

## Cork twisting Schoenflies problem

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ABSTRACT. The stable Andrews-Curtis conjecture in combinatorial group theory is the statement that every balanced presentation of the trivial group can be simplified to the trivial form by elementary moves corresponding to “handle-slides” together with “stabilization” moves. Schoenflies conjecture is the statement that the complement of any smooth embedding  $S^3 \hookrightarrow S^4$  is a pair of smooth balls. Here we suggest an approach to these problems by a cork twisting operation on contractible manifolds, and demonstrate it on the example of the first Cappell-Shaneson homotopy sphere.

### 1. Introduction

Let  $G(P) = \{x_1, x_2, \dots, x_n \mid r_1(x_1, \dots, x_n), \dots, r_n(x_1, \dots, x_n)\}$  be the trivial group for a balanced presentation  $P$ . Here balanced means the presentation has the same number of generators and relators. When there is no danger of confusion we will abbreviate  $r_j := r_j(x_1, \dots, x_n)$ . The presentation  $P$  is called *stably Andrews-Curtis trivial* (SAC-trivial in short) if by changing relators by the following finite number of the steps, and their inverses, we obtain the trivial presentation:

- (a)  $r_i \mapsto r_i r_j$  for some  $j \neq i$ .
- (b) Add a new generator  $x_{n+1}$  and a relation  $x_{n+1} \gamma$ , for any word  $\gamma \in G(P)$ .
- (c)  $r_i \mapsto r_i^{-1}$  or  $r_i \mapsto \gamma r_i \gamma^{-1}$ , where  $\gamma$  is any word in  $G(P)$ .

Fundamental group of any compact 2-complex gives such a presentation, so any compact contractible 4-manifold  $W$ , which is a *2-handlebody* (i.e., a handlebody consisting of handles of index  $\leq 2$ ) has such a presentation. Generators  $\{x_j\}$  correspond to the 1-handles, and the relations  $\{r_j\}$  correspond to the 2-handles. (a) corresponds to sliding 2-handles over each other, and (b) corresponds to introducing (or taking away) a cancelling pair of 1 and 2-handles. Call a pair of 2-handlebodies SAC *equivalent* if they are related by these two steps.

Not much is known about which presentations of the trivial group are SAC-trivial. In [AK] the following examples were introduced ( $n=0,1,\dots$ )

$$G(P_n) = \{x, y \mid xyx = yxy, x^{n+1} = y^n\}$$

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Partially supported by NSF grant DMS 0905917.

$G(P_n)$  is the trivial group since the first relation gives  $y = (yx)^{-1}x(yx)$ , so we have  $y^{n+1} = (yx)^{-1}x^{n+1}(yx) = (yx)^{-1}y^n(yx) = x^{-1}y^n x = x^{-1}x^{n+1}x = y^n$ , hence  $y = 1$  and  $x = 1$ . Gersten showed that  $G(P_2)$  is SAC-trivial ([Ge], [GST]), but it is not known whether the other  $G(P_n)$  are SAC-trivial. The case  $G(P_4)$  is particularly interesting, since it is the fundamental group presentation of a 2-handlebody of the first Cappell-Shaneson homotopy 4-ball  $W_0 = \Sigma_0 - B^4$  constructed in [AK], where  $\Sigma_0$  is the 2-fold covering space of the first Cappell-Shaneson exotic  $\mathbb{R}P^4$  defined in [CS].

The main reason topologists are interested in the SAC problem is its relation to the Schoenflies conjecture, which says “*The complement of a smoothly imbedded  $S^3 \hookrightarrow S^4$  is a disjoint union of two smooth 4-balls*”. So far, in dimension 4 only the topological version of this conjecture is known to be true ([B], [M]). If the presentation of  $\pi_1(W)$  of a smooth contractible 2-handlebody  $W^4$  is SAC-trivial then  $W \times [0, 1] = B^5$  (because in dimension 5 cancelling handles algebraically is equivalent canceling them geometrically), hence on the boundary this gives an imbedding  $W \hookrightarrow S^4$  via its double

$$D(W) := W \cup_{\partial} -W = \partial(W \times [0, 1]) = S^4$$

Then since  $\partial W = S^3$  the topological Schoenflies theorem implies  $W$  is homeomorphic to  $B^4$ . To apply this to the smooth Poincaré conjecture (PC), we first puncture a given smooth homotopy 4-sphere  $\Sigma$  to a homotopy 4-ball  $W = \Sigma - B^4$  and turn it to a 2-handlebody by canceling its 3-handles (if we can!), reducing it to a SAC problem. This is what is done for the first Cappell-Shaneson homotopy 4-ball  $W_0$  in [AK] (similar proof works for the other ones [G]). But there the associated SAC problem was bypassed by directly imbedding  $W_0 \hookrightarrow S^4$ , hence reducing the smooth PC to Schoenflies problem. In particular if the complement of  $W$  in  $S^4$  is  $B^4$  then  $W$  itself must be  $B^4$ . Figure 1 is the 3-handle free handlebody picture of  $W_0$ , which imbeds into  $S^4$  (Figure 28 of [AK]).

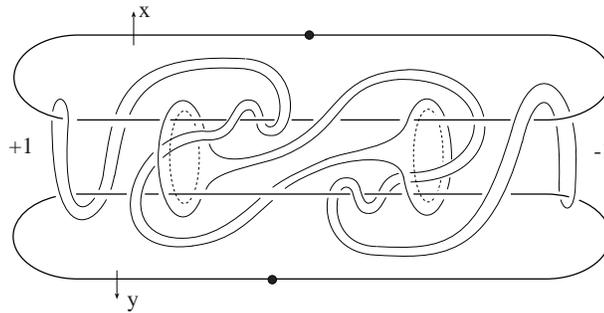


FIGURE 1.  $W_0$

In the end it turned out that all Cappell-Shaneson homotopy balls  $W_0, W_1, \dots$  ([CS]) are in fact diffeomorphic to  $B^4$ ; without even appealing to the Schoenflies problem ([G], [A1]). More specifically, the proofs proceed by first introducing canceling 2/3 handle pairs, then cancelling all the handles ending up with  $W_i = B^4$ ,  $i = 0, 1, 2, \dots$ . Here we revisit the Schoenflies problem by analyzing the approach of [AK] more closely, where another 2-handlebody  $W_0^*$  with  $\partial W_0^* = S^3$  was constructed so that

$$S^4 = W_0 \cup_{\partial} -W_0^*$$

We can reduce this Schoenflies problem to another Schoenflies problem which we can solve, i.e., by imbedding  $W_0^* \hookrightarrow S^4$  with complement  $B^4$

$$S^4 = W_0^* \cup_{\partial} B^4$$

which implies  $W_0^* = B^4$ , and so  $W_0 = B^4$ . Of course this last step is not new, it is just a case of proving some homotopy 4-balls are standard by introducing a single cancelling 2/3 handle pair ([G], [A1]). We stated it this way, to relate it to Schoenflies problem. Curiously the associated presentation of the fundamental group of  $W_0^*$  is  $G(P_2)$ , while  $\pi_1(W_0)$  is  $G(P_4)$ . The hope is, in general associating a contractible  $W$  another convenient “*twin*” contractible manifold  $W \mapsto W^*$  might help to resolve SAC triviality.

## 2. Flexible contractible manifolds

A flexible contractible 4-manifold is a smooth compact contractible 2-handlebody, where its 2-handles are represented by 0-framed unknotted curves (i.e., after erasing circles with dots we get a 0-framed unlink).

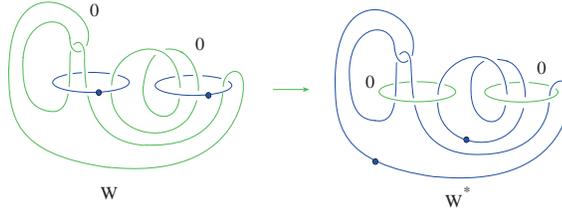


FIGURE 2. A flexible contractible manifold and its twin

We call the 2-handlebody  $W^*$  obtained from  $W$  by zero and dot exchanges of its handles ( $S^2 \times B^2 \leftrightarrow B^3 \times S^1$  exchanges in the interior of  $W$ ) the *twin of  $W$* . We call the operation  $W \mapsto W^*$  *cork twisting the flexible manifold  $W$* . Notice that this notion depends on the handles. Here we do not address the problem of how the twin of  $W$  changes after handle slides of  $W$ . It is clear that this decomposes the 4-sphere as  $S^4 = W \cup_{\partial} -W^*$ , i.e.,  $W$  imbeds into  $S^4$  with complement  $W^*$ .

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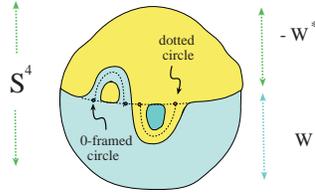


FIGURE 3. A flexible contractible manifold and its twin

3. The twin of  $W_0$

First by applying the diffeomorphism described in Section 2.3 of [A3], we identify the handlebody  $W_0$  of Figure 1 with Figure 4. This diffeomorphism is a combination of introducing and cancelling 1/2 handle pairs. Recall that the associated presentation of  $\pi_1(W_0)$  is  $G(P_4)$ .

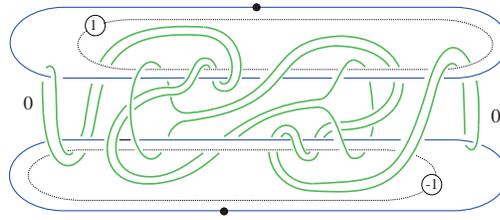


FIGURE 4.  $W_0$

**Theorem 3.1.** *The twin  $W_0^*$  of the 2-handlebody of  $W_0$  in Figure 4 is given by Figure 5, and furthermore the associated presentation of  $\pi_1(W_0^*)$  is  $G(P_2)$ .*

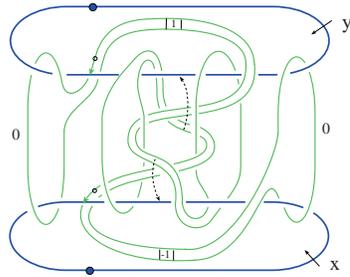


FIGURE 5.  $W_0^*$

*Proof.* Clearly Figure 6 is the twin of  $W_0$  in Figure 4. Notice that the two circles with dots do not link each other, since the affect of blowing down one  $+1$  is undone by blowing down another  $-1$  (before putting dots on them), of course during this process the framed circles representing the 2-handles changed appropriately.

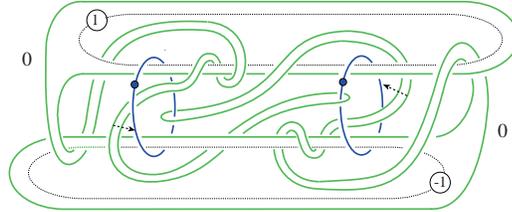


FIGURE 6.  $W_0^*$

By sliding 2-handles over the 1-handles (as indicated by dotted arrows) in Figure 6 we get Figure 7. Then isotoping the two dotted circles away from each other and after rotating the figure  $90^\circ$  we get Figure 5.

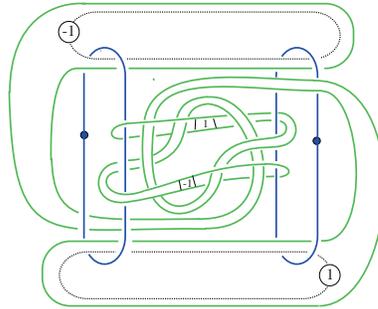


FIGURE 7.  $W_0^*$

Next we calculate the presentation of  $\pi_1(W_0^*)$  from Figure 5:

- (a)  $x^2yx^{-1}yx^{-1}y^{-1} = 1$
  - (b)  $y^{-2}x^{-1}yx^{-1}yx = 1$
- (a)  $\implies x^{-1}yxy^{-1} = yx^{-1}$ , and (b)  $\implies x^{-1}yxy^{-1} = y^{-1}xy$ . Hence  $xyx = yxy$ . Also  
 (a)  $\implies x^3yx^{-1}y = xyx = yxy \implies x^3y = yx^2$ . Hence  $x^3 = yx^2y^{-1} = (yxy^{-1})^2 = (x^{-1}yx)^2 = x^{-1}y^2x \implies x^3 = y^2$ . Hence we get the presentation  $G(P_2)$ .  $\square$

**Remark 3.2.** As shown in [AK], attaching a pair of 2-handles to  $\partial W_0$  along the dotted circles of Figure 1 gives  $\#_2(S^2 \times B^2)$ , which can be capped by a pair of 3-handles, which is just  $\#_2(B^3 \times S^1)$ . This describes an imbedding  $W \hookrightarrow S^4$ . The reader can check that turning these handle pairs upside down gives a handlebody of  $W_0^*$ .

**Theorem 3.3.**  $W_0^*$  smoothly imbeds into  $S^4$  with complement  $B^4$ , hence  $W_0^*$  (therefore  $W_0$ ) is diffeomorphic to  $B^4$ .

*Proof.* By sliding the 2-handles over the 1-handles of Figure 5 (as indicated by the dotted arrows in the figure) we get Figure 8. Again by applying the diffeomorphism of Section 2.3 of [A3] to this figure twice, we get Figure 9, then after the handle slides (indicated by dotted arrows) we get Figure 10, and then by an isotopy we get Figure 11.

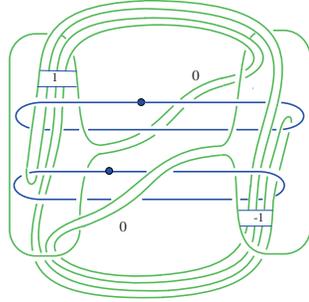


FIGURE 8.  $W_0^*$

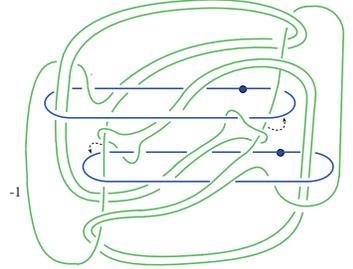


FIGURE 9.  $W_0^*$

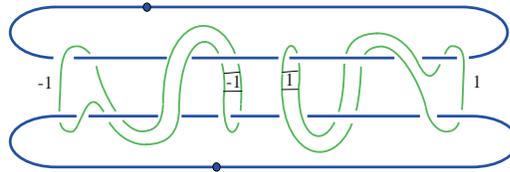


FIGURE 10.  $W_0^*$

We will now modify Figure 11 by a sequence of handle slides, and adding (and cancelling)  $1/2$  handle pairs to get a new 2-handlebody presentation of  $W_0^*$  which will have the required property. First by doing the handle slides indicated by the dotted arrows, we go from Figure 11 to Figures 12, and to the first picture of Figure 13. Then by cancelling a  $1/2$  handle pair we obtain the second picture of Figure 13, which is a ribbon 1-handle, induced from  $K \# -K$  where  $K$  is the trefoil knot, and a 2-handle.

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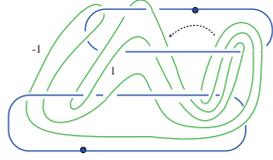


FIGURE 11.  $W_0^*$

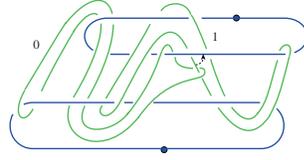


FIGURE 12.  $W_0^*$

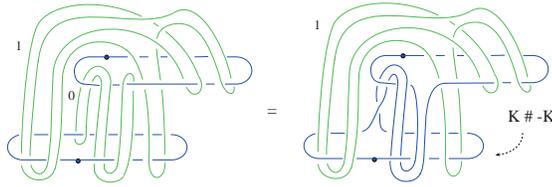


FIGURE 13.  $W_0^*$

The first picture of Figure 14 is the short hand of this handlebody (dotted line indicates the ribbon move giving the ribbon 1-handle in Figure 13). The second picture of Figure 14 is drawn after this ribbon move. Then doing the indicated handle slide to Figure 14 we get the handlebody  $C_2$ , where  $C_n$  is the handlebody described in Figure 15.

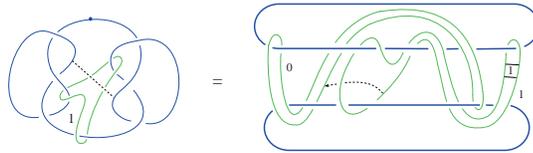


FIGURE 14.  $W_0^*$

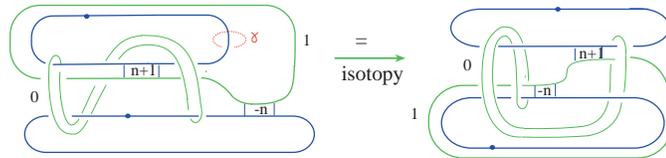


FIGURE 15.  $C_n$  and a useful isotopy

Notice  $C_n$  gives the presentation  $G(P_n)$  and is similar to  $H_{n,1}$  of [G], but it differs in the way its 0-framed 2-handle links the 1-handles (this provides us the useful isotopy of Figure 15). This difference is due to the fact that here we are getting a nonstandard ribbon which  $K\#-K$  bounds. Now the proof of the proposition follows from the following lemma whose proof is similar to the one in [G] for  $H(n, 1)$ .  $\square$

**Lemma 3.4.**  $C_n$  smoothly imbeds into  $S^4$  with complement  $B^4$ .

*Proof.* Attach a 2-handle to  $C_n$  along the loop  $\gamma$  with  $-1$  framing, as shown in the first picture of Figure 15 (this framing corresponds to 0-framing when viewed from  $S^3$ ). Denote this manifold by  $C_n + \gamma^{-1}$ . The steps of Figure 15 show the following equivalences by handle slides

$$C_n + \gamma^{-1} \cong C_{n-1} + \gamma^{-1} \dots \cong C_0 + \gamma^{-1}$$

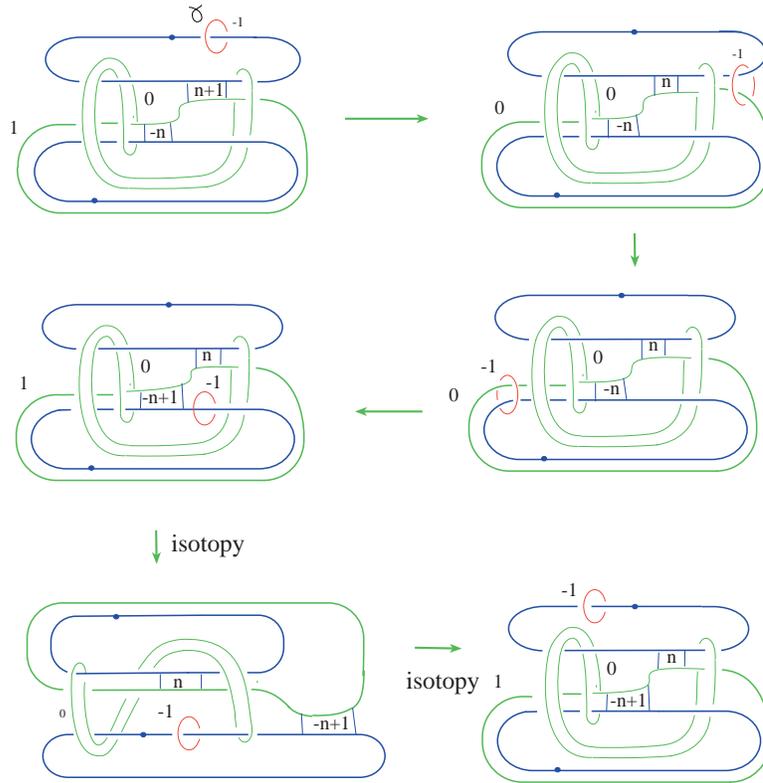


FIGURE 16.  $C_n + \gamma^{-1} \cong C_{n-1} + \gamma^{-1}$

Furthermore, Figure 17 shows  $C_0 + \gamma^{-1} \cong S^2 \times B^2$ , hence we can cap  $C_n + \gamma^{-1}$  with  $S^1 \times B^3$  to get  $S^4$ . Let  $N$  be the handlebody consisting of  $S^1 \times B^3$  union the dual of the 2-handle  $\gamma^{-1}$ .  $N$  is a contractible manifold with boundary  $S^3$ , consisting of a single pair of 1- and 2-handles, hence  $N \cong B^4$ . This fact follows from Property-R theorem of [Ga].  $\square$

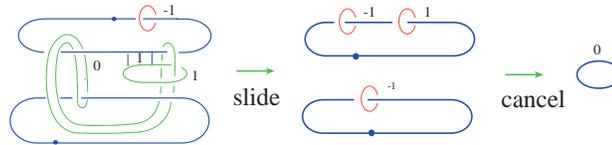


FIGURE 17.  $C_0 + \gamma^{-1} \cong S^2 \times B^2$

**Remark 3.5.** There is a certain dictionary relating AC-triviality of a 2-handlebody to its twin, which we didn't discussed here, opting directly dealing with Schoenflies problem. This is because the circle with dots can slide over each other just like 2-handles slide over each other (e.g., Section 1.2 of [A3]). We hope to address this in a future paper.

**Acknowledgements:** I want to thank Peter Teichner and Rob Kirby for stimulating my interest to the problems discussed here.

## References

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