_1$. With this understood, one obtains as argued in the previous paragraph that

$$\alpha \cdot c([\xi_\beta^-, h_-]) = -\hat{e}_1([A]).$$

Hence $e_1([A]) = -c([\xi_\beta^-, h_-]) = -(2\beta+1)/\alpha$. Analogously, $\Omega_\beta(h_-) \equiv \Omega_\beta$ is independent of $h_- \in \Lambda$, and is the 2-dimensional subspace defined by $e_1 = -(2\beta+1)/\alpha$. Since for $0 \leq \beta < \alpha$, $e_1 = -(2\beta+1)/\alpha$ takes no even integer values, Ω_β also misses the lattice of even integer coordinates in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$.

Finally, each Ω_β is isotopic to $\Omega_{0,-}$ in the complement of the lattice of even integer coordinates because $e_1 = -(2\beta+1)/\alpha$ lies in the interval $[-2 + \frac{1}{\alpha}, -\frac{1}{\alpha}]$, which contains no even integers and contains -1 , the value of the e_1 -coordinate of $\Omega_{0,-}$. □

With the preceding understood, the boundary map

$$\partial_\infty : \mathcal{M}_{M_+}(\xi^+, \hat{g}, \mu) \rightarrow H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$$

can be made transverse to each subspace $\Omega_\beta, \Omega_{0,-}$ by taking a generic choice of μ . Then the gluing theorem of Morgan, Mrowka and Szabó asserts that

$$SW_M(\mathcal{L}) = \#(\mathcal{M}_{M_+}(\xi^+, \hat{g}, \mu) \cap \partial_\infty^{-1}(\Omega_\beta))$$

and

$$SW_{M_0}(\phi(\mathcal{L})) = \#(\mathcal{M}_{M_+}(\xi^+, \hat{g}, \mu) \cap \partial_\infty^{-1}(\Omega_{0,-})),$$

where the right-hand sides stand for the algebraic intersections of $\mathcal{M}_{M_+}(\xi^+, \hat{g}, \mu)$ with $\partial_\infty^{-1}(\Omega_\beta)$ and $\partial_\infty^{-1}(\Omega_{0,-})$ respectively. The assertion $SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L}))$ follows from the fact that Ω_β is isotopic to $\Omega_{0,-}$ in the complement of the lattice of even integer coordinates in $H^1(N; \mathbb{R})/j_+2H^1(|M_+|; \mathbb{Z})$.

4. Proof of Theorem 1.1: case (ii)

The 3-torus N being essential in M (resp. M_0) means that there is a class ϖ in $H^2(|M|; \mathbb{R})$ whose restriction to $H^2(N; \mathbb{R})$ is non-trivial, which is easily seen to be equivalent to l being non-torsion in $H_1(|Y|; \mathbb{Z})$. The existence of such a ϖ is what is required in the setup of Taubes' gluing theorem in [22]. This said, we shall fix a covariantly constant 2-form ω_0 on N (with respect to a fixed flat metric) which represents the restriction of ϖ in $H^2(N; \mathbb{R})$.

With the preceding understood, let $\mathcal{L} \in S_M$ be given. We shall endow M with a Riemannian metric g such that in a regular neighborhood of N , $g = ds^2 + g_N$, where s is the normal coordinate in the regular neighborhood and g_N is a fixed flat metric on N . The Seiberg-Witten equations considered in this context take the following form

$$\begin{cases} D_A \psi &= 0 \\ P_+ F_A &= \tau(\psi \otimes \psi^*) - ir\omega \end{cases}$$

where ω is a self-dual 2-form and $r > 0$ is a fixed, sufficiently large constant. Moreover, in a regular neighborhood of N , $\omega = ds \wedge \theta + \omega_0$ where θ is the metric dual to ω_0 .

The idea of gluing is to analyze the effect on the moduli space of Seiberg-Witten equations when the cylindrical neck neighborhood of N gets metrically longer and longer. This requires knowledge about the corresponding L^2 -moduli spaces over the cylindrical-end orbifolds \hat{M}_+ , \hat{M}_- respectively.

Consider \hat{M}_+ first. Given any $\xi^+ \in S_{M_+}$, we let $\mathcal{M}(\xi^+)$ denote the space of solutions (A, ψ) of the above form of the Seiberg-Witten equations modulo gauge transformations, where (A, ψ) satisfies the following finite energy condition

$$\int_{\hat{M}_+} |F_A|^2 dvol < \infty.$$

(Here on the cylindrical end of \hat{M}_+ , one assumes that $\omega = ds \wedge \theta + \omega_0$.)

According to [22], Lemma 3.5, for each gauge equivalence class $[(A, \psi)] \in \mathcal{M}(\xi^+)$, the restriction of (A, ψ) to the slice $\{s\} \times N$ on the cylindrical end converges, up to a gauge transformation, exponentially fast to a (A_0, ψ_0) as $s \rightarrow \infty$, where A_0 is a trivial connection on $\det(\xi^+|_N)$ and ψ_0 is a non-zero, covariantly constant section of the spinor bundle on N . This said, each (A, ψ) determines a trivialization of $\xi^+|_N$, which is given by a homotopy class of isomorphisms from $\xi^+|_N$ to ξ_0 . Consequently, each gauge equivalence class $[(A, \psi)]$ determines an element $[(\xi^+, h_+)] \in \underline{S}_{M_+}$, and accordingly, there is a decomposition

$$\mathcal{M}(\xi^+) = \bigcup_{[(\xi^+, h_+)] \in \underline{S}_{M_+}} \mathcal{M}([(A, \psi)]).$$

Furthermore, for any constant C , there are only finitely many $[(\xi^+, h_+)]$ such that the moduli space $\mathcal{M}([(A, \psi)])$ is nonempty and

$$\int_{|M_+|} c(\xi^+, h_+) \cup \varpi \leq C.$$

Finally, each $\mathcal{M}([(A, \psi)])$ can be used to define a Seiberg-Witten invariant, which we will denote by $SW([(A, \psi)]) \in \mathbb{Z}$.

Similar discussions apply to \hat{M}_- , $\hat{M}_{0,-}$ as well. More precisely, given any $\xi^- \in S_{M_-}$, we have the corresponding L^2 -moduli space $\mathcal{M}(\xi^-)$ and a decomposition

$$\mathcal{M}(\xi^-) = \bigcup_{[(\xi^-, h_-)] \in \underline{S}_{M_-}} \mathcal{M}([(A, \psi)]),$$

and for $\xi^{0,-} \in S_{M_{0,-}}$, we have

$$\mathcal{M}(\xi^{0,-}) = \bigcup_{[(\xi^{0,-}, h_{0,-})] \in \underline{S}_{M_{0,-}}} \mathcal{M}([\xi^{0,-}, h_{0,-}]).$$

Finally, Seiberg-Witten invariants $SW([\xi_{\beta}^-, h_-]) \in \mathbb{Z}$, $SW([\xi^{0,-}, h_{0,-}]) \in \mathbb{Z}$ are defined using the moduli spaces $\mathcal{M}([\xi_{\beta}^-, h_-])$ and $\mathcal{M}([\xi^{0,-}, h_{0,-}])$ respectively.

With the preceding understood, Taubes' gluing theorem (cf. [22], Theorem 2.7) asserts that

$$SW_M(\mathcal{L}) = \sum_{([\xi^+, h_+], [\xi_{\beta}^-, h_-]) \in \psi^{-1}(\mathcal{L})} SW([\xi^+, h^+]) \cdot SW([\xi_{\beta}^-, h^-])$$

and

$$SW_M(\phi(\mathcal{L})) = \sum_{([\xi^+, h_+], [\xi^{0,-}, h_{0,-}]) \in \psi_0^{-1}(\phi(\mathcal{L}))} SW([\xi^+, h^+]) \cdot SW([\xi^{0,-}, h_{0,-}]),$$

where ψ , ψ_0 are the maps in Lemma 2.2.

By the definition of the map ϕ , it is clear that $([\xi^+, h_+], [\xi^{0,-}, h_{0,-}]) \in \psi_0^{-1}(\phi(\mathcal{L}))$ if and only if $[\xi^{0,-}, h_{0,-}] = \phi'([\xi_{\beta}^-, h_-])$ and $([\xi^+, h_+], [\xi_{\beta}^-, h_-]) \in \psi^{-1}(\mathcal{L})$. Thus the assertion $SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L}))$ follows immediately from the following lemma.

Lemma 4.1. *For any $[\xi_{\beta}^-, h_-] \in \underline{S}_{M_-}$, $SW([\xi_{\beta}^-, h_-]) = SW(\phi'([\xi_{\beta}^-, h_-]))$.*

Proof. According to Taubes [22], the Seiberg-Witten invariant of $M_{0,-} = D^2 \times T^2$ is given by the polynomial

$$t(1 - t^2)^{-1} = t + t^3 + t^5 + \dots$$

This can be equivalently stated as

$$SW([\xi^{0,-}, h_{0,-}]) = \begin{cases} 1 & \text{if } c([\xi^{0,-}, h_{0,-}]) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the lemma follows by showing that for any $[\xi_{\beta}^-, h_-] \in \underline{S}_{M_-}$,

$$SW([\xi_{\beta}^-, h_-]) = \begin{cases} 1 & \text{if } c([\xi_{\beta}^-, h_-]) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

because by the definition of the map ϕ' ,

$$c([\xi_{\beta}^-, h_-]) > 0 \text{ if and only if } c(\phi'([\xi_{\beta}^-, h_-])) > 0.$$

The strategy of the proof is based on the following observation: by Taubes' gluing theorem in [22], Lemma 4.1 must be true as long as $SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L}))$ holds for any one particular example of 4-orbifold M which has nonzero Seiberg-Witten invariant. For example, consider $M = T^2 \times \Sigma$ where Σ is the 2-torus with one singular point of multiplicity α . The corresponding orbifold M_0 is simply the 4-torus which has nonzero Seiberg-Witten invariant. For this 4-orbifold, the relation $SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L}))$ has been established in the previous section using the gluing theorem of Morgan, Mrowka and Szabó, which would then imply Lemma 4.1.

For a somewhat independent proof of the lemma, we can take $M = T^2 \times \Sigma$ where Σ is the 2-sphere with one singular point of multiplicity α . In this case, the claimed relation $SW_M(\mathcal{L}) = SW_{M_0}(\phi(\mathcal{L}))$ can be obtained using the wall-crossing formula in Li-Liu [14].

More concretely, since M has positive scalar curvature, the Seiberg-Witten invariant of M is given by the wall-crossing number, which equals (cf. Lemma 2.6 in [14])

$$\int_{T^{b_1}} c_{\frac{b_1}{2}}(V^+ - V_-).$$

Here $T^{b_1} = H^1(|M|; \mathbb{R})/H^1(|M|; \mathbb{Z})$ is the torus parametrizing the gauge equivalence classes of reducible solutions, and $V_+ - V_-$ is the index bundle of the family of Dirac operators parametrized by T^{b_1} . The key observation is that in this case M is Kähler, so that the Dirac operators are given by the d -bar operators (cf. [17]). The kernel and co-kernel of the d -bar operators, which are given by holomorphic sections of orbifold bundles over an orbifold with co-dimension 2 singularities, can be identified with the kernel and co-kernel of the corresponding d -bar operators for the de-singularization of the orbifold bundles over the de-singularization of the orbifold.

With this understood, let \mathcal{L} be a $Spin^c$ -structure on M whose spinor bundle is given by $(\mathbb{I} \oplus K_{\Sigma}^{-1}) \otimes E$, where E is an orbifold complex line bundle over Σ with Seifert invariant $(b, (\alpha, \beta))$, $0 \leq \beta < \alpha$. In this case, the de-singularization of E is the complex line bundle E_0 over \mathbb{S}^2 (the de-singularization of Σ) which has degree b . Thus the Seiberg-Witten invariant $SW_M(\mathcal{L})$, given by the wall-crossing number, is equal to the Seiberg-Witten invariant of M_0 associated to the $Spin^c$ -structure whose spinor bundle is $(\mathbb{I} \oplus K_{\mathbb{S}^2}^{-1}) \otimes E_0$ (which is given by the same wall-crossing number). Note that this $Spin^c$ -structure on M_0 is exactly $\phi(\mathcal{L})$. Finally, the Seiberg-Witten invariant of $M_0 = T^2 \times \mathbb{S}^2$ is nonzero, from which Lemma 4.1 follows. \square

5. Examples: Seifert 3-manifolds

Let $Y = \mathbb{S}^1 \times \Sigma$ where Σ is a Riemann surface of genus g . Then the Seiberg-Witten invariant of Y is given by the following polynomial

$$SW_Y(t) = (t^{-1} - t)^{2g-2}, \quad \text{where } t = PD(\mathbb{S}^1 \times \{pt\}).$$

In the polynomial $SW_Y(t)$, the coefficient of t^k is the Seiberg-Witten invariant $SW_Y(\xi)$ where the $Spin^c$ -structure ξ satisfies $c_1(\det \xi) = kt$. This formula can be obtained independently in two different ways. The first method is to identify SW_Y with the Seiberg-Witten invariant of the Kähler surface $T^2 \times \Sigma$ and then appeal to the formula in Brussee [7] or Friedman-Morgan [13]. The second method is to use Taubes' gluing theorem in [22] to run an induction on the genus g . The latter approach requires initial values for the cases where $g = 0$ or 1 , i.e., the Seiberg-Witten invariant of $T^2 \times \mathbb{S}^2$ or T^4 . The Seiberg-Witten invariant of $T^2 \times \mathbb{S}^2$ can be computed using the wall-crossing formula in [14], and the Seiberg-Witten invariant of T^4 follows from Taubes' work [21].

For our purpose here, we shall reinterpret the formula for $SW_Y(t)$ as follows. Note that Y has a canonical $Spin^c$ -structure ξ_0 , whose associated spinor bundle S_0 is given by

$\mathbb{I} \oplus K_{\Sigma}^{-1}$. Any relevant $Spin^c$ -structure ξ (i.e., with nonzero Seiberg-Witten invariant) may be identified with a complex line bundle D over Σ such that the associated spinor bundle of ξ is given by $D \oplus K_{\Sigma}^{-1} \otimes D$. With this understood, note that $c_1(\det \xi) = (2d - 2g + 2)t$ where d is the degree of D , and we can reinterpret the formula for $SW_Y(t)$ as

$$SW_Y(\xi) \equiv SW_Y(D) = \begin{cases} (-1)^d \binom{2g-2}{d} & \text{if } g \geq 1, d \in [0, 2g-2], \\ d+1 & \text{if } g = 0, d \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a straightforward application of Theorem 1.1, we shall compute the Seiberg-Witten invariant of the 3-orbifold $Y = \mathbb{S}^1 \times \Sigma$, where Σ is the 2-orbifold whose underlying surface $|\Sigma|$ has genus g and whose singular set consists of m points z_1, \dots, z_m with the associated multiplicities $\alpha_1, \dots, \alpha_m$. To this end, note that Y has a canonical $Spin^c$ -structure ξ_0 whose associated spinor bundle is $S_0 = \mathbb{I} \oplus K_{\Sigma}^{-1}$. Any $Spin^c$ -structure ξ with nonzero Seiberg-Witten invariant can be identified with an orbifold complex line bundle D over Σ in the sense that the associated spinor bundle of ξ is $S = D \oplus K_{\Sigma}^{-1} \otimes D$. Now observe that if a $Spin^c$ -structure ξ on Y is given by D over Σ , then the $Spin^c$ -structure $\phi(\xi)$ on $|Y| = \mathbb{S}^1 \times |\Sigma|$ is given by $|D|$ over $|\Sigma|$, where, if D is given by the Seifert invariant $(d, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$, with $0 \leq \beta_i < \alpha_i$, the complex line bundle $|D|$ over $|\Sigma|$ has degree d . With the preceding understood, Theorem 1.1 implies that

$$SW_Y(\xi) \equiv SW_Y(D) = SW_{|Y|}(|D|) = \begin{cases} (-1)^d \binom{2g-2}{d} & \text{if } g \geq 1, d \in [0, 2g-2], \\ d+1 & \text{if } g = 0, d \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Having been able to compute the Seiberg-Witten invariant of the 3-orbifolds $\mathbb{S}^1 \times \Sigma$, we shall next determine the Seiberg-Witten invariant of Seifert 3-manifolds Y , where $\pi : Y \rightarrow \Sigma$ is the unit sphere bundle of an orbifold complex line bundle E over Σ . We assume the Seifert invariant of E is $(e, (\alpha_1, e_1), \dots, (\alpha_m, e_m))$, where $0 < e_i < \alpha_i$, and $\gcd(e_i, \alpha_i) = 1$. We remark that if $b_1(Y) \geq 1$, then the Euler class of E must be torsion when the genus of $|\Sigma|$ is zero, i.e., $g = 0$.

The Seiberg-Witten invariant of Y can be identified with the Seiberg-Witten invariant of the 4-manifold $\mathbb{S}^1 \times Y$, which Seifert fibers over the 3-orbifold $\mathbb{S}^1 \times \Sigma$. The work of Baldrige [2] then allows us to compute the Seiberg-Witten invariant of $\mathbb{S}^1 \times Y$ in terms of that of $\mathbb{S}^1 \times \Sigma$. More precisely, according to [2], the relevant $Spin^c$ -structures on $\mathbb{S}^1 \times Y$ are pull-backs of $Spin^c$ -structures on $\mathbb{S}^1 \times \Sigma$. With this said, we shall let ξ_0 denote the $Spin^c$ -structure on $\mathbb{S}^1 \times Y$ which is the pull-back of the canonical $Spin^c$ -structure on $\mathbb{S}^1 \times \Sigma$ whose associated spinor bundle is given by $\mathbb{I} \oplus K_{\Sigma}^{-1}$. Then any relevant $Spin^c$ -structure ξ may be identified with a complex line bundle π^*D , where D is an orbifold complex line bundle over Σ . Note that the set of complex line bundles π^*D may be identified with the set of equivalence classes $[D]$, where $[D] = [D']$ if and only if $D - D' \equiv 0 \pmod{E}$.

With the preceding understood, work of Baldrige (cf. [2], Theorem C) implies that the Seiberg-Witten invariant of Y is given by the following formula

$$SW_Y(\xi) \equiv SW_Y([D]) = \sum_{D' \in [D]} SW_{\mathbb{S}^1 \times \Sigma}(D').$$

(cf. Mrowka, Ozsváth and Yu [19]; the special case of the above formula where Σ is non-singular has been discussed in [1].) It is useful to observe that if a $Spin^c$ -structure ξ corresponds to $[D]$, then $c_1(\det \xi) = \pi^*(c_1(2D - K_\Sigma))$. In terms of Poincaré duality, if D is given by the Seifert invariant $(d, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m))$ and if F, F_i denote the regular fiber class and the class of the exceptional fiber at z_i respectively, then

$$c_1(\det \xi) = (2d - 2g + 2)F + \sum_i (2\beta_i + 1 - \alpha_i)F_i.$$

We shall illustrate the above formula for Seifert 3-manifolds with the following three examples, which occupy the rest of this section.

Example 5.1. Let Y be the Seifert 3-manifold, where Σ is the 2-orbifold with $g = 5$, $\alpha_1 = 3$, $\alpha_2 = 5$, $\alpha_3 = 7$, and E has Seifert invariant $(1, (3, 2), (5, 3), (7, 5))$. Note that $H^2(Y; \mathbb{Z})$ has no 2-torsions, so that each $Spin^c$ -structure is uniquely determined by its determinant line bundle. With this understood, we shall compute the Seiberg-Witten invariant for the $Spin^c$ -structure ξ where

$$c_1(\det \xi) = -4F - 2F_2 - 4F_3.$$

It is easily seen that ξ corresponds to the equivalence class $[D]$ where D is given by the Seifert invariant $(2, (3, 1), (5, 1), (7, 1))$.

In order to determine the D' 's which belong to $[D]$, we first examine the Seifert invariants of kE , $k \in \mathbb{Z}$. Denote the Seifert invariant of kE by $(f_k, (3, a_k), (5, b_k), (7, c_k))$. It is easy to check that if $k < 0$, then $f_k \leq -4$, and if $k > 2$, then $f_k > 5$, and for $k = 2$, the Seifert invariant of kE is $(5, (3, 1), (5, 1), (7, 3))$. From this analysis, it follows that the only elements of $[D]$ which contribute nontrivially to the Seiberg-Witten invariant are $D_0 = D = (2, (3, 1), (5, 1), (7, 1))$, $D_1 = (4, (3, 0), (5, 4), (7, 6))$ and $D_2 = (7, (3, 2), (5, 2), (7, 4))$. Consequently,

$$SW_Y(\xi) = SW_Y([D]) = \sum_{i=0}^2 SW_{\mathbb{S}^1 \times \Sigma}(D_i) = \sum_{d=2,4,7} (-1)^d \binom{8}{d} = 90.$$

Example 5.2. In this example we consider $Y = \mathbb{S}^1 \times \mathbb{S}^2$, which is viewed as a Seifert 3-manifold over Σ , where Σ is the 2-orbifold whose underlying space is \mathbb{S}^2 , with two singular points of the same multiplicity α . The corresponding orbifold complex line bundle E has Seifert invariant $(-1, (\alpha, \beta), (\alpha, \alpha - \beta))$, where $0 < \beta < \alpha$, $\gcd(\alpha, \beta) = 1$.

We shall compute the Seiberg-Witten invariant of Y as a Seifert 3-manifold over Σ , where for simplicity we take $\alpha = 3$ and $\beta = 1$. For each d , there are 3 distinct equivalence classes $[D_{d,i}]$, $i = 0, 1, 2$, where $D_{d,0} = (d, (3, 0), (3, 0))$, $D_{d,1} = (d, (3, 0), (3, 1))$ and $D_{d,2} = (d, (3, 1), (3, 1))$. The $Spin^c$ -structures $\xi_{d,i}$ which correspond to $[D_{d,i}]$ satisfy

$c_1(\det \xi_{d,i}) = (6d + 2 + 2i)t$, where $t = PD(\mathbb{S}^1 \times \{pt\})$. In order to compute the Seiberg-Witten invariants, we determine the elements of each $[D_{d,i}]$. It is straightforward to check that the elements of $[D_{d,0}]$ have Seifert invariants

$$(d, (3, 0), (3, 0)), \quad (d-1, (3, 1), (3, 2)), \quad (d-1, (3, 2), (3, 1)),$$

the elements of $[D_{d,1}]$ have Seifert invariants

$$(d, (3, 0), (3, 1)), \quad (d, (3, 1), (3, 0)), \quad (d-1, (3, 2), (3, 2)),$$

and the elements of $[D_{d,2}]$ have Seifert invariants

$$(d, (3, 1), (3, 1)), \quad (d, (3, 2), (3, 0)), \quad (d, (3, 0), (3, 2)).$$

Consequently, for $d \geq 0$,

$$SW_Y(\xi_{d,i}) = SW_Y([D_{d,i}]) = \begin{cases} (d+1) + d + d = 3d + 1 & \text{if } i = 0, \\ (d+1) + (d+1) + d = 3d + 2 & \text{if } i = 1, \\ (d+1) + (d+1) + (d+1) = 3d + 3 & \text{if } i = 2, \end{cases}$$

and for $d < 0$, $SW_Y(\xi_{d,i}) = 0$ for $i = 0, 1, 2$.

The Seiberg-Witten invariant of $Y = \mathbb{S}^1 \times \mathbb{S}^2$ is given by the polynomial

$$SW_Y(t) = (t^{-1} - t)^{-2} = \sum_{m=1}^{\infty} mt^{2m}, \quad \text{where } t = PD(\mathbb{S}^1 \times \{pt\}).$$

It is clear that our computation gives the same result.

Example 5.3. In this example Y is the mapping torus of a periodic diffeomorphism $f : T^2 \rightarrow T^2$ of order 6, where f is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Topologically, Y is the 0-surgery on the trefoil knot. This description allows us to compute the Seiberg-Witten invariant in terms of the Alexander polynomial of the knot (cf. [16, 11]), and it is given by the following polynomial

$$SW_Y(t) = \frac{t^4 - t^2 + 1}{(1 - t^2)^2} = 1 + \sum_{m=1}^{\infty} mt^{2m},$$

where t is the Poincaré dual of the generator of $H_1(Y)$.

On the other hand, since f is periodic, Y can be also viewed as a Seifert 3-manifold over Σ , where Σ is the 2-orbifold whose underlying space is \mathbb{S}^2 and which has 3 singular points z_1, z_2, z_3 with associated multiplicities 2, 3 and 6. The corresponding orbifold complex line bundle E has Seifert invariant $(-2, (2, 1), (3, 2), (6, 5))$. We shall compute the Seiberg-Witten invariant of Y from this point of view.

First, note that E is torsion of order 6, with the Seifert invariant of kE given by

$$\begin{aligned} &(-2, (2, 1), (3, 2), (6, 5)), & \text{if } k = 1, \\ &(-1, (2, 0), (3, 1), (6, 4)), & \text{if } k = 2, \\ &(-1, (2, 1), (3, 0), (6, 3)), & \text{if } k = 3, \\ &(-1, (2, 0), (3, 2), (6, 2)), & \text{if } k = 4, \\ &(-1, (2, 1), (3, 1), (6, 1)), & \text{if } k = 5. \end{aligned}$$

With this understood, for each d , there are 6 distinct equivalence classes $[D_{d,i}]$, with $i = 0, 1, 2, 3, 4, 5$, where in terms of the Seifert invariant,

$$D_{d,i} = \begin{cases} (d, (2, 0), (3, 0), (6, 0)), & \text{if } i = 0, \\ (d, (2, 0), (3, 0), (6, 1)), & \text{if } i = 1, \\ (d, (2, 0), (3, 1), (6, 0)), & \text{if } i = 2, \\ (d, (2, 1), (3, 0), (6, 0)), & \text{if } i = 3, \\ (d, (2, 1), (3, 0), (6, 1)), & \text{if } i = 4, \\ (d, (2, 1), (3, 1), (6, 0)), & \text{if } i = 5. \end{cases}$$

Let $\xi_{d,i}$ be the $Spin^c$ -structure corresponding to $[D_{d,i}]$. Then one can easily check that $c_1(\det \xi_{d,i}) = (12d + 2i)t$. It is straightforward, though tedious, to write down the Seifert invariants of the elements of each $[D_{d,i}]$, with which one obtains

$$SW_Y(\xi_{d,i}) = SW_Y([D_{d,i}]) = \begin{cases} 0 & \text{if } d < 0, \\ 1 & \text{if } d = 0, i = 0, \\ 6d + i & \text{otherwise.} \end{cases}$$

It is clear that this is the same as given by the polynomial $SW_Y(t) = 1 + \sum_{m=1}^{\infty} mt^{2m}$.

6. Application: non-Kähler elliptic surfaces

We first extend the formula for the Seiberg-Witten invariants of Seifert 3-manifolds from the previous section to Seifert fibered 3-orbifolds. The proof of Theorem 1.4 follows easily from this with the help of Baldrige's theorem in [2].

Let $\pi : Y \rightarrow \Sigma$ be a Seifert fibered 3-orbifold where Σ has genus g and m singular points z_i with multiplicities α_i , $i = 1, \dots, m$, and where the corresponding orbifold complex line bundle E has Seifert invariant $(e, (\alpha_i, e_i))$. Here for each i , $0 \leq e_i < \alpha_i$, and e_i and α_i are not necessarily relatively prime. In fact, the components of the singular set of Y are given by exceptional fibers: the fiber over z_i lies in the singular set if and only if $\gcd(\alpha_i, e_i) > 1$, where in this case, $a_i \equiv \gcd(\alpha_i, e_i)$ is the associated multiplicity of the corresponding singular component.

As we shall see from the proof, the set of $Spin^c$ -structures of Y which have nonzero Seiberg-Witten invariant may be identified with a subset of equivalence classes $[D]$, where $[D] = [D']$ if and only if $D - D' \equiv 0 \pmod{E}$, such that under the identification, if ξ corresponds to $[D]$ and D has Seifert invariant $(d, (\alpha_i, \beta_i))$, one has

$$\det \xi = (2d - 2g + 2)F + \sum_i (2\beta_i + 1 - \alpha_i)F_i.$$

Here F stands for the complex line bundle whose first Chern class is Poincaré dual to a regular fiber, and for each i , if $a_i \equiv \gcd(\alpha_i, e_i) = 1$, F_i stands for the complex line bundle whose first Chern class is Poincaré dual to the exceptional fiber at z_i (which is not a singular component), and if $a_i \equiv \gcd(\alpha_i, e_i) > 1$, F_i stands for the orbifold complex line bundle defined as follows: in a neighborhood of the exceptional fiber at z_i (which is a singular component with multiplicity a_i), F_i is given by the \mathbb{Z}_{a_i} -equivariant complex line bundle $D^2 \times \mathbb{S}^1 \times \mathbb{C}$ with the standard \mathbb{Z}_{a_i} -action

$$\lambda \cdot (z, x, w) = (\lambda z, x, \lambda w), \quad \lambda = \exp(2\pi i/a_i), \quad (z, x) \in D^2 \times \mathbb{S}^1, w \in \mathbb{C},$$

and over the rest of Y , F_i is defined by extending the trivialization determined by the \mathbb{Z}_{a_i} -equivariant nonzero section $(z, x) \mapsto z$ along $\partial D^2 \times \mathbb{S}^1$.

Theorem 6.1. *The Seiberg-Witten invariant of Y is given by*

$$SW_Y(\xi) \equiv SW_Y([D]) = \sum_{D' \in [D]} SW_{\mathbb{S}^1 \times \Sigma}(D').$$

Proof. Without loss of generality, we can assume that $a_m > 1$, which means that the exceptional fiber at z_m is a singular component. We let Y_0 be the 3-orbifold obtained by removing the exceptional fiber at z_m from the singular set of Y . Then it follows easily that Y_0 is Seifert fibered over Σ_0 , where Σ_0 is the 2-orbifold obtained from Σ by changing the multiplicity of z_m to α_m/a_m . The corresponding orbifold complex line bundle E_0 for Y_0 has Seifert invariant

$$(e, (\alpha_1, e_1), \dots, (\alpha_{m-1}, e_{m-1}), (\alpha_m/a_m, e_m/a_m)).$$

In fact, let $\pi : Y \rightarrow \Sigma$ and $\pi_0 : Y_0 \rightarrow \Sigma_0$ denote the Seifert fibrations. Then the local models for π and π_0 near z_m are given as follows. Consider the \mathbb{Z}_{α_m} -action on $D^2 \times \mathbb{S}^1$ given by

$$\lambda \cdot (z, w) = (\lambda z, \lambda^{e_m} w), \quad \lambda = \exp(2\pi i/\alpha_m), \quad (z, w) \in D^2 \times \mathbb{S}^1.$$

The Seifert fibration π is given by the \mathbb{Z}_{α_m} -equivariant map $(z, w) \mapsto z$. Likewise, π_0 is given by the $\mathbb{Z}_{\alpha_m/a_m}$ -equivariant map $(z, w) \mapsto z$, where the $\mathbb{Z}_{\alpha_m/a_m}$ -action on $D^2 \times \mathbb{S}^1$ is given by

$$\lambda_0 \cdot (z, w) = (\lambda_0 z, \lambda_0^{e_m/a_m} w), \quad \lambda_0 = \lambda^{a_m} = \exp(2\pi a_m i/\alpha_m), \quad (z, w) \in D^2 \times \mathbb{S}^1.$$

Now let $\Sigma_-, \Sigma_{0,-}$ be regular neighborhoods of z_m in Σ, Σ_0 respectively. Then there are decompositions $\Sigma = \Sigma_+ \cup_\gamma \Sigma_-$ and $\Sigma_0 = \Sigma_+ \cup_\gamma \Sigma_{0,-}$, where γ is the link around z_m . These decompositions are compatible with the decompositions $Y = Y_+ \cup_T Y_-$ and $Y_0 = Y_+ \cup_T Y_{0,-}$, where T is the 2-torus over γ . Moreover, $\pi = \pi_0 : Y_+ \rightarrow \Sigma_+$.

With the preceding understood, we shall reduce the proof of the theorem for Y to that for Y_0 using Theorem 1.1, and inductively, the theorem is reduced to the case of Seifert 3-manifolds, which we have already established.

To this end, we associate to each orbifold complex line bundle D over Σ an orbifold complex line bundle D_0 over Σ_0 as follows. Suppose D has Seifert invariant $(d, (\alpha_i, \beta_i))$.

Then we define D_0 to be the one which has Seifert invariant

$$(d, (\alpha_1, \beta_1), \dots, (\alpha_{m-1}, \beta_{m-1}), (\alpha_m/a_m, [\beta_m/a_m])),$$

where $\beta_m = [\beta_m/a_m] \cdot a_m + b_m$, with $0 \leq b_m < a_m$. We claim that under the map ϕ in Theorem 1.1, the $Spin^c$ -structure corresponding to $[D]$ is sent to the $Spin^c$ -structure corresponding to $[D_0]$.

To see this, let ξ, ξ^0 be the $Spin^c$ -structures corresponding to $[D]$ and $[D_0]$ respectively. Then the associated spinor bundles are $S = \pi^*(D \oplus K_{\Sigma}^{-1} \otimes D)$ and $S^0 = \pi_0^*(D_0 \oplus K_{\Sigma_0}^{-1} \otimes D_0)$. Over Y_+ , S and S^0 are identified as follows. Consider D over Σ_- and D_0 over $\Sigma_{0,-}$. There are canonical trivializations of D and D_0 along γ , such that with these trivializations the relative first Chern class of D and D_0 over Σ_- and $\Sigma_{0,-}$ are β_m/α_m and $[\beta_m/a_m]/(\alpha_m/a_m)$ respectively. In fact, these trivializations are given by the equivariant nonzero sections $z \mapsto z^{\beta_m}$ and $z \mapsto z^{[\beta_m/a_m]}$ respectively. Now by the definition of D_0 , D is isomorphic to D_0 over Σ_+ with these trivializations on the boundary $\gamma = \partial\Sigma_+$. Likewise, K_{Σ}^{-1} and $K_{\Sigma_0}^{-1}$ are isomorphic over Σ_+ with certain canonical trivializations on γ , which are given by the equivariant nonzero section $z \mapsto z$. Note that with respect to these trivializations, the relative first Chern class of K_{Σ}^{-1} and $K_{\Sigma_0}^{-1}$ over Σ_- and $\Sigma_{0,-}$ are $1/\alpha_m$ and $1/(\alpha_m/a_m)$ respectively. Now with $\pi = \pi_0 : Y_+ \rightarrow \Sigma_+$, these isomorphisms between D, D_0 and $K_{\Sigma}^{-1}, K_{\Sigma_0}^{-1}$ over Σ_+ give an identification between S and S^0 over Y_+ , as a bundle with a fixed trivialization on $T = \partial Y_+$.

With this understood, the claim $\xi^0 = \phi(\xi)$ will follow by examining the relative first Chern classes of $\det S$ and $\det S^0$ over Y_- and $Y_{0,-}$ with respect to these trivializations on T . The relative first Chern classes can be calculated using the local models for π and π_0 near z_m and the equivariant nonzero sections defining these trivializations; their evaluations on the 2-disc $D^2 \times \{pt\}$ are (cf. [8], Lemma 3.6)

$$\frac{2\beta_m + 1}{a_m} = 2[\beta_m/a_m] + \frac{2b_m + 1}{a_m} \quad \text{for } \det S,$$

and

$$2[\beta_m/a_m] + 1 \quad \text{for } \det S_0.$$

By Lemma 2.3, the trivializations of $S|_{Y_-}$ and $S^0|_{Y_{0,-}}$ on T must be of the form $h_- : \xi|_T \rightarrow \xi_0$ and $h_{0,-} : \xi^0|_T \rightarrow \xi_0$, and moreover, by Definition 2.4, $\xi^0 = \phi(\xi)$ as claimed.

By Theorem 1.1, we have $SW_Y([D]) = SW_{Y_0}([D_0])$. Thus the proof of the theorem boils down to verifying

$$\sum_{D' \in [D]} SW_{\mathbb{S}^1 \times \Sigma}(D') = \sum_{D'_0 \in [D_0]} SW_{\mathbb{S}^1 \times \Sigma_0}(D'_0).$$

To see this, first note that under the correspondence $D \mapsto D_0$, $D' \in [D]$ if and only if $D'_0 \in [D_0]$ (where $D' \mapsto D'_0$), because $D' - D \equiv 0 \pmod{E}$ if and only if $D'_0 - D_0 \equiv 0 \pmod{E_0}$. Secondly, $SW_{\mathbb{S}^1 \times \Sigma}(D') = SW_{\mathbb{S}^1 \times \Sigma_0}(D'_0)$, because if D' is given by the Seifert

invariant $(d', (\alpha_i, \beta'_i))$, then D'_0 is given by

$$(d', (\alpha_1, \beta'_1), \dots, (\alpha_{m-1}, \beta'_{m-1}), (\alpha_m/a_m, [\beta'_m/a_m])),$$

so that both $SW_{\mathbb{S}^1 \times \Sigma}(D')$ and $SW_{\mathbb{S}^1 \times \Sigma_0}(D'_0)$ are equal to

$$\begin{cases} (-1)^{d'} \binom{2g-2}{d'} & \text{if } g \geq 1, d' \in [0, 2g-2], \\ d'+1 & \text{if } g=0, d' \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we remark that the above proof also shows that the equation

$$\det \xi = (2d - 2g + 2)F + \sum_i (2\beta_i + 1 - \alpha_i)F_i$$

follows, with the help of Proposition 2.6, from the corresponding equation for Seifert 3-manifolds. This finishes the proof of Theorem 6.1. \square

In the remaining part of this section, we give a proof of Theorem 1.4. To this end, let X be a minimal elliptic surface obtained from $E \times C$, where $E = \mathbb{C}/\Lambda$ and C is a curve of genus g , by doing logarithmic transforms on lifts $x_i \in \mathbb{C}$ of m_i -torsion points ξ_i modulo Λ , $i = 1, 2, \dots, n$. We remark that X is non-Kähler if and only if $\sum_i x_i \neq 0$. In this case, $b_2^+(X) = 2g$. If X is Kähler, i.e., $\sum_i x_i = 0$, then $b_2^+(X) = 2g + 1$. See [12] for details.

For our purpose here it is more convenient to describe X topologically as a principal T^2 -bundle over Σ , where Σ is the 2-orbifold canonically obtained from C as follows. Let t_i , $i = 1, \dots, n$, be the point of C over which the logarithmic transform on x_i is performed. Then the singular points of Σ consist of t_i with multiplicity m_i for all i where $m_i > 1$.

As we explained in the introduction, after fixing a basis $\mathbf{e}_1, \mathbf{e}_2$ of Λ , X gives rise to a pair of orbifold complex line bundles (or equivalently, principal \mathbb{S}^1 -bundles) E_1, E_2 over Σ . We shall give a more intrinsic description of E_1, E_2 here. Note that $\mathbf{e}_1, \mathbf{e}_2$ determine an identification of $T^2 = \mathbb{C}/\Lambda$ with $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}\mathbf{e}_1 \times \mathbb{R}/\mathbb{Z}\mathbf{e}_2$. Let $p_1, p_2 : T^2 \rightarrow \mathbb{S}^1$ be the corresponding projections onto the first and the second factor respectively, and let Y_1, Y_2 be the principal \mathbb{S}^1 -bundles over Σ induced by p_1, p_2 . Then E_1, E_2 are the orbifold complex line bundles associated to Y_1, Y_2 . Note that this process can be reversed: given a pair of principal \mathbb{S}^1 -bundles Y_1, Y_2 over Σ , we obtain X back as the pull-back bundle of $Y_1 \times Y_2 \rightarrow \Sigma \times \Sigma$ via the diagonal map $\Sigma \rightarrow \Sigma \times \Sigma$.

If we choose a different basis $\mathbf{e}'_1, \mathbf{e}'_2$, with E'_1, E'_2 being the corresponding orbifold complex line bundles, where

$$(\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{e}'_1, \mathbf{e}'_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}),$$

then $E'_1 = aE_1 + bE_2$, $E'_2 = cE_1 + dE_2$. In particular, the subgroup of orbifold complex line bundles generated by E_1, E_2 coincides with the subgroup generated by E'_1, E'_2 , which was denoted by Γ_X in the introduction.

Finally, X may be regarded as a principal \mathbb{S}^1 -bundle over the 3-orbifold Y_2 , with a bundle morphism $X \rightarrow Y_1$ which induces the Seifert fibration $\pi_2 : Y_2 \rightarrow \Sigma$. In other

words, the orbifold complex line bundle corresponding to the Seifert fibration $\pi : X \rightarrow Y_2$ is $\pi_2^* E_1$.

With the preceding understood, the following theorem is a slightly more general version of Theorem 1.4, where X is allowed to be Kähler.

Theorem 6.2. *Suppose $b_2^+ \geq 1$ (which means that X must be Kähler when $g = 0$). Then the subset of $Spin^c$ -structures which may have nonzero Seiberg-Witten invariant can be identified with the set $\{(D)\}$, where (D) is the orbit of an orbifold complex line bundle D under the action of Γ_X . With this understood,*

$$SW_X((D)) = \begin{cases} \sum_{D' \in (D), |D'| \in [0, 2g-2]} (-1)^{|D'|} \binom{2g-2}{|D'|} & \text{if } g \geq 1 \\ \sum_{D' \in (D), |D'| \geq 0} (|D'| + 1) & \text{if } g = 0. \end{cases}$$

Moreover, if \mathcal{L} is the $Spin^c$ -structure corresponding to (D) , then

$$c_1(\det \mathcal{L}) = (2d - 2g + 2)F + \sum_i (2s_i + 1 - m_i)F_i,$$

where F stands for a regular fiber and F_i for the fiber at t_i of the elliptic fibration on X , and $(d, (m_i, s_i))$ is the Seifert invariant of D .

We remark that when $b_2^+ = 1$, which happens if and only if $g = 0$ and X is Kähler, the Seiberg-Witten invariant of X is defined using the Taubes chamber, i.e., using the Kähler form to orient $H^{2,+}(X; \mathbb{R})$.

Proof. Applying Baldrige's theorem in [2] to the Seifert fibration $\pi : X \rightarrow Y_2$, and with Theorem 6.1, we see that the relevant $Spin^c$ -structures of X are parametrized by the set of equivalence classes $[[D]]$, where $[[D]] = [[D']]$ if and only if $[D]$ and $[D']$ differ by a multiple of $[E_1]$. On the other hand, recall from Theorem 6.1 that $[D]$ stands for the equivalence class of D where D, D' are considered equivalent if and only if $D - D' \equiv 0 \pmod{E_2}$, or equivalently, $\pi_2^* D = \pi_2^* D'$. It follows easily that the equivalence class $[[D]]$ coincides with the orbit (D) . Furthermore, by Baldrige's formula in [2] as well as the formula in Theorem 6.1,

$$\begin{aligned} SW_X((D)) &= \sum_{[D_1] \in [[D]]} SW_{Y_2}([D_1]) \\ &= \sum_{[D_1] \in [[D]]} \sum_{D_2 \in [D_1]} SW_{\mathbb{S}^1 \times \Sigma}(D_2) \\ &= \sum_{D' \in (D)} SW_{\mathbb{S}^1 \times \Sigma}(D'), \end{aligned}$$

from which the claimed formula for $SW_X((D))$ in Theorem 6.2 follows. Finally, the formula for $c_1(\det \mathcal{L})$ follows directly from the corresponding formula for Seifert fibered 3-orbifolds (cf. Theorem 6.1). \square

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