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Desingularizing special generic maps

Osamu Saeki and Masamichi Takase

Dedicated to Professor Shyuichi Izumiya on the occasion of his sixtieth birthday

Abstract. Let $f\colon M \to \mathbf{R}^p$ be a special generic map of a closed n-dimensional manifold M with $n \geq p \geq 1$. We study the condition for f to be factorized as $f = \pi \circ \eta$ for an immersion (or an embedding) $\eta\colon M \to \mathbf{R}^{n+1}$ and an orthogonal projection $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^p$. For various dimension pairs (n, p) we give answers to such a lifting problem. In particular, for the cases where p is equal to 1 and 2 we obtain complete results.

1. Introduction

In [13], Haefliger studied generic smooth maps of surfaces into the plane and found a condition for such a map, to be lifted to an immersion in codimension one, that is, to be factorized as the composition of an immersion into $\mathbf{R}^3$ and an orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ (see also [28, 29]); not every such generic map can be so lifted. Minoru Yamamoto [40] showed that a generic smooth map of a closed surface into $\mathbf{R}^2$ can always be lifted to an embedding into $\mathbf{R}^4$. For a given singular map f, such an immersion lift or an embedding lift into an appropriately higher dimensional space can be considered as a desingularization of f, and many authors, including Saito [34], Blank and Curley [2], Kushner, Levine and Porto [21], Burlet and Haab [8], and Levine [26], have studied similar lifting problems for various kinds of singular maps in other dimensions.

In this paper we study the condition for a special generic map $f\colon M \to \mathbf{R}^p$ of a closed n-dimensional manifold M to be factorized as $f = \pi \circ \eta$ for an immersion or an embedding $\eta\colon M \to \mathbf{R}^{n+1}$ and an orthogonal projection $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^p$. Here, a special generic map $M \to N$, $n = \dim M \geq \dim N = p \geq 1$, between smooth manifolds is a smooth map with only definite fold singularities, which have the normal form

$$(x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2).$$

For various pairs (n, p) of the dimensions, we give answers to such a lifting problem of special generic maps. In particular, when p is equal to 1 or 2, we obtain quite comprehensive results.

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We first introduce in §2 the necessary notation. In §3, we recall the important notion of a Stein factorization and study several related materials, which are frequently used in this paper. In §4, we show that every special generic map $f\colon M \to \mathbf{R}$, $n = \dim M \geq 1$, can be factorized as the composition of an immersion into $\mathbf{R}^{n+1}$ and an orthogonal projection. The lifting problem in this case relates to the space of immersions of S^{n-1} into $\mathbf{R}^n$, and in fact, the above result is a consequence of the fact that the standard embedding $i\colon S^{n-1} \to \mathbf{R}^n$ is regularly homotopic to $i \circ \psi$ for every orientation preserving diffeomorphism $\psi\colon S^{n-1} \to S^{n-1}$ (see [16]). We will give a short proof of this fact using the Smale–Hirsch theory [15, 36] and the well-known fact that the tangent bundle of a homotopy sphere is topologically equivalent to that of a standard sphere (for example, see [32]). Furthermore, by an argument using the graphing (spinning) construction for the above immersion space, we characterize those regular homotopy classes represented by immersions $M \to \mathbf{R}^{n+1}$ lifting a given special generic map $f\colon M \to \mathbf{R}$, in terms of the normal degrees. As to embedding lifts, it turns out that when $\dim M \geq 2$ the existence of an embedding lift $M \to \mathbf{R}^{n+1}$ depends only on the diffeomorphism type of the source manifold M. Namely we show that a special generic map $f\colon M \to \mathbf{R}$, $n = \dim M \geq 2$, factors through an embedding $M \to \mathbf{R}^{n+1}$ if and only if M is diffeomorphic to the standard sphere S^n.

In §5, we consider special generic maps into the plane. Let $f\colon M \to \mathbf{R}^2$ be a special generic map of a closed n-dimensional manifold ($n \geq 2$) into $\mathbf{R}^2$. We first show that if M is orientable, then such a map can always be factored through an immersion $M \to \mathbf{R}^{n+1}$. A key step of the proof relies on the fact that the inclusion of the space of certain *smooth* immersions $S^{n-2} \to \mathbf{R}^{n-1}$ into the corresponding space of locally flat *topological* immersions induces an injective homomorphism on their fundamental groups. When M is non-orientable, we show that a special generic map $f\colon M \to \mathbf{R}^2$ factors through an immersion $M \to \mathbf{R}^{n+1}$ if and only if $n = 2, 4, 8$ and the neighborhood of the singular point set of f in M is orientable. As for an embedding lift of f, we see that not every special generic map $f\colon M \to \mathbf{R}^2$ factors through an embedding $M \to \mathbf{R}^{n+1}$ (even when M is orientable), and when $\dim M \geq 3$ it turns out again that the existence of such an embedding lift depends only on the diffeomorphism type of the source manifold M. More precisely we have the following: a special generic map $f\colon M \to \mathbf{R}^2$, $n = \dim M \geq 3$, factors through an embedding $M \to \mathbf{R}^{n+1}$ if and only if M is diffeomorphic to S^n or the connected sum of some copies of $S^1 \times S^{n-1}$. The proof involves a study of a Schottky group acting on S^n and a computation of the inertia group of the connected sum of some copies of $S^1 \times S^{n-1}$. Yamamoto's theorem [40] mentioned above also plays a crucial role. Note that there exists a special generic map $f\colon S^2 \to \mathbf{R}^2$ which does not factor through any embedding $S^2 \to \mathbf{R}^3$ (but factors through an immersion $S^2 \to \mathbf{R}^3$, see Remark 5.8).

In §6, we compile various results for other particular dimension pairs of the source and target manifolds. For example, we show that for dimension pairs (n, p) with $p = n - 2$ or $n - 3$ and $p \leq 4$, a special generic map $f\colon M \to \mathbf{R}^p$, $n = \dim M$, can be factorized through an immersion $M \to \mathbf{R}^{n+1}$ if and only if the source manifold M is spin. We also show that when $n - p = 1$, such a factorization is possible if and only if the homology class represented by the singular point set vanishes in $H_{p-1}(M; \mathbf{Z})$. Moreover, the existence of an embedding lift and that of a deformation of an immersion lift to an embedding lift are studied. Finally, we show that a closed orientable 4-manifold admitting a special generic map into $\mathbf{R}^3$ can be embedded into $\mathbf{R}^5$ if and only if it is spin.

2. Notation

Throughout the paper, we use the following notation. First note that all manifolds and maps between them are supposed to be differentiable of class C^∞ unless otherwise specified. The (co)homology groups are with integer coefficients unless otherwise specified. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects. For a topological space X, the symbol "id_X" denotes the identity map of X.

For integers $n \geq p > 0$, let $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^p$ be the projection defined by

$$\pi(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_p).$$

Such a map π will be called the *standard projection*.

We let $\mathrm{Diff}(S^m)$ denote the group of diffeomorphisms of the m-sphere S^m and $\mathrm{Diff}_*(S^m)$ the subgroup of those which restricts to the identity on the southern hemisphere. Let Θ_m denote the group of homotopy m-spheres, that is, the group of h-cobordism classes of homotopy m-spheres.

We denote by $\mathrm{Imm}(S^m, \mathbf{R}^N)$ the space of smooth immersions of S^m into $\mathbf{R}^N$ equipped with the C^∞ topology and by $\mathrm{Imm}_*(S^m, \mathbf{R}^N)$ the corresponding space of those immersions which are the standard inclusion on the southern hemisphere and which map the northern hemisphere into the northern half space of $\mathbf{R}^N$. Let $\mathrm{Emb}(S^m, \mathbf{R}^N)$ and $\mathrm{Emb}_*(S^m, \mathbf{R}^N)$ be the corresponding subspaces consisting of smooth embeddings. Note that $\mathrm{Emb}_*(S^m, \mathbf{R}^N)$ can be identified with the space $\mathcal{K}_{N,m}$ of long knots — the space of smooth embeddings of $\mathbf{R}^m$ into $\mathbf{R}^N$ which are standard outside of a ball (see for example [4]). We often use these spaces in the codimension one case; note that Θ_m is isomorphic to $\pi_0\mathcal{K}_{m,m-1}$ which is isomorphic to $\pi_0\mathrm{Emb}_*(S^{m-1}, \mathbf{R}^m)$ for $m \geq 5$, according to [4, the remark after Theorem 5.4].

3. Stein factorization

Let us recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 3.1. Let $f\colon M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and x and x' belong to the same connected component of $f^{-1}(y)$. We define $W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation and denote the quotient map by $q_f\colon M \to W_f$. Then it is easy to see that there exists a unique continuous map $\bar{f}\colon W_f \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes. This commutative diagram is called the *Stein factorization* of f (see [26]).

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [7, 33]).

Proposition 3.2. *Let $f\colon M \to N$ be a proper special generic map between smooth manifolds with $n = \dim M > \dim N = p$. Then, we have the following.*

(1) *The set $S(f)$ of singular points of f is a submanifold of M of dimension $p-1$ and is closed as a subset of M.*
(2) *The quotient space W_f has the structure of a smooth p-dimensional manifold with boundary such that $\bar{f}\colon W_f \to N$ is an immersion.*
(3) *The quotient map $q_f\colon M \to W_f$ restricted to $S(f)$ is a diffeomorphism onto ∂W_f.*
(4) *If M is connected, then the quotient map q_f restricted to $M \setminus S(f)$ is a smooth fiber bundle over $\operatorname{Int} W_f$. Furthermore, if $S(f) \neq \emptyset$, then the fiber is the standard $(n-p)$-dimensional sphere S^{n-p}.*

Note that the structure group of the fiber bundle $q_f\colon M \setminus S(f) \to \operatorname{Int} W_f$ mentioned in Proposition 3.2 (4) for the case of $S(f) \neq \emptyset$ is the group $\mathrm{Diff}(S^{n-p})$ in general. It is known that the natural inclusion $O(n-p+1) \hookrightarrow \mathrm{Diff}(S^{n-p})$ induces a weak homotopy equivalence for $n-p = 1, 2, 3$ (see [14, 37]). Based on this fact, the first author has shown in [33] that there exists a D^{n-p+1}-bundle $E \to W_f$ whose structure group is $O(n-p+1)$ such that ∂E is diffeomorphic to M, provided that $n-p = 1, 2$ or 3.

In §6, we will need the following lemma.

Lemma 3.3. *Let $\beta\colon E_0 \to B$ be an oriented S^1-bundle over a connected CW complex with structure group $SO(2)$. Suppose that there exists a continuous map $F\colon E_0 \to \mathbf{R}^2$ such that on each fiber of β, F is a smooth immersion of a circle into $\mathbf{R}^2$ of winding number ± 1. Then the S^1-bundle β is trivial.*

Proof. By using the normalized differential of F restricted to each S^1-fiber, we can construct a continuous map $F'\colon E_0 \to S^1$ which is of degree 1 on each fiber of β.

Let $\widetilde{\beta}\colon \widetilde{E} \to B$ be the 2-dimensional vector bundle associated with β. Set $\widetilde{E}^* = \widetilde{E} \setminus B$, where B is identified with the zero section of $\widetilde{\beta}$. Then the continuous map F' naturally extends to a continuous map $\widetilde{F'}\colon (\widetilde{E}, \widetilde{E}^*) \to (\mathbf{R}^2, \mathbf{R}^2 \setminus \{0\})$.

Let us consider the commutative diagram:

$$\begin{tikzcd} H^2(\widetilde{E}, \widetilde{E}^*) \arrow[r, "i_1^*"] & H^2(\widetilde{E}) & H^2(B) \arrow[l, "\widetilde{\beta}^*"'] \\ H^2(\mathbf{R}^2, \mathbf{R}^2 \setminus \{0\}) \arrow[u, "\widetilde{F'}^*"] \arrow[r, "i_2^*"] & H^2(\mathbf{R}^2) \arrow[u, "\widetilde{F'}^*"] & H^2(*), \arrow[l, "c_2^*"'] \arrow[u, "c_1^*"'] \end{tikzcd}$$

where $c_1\colon B \to *$ and $c_2\colon \mathbf{R}^2 \to *$ are the constant maps, $i_1\colon \widetilde{E} \to (\widetilde{E}, \widetilde{E}^*)$ and $i_2\colon \mathbf{R}^2 \to (\mathbf{R}^2, \mathbf{R}^2 \setminus \{0\})$ are the inclusion maps, and $\widetilde{\beta}^*$ and c_2^* are isomorphisms. Note that

$$(\widetilde{\beta}^*)^{-1} \circ i_1^* \circ \widetilde{F'}^*(u) \in H^2(B)$$

for the natural generator u of $H^2(\mathbf{R}^2, \mathbf{R}^2 \setminus \{0\}) \cong \mathbf{Z}$ coincides with the Euler class e of the S^1-bundle β, since $\widetilde{F'}^*(u)$ coincides with the Thom class of $\widetilde{\beta}$. Since $H^2(*) = 0$ and the above diagram is commutative, we see that $e = 0$, which implies that β is a trivial S^1-bundle. □

Remark 3.4. It would be an interesting problem to describe the Euler class e of an oriented S^1-bundle in terms of the space $C^\infty(S^1, \mathbf{R}^2)$. Note that Kazarian [20] has obtained some results in terms of $C^\infty(S^1, \mathbf{R})$.

4. Function case

In this section, we consider the lifting problem of *special generic functions*, namely, special generic maps into $\mathbf{R}$. Note that a closed connected n-dimensional manifold M admits a special generic function if and only if it is diffeomorphic to a twisted sphere, that is, the union of two n-dimensional disks attached along their boundaries. In view of works of Cerf, Smale et al., this condition is satisfied if and only if

(i) M is diffeomorphic to the standard n-sphere for $n \leq 6$ (where every twisted sphere is diffeomorphic to the standard sphere),

(ii) M is a homotopy n-sphere for $n \geq 7$.

Our first result in this section is the following.

Theorem 4.1. *Let $f\colon M \to \mathbf{R}$ be a special generic function of a closed n-dimensional manifold M with $n \geq 1$. Then, there always exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ such that $f = \pi \circ \eta$, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}$ is the standard projection.*

Proof. We may assume that M is connected. When $n = 1$, f is a Morse function on the circle and it is almost straightforward to prove the required result. When $n = 2$, the result is proven by Burlet and Haab [8].

Suppose that $n > 1$. Then, the special generic map f is a Morse function with exactly two critical points. Let x_+ (or x_-) be the maximum (resp. minimum) point in M of f and set $a_\pm = f(x_\pm)$. Then, $f(M) = [a_-, a_+]$ and for each $y \in (a_-, a_+)$, $f^{-1}(y)$ is diffeomorphic to the standard $(n-1)$-sphere S^{n-1}.

For $\varepsilon > 0$ sufficiently small, set $D_- = f^{-1}([a_-, a_- + \varepsilon])$ and $D_+ = f^{-1}([a_+ - \varepsilon, a_+])$. By the Morse lemma, they are diffeomorphic to the n-dimensional (unit) disk, and $f|_{D_\pm}$ is identified with the function

$$g_\pm \colon (x_1, x_2, \ldots, x_n) \longmapsto \mp\varepsilon(x_1^2 + x_2^2 + \cdots + x_n^2) + a_\pm.$$

Let us consider the embedding $\eta_\pm \colon D_\pm \to \mathbf{R} \times \mathbf{R}^n$ defined by $\eta_\pm = (g_\pm, i_\pm)$, where $i_\pm \colon D_\pm = D^n \to \mathbf{R}^n$ is the standard embedding as the unit disk. Note that $\pi \circ \eta_\pm = f|_{D_\pm}$ (see Figure 1).

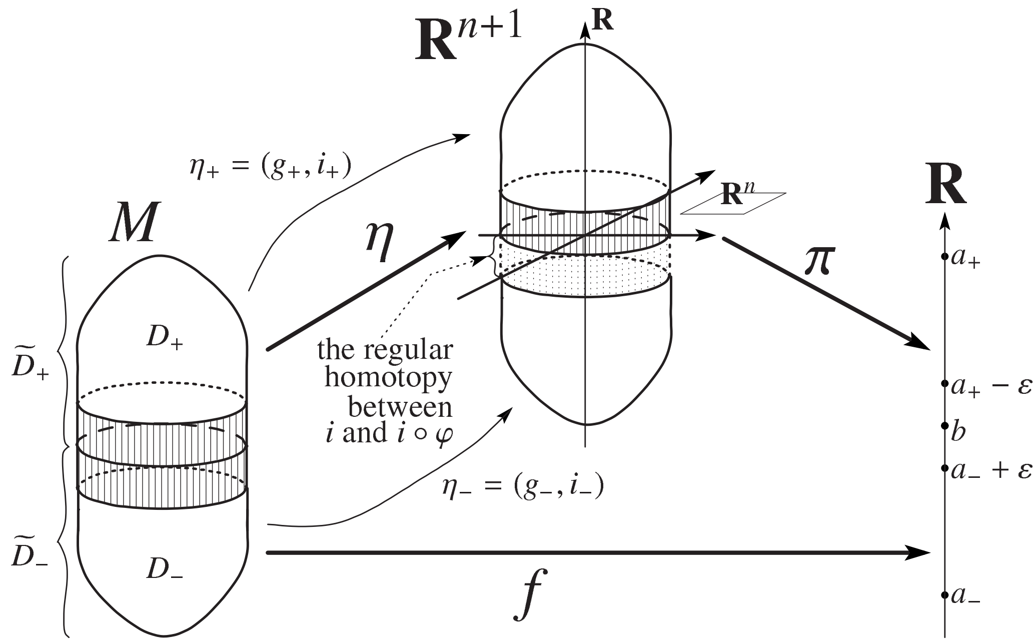

FIGURE 1. Lifting the special generic function f

Set $b = (a_- + a_+)/2$. Then, both $\widetilde{D}_- = f^{-1}([a_-, b])$ and $\widetilde{D}_+ = f^{-1}([b, a_+])$ are diffeomorphic to D^n. Let $\varphi \colon \partial\widetilde{D}_- = S^{n-1} \to S^{n-1} = \partial\widetilde{D}_+$ be the identification diffeomorphism. By modifying the identification $\widetilde{D}_- = D^n$ if necessary, we may assume that $\varphi \colon S^{n-1} \to S^{n-1}$ preserves the orientation. (Note that then $M = \widetilde{D}_+ \cup_\varphi (-\widetilde{D}_-)$.)

The following lemma is a consequence of [16, Theorem 2]. For completeness, we give a short proof here.

Lemma 4.2. *Let $i\colon S^{n-1} \to \mathbf{R}^n$ be the standard embedding, $n \geq 1$. Then, for any orientation preserving diffeomorphism $\psi\colon S^{n-1} \to S^{n-1}$, the immersions i and $i \circ \psi$ are regularly homotopic.*

Proof. Let $\mathrm{Mono}(TS^{n-1}, T\mathbf{R}^n)$ be the space of all fiberwise monomorphisms

$$TS^{n-1} \longrightarrow T\mathbf{R}^n,$$

where TS^{n-1} and $T\mathbf{R}^n$ are the tangent bundles of S^{n-1} and $\mathbf{R}^n$ respectively. By the Smale–Hirsch theory (see [15, 36]), it suffices to show that the fiberwise monomorphisms di and $di \circ d\psi$ lie in the same connected component of $\mathrm{Mono}(TS^{n-1}, T\mathbf{R}^n)$. Since $TS^{n-1} \oplus \tau^1$ and $T\mathbf{R}^n$ are trivial, we see that $\mathrm{Mono}(TS^{n-1}, T\mathbf{R}^n)$ is homotopy equivalent to the space $\mathrm{Map}(S^{n-1}, SO(n))$ of continuous maps $S^{n-1} \to SO(n)$, where τ^1 is the trivial line bundle. Therefore, di and $di \circ d\psi$ lie in the same connected component if and only if the tangent bundles of the homotopy n-spheres $D^n \cup_i (-D^n)$ and $D^n \cup_{i \circ \psi} (-D^n)$ are equivalent. Since the last statement is known to be true (see [32, (1.1) Lemma]), we obtain the desired result. □

By the above lemma, $i\colon S^{n-1} \to \mathbf{R}^n$ and $i \circ \varphi\colon S^{n-1} \to \mathbf{R}^n$ are regularly homotopic.

Let us consider the embedding $\eta_-|_{\partial D_-}\colon \partial D_- = S^{n-1} \to \{a_- + \varepsilon\} \times \mathbf{R}^n$, which can be identified with i. Then, by using the trace of the regular homotopy between i and $i \circ \varphi$ as above, we can construct an immersion $\widetilde{\eta}_-\colon \widetilde{D}_- \to [a_-, b] \times \mathbf{R}^n$ such that $\widetilde{\eta}_-|_{D_-} = \eta_-$ and that $\widetilde{\eta}_-|_{\partial \widetilde{D}_-}$ is identified with $i \circ \varphi$. Then, we can extend the immersions $\widetilde{\eta}_-$ and η_+ to obtain an immersion $\eta\colon \widetilde{D}_+ \cup_\varphi (-\widetilde{D}_-) = M \to [a_-, a_+] \times \mathbf{R}^n \subset \mathbf{R}^{n+1}$ such that $\pi \circ \eta = f$. This completes the proof. □

For a closed oriented n-dimensional manifold M and an immersion $\eta\colon M \to \mathbf{R}^{n+1}$, its Gauss map $\gamma\colon M \to S^n$ is defined by associating to each point $x \in M$ the unit vector $\gamma(x)$ in $\mathbf{R}^{n+1}$ which is oriented normal to the image $d\eta_x(T_xM)$ of the differential of η at x, where T_xM denotes the oriented tangent space of M at x. The degree of the Gauss map $\gamma\colon M \to S^n$ is called the *normal degree* of η, which is a regular homotopy invariant. Theorem 4.1 can be refined as follows.

Theorem 4.3. *Let $f\colon M \to \mathbf{R}$ be a special generic function of a closed connected n-dimensional manifold M with $n \geq 2$, and let $\eta_0\colon M \to \mathbf{R}^{n+1}$ be an immersion. Then, there exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ regularly homotopic to η_0 such that $f = \pi \circ \eta$ if and only if the normal degree of η_0 is equal to*

$$\begin{cases} \pm 1, & n \neq 3, 7, \\ \pm 1 \text{ or } 0, & n = 3, 7, \end{cases}$$

where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}$ is the standard projection.

Remark 4.4. It is known that the set $D(n)$ of possible values for the normal degrees of immersions $M \to \mathbf{R}^{n+1}$ of a homotopy n-sphere M is given as follows:

$$D(n) = \begin{cases} \{1\}, & n \text{ is even}, \\ \mathbf{Z}, & n = 1, 3, 7, \\ \mathbf{Z} \setminus 2\mathbf{Z}, & n \text{ is odd and } n \neq 1, 3, 7. \end{cases}$$

(See [3, Theorem 3] and [15, 30, 36], where we also need the celebrated result of Adams [1] that S^n is parallelizable if and only if $n = 1, 3, 7$.) Therefore, when n is even in the above theorem, an arbitrary immersion η_0 has the normal degree 1, and hence we can always find an immersion lift η regularly homotopic to η_0.

Proof of Theorem 4.3. Suppose that an immersion lift η as in the theorem exists. Recall that π is the projection to the first coordinate. The vector $\pm(1, 0, 0, \dots, 0) \in \mathbf{R}^{n+1}$ is normal to the image of η exactly at the images of the two critical points of f. Using the fact that the critical points are nondegenerate, we can easily check that the Gauss map of η has $(1, 0, 0, \dots, 0) \in S^n$ as a regular value. Thus, the normal degree of η must be equal to 0, ± 1 or ± 2.

Let $x_\pm \in M$ be the two critical points of f with $a_- = f(x_-) < f(x_+) = a_+$. Note that $f(M) = [a_-, a_+]$. Let $\pi' \colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ be the projection to the last n coordinates. Since the differential of $\pi' \circ \eta \colon M \to \mathbf{R}^n$ at $x_\pm$ is of rank n, $\pi' \circ \eta$ is an embedding near $x_\pm$. Taking $\varepsilon > 0$ sufficiently small, set $D_- = f^{-1}[a_-, a_- + \varepsilon]$ and $D_+ = f^{-1}[a_+ - \varepsilon, a_+]$. By the Morse lemma, $D_\pm$ are diffeomorphic to the n-dimensional disk. Furthermore, we may assume that $\pi' \circ \eta|_{D_\pm}$ are embeddings. Since $f|_{M \setminus (\mathrm{Int}\, D_- \cup \mathrm{Int}\, D_+)}$ is a submersion from $M \setminus (\mathrm{Int}\, D_- \cup \mathrm{Int}\, D_+)$ to $[a_- + \varepsilon, a_+ - \varepsilon]$, $M \setminus (\mathrm{Int}\, D_- \cup \mathrm{Int}\, D_+)$ is diffeomorphic to $S^{n-1} \times [a_- + \varepsilon, a_+ - \varepsilon]$. As η is an immersion, for $t \in [a_- + \varepsilon, a_+ - \varepsilon]$, $\bar{\eta}_t = \eta|_{f^{-1}(t)}$ can be regarded as a regular homotopy between the embeddings $\bar{\eta}_{a_-+\varepsilon}$ and $\bar{\eta}_{a_+-\varepsilon}$. Note that, by virtue of Lemma 4.2, $\bar{\eta}_{a_-+\varepsilon}$ is regularly homotopic to the standard embedding $i \colon S^{n-1} \to \mathbf{R}^n$, while $\bar{\eta}_{a_+-\varepsilon}$ is regularly homotopic to i or $r \circ i$, where $r \colon \mathbf{R}^n \to \mathbf{R}^n$ is a reflection.

If the normal degree of η is equal to 0 or ± 2, then by an orientation reason, i must be regularly homotopic to $r \circ i$. Then by [1, 36] (see also [16, Proposition 3] or [27, Corollary 2]), n must be equal to 3 or 7. In this case, since n is odd, the contributions to the degree of the Gauss map at the points $x_\pm$ have opposite signs, and hence the normal degree must be equal to 0.

Conversely, suppose that the normal degree of η_0 satisfies the condition in the theorem. Let us define the *graphing map*

$$\overline{\mathrm{gr}} \colon \Omega\mathrm{Imm}_*(S^{n-1}, \mathbf{R}^n) \longrightarrow \mathrm{Imm}_*(S^n, \mathbf{R}^{n+1}) \tag{4.1}$$

as follows, where Ω indicates the loop space (see §2 for the definition of $\mathrm{Imm}_*(S^m, \mathbf{R}^N)$). Let $h \colon [0, 1] \to \mathrm{Imm}_*(S^{n-1}, \mathbf{R}^n)$ be a loop such that $h(0) = h(1)$ is the standard inclusion $i \colon S^{n-1} \to \mathbf{R}^n$. Consider the map $\widetilde{h} \colon S^{n-1} \times [0, 1] \to \mathbf{R}^n \times [0, 1]$ defined by $\widetilde{h}(x, t) = (h(t)(x), t)$ for $(x, t) \in S^{n-1} \times [0, 1]$. Capping-off $\widetilde{h}$ by "standard" embeddings of disks, we get an immersion $S^n \to \mathbf{R}^{n+1}$ after a suitable smoothing. By identifying the union of the capped disks and (southern hemisphere of S^{n-1}) $\times [0,1]$ with the southern hemisphere of S^n, the immersion thus obtained can be regarded as an element of $\mathrm{Imm}_*(S^n, \mathbf{R}^{n+1})$, which we denote by $\overline{\mathrm{gr}}(h)$. Note that the graphing map appears in [4] in the context of embeddings.

Now we have the commutative diagram

$$\begin{array}{ccccc}
\pi_1\mathrm{Imm}_*(S^{n-1}, \mathbf{R}^n) & \xrightarrow{\overline{\mathrm{gr}}_*} & \pi_0\mathrm{Imm}_*(S^n, \mathbf{R}^{n+1}) & \xrightarrow{\delta} & \mathbf{Z} \\
\cong\downarrow & & \downarrow\cong & & \downarrow\cong \\
\pi_n(SO(n)) & \xrightarrow{\iota_*} & \pi_n(SO(n+1)) & \longrightarrow & \pi_n(S^n),
\end{array}$$

where $\iota\colon SO(n) \to SO(n+1)$ is the standard inclusion, the second row is a part of the homotopy exact sequence associated with the fibration $SO(n) \to SO(n+1) \to S^n$, and the first two vertical maps are isomorphisms by virtue of the Smale–Hirsch theory. Note that the map δ above is given by (normal degree) $-\, 1$.

Let $\eta_1\colon M \to \mathbf{R}^{n+1}$ be an immersion lift of f constructed in the proof of Theorem 4.1. Note that its normal degree is equal to 1. We may assume that η_0 is obtained by the connected sum of an immersion $\eta_0'\colon S^n \to \mathbf{R}^{n+1}$ and η_1. Note that the normal degree of η_0' is equal to that of η_0. By modifying η_0' by regular homotopy, we may further assume that it is an element of $\mathrm{Imm}_*(S^n, \mathbf{R}^{n+1})$. If η_0' has normal degree 1, then its regular homotopy class is in the image of $\overline{\mathrm{gr}}_*$. Then, in the proof of Theorem 4.1, by choosing the regular homotopy between i and $i \circ \varphi$ appropriately, we see that there exists a desired immersion lift η of f in the regular homotopy class of η_0. When η_0 has normal degree -1, which can occur only when n is odd by Remark 4.4, its reflection is regularly homotopic to an immersion lift of f. Then, its reflection with respect to an appropriate coordinate hyperplane gives a desired immersion lift of f.

Let us now consider the case where $n = 3$ or 7 and the normal degree of η_0 vanishes. Since i and $r \circ i\colon S^{n-1} \to \mathbf{R}^n$ are regularly homotopic, we can construct an immersion lift η_1 of f as in the proof of Theorem 4.1 such that the normal degree of η_1 is zero. We may assume that η_0 is obtained by the connected sum of η_1 and an immersion $\eta_0'\colon S^{n-1} \to \mathbf{R}^n$. Note that the normal degree of η_0' is equal to 1. Then, by the same argument as in the previous paragraph, we obtain the desired result.

This completes the proof. $\square$

Remark 4.5. When $n = 1$, we can easily prove the following. Let $f\colon S^1 \to \mathbf{R}$ be a Morse function with $2k$ critical points, and let $\eta_0\colon S^1 \to \mathbf{R}^2$ be an immersion. Then, there exists an immersion $\eta\colon S^1 \to \mathbf{R}^2$ regularly homotopic to η_0 such that $f = \pi \circ \eta$ if and only if the absolute value of the normal degree of η_0 is less than or equal to k, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}$ is the standard projection.

As to the existence of an embedding lift, we have the following.

Theorem 4.6. *Let $f\colon M \to \mathbf{R}$ be a special generic function of a closed connected n-dimensional manifold M with $n \geq 2$. Then, there exists an embedding $\eta\colon M \to \mathbf{R}^{n+1}$*

such that $f = \pi \circ \eta$ if and only if M is diffeomorphic to the standard n-sphere S^n, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}$ is the standard projection.

Proof. Note that M is the union of two n-dimensional disks attached along their boundaries, since it admits a special generic map $f\colon M \to \mathbf{R}$. Accordingly, as stated at the beginning of this section, M is diffeomorphic to S^n for $n \leq 6$ and is a homotopy n-sphere for $n \geq 7$. Suppose that there exists an embedding $\eta\colon M \to \mathbf{R}^{n+1}$. Then, the closure of the bounded component of $\mathbf{R}^{n+1} \setminus \eta(M)$ is a contractible n-dimensional manifold bounded by $\eta(M) \cong M$. Therefore, M is diffeomorphic to S^n for $n \geq 7$ (see [31]).

Conversely, suppose that M is diffeomorphic to S^n. Then the identification diffeomorphism $\varphi\colon \partial\widetilde{D}_- = S^{n-1} \to S^{n-1} = \partial\widetilde{D}_+$ in the proof of Theorem 4.1 should be isotopic to the identity for $n \neq 5$ (see [9, 10, 37]). Thus, i and $i \circ \varphi$ are isotopic as embedding maps. For $n = 5$ we can still see that i and $i \circ \varphi$ are isotopic, since φ is chosen to be orientation-preserving, so that $i \circ \varphi$ is isotopic to a "based" embedding lying in $\pi_0\mathrm{Emb}_*(S^4, \mathbf{R}^5) \cong \pi_0\mathcal{K}_{5,4} \cong \Theta_5 = \{0\}$ and hence is isotopic to i (see [4, the remark after Theorem 5.4]). Therefore, by using an isotopy between i and $i \circ \varphi$, we can construct an embedding $\eta\colon M \to \mathbf{R} \times \mathbf{R}^n$ such that $\pi \circ \eta = f$ as in the proof of Theorem 4.1. This completes the proof. $\square$

Remark 4.7. It is an interesting combinatorial problem to determine those Morse functions on the circle that can be lifted to an embedding into the plane. Minoru Yamamoto [41] recently gave a complete solution to this problem.

Remark 4.8. Two smooth maps $f_i\colon M_i \to N_i$, $i = 0, 1$, between smooth manifolds are said to be *right-left equivalent* if there exist diffeomorphisms $\Psi\colon M_0 \to M_1$ and $\psi\colon N_0 \to N_1$ which make the following diagram commutative:

$$\begin{array}{ccc} M_0 & \xrightarrow{f_0} & N_0 \\ {\scriptstyle\Psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ M_1 & \xrightarrow{f_1} & N_1. \end{array}$$

We can show that for a given manifold M, the right-left equivalence class of a special generic function $f\colon M \to \mathbf{R}$ as in Theorem 4.1 is unique if $n \geq 2$ and $n \neq 5$. In particular, if M is diffeomorphic to S^n, $n \geq 2$, $n \neq 5$, then f is right-left equivalent to the standard height function $S^n \to \mathbf{R}$.

Note that in the case $n = 5$, where Cerf's result [10] is inapplicable, we cannot specify the identification diffeomorphism $\varphi\colon S^{n-1} \to S^{n-1}$, used in the proof of Theorem 4.1, from the diffeomorphism type of the source manifold of the function (even though we see that it is diffeomorphic to S^5).

5. The case of special generic maps into $\mathbf{R}^2$

In this section, we consider special generic maps of closed n-dimensional manifolds into $\mathbf{R}^2$ and study the existence of their immersion or embedding lifts into $\mathbf{R}^{n+1}$.

We first recall that by [33] a closed connected n-dimensional manifold M, $n \geq 2$, admits a special generic map $f\colon M \to \mathbf{R}^2$ if and only if M is diffeomorphic to one of the following manifolds:

(1) for $n \leq 6$,

$$\begin{aligned} &\overset{k}{\sharp}\left(S^1 \times S^{n-1}\right), && k \geq 0,\\ \left(\overset{k-1}{\sharp}(S^1 \times S^{n-1})\right) &\sharp \left(S^1 \tilde{\times} S^{n-1}\right), && k \geq 1, \end{aligned}$$

(2) for $n \geq 7$,

$$\begin{aligned} \left(\overset{k}{\underset{j=1}{\sharp}}(S^1 \times \Sigma_j^{n-1})\right) &\sharp\, \Sigma^n, && k \geq 0,\\ \left(\overset{k-1}{\underset{j=1}{\sharp}}(S^1 \times \Sigma_j^{n-1})\right) \sharp \left(S^1 \tilde{\times} \Sigma_k^{n-1}\right) &\sharp\, \Sigma^n, && k \geq 1, \end{aligned}$$

where each Σ_j^{n-1} (or Σ^n) is a homotopy $(n-1)$-sphere (resp. a homotopy n-sphere), $S^1 \tilde{\times} \Sigma^{n-1}$ denotes a non-orientable Σ^{n-1}-bundle over S^1 for a homotopy $(n-1)$-sphere Σ^{n-1}, $\sharp$ denotes the connected sum, and the connected sum over the empty set is assumed to be the standard n-sphere S^n.

When the source manifold is orientable, we have the following.

Theorem 5.1. *Let $f\colon M \to \mathbf{R}^2$ be a special generic map of a closed orientable n-dimensional manifold M, $n \geq 2$. Then there always exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ such that $f = \pi \circ \eta$, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^2$ is the standard projection.*

Proof. For $n = 2$, since the manifold M is orientable and f has no cusp singularities, the theorem is a consequence of Haefliger's result [13].

Let us consider the case $n \geq 3$. We may assume that M is connected. Then, the quotient space W_f in the Stein factorization of f has the structure of a smooth orientable compact connected surface with nonempty boundary. Now since W_f is naturally immersed in $\mathbf{R}^2$ via $\bar{f}$ by Proposition 3.2, we will arrange to construct below an immersion of M into $W_f \times \mathbf{R}^{n-1}$ so that it lifts $q_f\colon M \to W_f$; then such an immersion composed with $(\bar{f}, \mathrm{id})\colon W_f \times \mathbf{R}^{n-1} \to \mathbf{R}^2 \times \mathbf{R}^{n-1} = \mathbf{R}^{n+1}$ will be a desired immersion η. Let

$$W_f = h^0 \cup \left(\bigcup_{j=1}^{k} h_j^1\right) \tag{5.1}$$

be a handlebody decomposition of the surface W_f, where $h^0 \cong D^2$ is a 2-dimensional 0-handle and $h_j^1 \cong [-1,1] \times [-1,1]$ are 2-dimensional 1-handles attached to h^0 (see Figure 2).

Let us identify each h_j^1 with $I \times J$, where $I = J = [-1,1]$ and h_j^1 is attached to h^0 along $I \times \partial J$. It is easy to observe that q_f restricted to $M_{j,t} = q_f^{-1}(I \times \{t\})$ for $t \in J$, is a special generic map of a smooth closed connected $(n-1)$-dimensional manifold

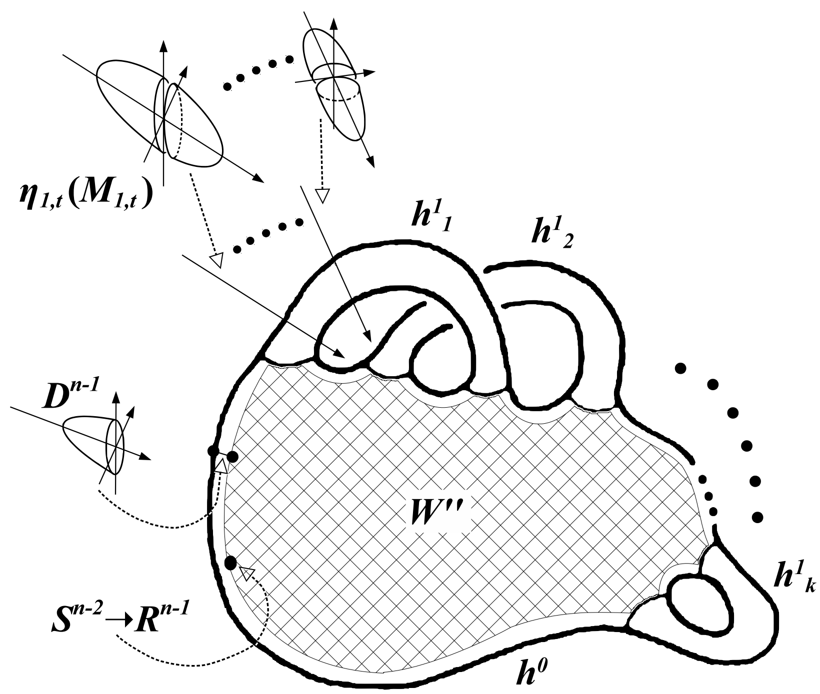

FIGURE 2. Lifting $q_f\colon M \to W_f$

into $I \times \{t\}$, where $I \times \{t\} \subset I \times J = h^1_j$. Hence $M_{j,t}$ is a homotopy $(n-1)$-sphere and there exists an immersion $\eta_{j,t}\colon M_{j,t} \to (I \times \{t\}) \times \mathbf{R}^{n-1}$ which lifts $q_f|_{M_{j,t}}$ by Theorem 4.1. Furthermore, by virtue of a kind of Thom's second isotopy lemma (see [17, 18]), we may regard $\{q_f|_{M_{j,t}}\}_{t \in J}$ as a trivial family of special generic functions. Therefore, we can choose $\{\eta_{j,t}\}_{t \in J}$ as a trivial family of immersions, and hence we can construct a smooth immersion $\eta_j\colon M_j \to h^1_j \times \mathbf{R}^{n-1}$ that lifts $q_f|_{M_j}\colon M_j \to h^1_j$, where $M_j = q_f^{-1}(h^1_j) \cong M_{j,0} \times J$.

Let C be a small closed collar neighborhood of ∂W_f in W_f. By virtue of the construction of $\eta_{j,t}$ in Theorem 4.1, together with the assumption that M is orientable, we can construct an immersion $\eta'\colon M' \to W' \times \mathbf{R}^{n-1}$ that lifts $q_f|_{M'}$, where

$$W' = C \cup \left(\bigcup_{j=1}^{k} h^1_j \right),$$

$M' = q_f^{-1}(W')$ and $\eta'|_{M_j} = \eta_j$ for each j. In fact, η' can be chosen so that it is an embedding on $q_f^{-1}(C)$. Recall that $q_f^{-1}(C)$ has the structure of a D^{n-1}-bundle over ∂W_f with structure group $SO(n-1)$ by [23]. See (5.2) below.

Let W'' be the closure of $W_f \setminus W'$ and set $M'' = q_f^{-1}(W'')$. Note that the restriction $q_f|_{M''}\colon M'' \to W''$ is a smooth fiber bundle with fiber the standard sphere S^{n-2}. Since $W'' \cong D^2$, it is a trivial fiber bundle.

Let us consider the immersion $\eta'|_{\partial M''}\colon \partial M'' \to \partial W'' \times \mathbf{R}^{n-1}$, which can be regarded as a family of immersions $S^{n-2} \to \mathbf{R}^{n-1}$ parameterized by $\partial W'' \cong S^1$. By the construction of η', we see that $\eta'|_{\partial M''}$, as an element in $\pi_1 \mathrm{Imm}(S^{n-2}, \mathbf{R}^{n-1})$, is represented by a family of embeddings whose images coincide with that of the standard embedding. In other words, there exists a family of diffeomorphisms $\Phi_t\colon S^{n-2} \to S^{n-2}$, $t \in [0,1]$, such that $\eta'|_{\partial M''} \in \pi_1 \mathrm{Imm}(S^{n-2}, \mathbf{R}^{n-1})$ corresponds to $\{i \circ \Phi_t\}_{t\in[0,1]}$, where $i\colon S^{n-2} \to \mathbf{R}^{n-1}$ is the standard embedding. In the following, we may assume that Φ_0 and Φ_1 are the identity of S^{n-2}, so that we can consider Φ_t to be a loop in $\mathrm{Diff}(S^{n-2})$.

It is known that $\mathrm{Diff}(S^{n-2})$ is weakly homotopy equivalent to $\mathrm{Diff}_*(S^{n-2}) \times O(n-1)$ (for example, see [9, Appendice] and also §2). Therefore, we have that $\pi_1(\mathrm{Diff}(S^{n-2}))$ is isomorphic to $\pi_1(\mathrm{Diff}_*(S^{n-2})) \times \pi_1(SO(n-1))$. Note that the $SO(n-1)$-factor corresponds to rotations of S^{n-2}, which naturally extend to rotations of D^{n-1}. Recall that the structure group of the D^{n-1}-bundle

$$q_f^{-1}(C) \xrightarrow{q_f} C \xrightarrow{\mathrm{pr}} \partial W_f \tag{5.2}$$

is reduced to $SO(n-1)$, where $\mathrm{pr}\colon C \cong \partial W_f \times [0,1] \to \partial W_f$ is the projection to the first factor (see [33]). Therefore, if $\alpha = \{\Phi_t\}_{t\in[0,1]} \in \pi_1(\mathrm{Diff}(S^{n-2}))$ corresponds to an element of the form

$$(0, *) \in \pi_1(\mathrm{Diff}_*(S^{n-2})) \times \pi_1(SO(n-1)) \cong \pi_1(\mathrm{Diff}(S^{n-2})),$$

then we can modify η' by rotating some disk fibers of (5.2) so that η' can be extended to an immersion lift of q_f.

Thus, in order to show that the family of immersions $\eta'|_{\partial M''}$ extends to an immersion lift $M'' \to W'' \times \mathbf{R}^{n-1}$ of $q_f|_{M''}\colon M'' \to W''$, we have only to check that the loop of "based" embeddings $\{i \circ \Phi_t\}_{t\in[0,1]}$, when Φ_t is a loop in $\mathrm{Diff}_*(S^{n-2})$, represents the trivial element in $\pi_1 \mathrm{Imm}_*(S^{n-2}, \mathbf{R}^{n-1})$ ($\cong \pi_{n-1}(SO(n-1))$). Since the family $\{i \circ \Phi_t\}_{t\in[0,1]}$ actually determines an element of $\pi_1 \mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1})$, it is straightforward from Lemma 5.2 below that it hits the trivial element in $\pi_1 \mathrm{Imm}_*(S^{n-2}, \mathbf{R}^{n-1})$ for $n \geq 6$. For $n = 3, 4$ and 5, $\mathrm{Diff}_*(S^{n-2})$ is contractible ([14, 37]) and hence the family $\{i \circ \Phi_t\}_{t\in[0,1]}$ is automatically trivial as an element of $\pi_1 \mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1})$. Note that the claim for $n = 4$ and 5 can also be deduced from Theorem 6.1 in the following section (see Remark 6.2). □

Lemma 5.2. *For $n \geq 6$, the natural homomorphism*

$$\pi_1 \mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \longrightarrow \pi_1 \mathrm{Imm}_*(S^{n-2}, \mathbf{R}^{n-1}),$$

induced by the inclusion, is trivial.

Proof. By [23, Theorem A (t/d)] (note that the argument in [23, Example in p. 146] is valid still in the codimension one case, see [11, 39]), $\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1})$ sits in the exact sequence

$$\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \longrightarrow \pi_1\mathrm{Imm}_*(S^{n-2}, \mathbf{R}^{n-1}) \xrightarrow{b_*} \pi_1\mathrm{Imm}_*^t(S^{n-2}, \mathbf{R}^{n-1}),$$

where $\mathrm{Imm}_*^t(S^{n-2}, \mathbf{R}^{n-1})$ is the space of locally flat topological immersions corresponding to $\mathrm{Imm}_*(S^{n-2}, \mathbf{R}^{n-1})$ and the two arrows are the homomorphisms induced by the inclusions. Furthermore, by Smale, Lees and Lashof ([36, 25, 22]), we see that the map b_* corresponds to the natural homomorphism

$$\pi_{n-1}(SO(n-1)) \longrightarrow \pi_{n-1}(\mathrm{Top}(n-1)),$$

where $\mathrm{Top}(n-1)$ denotes the space of germs of the homeomorphisms of $(\mathbf{R}^{n-1}, 0)$ to itself (e.g., see [5, §3.2 in Part II]; see also [23, Remark in the middle of p. 147]). Since this map is injective by [6, Proposition 5.4 (iv)], the result follows. □

Remark 5.3. In the above proof, the family $\{i \circ \Phi_t\}_{t\in[0,1]}$ may not necessarily represent the trivial element of $\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1})$ in general. More precisely, for each element $\sigma \in \pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1})$, $n \geq 6$, we can choose a special generic map for which the family $\{i \circ \Phi_t\}_{t\in[0,1]}$ determined as in the above proof represents σ. This is seen as follows.

First, by [4, Proposition 5.3], the group $\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \cong \pi_1\mathcal{K}_{n-1,n-2}$ is known to be isomorphic to $\pi_0\mathcal{K}_{n-1,n-1} \cong \pi_0\mathrm{Diff}_+(S^{n-1})$, where $\mathrm{Diff}_+(S^{n-1})$ denotes the group of orientation preserving diffeomorphisms of S^{n-1}, which has the homotopy type of $SO(n) \times \mathcal{K}_{n-1,n-1}$ (see [4, p. 47]). Furthermore, the group $\pi_0\mathrm{Diff}_+(S^{n-1})$ is isomorphic to the group Θ_n of homotopy n-spheres.

Now, let us choose the homotopy n-sphere Σ^n corresponding to σ by the above isomorphism. Then, for an arbitrary set of homotopy $(n-1)$-spheres $\Sigma_1^{n-1}, \Sigma_2^{n-1}, \ldots, \Sigma_k^{n-1}$, $k \geq 0$, there exists a special generic map

$$f\colon \left(\mathop{\sharp}_{j=1}^{k} (S^1 \times \Sigma_j^{n-1})\right) \sharp\, \Sigma^n \longrightarrow \mathbf{R}^2,$$

for which (each $M_{j,t}$ in the above proof corresponds to Σ_j^{n-1} and) the family $\{i \circ \Phi_t\}_{t\in[0,1]}$ represents the element σ corresponding to Σ^n in $\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \cong \Theta_n$.

Remark 5.4. The natural map

$$\mathrm{Emb}_*(S^m, \mathbf{R}^N) \longrightarrow \mathrm{Imm}_*(S^m, \mathbf{R}^N)$$

is the so-called Smale–Hirsch map [4, §2]. We have seen in Lemma 5.2 that in the codimension one case ($N = m + 1 \geq 5$) the Smale–Hirsch map induces the trivial homomorphism between the fundamental groups. It seems that the argument there may be applicable for the cases of some other dimensions and codimensions as well.

In the case of special generic maps of non-orientable manifolds, we have the following.

Theorem 5.5. *Let $f\colon M \to \mathbf{R}^2$ be a special generic map of a closed non-orientable n-dimensional manifold M, $n \geq 2$. Then, there exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ such that $f = \pi \circ \eta$ for the standard projection $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^2$ if and only if $n = 2, 4$ or 8, and the tubular neighborhood of the singular point set $S(f)$ in M is orientable.*

Proof. Since f has no cusp singularities, when $n = 2$ the theorem is a direct corollary of [13, Théorème 1]. We assume that $n > 2$ below.

First, suppose that an immersion lift $\eta\colon M \to \mathbf{R}^{n+1}$ exists. The collar neighborhood C of ∂W_f in W_f can be identified with an I'-bundle over ∂W_f, where $I' = [0, 1]$. For each singular point $p \in S(f)$, let $I'_p \cong [0, 1]$ be the fiber over $q_f(p) \in \partial W_f$ of the collar neighborhood. Then $D_p = q_f^{-1}(I'_p)$ is an $(n-1)$-dimensional disk centered at p. Since $\eta|_{D_p}$ is an immersion and $\pi \circ \eta|_{D_p}$ has a critical point at p, $\pi' \circ \eta|_{D_p}\colon D_p \to \mathbf{R}^{n-1}$ must be an immersion near p, where $\pi'\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n-1}$ is the projection defined by $\pi'(x_1, x_2, \ldots, x_{n+1}) = (x_3, x_4, \ldots, x_{n+1})$. Therefore, by compactness of $S(f)$, we may assume that each $(n-1)$-dimensional disk fiber of N is embedded into $\mathbf{R}^{n-1}$ by $\pi' \circ \eta|_N\colon N \to \mathbf{R}^{n-1}$, where $N = q_f^{-1}(C)$ is a small closed tubular neighborhood of $S(f)$ in M. This implies that N is orientable.

Since M is non-orientable, there exists a simple closed curve γ embedded in $\operatorname{Int} W_f$ such that $\Gamma = q_f^{-1}(\gamma)$ is a non-orientable S^{n-2}-bundle over $\gamma \cong S^1$. For the immersion lift $\eta\colon M \to \mathbf{R}^{n+1}$ of f, $\pi' \circ \eta$ restricted to Γ is an immersion on each S^{n-2}-fiber. This implies that the standard embedding of S^{n-2} into $\mathbf{R}^{n-1}$ is regularly homotopic to its precomposition with an orientation-reversing diffeomorphism, and hence that $n = 4$ or 8 by [1, 16, 27, 36].

Conversely, suppose $n = 4$ or 8 and that the tubular neighborhood N of $S(f)$ in M is orientable. We may assume that M is connected. Then we can find an embedding $\eta_1\colon N \to W_f \times \mathbf{R}^{n-1}$ such that $p_1 \circ \eta_1 = q_f|_N\colon N \to C$, where $p_1\colon W_f \times \mathbf{R}^{n-1} \to W_f$ is the projection to the first factor. Then, using the notation as in the proof of Theorem 5.1, we observe that, for each j, q_f restricted to $M_{j,t} = q_f^{-1}(I \times \{t\})$, $t \in J$, is a special generic map into $I \times \{t\}$, and that its immersion lift has already been constructed over the small disk neighborhoods of its critical points. Note that the immersion lift may or may not respect the orientations of the disks induced by an orientation of $M_{j,t}$. Nevertheless, we can extend the embedding η_1 to an immersion lift of q_f over each 1-handle h^1_j of W_f, since the standard embedding $S^{n-2} \to \mathbf{R}^{n-1}$ is regularly homotopic to its reflection by our assumption on n. We thus obtain an immersion $\eta'\colon M' \to W' \times \mathbf{R}^{n-1}$ that lifts $q_f|_{M'}$, where M' and W' are as in the proof of Theorem 5.1. Then, we can apply the rest of the argument in the proof of Theorem 5.1 to obtain a desired immersion lift of f. $\square$

We do not know if we have a statement corresponding to Theorem 4.3 or not for special generic maps into $\mathbf{R}^2$.

For embedding lifts, we have the following. Compare it with Theorem 4.6.

Theorem 5.6. *Let $f\colon M \to \mathbf{R}^2$ be a special generic map of a closed connected n-dimensional manifold M with $n \geq 3$. Then, there exists an embedding $\eta\colon M \to \mathbf{R}^{n+1}$*

such that $f = \pi \circ \eta$ if and only if M is diffeomorphic to S^n or the connected sum of a finite number of copies of $S^1 \times S^{n-1}$, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^2$ is the standard projection.

Proof. Suppose that an embedding lift η as in the theorem exists. Note first then, that the source manifold must be orientable. Then, if $n \le 6$ the source manifold must be diffeomorphic to S^n or the connected sum of a finite number of copies of $S^1 \times S^{n-1}$ (see the remark at the beginning of this section). Assume $n \ge 7$ and let $\pi'\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n-1}$ be the projection defined by $\pi'(x_1, x_2, \ldots, x_{n+1}) = (x_3, x_4, \ldots, x_{n+1})$. Then we see that the map $\widetilde{\eta} = (q_f, \pi' \circ \eta)\colon M \to W_f \times \mathbf{R}^{n-1}$ is an embedding. Let us consider the handlebody decomposition (5.1) of W_f and use the same notation as in the proof of Theorem 5.1. For $I_{j,t} = I \times \{t\} \subset h_j^1$, $t \in J$, set $M_{j,t} = q_f^{-1}(I_{j,t})$. Then, $q_f|_{M_{j,t}}\colon M_{j,t} \to I_{j,t}$ is a special generic function and $\widetilde{\eta}|_{M_{j,t}}\colon M_{j,t} \to I_{j,t} \times \mathbf{R}^{n-1}$ is an embedding lift. Therefore, $M_{j,t}$ must be diffeomorphic to the standard sphere S^{n-1}.

Furthermore, $\widetilde{\eta}(M_{j,t})$ bounds an n-dimensional disk in $I_{j,t} \times \mathbf{R}^{n-1}$. Let $\widehat{M}$ be the closed n-dimensional manifold obtained by attaching the n-dimensional disks bounded by $M_{j,t}$, $j = 1, 2, \ldots, k$ and $t = \pm 1$, to $\widetilde{\eta}(q_f^{-1}(h^0))$. Then, it is a homotopy n-sphere embedded in $h^0 \times \mathbf{R}^{n-1}$. Therefore, $\widehat{M}$ is diffeomorphic to the standard n-sphere and bounds an $(n+1)$-dimensional disk in $h^0 \times \mathbf{R}^{n-1}$. Thus, $\widetilde{\eta}(M)$ is the boundary of a compact orientable $(n+1)$-dimensional manifold obtained by attaching k copies of $D^n \times J$, i.e., k copies of a 1-handle, to D^{n+1}. Hence, M is diffeomorphic to S^n or the connected sum of k copies of $S^1 \times S^{n-1}$.

Conversely, suppose that M is diffeomorphic to $\sharp^k (S^1 \times S^{n-1})$, where $k \ge 0$ and it denotes S^n when $k = 0$. When $n = 3, 4$ or 5, the construction of the immersion lift in the proof of Theorem 5.1 can in fact yield an embedding lift, since there the loop $\{i \circ \Phi_t\}_{t \in [0,1]}$ is trivial as an element of $\pi_1 \mathrm{Diff}_*(S^{n-2})$.

Assume that $n \ge 6$. Recall that a special generic map f determines (depending on the choice of a handlebody decomposition of W_f) the decomposition

$$M \cong \left(\mathop{\sharp}_{j=1}^{k} (S^1 \times \Sigma_j^{n-1}) \right) \sharp\, \Sigma^n,$$

where each Σ_j^{n-1} is a homotopy $(n-1)$-sphere which corresponds to $M_{j,t}$ appearing in the construction of the immersion lift in the proof of Theorem 5.1, and Σ^n corresponds to the family $\{i \circ \Phi_t\}_{t \in [0,1]}$ under the isomorphism $\pi_1 \mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \cong \Theta_n$ (see Remark 5.3).

Now, each homotopy $(n-1)$-sphere $\Sigma_j^{n-1} \cong M_{j,t}$ is embedded in M. Since the universal cover of $M \cong \sharp^k (S^1 \times S^{n-1})$ can be smoothly embedded in S^n as the orbit of a fundamental domain of a Schottky group, isomorphic to $\pi_1(M)$, acting on S^n (see e.g. pages 508–510 in [19]), we see that each Σ_j^{n-1} is diffeomorphic to the standard sphere S^{n-1}. Thus, we see that

$$\left(\overset{k}{\sharp} (S^1 \times S^{n-1}) \right) \sharp\, \Sigma^n \cong \overset{k}{\sharp} (S^1 \times S^{n-1}),$$

in other words, Σ^n lies in the inertia group of $\sharp^k(S^1 \times S^{n-1})$, where recall that for a closed oriented n-dimensional manifold N the inertia group $I(N)$ is the subgroup of Θ_n consisting of those homotopy n-spheres Σ such that N is diffeomorphic to $N \sharp \Sigma$. In Lemma 5.7 below, we will prove that the group $I(\sharp^k(S^1 \times S^{n-1}))$ is trivial. Therefore, in turn we see that Σ^n is diffeomorphic to the standard sphere S^n.

In view of the construction of the immersion lift in the proof of Theorem 5.1, "$\Sigma_j^{n-1} \cong S^{n-1}$" implies that each η_j and hence η' can be chosen to be embeddings. The obstruction to extending this "partial embedding lift" η' into $W_f \times \mathbf{R}^{n-1}$ on the whole M coincides with Σ^n sitting in $\pi_1\mathrm{Emb}_*(S^{n-2}, \mathbf{R}^{n-1}) \cong \Theta_n$ (see Remark 5.3). Therefore, "$\Sigma^n \cong S^n$" ensures the existence of an embedding of M into $W_f \times \mathbf{R}^{n-1}$ which lifts $q_f\colon M \to W_f$.

Hence, all we have to show now is that there is an embedding of $W_f \times \mathbf{R}^{n-1}$ into $\mathbf{R}^{n+1}$ whose composition with an orthogonal projection $\mathbf{R}^{n+1} \to \mathbf{R}^2$ coincides with the map $\bar{f}\colon W_f \to \mathbf{R}^2$. Considering $n \geq 3$, this is seen as follows. By obviously pasting two copies of the immersion $\bar{f}\colon W_f \to \mathbf{R}^2$ of the surface W_f along the boundary, we obtain the stable (fold) map $\bar{f} \cup \bar{f}$ of the closed surface $W_f \cup W_f$ into $\mathbf{R}^2$. By Minoru Yamamoto's theorem [40], this map $\bar{f} \cup \bar{f}$, and hence $\bar{f}$, factors through an embedding into $\mathbf{R}^4$. Since such an embedding $W_f \to \mathbf{R}^4$ has the trivial normal bundle, it extends to an embedding $W_f \times \mathbf{R}^2 \to \mathbf{R}^4$. By taking the product with $\mathbf{R}^{n-3}$, we obtain a desired embedding $W_f \times \mathbf{R}^{n-1} \to \mathbf{R}^{n+1}$.

This completes the proof. □

Lemma 5.7. *The inertia group of the connected sum of a finite number of copies of $S^1 \times S^{n-1}$, $n \geq 5$, is trivial.*

Proof. For $n = 5$ and 6, there is nothing to prove since Θ_n is trivial. When $n \geq 7$, since the connected sum $\sharp^k(S^1 \times S^{n-1})$ of k copies of $S^1 \times S^{n-1}$ ($k \geq 1$) can be embedded in $\mathbf{R}^{n+1}$, we can apply the argument of [35, Proposition and Remark in §1.2]. Namely, it suffices to show that each automorphism on the fundamental and homology groups of $\sharp^k(S^1 \times S^{n-1})$ is induced by a diffeomorphism. Furthermore, by Poincaré duality, we have only to show that each automorphism on the fundamental group is induced by a diffeomorphism, but this is proven by [24, Lemma 2 and Remark on p. 340]. □

Remark 5.8. In contrast to Theorem 5.6, there exists a special generic map $f\colon S^2 \to \mathbf{R}^2$ which cannot be factored through any embedding $\eta\colon S^2 \to \mathbf{R}^3$ (but which can be factored through an immersion by [13] or by Theorem 5.1). For example, we can check that the map ("S^1-immersion" in the terminology there) constructed by Èliašberg by using [12, Figure 4 in Appendix 1] is such a case (see [41] for details and other examples).

Remark 5.9. In Theorem 5.6, the condition for the source manifold to be "standard" is not a necessary condition for the embeddability in codimension one, as indicated by the following example.

Let Σ^{2k-1} be a homotopy $(2k-1)$-sphere, $k \geq 4$, which bounds a parallelizable $2k$-dimensional manifold P^{2k}. We may assume that P^{2k} is $(k-1)$-connected, so that

P^{2k} has the homotopy type of a k-dimensional cell complex. Thus, P^{2k} and hence Σ^{2k-1} can be embedded in $\mathbf{R}^{2k+1}$. Since an embedding $\Sigma^{2k-1} \to \mathbf{R}^{2k+1}$ has trivial normal bundle, there exists an embedding of $M = S^1 \times \Sigma^{2k-1}$ into $\mathbf{R}^{2k+1}$ (as the boundary of a tubular neighborhood of the embedded Σ^{2k-1}).

Namely, such a manifold M, which admits a special generic map into $\mathbf{R}^2$ by [33], can be embedded into $\mathbf{R}^{2k+1}$, even if it is not "standard", that is, not diffeomorphic to $S^1 \times S^{2k-1}$. We also note that by Theorem 5.6, if M is not standard, then no embedding of M into R^{2k+1} can be projected to a special generic map into R^2.

6. Further results

In this section, we study the lifting problem of special generic maps of low dimensional manifolds. First we show the following.

Theorem 6.1. *Let $f\colon M \to \mathbf{R}^p$ be a special generic map of a closed orientable n-dimensional manifold M with $n - p = 2$ or 3 and $p \leq 4$, and $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^2$ the standard projection. Then, there exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ such that $f = \pi \circ \eta$ if and only if M is spin, i.e., the second Stiefel–Whitney class $w_2(M) \in H^2(M; \mathbf{Z}_2)$ of M vanishes.*

Remark 6.2. Those dimension pairs (n, p) which satisfy the conditions of the above theorem are $(3, 1)$, $(4, 1)$, $(4, 2)$, $(5, 2)$, $(5, 3)$, $(6, 3)$, $(6, 4)$ and $(7, 4)$. For (n, p) equal to $(3, 1), (4, 1), (4, 2)$ and $(5, 2)$, a closed orientable manifold M as above has vanishing second cohomology and hence M is always spin.

Proof of Theorem 6.1. If there exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$, then M is stably parallelizable, since M is orientable of dimension n. Therefore, M is spin.

Conversely, suppose that M is spin. By [33], there exists a linear D^{n-p+1}-bundle $\xi\colon E \to W_f$ such that $\partial E \cong M$, since $n - p \leq 3$. Furthermore, modifying the map ξ slightly near $\xi^{-1}(\partial W_f)$, we can construct a smooth map $\widehat{\xi}\colon E \to W_f$ such that

(1) $\widehat{\xi}|_{\partial E}$ can be identified with q_f,
(2) $\widehat{\xi}^{-1}(\partial W_f) = S(f)$, and
(3) $\widehat{\xi}|_{E \setminus S(f)} : E \setminus S(f) \to \operatorname{Int} W_f$ is a linear D^{n-p+1}-bundle.

Let us consider the cohomology exact sequence for the pair $(E, \partial E) = (E, M)$:

$$H^2(E, M; \mathbf{Z}_2) \longrightarrow H^2(E; \mathbf{Z}_2) \xrightarrow{i^*} H^2(M; \mathbf{Z}_2),$$

where $i\colon M \to E$ is the inclusion map. Note that $H^2(E, M; \mathbf{Z}_2)$, $H_{n-1}(E; \mathbf{Z}_2)$, and $H_{n-1}(W_f; \mathbf{Z}_2)$ are trivial since $n - 1 \geq p$ and W_f is a p-dimensional manifold each of whose components has nonempty boundary. Thus, $i^*\colon H^2(E; \mathbf{Z}_2) \to H^2(M; \mathbf{Z}_2)$ is injective. By assumption, we have $w_2(M) = 0$ and hence we have $w_2(E) = 0$ by the naturality of the Stiefel–Whitney class.

On the other hand, since $TE \cong \xi^*(TW_f \oplus \zeta)$ and W_f is parallelizable, we have $w_2(E) = \xi^* w_2(\zeta)$, where ζ is the $(n - p + 1)$-plane bundle over W_f associated with ξ. As $\xi^*\colon H^2(W_f; \mathbf{Z}_2) \to H^2(E; \mathbf{Z}_2)$ is an isomorphism, we see that $w_2(\zeta) = 0$. Note that

W_f has the homotopy type of a CW complex of dimension $p-1 \leq 3$. Consequently ζ is a trivial $(n-p+1)$-plane bundle, since it is orientable, its second Stiefel–Whitney class vanishes, $n-p+1 \geq 3$, and $\pi_2(SO(n-p+1)) = 0$.

Therefore, by thickening the immersion $\bar{f} : W_f \to \mathbf{R}^p \subset \mathbf{R}^{n+1}$, we can construct an immersion $\widetilde{\eta}\colon E \to \mathbf{R}^{n+1}$ such that $\pi \circ \widetilde{\eta} = \bar{f} \circ \widehat{\xi}$. The immersion obtained as the restriction to $\partial E = M$, $\eta = \widetilde{\eta}|_{\partial E}$, is a desired lift of f. This completes the proof. $\square$

Let us now consider the case $n-p=1$. Let $f\colon M \to \mathbf{R}^p$ be a special generic map with $n = \dim M > p$. We fix an orientation of the target $\mathbf{R}^p$ once and for all. Since the map $\bar{f}\colon W_f \to \mathbf{R}^p$ is an immersion, W_f can be oriented so that $\bar{f}$ preserves the orientations. Then its boundary ∂W_f has the induced orientation. We orient $S(f)$ so that $q_f|_{S(f)}\colon S(f) \to \partial W_f$ is an orientation preserving diffeomorphism.

Theorem 6.3. *Let $f\colon M \to \mathbf{R}^p$ be a special generic map of a closed orientable n-dimensional manifold M with $n-p=1$ and $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^p$ the standard projection. Then, there exists an immersion $\eta\colon M \to \mathbf{R}^{n+1}$ such that $f = \pi \circ \eta$ if and only if the homology class $[S(f)] \in H_{p-1}(M)$ represented by $S(f)$ vanishes.*

Proof. We may assume that M is connected. Then the map

$$q_f|_{M \setminus S(f)}\colon M \setminus S(f) \longrightarrow \operatorname{Int} W_f \tag{6.1}$$

is a smooth S^1-bundle, whose structure group can be reduced to $SO(2)$.

First suppose that there exists an immersion η such that $f = \pi \circ \eta$. Define the new function $F\colon M \setminus S(f) \to \mathbf{R}^2$ by $F = \pi' \circ \eta|_{M \setminus S(f)}$, where $\pi'\colon \mathbf{R}^{n+1} \to \mathbf{R}^2$ is the projection defined by

$$\pi'(x_1, x_2, \ldots, x_{n+1}) = (x_n, x_{n+1}).$$

Then, for each $y \in \operatorname{Int} W_f$, the map

$$F|_{q_f^{-1}(y)}\colon q_f^{-1}(y) \longrightarrow \mathbf{R}^2 \tag{6.2}$$

is an immersion of a circle into the plane, since η is an immersion.

Let us show that the winding number of the immersion (6.2) is equal to ± 1. Take a point $x_0 \in S(f)$ and set $y_0 = q_f(x_0)$. Let J be a small arc embedded in W_f which intersects ∂W_f transversely exactly at y_0. Then, $D = q_f^{-1}(J)$ is a small 2-disk which intersects $S(f)$ transversely exactly at x_0. Note that $f = \pi \circ \eta$ and the differential of $f|_D$ at x_0 vanishes. Therefore, the differential of $\pi' \circ \eta|_D$ at x_0 must have rank two, since $\eta|_D$ is an immersion. By taking J smaller, we may assume that $\eta|_D$ is an embedding by the implicit function theorem. Thus, the immersion (6.2) is an embedding for the end point y of J different from y_0. In particular, its winding number is equal to ± 1. As M is connected, so is W_f. Therefore, $\operatorname{Int} W_f$ is also connected. Hence, the immersions (6.2) are all regularly homotopic. Consequently, the winding number of the immersion (6.2) is always equal to ± 1.

Therefore, by Lemma 3.3, the S^1-bundle (6.1) is trivial. Consequently, there exists a topological embedding $s\colon W_f \to M$ such that $q_f \circ s = \mathrm{id}_{W_f}$. Note that $S(f)$ coincides with the oriented boundary of $s(W_f)$, which implies that $[S(f)] \in H_{p-1}(M)$ vanishes.

Conversely, suppose that $[S(f)] \in H_{p-1}(M)$ vanishes. Let us show that the S^1-bundle (6.1) is trivial.

By [33], there exists a linear D^2-bundle $\xi\colon E \to W_f$ with $\partial E \cong M$ and a smooth map $\widehat{\xi}\colon E \to W_f$ with $\widehat{\xi}|_{\partial E}$ being identified with $q_f\colon M \to W_f$ as in the proof of Theorem 6.1. Let C be a small collar neighborhood of ∂W_f in W_f and set $N_S = q_f^{-1}(C)$, which is a tubular neighborhood of $S(f)$ in M. Note that N_S can be identified with $\xi^{-1}(\partial W_f)$. Set $M' = M \setminus \operatorname{Int} N_S$. Then, the Euler class of the linear D^2-bundle ξ is the image of the Thom class u by the composition

$$H^2(E, M') \xrightarrow{i^*} H^2(E) \cong H^2(W_f),$$

where $i\colon E \to (E, M')$ is the inclusion map. Let us observe that $H^2(E, M)$, $H_p(E)$, and $H_p(W_f)$ are trivial. In view of the exact sequence

$$H^2(E, M) \longrightarrow H^2(E) \xrightarrow{j^*} H^2(M),$$

where $j\colon M \to E$ is the inclusion map, in order to show that the bundle ξ is trivial, we have only to show that $j^*i^*u = 0$ in $H^2(M)$.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^2(E, M') & \xrightarrow{k^*} & H^2(M, M') & \xrightarrow{\cong} & H^2(N_S, \partial N_S) & \xrightarrow{\cong} & H_{p-1}(N_S) \\
{\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle \ell^*} & & & & \big\downarrow{\scriptstyle h_*} \\
H^2(E) & \xrightarrow{j^*} & H^2(M) & \xrightarrow{=} & H^2(M) & \xrightarrow{\cong} & H_{p-1}(M),
\end{array}$$

where $k\colon (M, M') \to (E, M')$, $\ell : M \to (M, M')$ and $h\colon N_S \to M$ are the inclusion maps, the second map in the upper row is the excision isomorphism, and the other two isomorphisms correspond to the Poincaré–Lefschetz duality.

By the naturality of the Thom class, the image of $u \in H^2(E, M')$ in $H^2(N_S, \partial N_S)$ is the Thom class of the D^2-bundle $N_S \to S(f)$. Therefore, we see that j^*i^*u considered as an element of $H_{p-1}(M)$ coincides with $\pm[S(f)]$. By our assumption this vanishes, and hence the bundles ξ and (6.1) are trivial. By this, we can construct a desired lift of f in exactly the same way as in the last paragraph of the proof of Theorem 6.1. This completes the proof. $\square$

Remark 6.4. For a special generic map $f\colon M \to \mathbf{R}^p$, it is known that the $\mathbf{Z}_2$-homology class $[S(f)]_2 \in H_{p-1}(M; \mathbf{Z}_2)$ represented by $S(f)$ is Poincaré dual to the Stiefel–Whitney class $w_{n-p+1}(M)$ (see [38]). Therefore, if the condition in Theorem 6.3 is satisfied, then the source manifold M is spin.

Remark 6.5. When $(n, p) = (3, 2)$, by Theorem 5.1, every special generic map $f\colon M \to \mathbf{R}^2$ of a closed orientable 3-manifold M can be lifted to an immersion $M \to \mathbf{R}^4$. In fact, we can show that, in such a case, $[S(f)] \in H_1(M)$ always vanishes.

For embedding lifts, we have the following.

Proposition 6.6. *If $2p < n+1$ in Theorems* 6.1 *or* 6.3*, then we can take an embedding as the lift η of f.*

Proof. Since $\bar{f}\colon W_f \to \mathbf{R}^p$ can be lifted to an embedding into $\mathbf{R}^{n+1}$, the result follows. □

Remark 6.7. The case $(n, p) = (6, 3)$, which is covered by Proposition 6.6, is not included in Theorems 4.6 and 5.6. In this case, in view of [33, Proposition 6.13], we see the following. For a special generic map $f\colon M \to \mathbf{R}^3$ of a closed simply-connected 6-dimensional manifold M, there exists an embedding $\eta\colon M \to \mathbf{R}^7$ such that $f = \pi \circ \eta$ if and only if M is diffeomorphic to S^6 or the connected sum of a finite number of copies of $S^2 \times S^4$, where $\pi\colon \mathbf{R}^7 \to \mathbf{R}^3$ is the standard projection. Compare this with Theorems 4.6 and 5.6.

We also have the following.

Theorem 6.8. *In Theorem* 6.1*, or in Theorem* 6.3 *with $p \leq 3$, there exists a regular homotopy of immersions $\eta_t\colon M \to \mathbf{R}^{n+1}$, $t \in [0, 1]$, with $f = \pi \circ \eta_0$ such that $f_t = \pi \circ \eta_t$ is a special generic map for each $t \in [0, 1]$ and η_1 is an embedding.*

Proof. Since W_f is a smooth p-dimensional manifold each of whose components has nonempty boundary, there exists a $(p-1)$-dimensional subcomplex V in W_f such that W_f can be isotopically deformed to a small regular neighborhood $\widetilde{V}$ of V.

Let us consider the immersion $\bar{f}\colon W_f \to \mathbf{R}^p = \mathbf{R}^p \times \{0\} \subset \mathbf{R}^p \times \mathbf{R}^{n-p+1}$. Since $(p-1) + (p-1) < n+1$, by a small regular homotopy, we can deform $\bar{f}$ so that it is an embedding on $\widetilde{V}$. Therefore, by a further regular homotopy, say $\{\bar{f}_t\}$, $\bar{f}$ can be deformed to an embedding $\bar{f}_1\colon W_f \to \mathbf{R}^{n+1}$.

In the proofs of Theorems 6.1 and 6.3, we can construct the immersion $\widetilde{\eta}$ of E in such a way that its image is contained in a small regular neighborhood of $\bar{f}(W_f) \subset \mathbf{R}^p \times \{0\}$ and each fiber of $\widehat{\xi}\colon E \to W_f$ is mapped to a space of the form $\{*\} \times \mathbf{R}^{n-p+1}$. Therefore, we can deform $\widetilde{\eta}$ by a regular homotopy that is compatible with $\{\bar{f}_t\}$, say $\{\widetilde{\eta}_t\}$, so that the resulting immersion $\widetilde{\eta}_1$ is an embedding and that $\pi \circ \widetilde{\eta}_t$ restricted to ∂E is a special generic map for each t. The restriction $\eta_t = \widetilde{\eta}_t|_{\partial E}$ is the desired regular homotopy. This completes the proof. □

Theorem 6.9. *Let $f\colon M \to \mathbf{R}^3$ be a special generic map of a closed orientable* 4*-dimensional manifold M. Then, M can be embedded into $\mathbf{R}^5$ if and only if M is spin.*

Proof. If M can be embedded into $\mathbf{R}^5$, then it is clearly spin.

Conversely, suppose that M is spin. There exists a linear D^2-bundle $\xi\colon E \to W_f$ such that $\partial E \cong M$. By the proof of Theorem 6.1, we have $w_2(\zeta) = 0$, where ζ is the 2-plane bundle associated with ξ. Let us show that E is diffeomorphic to $W_f \times D^2$.

Let $e(\zeta) \in H^2(W_f)$ be the Euler class of ζ. Since its modulo two reduction coincides with $w_2(\zeta)$, which vanishes, there exists a cohomology class $e' \in H^2(W_f)$ with $e(\zeta) = 2e'$. Let $A = \bigcup \alpha_i$ be a finite disjoint union of properly embedded oriented arcs α_i in W_f

which is Poincaré dual to e'. Let $D_i (\subset \operatorname{Int} W_f)$ be a small 2-disk fiber of the tubular neighborhood L_i of α_i in W_f. Note that $L_i \cong D_i \times [-1,1]$.

If we remove the union of 2-handles $\bigcup(\xi^{-1}(L_i)) \cong \bigcup(D_i \times [-1,1] \times D^2)$ from the 5-dimensional manifold E and paste it back by another diffeomorphism corresponding to -2 times the generator of $\pi_1(SO(2))$ for each i, then we obtain a trivial D^2-bundle over W_f, where $SO(2)$ is the group of rotations of the last D^2-factor of each $D_i \times [-1,1] \times D^2$.

On the other hand, by the homotopy exact sequence associated with the fiber bundle $SO(2) \longrightarrow SO(3) \to SO(3)/SO(2) = S^2$,

$$0 \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(SO(2)) \longrightarrow \pi_1(SO(3)) \longrightarrow 0,$$

we see that the corresponding element in $\pi_1(SO(3))$ vanishes. This means that the attaching diffeomorphism of each of the 5-dimensional 2-handles is isotopic to the original attaching map. Therefore, E is diffeomorphic to $W_f \times D^2$.

Consequently, there exists an immersion $\widetilde{\eta}\colon E \to \mathbf{R}^5$ such that $\pi \circ \widetilde{\eta} = \bar{f} \circ p'$, where $p'\colon E \cong W_f \times D^2 \to W_f$ is the projection to the first factor.

Since W_f has a handlebody decomposition consisting of 0-, 1- and 2-handles, so does E. Thus, we can modify the immersion $\widetilde{\eta}$ by a regular homotopy so that we get an embedding $E \to \mathbf{R}^5$. Therefore, $M \cong \partial E$ can be embedded into $\mathbf{R}^5$. $\square$

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Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: `saeki@imi.kyushu-u.ac.jp`

Faculty of Science and Technology, Seikei University, 3-3-1 Kichijoji-kitamachi, Musashino, Tokyo 180-8633, Japan

E-mail address: `mtakase@st.seikei.ac.jp`