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A plug with infinite order and some exotic 4-manifolds

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Abstract. It is known that a closed, orientable, simply-connected 4-dimensional manifold X' which is an exotic copy of X can be obtained by removing a submanifold C (called a cork) of X and regluing C back. We can assume that C is contractible and the gluing map is an involution. In this paper we define *corks and plugs with order* $p \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and we show a plug (P, φ) with infinite order which produces "a crossing change" of Fintushel-Stern's knot surgery where (P, φ^2) is a (generalized) cork with infinite order.

1. Introduction

1.1. Smooth structures

Let X, X' be two smooth manifolds. If X' is homeomorphic but non-diffeomorphic to X, then we say that X and X' are *exotic (or exotic pair)*. Here we define cork and plug according to [AY1].

Definition 1.1 (Cork)**.** Let C be a compact, contractible, Stein 4-manifold with boundary and $\tau : \partial C \to \partial C$ a diffeomorphism with $\tau^2 = \mathrm{id}$. (C, τ) is called a cork, if the map τ cannot extend to any diffeomorphism of C.

We call (C, τ) a cork of a 4-manifold X, if there exists an embedding $i : C \hookrightarrow X$ such that when removing $C \subset X$ and regluing C via the diffeomorphism τ, the resulting manifold $X' = (X - C) \cup_\tau C$ is exotic to X. This deformation $X \rightsquigarrow X'$ is called a cork twist and presented as (C, τ, i).

The cut and paste notation like $(X - Z) \cup_\varphi Y$ will be defined in Section 1.2. When the embedding i of the cork C is clear from the context, the cork twist is denoted simply by (C, τ).

As proven in [AM, CFHS, M], any *closed*, *simply-connected* exotic pair can be obtained by a cork twist (C, τ, i). Thus, how to find corks for every exotic pair is a next natural problem.

Akbulut and Yasui in [AY1] introduced another type of a construction (P, τ) to give an exotic pair.

Key words and phrases. 4-manifold, exotic structure, cork, plug, Fintushel-Stern's knot-surgery.

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Definition 1.2 (Plug)**.** Let (P, τ) be a compact, Stein 4-manifold P with boundary and $\tau : \partial P \to \partial P$ a diffeomorphism with $\tau^2 = \mathrm{id}$. We assume that τ cannot be extended to a homeomorphism $P \to P$. We call (P, τ) a plug if there exist a smooth 4-manifold X and an embedding $i : P \hookrightarrow X$ such that when removing $P \subset X$ and regluing P via the map τ, the resulting manifold $X' = (X - P) \cup_\tau P$ is exotic to X. This deformation $X \rightsquigarrow X'$ is called a plug twist and presented as (P, τ, i).

Recently many smooth structures have been constructed by using corks and plugs as in [A2, AY1, AY2, AY3, AY4, GS]. Since the main idea for the existence of cork is due to the failure of smooth 4-dimensional h-cobordism theorem, naturally the self-diffeomorphism of the boundary of corks and plugs must be an involution. For example, as appeared in [A1, BG], a cork C for an exotic pair $E(2)\#\overline{\mathbb{C}P^2}$ and $\#^3\mathbb{C}P^2\#^{20}\overline{\mathbb{C}P^2}$ is given by a Mazur type 4-manifold with a symmetric framed link diagram. The involution is the one induced by the symmetry of the framed link. Here $E(n)$ is the elliptic fibration with $12n$ Lefschetz singularities.

Let $T \subset X$ be an embedded torus with the trivial normal bundle. Let K be a knot in S^3. M_K is the 0-surgery along K and T_m is a torus $m \times S^1$ in $M_K \times S^1$, where m is the meridian of K. *(Fintushel-Stern's) knot-surgery* [FS1] is defined to be

$$X_K = (M_K \times S^1)\#_{T_m = T} X.$$

See Section 1.2 for the notation for the fiber-sum.

We give a brief review about the (2-component) link-surgery in [FS1]. Let U_1, U_2 be two 4-manifolds and $T_i \subset U_i$ embedded tori with trivial neighborhoods. Let $L = K_1 \cup K_2$ be a 2-component link. Let

$$\alpha_L : \pi_1(S^3 - L) \to \mathbb{Z}$$

be a homomorphism satisfying $\alpha_L(m_i) = 1$, where m_i is the meridian curve of K_i. Let M_L be the $\alpha(\ell_i)$-surgery of L, where ℓ_i is the longitude of K_i. Let $\mathcal{U} = (U_1, U_2)$ denote a collection of 4-manifolds U_i and the embedded tori T_i, and $T_{m_i} = m_i \times S^1$, where m_i are the meridians of K_i in M_L. Then, we denote by $\mathcal{U}_L$ the following:

$$\mathcal{U}_L = U_1 \#_{T_1 = T_{m_1}} (M_L \times S^1) \#_{T_{m_2} = T_2} U_2.$$

In the case of $U = U_1 = U_2$, we write as $\mathcal{U}_L = U_L$. We call $\mathcal{U}_L$ *the link-surgery by the link* L. This link-surgery operation can be generalized to an n-component link version.

Our motivation is to construct "a cork (or plug)" representing the knot-surgery or link-surgery. Since the knot-surgery gives infinitely many exotic structures, we, naturally, consider corks and plugs with infinite order. In the next subsection we will define such corks and plugs with order p, where p is greater than or equal to 2 or ∞.

One of the main aims in this paper is to give an example of a (generalized) cork with infinite order. It is the square (P, φ^2) of a plug (P, φ) with infinite order. Let X be a 4-manifold as described in Figure 1. This manifold is diffeomorphic to a neighborhood of Kodaira's singular fiber with type III (see Table 1.9 in [HKK]). We denote the neighborhood of the fiber by V.

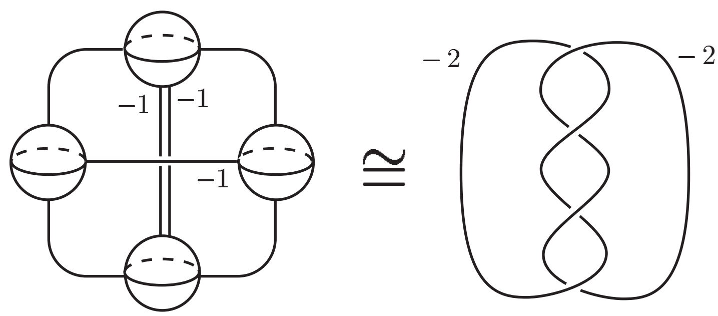

FIGURE 1. Kodaira's singular fibration V of type III

Remark 1.3. Throughout this paper, unlabeled links in the handle diagrams of 4-manifolds or 3-manifolds stand for 0-framed 2-handles or 0-surgeries respectively.

Remark 1.4. The handle decompositions of knot-surgery and link-surgery are based on the descriptions in [A3, T1, T2].

Let P be the simply-connected 4-manifold with the handle decomposition given in FIGURE 2. We will define a diffeomorphism $\varphi : \partial P \to \partial P$ in Section 3.

Theorem 1.5. *The (P, φ)-twist is a plug with infinite order.*

This plug (P, φ) gives in some sense a crossing change concerning Fintushel-Stern's knot-surgery.

Theorem 1.6. *The (P, φ^2)-twist is a generalized cork.*

This meas the (P, φ)-twist cannot extend to the inside of P as a homeomorphism, but the square can extend to the inside of P as a homeomorphism.

In general we show the following theorem.

Theorem 1.7. *Let X be a 4-manifold containing V and K a knot in S^3. Let X_K be the knot-surgery along the general torus fiber in V. Suppose that K' is one crossing change of K. Then there exists an embedding $P \hookrightarrow X_K$ such that the plug twist of X_K is $X_{K'}$.*

Let K^n be an n-times crossing change of a knot K. Then the infinitely many exotic 4-manifolds X_{K^n} are obtained by iterating the plug twist (P, φ).

Variations and application of the plug (P, φ) can be seen in the forthcoming paper [T3].

1.2. Notation on cut-and-paste.

Here we define a notation on cut-and-paste operations on manifolds.

Definition 1.8 (Cut-and-paste)**.** Let X be an n-manifold and $i : Z \hookrightarrow X$ an embedding of compact, codimension-0 submanifold Z with boundary. Here the boundary $\partial Z = \Sigma$

is a smooth $(n-1)$-manifold. Let Y be an n-manifold with a fixed diffeomorphism map $\Sigma \to \partial Y$. Let φ be a map $\Sigma \to \Sigma$. The manifold obtained by identifying $\partial(X-Z)$ and ∂Y via the map φ is denoted as follows:

$$(X-Z) \cup_{\varphi} Y.$$

In particular, if $Y = Z$, then the surgery $(X-Z) \cup_{\varphi} Z$ is called (Z, φ, i)-twist of X. If the embedding i is clear from the context, we call the twist a (Z, φ)-twist.

Let X, Y be 4-manifolds with tori $T_1 \subset X, T_2 \subset Y$ with trivial neighborhoods. Then, the resulting manifold by doing the fiber-sum of X and Y as defined in [GS] is written as follows:

$$X \#_{T_1=T_2} Y := (X - \nu(T_1)) \cup_{\varphi} (Y - \nu(T_2)),$$

where $\nu(\cdot)$ stands for the neighborhood and φ is a fiber-preserving diffeomorphism. In fact, the diffeomorphism type depends on the diffeomorphism φ.

2. Cork and Plug with order p.

In Section 1.1 we gave the original definition of corks and plugs. Here we define notions of *corks and plugs with order* p.

Definition 2.1 (Cork with order p)**.** Let (C, φ) be a pair of a compact, contractible, Stein 4-manifold C with boundary and a diffeomorphism $\varphi : \partial C \to \partial C$. Let p be an integer with $p \geq 2$ or infinity. The pair (C, φ) is called a cork with order p if it satisfies the following properties:

(1) The order of φ is p.
(2) For any $0 < k < p$ the map φ^k cannot be extended to $\varphi^k : C \to C$ as a diffeomorphism.

If there exist p mutually exotic 4-manifolds $\mathcal{X} = \{X, X_1, X_2, \cdots, X_{p-1}\}$ (in the case of $p < \infty$) or $\mathcal{X} = \{X, X_1, X_2, \cdots\}$ (in the case of $p = \infty$) such that an embedding $i : C \hookrightarrow X$ and a diffeomorphism $\varphi : \partial C \to \partial C$ produce the 4-manifolds

$$X_k = [X - C] \cup_{\varphi^k} C,$$

then (C, φ, i) is a cork with order p of $\mathcal{X}$.

We also treat a non-contractible cork C as appeared in [AY1]. It is called *a generalized cork with order* p. The regluing map φ for any generalized cork must extend to a homeomorphism $C \to C$.

Definition 2.2 (Plug with order p)**.** Let (P, φ) be a pair of a compact Stein 4-manifold P with boundary and a diffeomorphism $\varphi : \partial P \to \partial P$. Let p be an integer with $p \geq 2$ or infinity. The pair (P, φ) is called a plug with order p if it satisfies the following properties:

(1) The order of φ is p.
(2) The map φ cannot be extended to any self-homeomorphism of P.

(3) There exists a 4-manifold X containing P such that for any $0 < k < p$, X and
$$X_k = [X - P] \cup_{\varphi^k} P$$
are p mutually exotic manifolds.

The classical cork or plug corresponds to the $p = 2$ case in the definition above. Although we give an example of an infinite order plug, it is unknown whether there exists a finite order plug with $p > 2$ or not.

Question 2.3. Does there exist any plug or cork (Q, ψ) with finite order p $(3 \leq p < \infty)$?

3. A plug (P, φ) with infinite order.

We construct a plug (P, φ) with infinite order. We verify that the pair satisfies all of the properties (1), (2), and (3) for (P, φ) to be a plug with infinite order.

3.1. The definition of P and the topological invariants.

Let P be the compact 4-manifold admitting the handle decomposition as in Figure 2. The manifold P is also obtained by attaching three -1-framed 2-handles on $V_2 \times S^1$,

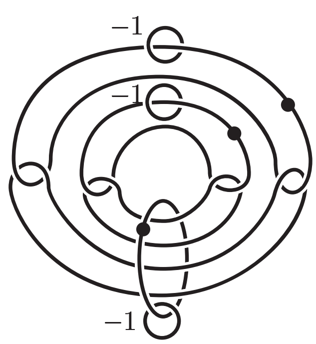

Figure 2. A handle decomposition of P.

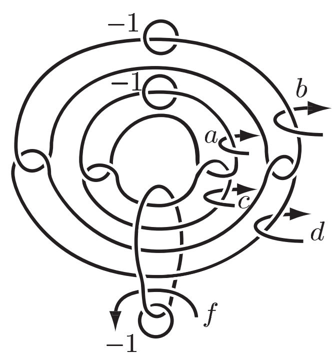

Figure 3. The generators in $\pi_1(\partial P)$.

where V_2 is the genus 2 handlebody. Since the three 2-handles kill the three generators in the $\pi_1(V_2 \times S^1)$, P is simply-connected. Thus, we have $H_2(P) \cong \mathbb{Z}^2$. The presentation of $\pi_1(\partial P)$ is the following:

$$\begin{aligned}\pi_1(\partial P) &= \langle a, b, c, d, f \,|\, [f, b^{-1}], [f, a^{-1}], a[f, c^{-1}], b[f, d^{-1}], \\ &\qquad f^{-1}[c^{-1}, f][c, f][d^{-1}, f][d, f] \rangle, \\ &\cong \langle c, d, f \,|\, [f, [f, d^{-1}]], [f, [f, c^{-1}]], f^{-1}[c^{-1}, f][c, f][d^{-1}, f][d, f] \rangle,\end{aligned}$$

where we use generators a, b, c, d and f in Figure 3 and the symbol $[x, y]$ stands for the commutator $xyx^{-1}y^{-1}$. Therefore we get $H_1(\partial P) \cong \mathbb{Z}^2$.

3.2. A Stein structure on P.

In order for a 4-manifold X to admit a Stein structure, we must show that it has a specific handle decomposition. That is, as in [E] or [GS], X is a 2-handlebody, and the attaching spheres of 2-handles have Legendrian presentation in $\#^n S^2 \times S^1 = \partial(\natural^n D^3 \times S^1)$ such that each framing is $tb-1$, where tb stands for the Thurston-Bennequin invariant of the Legendrian knot. To show that the 4-manifold P is a plug we must show that P is a compact Stein manifold. Canceling the two 1-handles, P admits the handle decomposition as in Figure 4. The Thurston-Bennequin invariants of the components are all 1. Hence, this handle decomposition satisfies Eliashberg's condition above. Therefore we get the following proposition:

Proposition 3.1. *P admits a Stein structure.*

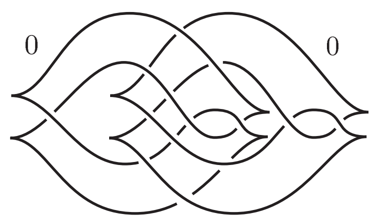

Figure 4. A Stein structure over P.

3.3. A diffeomorphism φ on ∂P.

We define a diffeomorphism φ on ∂P. Sliding and isotopy in Figure 5 gives a diffeomorphism on ∂P and we call it φ. Hence the generators c, d and f in $\pi_1(\partial P)$ are mapped via φ to the following respectively:

$$(Xb)^{-1}a(Xb)a^{-1}c(Xb)$$
$$d(Xd)^{-1}a(Xd)$$
$$\text{and } f,$$

where $X = [b^{-1}, d]$.

Lemma 3.2. *The map φ acts on $H_1(\partial P)$ and $H_2(\partial P)$ trivially.*

Proof. Taking the abelianization of the images of c, d and f via φ, we see that the map $\varphi_* : H_1(\partial P) \to H_1(\partial P)$ is the identity. Hence, through the Poincaré dual, the action on H_2 is also the identity. $\square$

The map φ trivially acts on the homology, but the order of φ is infinite. This will be proven in the next subsection.

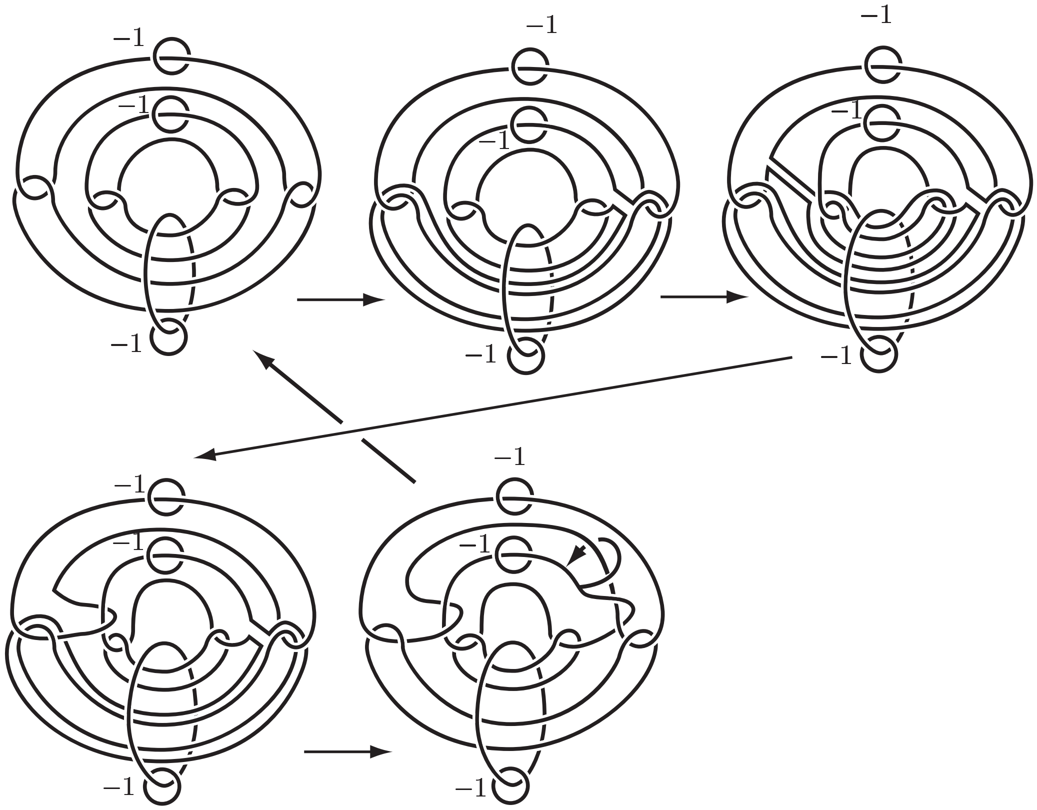

FIGURE 5. The diffeomorphism φ.

3.4. Infinite exotic structures which (P, φ^i)-twists give.

In this subsection we show that there exists a compact 4-manifold X containing P such that infinitely many surgeries

$$(X - P) \cup_{\varphi^i} P$$

give infinitely many exotic structures on X. This is the most essential part among the properties of this plug. As a corollary, φ has infinite order in the mapping class group on ∂P.

Proposition 3.3. *Let K_n be a twist knot in S^3 given by the diagram in* FIGURE *6. Let V be a neighborhood of singular fiber of Kodaira's type III and V_{K_n} the knot-surgery of V. Then, there exists $P \hookrightarrow V_{K_n}$ $(n \geq 0)$ such that the resulting manifolds*

$$(V_{K_n} - P) \cup_{\varphi} P = V_{K_{n+1}},$$

$$(V - P) \cup_{\varphi^n} P = V_{K_n}$$

and the order of φ is infinite. In particular, $\{V_{K_n}\}$ contains infinitely many exotic copies of V.

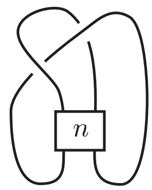

FIGURE 6. The diagram of K_n, where the box with n in the picture represents n full twists.

Proof. Let K_n be the $2n$-twist knot as in FIGURE 6. The neighborhood V is the left of FIGURE 7. Performing Fintushel-Stern's knot-surgery [FS1] on V, we get the deformation

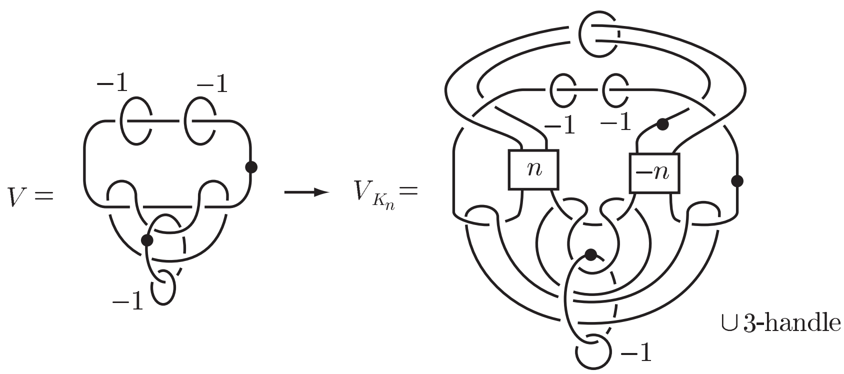

FIGURE 7. Fintushel-Stern's knot-surgery of X.

in FIGURE 7. Sliding one of the two -1-framed 2-handles over 0-framed 2-handle in the diagram, and removing the top 0-framed 2-handle, we get the handle diagram of P in V_{K_n} (see FIGURE 8).

Here, we perform the (P, φ)-twist of V_{K_n} along P as follows:

$$(V_{K_n} - P) \cup_{\varphi} P.$$

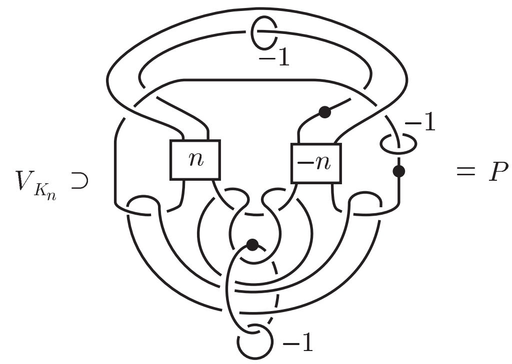

FIGURE 8. The submanifold $P \subset V_{K_n}$.

We keep track of the handle moves in FIGURE 5 to V_{K_n}. In FIGURE 9, any attaching spheres of 3-handles are not described. However, due to [Tr], via the (P,φ)-twist, the extension to the 3-handles are unique.

As a result, it turns out that the surgery adds the full twist to the diagram of V_{K_n} (see FIGURE 9). Consequently, we get

$$V_{K_n} \to (V_{K_n} - P) \cup_\varphi P = V_{K_{n+1}}.$$

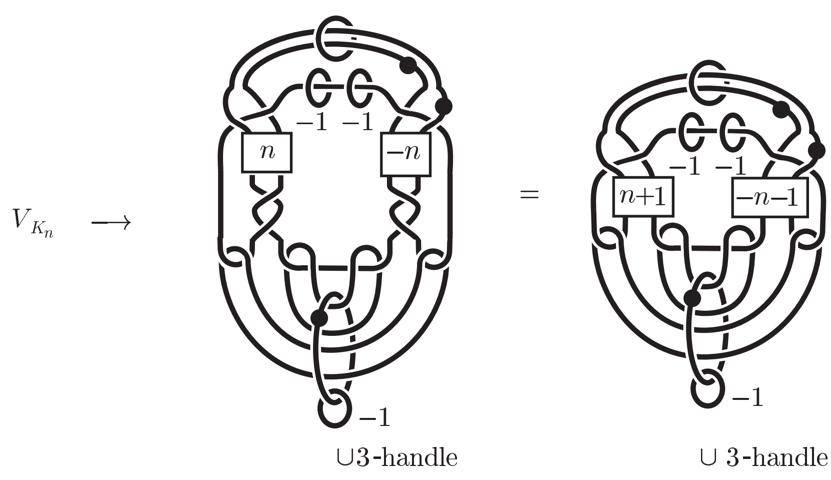

FIGURE 9. The (P,φ)-twist gives $V_{K_n} \to (V_{K_n} - P) \cup_\varphi P = V_{K_{n+1}}$.

FIGURE 10 determines a diffeomorphism $\phi_n : \partial V \to \partial V_{K_n}$. More precisely, FIGURE 10 itself gives a diffeomorphism on $\partial V \# S^2 \times S^1$, however the part of $S^2 \times S^1$ does not change. For, ∂V is a prime manifold, because ∂V is a torus bundle over S^1 and the universal cover is $\mathbb{R}^3$, in particular, $\pi_2(\partial V) = 0$. Further, it is well-known that the separating sphere of the connected-sum of two prime 3-manifolds is unique up to isotopy. Thus any diffeomorphism $\partial V \# S^2 \times S^1$ gives a diffeomorphism on ∂V uniquely.

The (P, φ)-twist $V_{K_n} \rightsquigarrow V_{K_{n+1}}$ above induces the boundary diffeomorphism $\phi_{n+1} \circ \phi_n^{-1}$ between ∂V_{K_n} and $\partial V_{K_{n+1}}$. The diffeomorphism is the identity map via the diffeomorphisms ϕ_n, ϕ_{n+1}. Note that in the diffeomorphism, the same handle move process as φ up to isotopy.

Actually, $\phi_n : \partial V \to \partial V_{K_n}$ is the same map as the boundary diffeomorphism from knot-surgery, as in FIGURE 5 in [A3]. Thus, we can get the result by (P, φ)-twist as follows:

$$E(2)_{K_n} \to (E(2)_{K_n} - P) \cup_\varphi P \cong E(2)_{K_{n+1}}.$$

It is well-known that the knot-surgery $E(2)_K$ gives exotic structures. The Seiberg-Witten invariant formula on Fintushel-Stern's knot-surgery in [FS1] says that:

$$SW_{E(2)_{K_n}} = -nt + 2n + 1 - nt^{-1}.$$

This implies that $E(2)_{K_n}$ are infinitely many exotic 4-manifolds. Thus, the (P, φ^i)-twist of $E(2) = E(2)_{K_0}$ gives rise to infinitely many mutually exotic 4-manifolds $\{E(2)_{K_i} | i \in \mathbb{Z}\}$. Hence, $X = E(2) = E(2)_{K_0}$ satisfies this proposition. In particular, the map φ has infinite order. □

As a corollary we get the following:

Corollary 3.4. *Let X be a 4-manifold containing V. Let K_n be a twist knot as in FIGURE 6. Then, the plug twist (P, φ) of $V_{K_n} \subset X_{K_n}$ produces $X_{K_{n+1}}$.*

Proof. The (P, φ)-twist of V_{K_n} gives $V_{K_{n+1}}$. The diffeomorphism $\partial V_{K_n} \to \partial V_{K_{n+1}}$ via the twist is the same map as the gluing map of the knot-surgery as is presented in FIGURE 5 of [A3]. Thus, the manifold X_{K_n} is changed to $X_{K_{n+1}}$ by (P, φ)-twist. □

3.5. Extendability as a homeomorphism.

Lastly we show that φ cannot be extended to a self-homeomorphism on P. To prove that, we use Boyer's theorem in [B]. First, we prepare with the following proposition.

Proposition 3.5. *The twisted doubles $D_{\varphi^n}(P) := P \cup_{\varphi^n} \overline{P}$ have the following properties:*

$$D_{\varphi^n}(P) \text{ is } \begin{cases} \text{spin} & : n \text{ even} \\ \text{not-spin} & : n \text{ odd.} \end{cases}$$

Since any double $D_{\varphi^n}(P)$ is simply-connected, we may see whether the intersection form $Q_{D_{\varphi^n}(P)}$ is even or odd. Note that the intersection form of P is even.

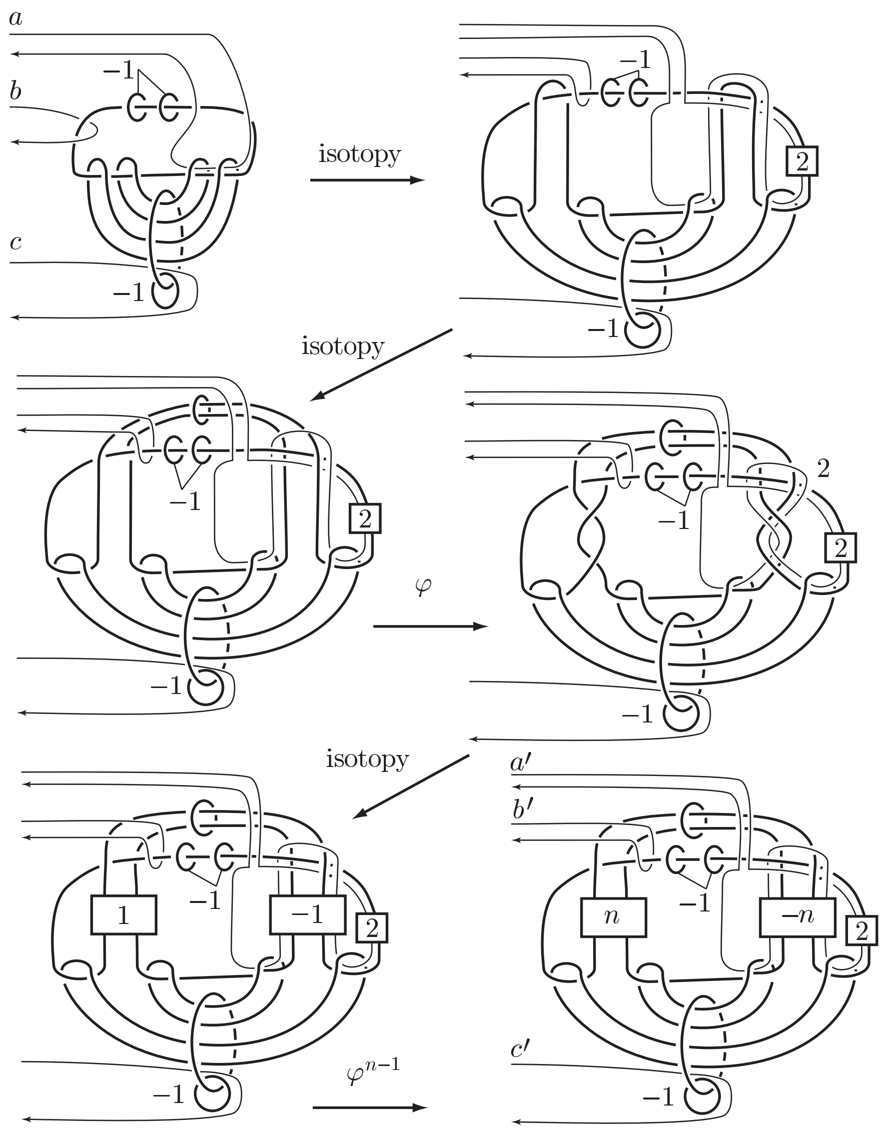

FIGURE 10. The diffeomorphism $\phi_n : \partial V \to \partial V_{K_n}$.

Proof. First the double $D(P) := P \cup_{\mathrm{id}} \overline{P}$ is diffeomorphic to $\#^2 S^2 \times S^2$ by the handle calculus in FIGURE 11. The twisted double $D_\varphi(P)$ is diffeomorphic to $\#^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ as in FIGURE 12.

In general, $D_{\varphi^n}(P)$ is diffeomorphic to the manifold given by the left diagram in FIGURE 13. The change of the framing of the bottom 2-handle in this picture to the 0-framing gives the manifold D_n (the right of FIGURE 13). Since D_n is the diagram of some closed 4-manifold, the change is a kind of homeomorphic deformation. Furthermore, the application of the local move in FIGURE 14 to the D_n yields FIGURE 15. Namely, we get the classification:

$$D_n \cong \begin{cases} \#^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) & \text{if } n \text{ odd} \\ \#^2 S^2 \times S^2 & \text{if } n \text{ even.} \end{cases}$$

Therefore, $D_{\varphi^n}(P)$ is spin or non-spin for n is even or odd respectively. $\square$

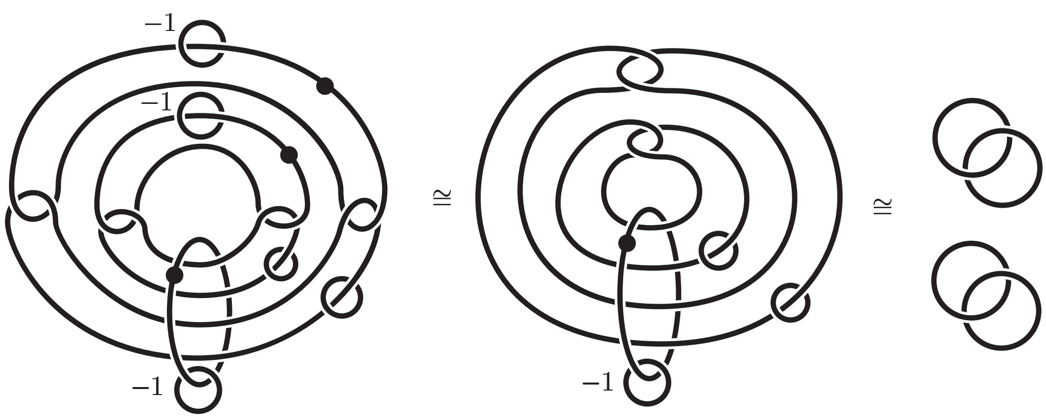

FIGURE 11. $D(P) = P \cup_{\mathrm{id}} \overline{P}$.

The problem of whether $D_{\varphi^n}(P)$ is diffeomorphic to D_n or not is remaining.

Question 3.6. Is $D_{\varphi^n}(P)$ for $|n| \geq 2$ an exotic $\#^2 S^2 \times S^2$ or $\#^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$?

Applying Proposition 3.5 to (iii) in (0.8) Proposition in Boyer's paper [B], we get the following:

Corollary 3.7. *For any integer n the diffeomorphism $\varphi^n : \partial P \to \partial P$*

$$\begin{cases} \textit{cannot be extended to a self-homeomorphism on } P & : n \textit{ odd} \\ \textit{can be extended to a self-homeomorphism on } P & : n \textit{ even.} \end{cases}$$

In other words the following holds:

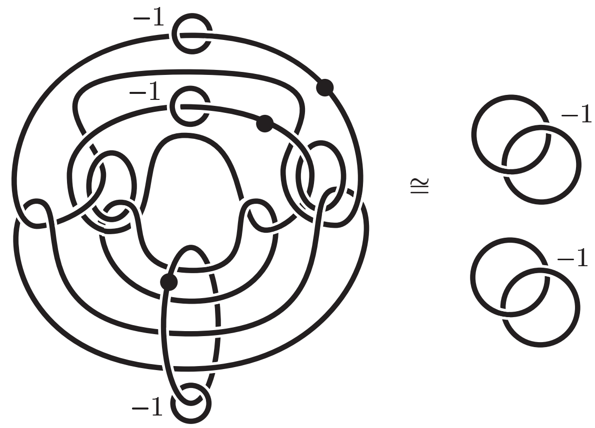

FIGURE 12. $D_\varphi(P) = P \cup_\varphi \overline{P}$.

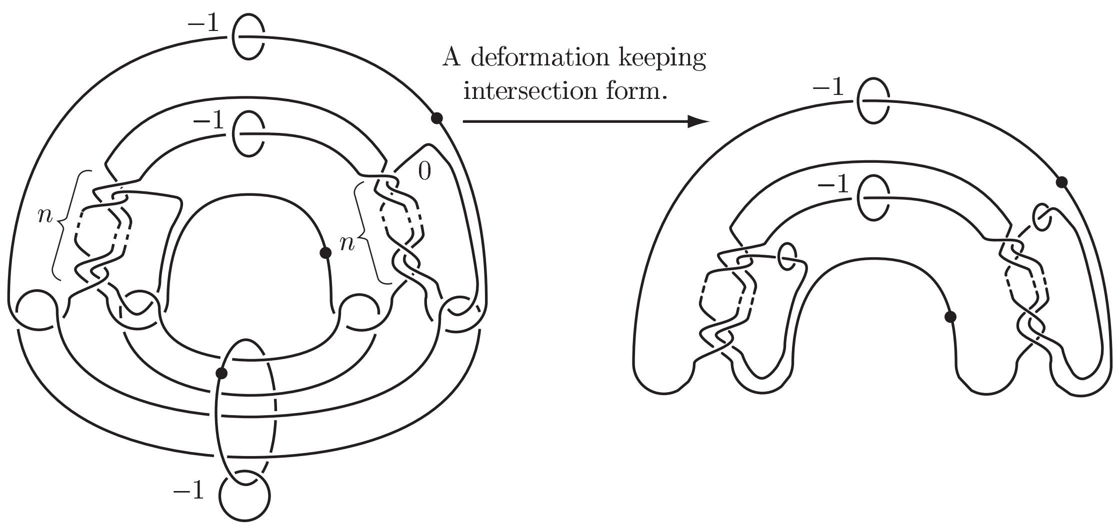

FIGURE 13. $D_{\varphi^n}(P) \to D_n$ (homeomorphism).

Corollary 3.8. *For each non-zero integer m, each (P, φ^{2m})-twist is a generalized cork with infinite order and each (P, φ^{2m+1}) is a plug with infinite order.*

In particular, this corollary proves Theorem 1.6.

FIGURE 14. A local move which gives a diffeomorphism.

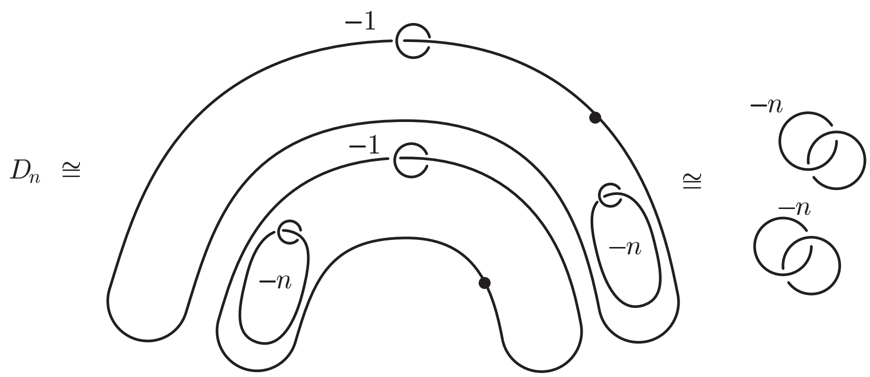

FIGURE 15. Diffeomorphism types of D_n.

Proof of Theorem 1.5.
We show the (P, φ)-twist satisfies the condition of a plug with infinite order. Proposition 3.1 shows P is a Stein manifold. Proposition 3.3 shows φ has infinite order and (P, φ)-twists produce infinitely many exotic manifolds. Corollary 3.7 shows that φ cannot extend inside P. □

Proof of Theorem 1.7.
Let D be a diagram of a knot K in S^3. Given a crossing point in D as in FIGURE 16, we move the point in the bottom position in the picture by isotopy. Second, we move the local minimum points and local maximal points in the bottom and top positions respectively. The resulting picture is a plate presentation of K that the crossing point is a bottom position in the braid. This presentation can give the handle decomposition of $[S^3 - \nu(K)] \times S^1$. For the way to describe any knot-surgery and $[S^3 - \nu(K)] \times S^1$ one can see [A3, T2]. V_K is $[S^3 - \nu(K)] \times S^1$ attaching the 3 vanishing cycles. See FIGURE 17 for

the embedding $P \hookrightarrow V_K$. Applying the (P, φ)-twist (the process in FIGURE 5) to $P \subset V_K$, we obtain $V_{K'}$, where K' is the crossing change of K.

We show that the (P, φ)-twist deforms X_K into $X_{K'}$. We give $\phi_K : \partial V \to \partial V_K$ in the similar way to FIGURE 10. The map ϕ_K is defined by using a plate presentation of K, however this is independent of the plate presentation of K. Suppose that ϕ_K and ϕ'_K are two boundary diffeomorphisms for different presentations. Here there exists an isotopy from ∂V_K to ∂V_K such that $\phi_K(b)$ and $\phi_K(c)$ are isotopic to $\phi'_K(b)$ and $\phi'_K(c)$, because we can make an isotopy between two presentations with $\phi_K(b)$ and $\phi_K(c)$ fixed. Thus $\phi'^{-1}_K \circ \phi_K(b) = b$ and $\phi'^{-1}_K \circ \phi_K(c) = c$ hold up to isotopy.

The maps ϕ_K and ϕ'_K induces two fibrations $\pi, \pi' : \partial V_K \to S^1$ respectively coming from $\partial V \to S^1$. As mentioned above, $\phi'^{-1}_K \circ \phi_K$ is a bundle isomorphism $\partial V \to \partial V$ fixing the fibers. Here $\phi'^{-1}_K \circ \phi_K$ induces a diffeomorphism $S^1 \to S^1$ it is isotopic to identity. The mapping class group of S^1 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the isotopy class is determined by the change of the orientation of S^1. Actually, the linking number of the curve a in FIGURE 10 with the circle representing the 0-framed 2-handles is preserved for ϕ_K and ϕ'_K. Thus, the map $\phi'^{-1}_K \circ \phi_K$ is reduced to the identity $S^1 \to S^1$. Therefore, $\phi'^{-1}_K \circ \phi_K$ is the identity map up to isotopy. In particular, $\phi_K = \phi'_K$ holds up to isotopy.

$$\begin{tikzcd}
\partial V \arrow[r, "\phi_K"] \arrow[d, "\pi_0"'] & \partial V_K \arrow[d, "\pi"', "\pi'"] & \partial V \arrow[l, "\phi'_K"'] \arrow[d, "\pi_0"] \\
S^1 \arrow[r] & S^1 & S^1 \arrow[l]
\end{tikzcd}$$

This diffeomorphism gives a diffeomorphism on $\partial V \# S^2 \times S^1 \# \cdots \# S^2 \times S^1$. Any diffeomorphism of $\partial V \# S^2 \times S^1 \# \cdots \# S^2 \times S^1$ can give a diffeomorphism on ∂V uniquely, because any attachment of the 3-handles in V_K is unique (due to [Tr]). In particular, the sphere separating as ∂V and $S^2 \times S^1 \# \cdots \# S^2 \times S^1$ is uniquely determined. Consequently, this ϕ_K does not depend on the way of the plate presentation of K.

Therefore, $\phi_K \circ \phi^{-1}_{K'} : \partial V_{K'} \to \partial V_K$ is the same handle move as that of φ. The diffeomorphism $\partial V \to \partial V_K$ is the diffeomorphism induced by knot-surgery. See FIGURE 5 in [A3]. This implies $X_K \to X_{K'}$. $\square$

Remark 3.9. Let L_1, L_2 be two unknotting number 1 knots. Then there exist embeddings $\iota_{L_1}, \iota_{L_2} : P \hookrightarrow E(2)$ such that the plug twists give the knot-surgery $E(2) \rightsquigarrow E(2)_{L_1}$ and $E(2) \rightsquigarrow E(2)_{L_2}$. Therefore the embeddings ι_{L_1} and ι_{L_2} are not smoothly isotopic each other, if $\Delta_{L_1} \neq \Delta_{L_2}$.

Remark 3.10. Let X be a manifold containing V. Note that the knot-surgery X_{K_n} along a general fiber of V is the plug twist (P, φ^n, i) for odd n. On the other hand, since K_n is an unknotting number 1 knot, there exists another plug twist of X by $(P, \varphi^{-1}, \kappa_n)$

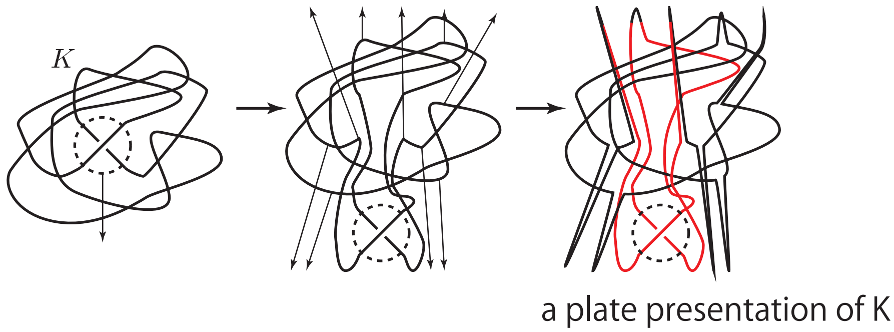

FIGURE 16. A presentation of any knot with a given crossing in a bottom position.

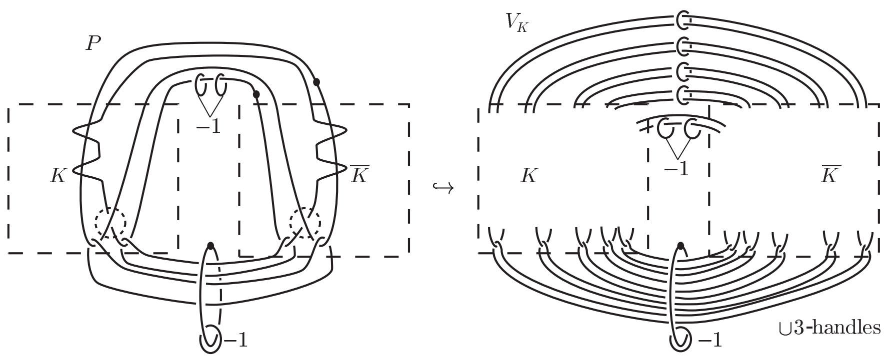

FIGURE 17. $P \hookrightarrow V_K$

such that the plug twist gives X_{K_n}.

$$\begin{tikzcd} X \arrow[r, "{(P,\varphi^{-1},\kappa_n)}"] \arrow[dr, "{(P,\varphi^{n},i)}"'] & X_{K_n} \\ & X_{K_n} \end{tikzcd}$$

4. Some questions.

Here we summarize the several questions.

Question 2.3. *Does there exist any plug or cork (Q, ψ) with finite order p $(3 \leq p < \infty)$?*

Question 3.6. *Is $D_{\varphi^n}(P)$ for $|n| \geq 2$ an exotic $\#^2 S^2 \times S^2$ or $\#^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$?*

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