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Naturality of FHT isomorphism

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Abstract. Freed, Hopkins and Teleman constructed an isomorphism (we call it FHT isomorphism) between twisted equivariant K-theory of compact Lie group G and the "Verlinde ring" of the loop group of G ([FHT1] [FHT2] [FHT3]). However, naturality of the isomorphism with respect to group homomorphisms has not been verified. We construct induced homomorphisms $f^{\#} : K_T^{\tau+\dim(T)}(T) \to K_S^{f^*\tau+\dim(S)}(S)$ and $f^! : R^{\tau}(LT) \to R^{f^*\tau}(LS)$ for $f : S \to T$ whose tangent map is injective. τ is a positive central extension of the loop group of T, so that FHT isomorphism is a natural transformation between two objects. In fact, we construct another object *char* and verify that three objects are naturally isomorphic with respect to $f^{\#}$, $f^!$ and $char(f)$ we introduce.

Moreover, we extend these constructions for K-theory and *char* to compact connected Lie group with torsion-free π_1 and homomorphism $f : H \to G$ satisfying the "decomposable condition", and verify that they are isomorphic. This is a generalization of naturality of $K_G^{\tau+\mathrm{rank}(G)}(G) \cong char(G, \tau)$ verified in [FHT1].

Introduction

Study of loop groups of compact Lie groups is extensively developed especially from the view point of representation theory. On the other hand, twisted K-theory was introduced by Donovan and Karoubi ([Ka]), and interest in it was rekindled by its appearance in string theory. Freed, Hopkins and Teleman connected them ([FHT2]). In this paper, we study functorial aspects of these two objects.

Let us start from the positive energy representation theory of loop groups. A positive energy representation reflects S^1 symmetry of loop groups and satisfies a certain finiteness. S^1 acts on loop groups via transformation of the parameter. When we make S^1 act on loop groups, we write S^1 as $\mathbb{T}_{\mathrm{rot}}$. Let G be a compact connected Lie group with torsion-free π_1 and $\mathcal{H}$ be a separable complex Hilbert space as a representation space.

Definition 0.1 ([PS]). *A positive energy representation is a continuous projective representation $\rho : LG \to PU(\mathcal{H})$ satisfying the following conditions.*

(1) *The action of $\mathbb{T}_{\mathrm{rot}}$ on LG lifts to the associated central extension by $U(1)$*

$$LG \times_{PU(\mathcal{H})} U(\mathcal{H}).$$

Therefore, we can define a new Lie group $(LG \times_{PU(\mathcal{H})} U(\mathcal{H})) \rtimes \mathbb{T}_{\mathrm{rot}}$.

Key words and phrases. Loop groups, twisted equivariant K-theory, representation theory.

(2) *ρ lifts to $\widehat{\rho} : (LG \times_{PU(\mathcal{H})} U(\mathcal{H})) \rtimes \mathbb{T}_{\mathrm{rot}} \to U(\mathcal{H})$.*

(3) *When we decompose $\mathcal{H}$ by the weight of $\mathbb{T}_{\mathrm{rot}}$, $\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$, $\mathcal{H}_n$ is finite dimension for all $n \in \mathbb{Z}$ and $\mathcal{H}_n = 0$ for all $n \ll 0$.*

Remark 0.2. *A central extension τ is also called level. A representation is at level τ if the induced central extension $LG \times_{PU(\mathcal{H})} U(\mathcal{H})$ is isomorphic to LG^τ.*

The direct sum of two representations at level τ is also at level τ. So finitely reducible positive energy representations at level τ form a semigroup under the direct sum.

Definition 0.3. *The representation group $R^\tau(LG)$ of LG at level τ is the Grothendieck completion of this semigroup.*

Our second main object is twisted equivariant K-theory. Let G be a compact Lie group, X be a topological space. We suppose that G acts on X continuously. Let τ be a G-equivariant twisting over X and $\{K_G^{\tau+k}(X)\}_{k\in\mathbb{Z}}$ be the τ-twisted K-groups, which are defined later.

Let G have torsion-free π_1 and τ be a "positive" central extension of LG, which is defined in Section 1. A central extension τ defines a G-equivariant twisting over G ([AS]), where G acts on itself by conjugation. Freed, Hopkins and Teleman constructed an isomorphism between $R^\tau(LG)$ and $K_G^{\tau+\sigma+\mathrm{rank}(G)}(G)$, where σ is a special central extension of LG, so called the spin extension.

Theorem 0.4 ([FHT2])**.** *If G is a compact connected Lie group with torsion-free π_1, we have an isomorphism*

$$FHT_G : R^\tau(LG) \to K_G^{\tau+\sigma+\mathrm{rank}(G)}(G).$$

We call it FHT isomorphism.

However, naturality of this isomorphism has not been verified. In this paper, we study naturality of FHT isomorphisms for tori.

Let us consider pull backs for two objects, K-theory and representation groups. Let $f : S \to T$ be a smooth group homomorphism between two tori T and S. We suppose that the tangent map df is injective, we call such a map a local injection. We can pull back a representation along the induced homomorphism $Lf : LS \to LT$. However, the pull back of finitely reducible representations is not finitely reducible when $\dim(T) > \dim(S)$ (Corollary 3.5), that is, $f^* : R^\tau(LT) \to R^{f^*\tau}(LS)$ cannot be defined by use of the pull back of representations. In the twisted equivariant K-theory, the K-theoretical pull back must be zero if $\dim(T) - \dim(S)$ is odd (Corollary 2.5).

Motivated by these observations, we modify pull backs so that the map FHT_T is a natural transformation of two objects. That is, we construct "induced homomorphisms"

$$f^\# : K_T^{\tau+\dim(T)}(T) \to K_S^{f^*\tau+\dim(S)}(S)$$

$$f^! : R^\tau(LT) \to R^{f^*\tau}(LS)$$

so that the following theorem holds.

Theorem 0.5. *The following diagram commutes.*

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{f^\#} & K_S^{f^*\tau+\dim(S)}(S) \\ {\scriptstyle FHT_T}\downarrow & & {\scriptstyle FHT_S}\downarrow \\ R^\tau(LT) & \xrightarrow{f^!} & R^{f^*\tau}(LS) \end{array}$$

This theorem is verified by use of other objects $char(T,\tau)$, $char(S,f^*\tau)$ and homomorphism $char(f)$, which are defined later.

In fact, we verify the following two theorems. Vertical arrows are defined later.

Theorem 0.6. *The following diagram commutes.*

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{f^\#} & K_S^{f^*\tau+\dim(S)}(S) \\ {\scriptstyle M.d._T}\downarrow & & {\scriptstyle M.d._S}\downarrow \\ char(T,\tau) & \xrightarrow{char(f)} & char(S,f^*\tau) \end{array}$$

Theorem 0.7. *The following diagram commutes.*

$$\begin{array}{ccc} R^\tau(LT) & \xrightarrow{f^!} & R^{f^*\tau}(LS) \\ {\scriptstyle l.w._T}\downarrow & & {\scriptstyle l.w._S}\downarrow \\ char(T,\tau) & \xrightarrow{char(f)} & char(S,f^*\tau) \end{array}$$

We can extend Theorem 0.6 to $f : H \to G$ satisfying the decomposable condition (Section 2), where H and G can be not tori.

Theorem 0.8. *The following diagram commutes.*

$$\begin{array}{ccc} K_G^{\tau+\mathrm{rank}(G)}(G) & \xrightarrow{f^\#} & K_H^{f^*\tau+\mathrm{rank}(H)}(H) \\ {\scriptstyle M.d._G}\downarrow & & {\scriptstyle M.d._H}\downarrow \\ char(G,\tau) & \xrightarrow{char(f)} & char(H,f^*\tau) \end{array}$$

Let us explain the details and the backgrounds of each sides.

Firstly, we recall Cartan-Weyl theory for compact connected Lie group G. Let T be a chosen maximal torus of G and Λ_T the set of isomorphism classes of irreducible representations of T. We have known an explicit description of the representation group $R(G)$ (we do not deal with the product), that is,

$$R(G) \cong \mathbb{Z}[\Lambda_T/W(G)],$$

where $W(G)$ is the Weyl group of G and $\mathbb{Z}[\Lambda_T/W(G)]$ is the $\mathbb{Z}$ free module generated by the set $\Lambda_T/W(G)$.

Let us rewrite these isomorphisms and pull back of representations along a homomorphism in terms of K-theory over discrete sets. Obviously the following isomorphisms hold

$$\mathbb{Z}[\Lambda_T] \cong K(\Lambda_T), \mathbb{Z}[\Lambda_T/W(G)] \cong K(\Lambda_T/W(G))$$

where Λ_T has the discrete topology. Let H and G be connected Lie groups and $f : H \to G$ be a homomorphism of Lie groups. $S \subseteq H$ and $T \subseteq G$ are chosen maximal tori respectively. We can assume that $f(S) \subseteq T$. We use the same symbol to represent the restriction of f to S. f induces a homomorphism

$$\begin{tikzcd} \mathfrak{t}^* \arrow[r, "{}^t df"] & \mathfrak{s}^* \\ \Lambda_T \arrow[u] \arrow[r, "{}^t df"] & \Lambda_{S'} \arrow[u] \end{tikzcd}$$

Let us notice the following commutative diagrams.

$$\begin{tikzcd} R(T) \arrow[r, "f^*"] \arrow[d] & R(S) \arrow[d] \\ K(\Lambda_T) \arrow[r, "({}^t df)_!"] & K(\Lambda_S) \end{tikzcd} \quad \begin{tikzcd} R(G) \arrow[r, "f^*"] \arrow[d] & R(H) \arrow[d] \\ R(T) \arrow[r, "f^*"] & R(S) \end{tikzcd}$$

$$\begin{tikzcd} R(G) \arrow[r] \arrow[d, "\cong"'] & R(T) \arrow[d, "\cong"'] \\ K(\Lambda_T/W(G)) \arrow[r] & K(\Lambda_T) \end{tikzcd}$$

If we notice that the restriction $R(G) \to R(T)$ is injective, we can compute $f^* : R(G) \to R(H)$ by use of the following diagram.

$$\begin{tikzcd} R(G) \arrow[rrr, "f^*"] \arrow[dr, "\cong"] \arrow[ddd, hook] & & & R(H) \arrow[dl, "\cong"'] \arrow[ddd, hook] \\ & K(\Lambda_T/W(G)) \arrow[r] \arrow[d, hook] & K(\Lambda_S/W(H)) \arrow[d, hook] & \\ & K(\Lambda_T) \arrow[r, "({}^t df)_!"] & K(\Lambda_S) & \\ R(T) \arrow[rrr, "f^*"] \arrow[ur] & & & R(S) \arrow[ul] \end{tikzcd}$$

That is, f^* can be computed by the correspondence of orbits under ${}^t df$. The third object *char* is an analogue of $K(\Lambda_T/W(G))$ and the induced homomorphism $char(f)$ is also an analogue of the correspondence of $W(G)$-orbits.

Positive energy representations at level τ have been classified as follows ([PS]). The symbol $l.w.$ comes from the lowest weight.

Theorem 0.9 ([PS]). *We have an isomorphism*

$$l.w._T : R^\tau(LT) \xrightarrow{\cong} char(T, \tau),$$

Our main tool for the construction of $f^!$ is the explicit description of the above isomorphism $l.w._T$.

Secondly, we explain the details on twisted equivariant K-theory.

Twisted K-theory was introduced in order to extend Thom isomorphism for non K-orientable vector bundle ([Ka]). Later, it was extended by Rosenberg. It has an equivariant version just like untwisted cases.

Let us start from untwisted equivariant K-theory. Equivariant K-theory is concerned with representation theory ([Se]). Throughout this paper, we deal with compactly supported K-theory. Let X be a topological space and G be a Lie group acting on X. If X is a compact Hausdorff space, isomorphism classes of G-equivariant vector bundles form a semigroup under the direct sum, and equivariant K-group $K_G(X)$ is defined by the Grothendieck completion of this semigroup. Generally, $K_G(X)$ is the set of G-equivariant homotopy classes of G-equivariant family of Fredholm operators parametrized by X. For example, when the action is trivial, $K_G(X) \cong K(X) \otimes R(G)$, and when it is free, $K_G(X) \cong K(X/G)$.

Our second main object is twisted version of it. Before that, let us explain non-equivariant twisted K-theory. In the ordinary K-theory, a classifying space is the set of Fredholm operators of Hilbert space $\mathcal{H}$, $Fred(\mathcal{H})$. That is,

$$K(X) = [X, Fred(\mathcal{H})] = \pi_0(\Gamma(X, X \times Fred(\mathcal{H}))),$$

where $X \times Fred(\mathcal{H})$ is the trivial bundle. We can twist this trivial bundle by an $Aut(Fred(\mathcal{H})) = PU(\mathcal{H})$-principal bundle. A twisting over X is an element of the group $H^3(X;\mathbb{Z})$. It classifies $PU(\mathcal{H})$ principal bundles over X, since the functor H^3 has the classifying space $K(\mathbb{Z}, 3)$, which is also a classifying space $BPU(\mathcal{H})$. Therefore, if $\tau \in H^3(X;\mathbb{Z})$ is given, we have an associated $PU(\mathcal{H})$ principal bundle P. Then we have a classifying bundle $Fred(P) = P \times_{PU(\mathcal{H})} Fred(\mathcal{H})$ of twisted K-group $K^{\tau+0}(X)$. Twisted equivariant K-theory is the equivariant version of it. Let G act on X continuously.

Definition 0.10. *The space of sections is defined by*

$$\Gamma_{c,G}(X, Fred(P)) := \{s : X \to Fred(P) | supp(s)/G \text{ is compact, } G\text{-equivariant}\},$$

where $supp(s) := \{x \in X | s_x \text{ is not invertible}\}$ *is the support of* s.

Twisted equivariant K*-theory is defined by*

$$K_G^{\tau+0}(X) := \pi_0(\Gamma_{c,G}(X, Fred(P)))$$

$$K_G^{\tau+k}(X) = K_G^{\tau-k}(X) := K_G^{p^*\tau+0}(X \times \mathbb{R}^k),$$

where $p : X \times \mathbb{R}^k \to X$ *is the projection onto the first factor.*

Since it is a generalized cohomology theory for G-spaces ([AS]), we have some tools to compute. In fact, Freed, Hopkins and Teleman computed $K_G^{\tau+k}(G)$ using spectral sequences ([FHT1]) or "Mackey decompositions" ([FHT3]) for τ which is induced by a positive central extension of LG, where G acts on itself by conjugation. For simplicity, we deal with tori here. While this action of T on itself is trivial, an action on a projective bundle can be non-trivial.

Theorem 0.11 ([FHT1], [FHT3])**.** *The following isomorphism holds.*

$$K_T^{\tau+k}(T) \cong \begin{cases} 0 & (k \equiv \dim(T) + 1 \mod 2) \\ char(T, \tau) & (k \equiv \dim(T) \mod 2). \end{cases}$$

We write the isomorphism of the above theorem as

$$M.d._T : K_T^{\tau+\dim(T)}(T) \to char(T, \tau).$$

"$M.d.$" comes from a "Mackey decomposition".

From Theorem 0.9 with 0.11, $R^\tau(LT)$ is isomorphic to $K_T^{\tau+\dim(T)}(T)$. However, Freed, Hopkins and Teleman **CONSTRUCTED** an isomorphism

$$FHT_T : R^\tau(LT) \to K_T^{\tau+\dim(T)}(T)$$

by use of a family of "Dirac operators". This construction is very interesting because Dirac operators are very important objects for geometrician. However, they verified that FHT_T is an isomorphism by verifying that

$$FHT_T = (M.d._T)^{-1} \circ l.w._T.$$

From this fact, Theorem 0.7 and Theorem 0.6 imply our main result Theorem 0.5.

This paper consists of 4 sections.

In Section 1, we prepare some notions about central extensions of loop groups and study the twisted character group $char$. We define the induced homomorphism $char(f)$, which is an analogue of pull back of representations of tori. This object is the most useful for computation. We deal with not only tori but also compact connected Lie groups with torsion-free π_1.

In Section 2, we study twisted equivariant K-theory and define $f^\#$ so that it has naturality with $char(f)$. We deal with not only tori but also compact connected Lie groups with torsion-free π_1.

In Section 3, we study representation group of loop group of a torus. We define $f^!$ so that it holds naturality with $char(f)$.

In Section 4, we verify the main theorem. It is done by just a diagram chasing.

Contents

1. Twisted character group

In this chapter, we deal with the easiest object *char*. Though *char* did not appear explicitly in [FHT2], they verified their main theorem using *char* essentially.

1.1. Central extensions of loop groups and twisted character group

In this section, we explain some notations about central extensions of loop groups and representation theory of tori to define the group *char*. Then, we define *char* for not only tori but also more general Lie groups.

1.1.1. Central extensions of loop groups

Let G be a compact connected Lie group with torsion-free π_1. Let LG be the group of smooth maps from S^1 to G, the loop group of G. Its topology is defined so that l_n converges to l if and only if all derivatives $\frac{d^k l_n}{d\theta^k}$ converges to $\frac{d^k l}{d\theta^k}$ uniformly. This is one of the simplest infinite dimensional Fréchet Lie groups. S^1 acts on LG via transformation of parameter, that is, $(\theta_0^* l)(\theta) := l(\theta - \theta_0)$ $(l \in LG)$. When we make S^1 act on loop groups, we write S^1 as $\mathbb{T}_{\mathrm{rot}}$. So we can define a new Lie group $LG \rtimes \mathbb{T}_{\mathrm{rot}}$.

The Lie algebra of LG is the loop algebra $L\mathfrak{g} := C^\infty(S^1, \mathfrak{g})$. Its addition, scalar multiplication and Lie bracket are induced by ones of $\mathfrak{g}$. We write the Lie algebra of $LG \rtimes \mathbb{T}_{\mathrm{rot}}$ as $L\mathfrak{g} \oplus i\mathbb{R}_{\mathrm{rot}}$. If d is an infinitesimal generator of the action of $\mathbb{T}_{\mathrm{rot}}$, $[d, \beta] = \frac{d\beta}{d\theta}$ for $\beta \in L\mathfrak{g}$.

We need the following definition in order to define positive energy representations of loop groups, because positive energy representations must be projective unless it is trivial.

Definition 1.1 ([FHT2])**.** *A central extension*

$$1 \to U(1) \xrightarrow{i} LG^\tau \xrightarrow{p} LG \to 1$$

is admissible if

(1) *The action of* $\mathbb{T}_{\mathrm{rot}}$ *lifts to* LG^τ*. Therefore, we can define a new Lie group* $LG^\tau \rtimes \mathbb{T}_{\mathrm{rot}}$.

(2) $LG^\tau \rtimes \mathbb{T}_{\mathrm{rot}}$ *acts on* $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{\mathrm{rot}}$ *via the adjoint action. There exists an* $LG^\tau \rtimes \mathbb{T}_{\mathrm{rot}}$*-invariant symmetric bilinear form* $\langle\langle \cdot, \cdot \rangle\rangle_\tau$ *on* $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{\mathrm{rot}}$ *such that* $\langle\langle K, D \rangle\rangle_\tau = -1$*, where* K *is the infinitesimal generator of* $i(U(1))$*, and* D *is any element of* $L\mathfrak{g}^\tau \oplus i\mathbb{R}_{\mathrm{rot}}$ *projecting to* $i \in i\mathbb{R}_{\mathrm{rot}}$*. If the restriction of* $\langle\langle \cdot, \cdot \rangle\rangle_\tau$ *to* $L\mathfrak{g}$ *is positive definite,* τ *is called a positive central extension.*

Remark 1.2 ([FHT2] Lemma 2.18)**.** *Bilinear form* $\langle\langle \cdot, \cdot \rangle\rangle_\tau$ *determines a splitting* $L\mathfrak{g} \to L\mathfrak{g}^\tau \oplus i\mathbb{R}_{\mathrm{rot}}$*. So we can restrict* $\langle\langle \cdot, \cdot \rangle\rangle_\tau$ *to* $L\mathfrak{g}$ *.*

Definition 1.3. *Let* $f : H \to G$ *be a smooth group homomorphism, then the homomorphism* $Lf : LH \to LG$ *is defined by* $l \mapsto f \circ l$*. We can pull back a central extension* τ *of* LG *along* Lf *and we write it as* $f^*\tau$*. That is, the following diagram commutes.*

$$
\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r, "i"] & LG^\tau \arrow[r, "p"] & LG \arrow[r] & 1 \\
1 \arrow[r] \arrow[u, equal] & U(1) \arrow[r, "i"] \arrow[u, equal] & LH^{f^*\tau} \arrow[r, "p"] \arrow[u, "\widetilde{Lf}"] & LH \arrow[r] \arrow[u, "Lf"] & 1 \arrow[u, equal]
\end{tikzcd}
$$

One can verify the following easily, that is, one can construct $\mathbb{T}_{\mathrm{rot}}$-action on $LH^{f^*\tau}$ and bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{f^*\tau}$.

Lemma 1.4. *If* τ *is admissible, so is* $f^*\tau$*.*

Moreover, we can construct the tensor product of two central extensions.

Lemma 1.5 ([FHT2] Section 2.2)**.** *Let* τ_1 *and* τ_2 *be admissible central extensions of* LG*. Then we can define the tensor product between them by*

$$LG^{\tau_1+\tau_2} := LG^{\tau_1} \otimes LG^{\tau_2} = LG^{\tau_1} \times_{LG} LG^{\tau_2}.$$

Then $\tau_1 + \tau_2$ *is also admissible and the invariant bilinear form can be defined by*

$$\langle\langle \cdot, \cdot \rangle\rangle_{\tau_1+\tau_2} = \langle\langle \cdot, \cdot \rangle\rangle_{\tau_1} + \langle\langle \cdot, \cdot \rangle\rangle_{\tau_2}.$$

When G is a torus T, admissibility implies Lemma 1.8. In order to describe it, we need some terminologies about tori or their representations.

$\Pi_T := \ker(\exp : \mathfrak{t} \to T) \cong \pi_1(T) \cong H_1(T)$ is a lattice in $\mathfrak{t}$.

$\Lambda_T := \mathrm{Hom}(\Pi_T, \mathbb{Z}) \cong \mathrm{Hom}(T, U(1)) \cong H^1(T)$ is the character group. We can regard it as the set of irreducible representations of T from Schur's lemma (any irreducible representation of commutative group is 1 dimensional).

Lemma 1.6 ([FHT2] Proposition 2.27)**.** LT *has a canonical decomposition* $LT \cong T \times \Pi_T \times U$*, where* T *corresponds to the set of initial values of loops,* Π_T *corresponds to the set of "rotation numbers"(naturally isomorphic to* $\pi_1(T) \cong \pi_0(LT)$*) and*

$$U := \exp\left\{\beta : S^1 \to \mathfrak{t} \;\middle|\; \int_{S^1} \beta(s)ds = 0\right\}$$

is the set of derivatives of contractible loops whose initial values are 0*.*

Let τ *be an admissible central extension of* LT*, then the above decomposition is inherited partially, that is,* $LT^\tau \cong (T \times \Pi)^\tau \boxtimes U^\tau = (T \times \Pi)^\tau \times U^\tau / U(1)$

This lemma allows us to define very important object $\kappa^\tau : \Pi_T \to \Lambda_T$ and describe some formulae. κ^τ is defined by the following homomorphism $\widetilde{\kappa^\tau} : \Pi_T \to \mathrm{Hom}(T, U(1))$ and the isomorphism $\Lambda_T \cong \mathrm{Hom}(T, U(1))$.

Definition 1.7 ([FHT2] Proposition 2.27)**.** *A homomorphism* $\widetilde{\kappa^\tau} : \Pi_T \to \mathrm{Hom}(T, U(1))$ *is defined by*

$$\widetilde{\kappa^\tau}(X)(t) := \widetilde{\phi}_X \widetilde{t} \widetilde{\phi}_X^{-1} \widetilde{t}^{-1} \in i(U(1)) \subseteq LT^\tau,$$

where $\widetilde{t}$ *is a chosen lift of* $t \in T$*, and* $\widetilde{\phi}_X$ *is a chosen lift of* $LT \ni \phi_X$ *which is a geodesic loop corresponding to* $X \in \Pi_T \cong \pi_1(T)$ *whose initial value is* 0*. This definition is independent of the choices of lifts because they are determined up to* $i(U(1))$ *which is in the center of* LT^τ*.*

This definition means that κ^τ represents how non-commutativity is caused by the central extension.

Lemma 1.8 ([FHT2] Proposition 2.27)**.** κ^τ *determines a symmetric bilinear form* $\langle n, m \rangle_\tau = \kappa^\tau(n)(m)$ *on* Π_T *for* $n, m \in \Pi_T$*.*

Moreover, when we extend $\langle \cdot, \cdot \rangle_\tau$ *to* $\mathfrak{t}$ *linearly,* $\langle v, w \rangle_\tau = \langle \langle v, w \rangle \rangle_\tau$*.*

Proposition 1.9. *If a central extension* τ *of* LT *is positive, the bilinear form* $\langle \cdot, \cdot \rangle_\tau$ *is positive definite.*

Even if G is not a torus, we can take a maximal torus $i : T \hookrightarrow G$ and construct $\kappa^\tau : \Pi_T \to \Lambda_T$ and bilinear form $\langle \cdot, \cdot \rangle_\tau$. If τ is a positive central extension of LG, the restriction $i^*\tau$ is also positive.

Let us define the twisted character group $char(G, \tau)$ for a pair G and τ, where G is a compact connected Lie group with torsion-free π_1 and τ is a positive central extension of LG. In order to carry it out, we need some terminology.

We can define central extensions G^τ and T^τ by the restrictions of LG^τ to G and T. Then, we can verify that T^τ is a maximal torus of G^τ. So we can define a τ-twisted representation of T^τ. A τ-twisted representation is a homomorphism $\rho : T^\tau \to U(V)$ such that $\rho \circ i(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}\theta}\mathrm{id}_V$, where V is a representation space. Λ_T^τ is the set of irreducible τ-twisted representations. It can be thought as the set of $\lambda : \Pi_{T^\tau} \to \mathbb{Z}$ such that the composition of $\mathbb{Z} = \Pi_{U(1)} \xrightarrow{di} \Pi_{T^\tau} \xrightarrow{\lambda} \mathbb{Z}$ is identity.

We can define an action of Λ_T on Λ_T^τ by the tensor product of the representations. We write it as $\lambda + \lambda'$ ($\lambda \in \Lambda_T$, $\lambda' \in \Lambda_T^\tau$). The tensor product of the representations is defined by

$$(\lambda + \lambda')(t) = \lambda(p(t)) \cdot \lambda'(t),$$

where $t \in T^\tau$. Since $p \circ i(z) = 1$, $\lambda + \lambda'$ is also a τ-twisted representation.

The Weyl group of G and G^τ are naturally isomorphic. We write them as $W(G)$. Since $W(G)$ acts on Π_T, we can define the semi direct product

$$W^e_{\mathit{aff}}(G) := \Pi_T \rtimes W(G),$$

we call it the extended affine Weyl group. It is the Weyl group of LG^τ whose maximal torus is T^τ. Since $W(G)$ acts on Π_{T^τ}, $W(G)$ acts on Λ^τ_T by the dual action. And Π_T acts on Λ^τ_T through the homomorphism κ^τ and $\Lambda_T \curvearrowright \Lambda^\tau_T$ which is the dual of Π_{T^τ}. Therefore, $W^e_{\mathit{aff}}(G)$ acts on Λ^τ_T. We write the orbit space as $\Lambda^\tau_T/\kappa^\tau(W^e_{\mathit{aff}}(G))$.

$\mathbb{Z}[(\Lambda^\tau_T/\kappa^\tau(W^e_{\mathit{aff}}(G)))_{reg}]$ is the free $\mathbb{Z}$ module generated by the set of $W^e_{\mathit{aff}}(G)$-regular orbits $\Lambda^\tau_T/\kappa^\tau(W^e_{\mathit{aff}}(G))$. A regular orbit is an orbit whose stabilizer is trivial.

Definition 1.10.

$$\mathit{char}(G,\tau) := \mathbb{Z}[(\Lambda^\tau_T/\kappa^\tau(W^e_{\mathit{aff}}(G)))_{reg}]$$

Remark 1.11. *$\mathit{char}(G,\tau)$ is isomorphic to the twisted K-group $K_G^{\tau+\mathrm{rank}(G)}(G)$ and $R^\tau(LG)$ ([FHT2]). Thanks to positivity of the central extension, the orbit space $\Lambda^\tau_T/\kappa^\tau(W^e_{\mathit{aff}}(G))$ is a finite set.*

1.1.2. The decomposable condition

We deal with non-commutative cases after tori. As in the case of tori, we prove the theorems for special cases and later we extend them by use of decomposition of homomorphisms into easy pieces, inclusions into the first factor and finite coverings. So we should decompose homomorphisms between general groups. In order to do it, as far as we know, we should assume the following condition.

Let G and H be a compact connected Lie group with torsion-free π_1, $f : H \to G$ be a smooth group homomorphism such that the tangent map is injective, and τ be a positive central extension of LG. Take maximal tori S and T of H and G respectively. We can assume $f(S) \subseteq T$. That is, the following diagram commutes.

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle k} \\ S & \xrightarrow{f|_S} & T \end{array}$$

Definition 1.12. *Let $f : H \to G$ be a smooth group homomorphism. f satisfies the decomposable condition if and only if the followings are satisfied.*

(1) *$f^*\tau$ is also positive.*

(2) *$H/\ker(f)$ has torsion-free π_1.*

(3) *$[df(\mathfrak{h}), \mathfrak{s}^\perp] = 0$, where $\mathfrak{s}^\perp$ is the orthogonal complement of $df(\mathfrak{s})$ in $\mathfrak{t}$ under the bilinear form $\langle\cdot,\cdot\rangle_\tau$ defined in Lemma 1.8. We call this condition the "local condition".*

Remark 1.13. *Condition (1) and (2) guarantee that we can deal with H and $H/\ker(f)$ with the same formulae.*

Condition (3) guarantees that $H/\ker(f) \times S^\perp$ is also a group, where $S^\perp \subseteq T$ is the torus whose Lie algebra is $\mathfrak{s}^\perp$. That $S^\perp$ is indeed a torus is verified in Corollary 1.26.

If f satisfies this condition, df is automatically injective. It follows from Lemma 1.17.

The following examples satisfy the decomposable condition.

Example 1.14. (*a*) *Any injection f when* $\mathrm{rank}(H) = \mathrm{rank}(G)$.
(*b*) *Any local injection f satisfying* (2) *when* $\mathrm{rank}(H) = \mathrm{rank}(G)$.
(*c*) *Any local injection f when H is a torus.*
(*d*) $U(n) \hookrightarrow U(n+N)$ *defined by*

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We prepare some lemmas to define $char(f)$. The terminology $S^{\perp}$ is justified later and we admit that $S^{\perp}$ is truly a torus here.

Lemma 1.15. *If f satisfies the local condition, the normalizer of S in H is mapped to one of T in G.*

Proof. Since H is compact and connected, $\exp : \mathfrak{h} \to H$ is surjective, that is, for any $h \in H$, there exists $v \in \mathfrak{h}$ such that $h = \exp(v)$. In the same way, for $t \in S^{\perp}$, there exists $\widetilde{t} \in \mathfrak{s}^{\perp}$ such that $\exp(\widetilde{t}) = t$. From Campbell-Hausdorff's formula

$$f(h)tf(h)^{-1} = f(\exp(v))\exp(\widetilde{t})f(\exp(-v))$$

$$= \exp(df(v) + \widetilde{t} + \frac{1}{2}[df(v), \widetilde{t}] + \cdots)f(\exp(-v)) = \exp(df(v) + \widetilde{t})\exp(-df(v))$$

$$= \exp(df(v) + \widetilde{t} - df(v) + \frac{1}{2}[df(v) + \widetilde{t}, -df(v)] + \cdots)$$

$$= \exp(\widetilde{t}) = t.$$

Therefore, $f(h)$ and t commute. Moreover, for any $x \in T$, there exist $s \in S$ and $t \in S^{\perp}$ such that $x = f(s) \cdot t$, which is verified at Proposition 1.27. Therefore, for any $n \in N(S)$ and $x \in T$, the following holds.

$$f(n)xf(n^{-1}) = f(n)(f(s)t)f(n^{-1}) = f(n)f(s)f(n^{-1})t$$

$$= f(nsn^{-1})t \in T.$$

□

From this lemma, we can define a homomorphism $f_* : W(H) \to W(G)$, where $W(H)$ and $W(G)$ are the Weyl groups of H and G respectively.

Lemma 1.16. *f_* is an injection.*

Proof. Suppose that $n \in N(S)$ and $[n] \in \ker(f_*)$. Then, $Ad(f(n))$ defines an automorphism of $\mathfrak{t}$. From the assumption that $[n] \in \ker(f_*)$, $Ad(f(n)) = \mathrm{id}_{\mathfrak{t}}$. Since $Ad(f(n))|_{\mathfrak{s}} = Ad(n)$, $n \in S$. We obtain the conclusion. □

1.2. Induced homomorphisms

In this section, we construct the induced homomorphism $char(f) : char(T,\tau) \to char(S,f^*\tau)$ for a local injection $f : S \to T$ and study it. The point is that f can be rewritten **canonically** as the composition of finite coverings and inclusions into the first factor of the direct product of two tori thanks to positivity of the central extension, and that induced homomorphisms are compatible with this decomposition. Therefore it is sufficient to deal with only two special cases, inclusions and finite coverings.

Before the construction of $char(f)$, as preliminaries, let us verify the following formulae related with central extensions. Let $f : S \to T$ be a local injection, that is, the tangent map df is injective. Then we have the associated homomorphism $Lf : LS \to LT$ and $\widetilde{Lf} : LS^{f^*\tau} \to LT^\tau$. One can verify the following easily by the definition.

Lemma 1.17.

$$\kappa^{f^*\tau} = {}^t df \circ \kappa^\tau \circ df$$

It is used so many times in this paper.

One can verify the following in the same way. Let T_1 and T_2 be tori, $T := T_1 \times T_2$, $i_j : T_j \to T$ be the natural inclusion into the j'th factor, $p_j : T \to T_j$ be the projection onto the j'th factor and τ_j be an admissible central extension of LT_j $(j = 1, 2)$. Then $\tau = p_1^*\tau_1 + p_2^*\tau_2$ is an admissible central extension of LT. In this case we have a canonical isomorphisms $LT \cong LT_1 \times LT_2$ and $LT^\tau \cong LT_1^{\tau_1} \boxtimes LT_2^{\tau_2} = LT_1^{\tau_1} \times_{U(1)} LT_2^{\tau_2}$. Therefore, for any $l_1, l_1' \in LT_1^{\tau_1}$, $l_2, l_2' \in LT_2^{\tau_2}$, $l_1 \boxtimes l_2 \cdot l_1' \boxtimes l_2' = l_1 l_1' \boxtimes l_2 l_2'$. Moreover, $LT^\tau \cong U_1^{\tau_1} \boxtimes U_2^{\tau_2} \boxtimes (T_1 \times \Pi_{T_1})^{\tau_1} \boxtimes (T_2 \times \Pi_{T_2})^{\tau_2}$. This commutativity implies the following.

Lemma 1.18. *$\kappa^\tau : \Pi_{T_1\times T_2} \to \Lambda_{T_1\times T_2}$ is the composition of*

$$\Pi_{T_1\times T_2} \xrightarrow{dp_1 \oplus dp_2} \Pi_{T_1} \oplus \Pi_{T_2} \xrightarrow{\kappa^{\tau_1}\oplus\kappa^{\tau_2}} \Lambda_{T_1} \oplus \Lambda_{T_2} \xrightarrow{{}^t dp_1 \oplus {}^t dp_2} \Lambda_{T_1\times T_2}.$$

Formally

$$\kappa^\tau = \begin{pmatrix} \kappa^{\tau_1} & 0 \\ 0 & \kappa^{\tau_2} \end{pmatrix}$$

1.2.1. Definition of induced homomorphism $char(f)$

Let $f : S \to T$ be a local injection and τ be a positive central extension of LT. We define $char(f)([\lambda]_T)$ for $\lambda \in \Lambda_T^\tau$, where $[\lambda]_T$ means the Π_T-orbit including λ. Let us consider the set $\{{}^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\}$ and decompose it as the finite union of Π_S-orbits. That is,

$$\{{}^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\} = \coprod_{i=1}^N \{\mu_i + \kappa^{f^*\tau}(m) | m \in \Pi_S\}.$$

Since $\Lambda_S^{f^*\tau}/\kappa^{f^*\tau}(\Pi_S)$ is a finite set, truly $N < \infty$.

Definition 1.19.

$$char(f)([\lambda]_T) := \sum_{i=1}^{N} [\mu_i]_S$$

Lemma 1.20. *It is well-defined.*

Proof. It is sufficient to verify that $\{{}^t df(\lambda + \kappa^\tau(n)) | n \in \Pi_T\}$ is Π_S-invariant. We can verify it using the following. Of course we use Lemma 1.17.

$$\begin{aligned} &{}^t df(\lambda + \kappa^\tau(n)) + \kappa^{f^*\tau}(l) \\ = &{}^t df(\lambda + \kappa^\tau(n + df(l))) \in \{{}^t df(\lambda + \kappa^\tau(m)) | m \in \Pi_S\}. \end{aligned}$$

□

We can easily compute the above homomorphism in some cases.

Lemma 1.21. *Let $i_1 : T_1 \to T_1 \times T_2$ be the natural inclusion into the first factor and $\tau = p_1^*\tau_1 + p_2^*\tau_2$, where τ_j is a positive central extension of LT_j and $p_j : T_1 \times T_2 \to T_j$ is the natural projection onto the j'th factor. Then,*

$$char(i_1)([\lambda]_{T_1 \times T_2}) = [{}^t di_1(\lambda)]_{T_1}.$$

Proof. Let us recall Lemma 1.18, that is, $\kappa^\tau = {}^t dp_1 \circ \kappa^{\tau_1} \circ dp_1 + {}^t dp_2 \circ \kappa^{\tau_2} \circ dp_2$ when $\tau = p_1^*\tau_1 + p_2^*\tau_2$.

$$\begin{aligned} &\{{}^t di_1(\lambda) + \kappa^{\tau_1}(n_1) | n_1 \in \Pi_{T_1}\} \times \{{}^t di_2(\lambda) + \kappa^{\tau_2}(n_2) | n_2 \in \Pi_{T_2}\} \\ \xrightarrow{\cong} &\{{}^t dp_1 \circ {}^t di_1(\lambda) + {}^t dp_2 \circ {}^t di_2(\lambda) + {}^t dp_1 \circ \kappa^{\tau_1} \circ dp_1(n) + {}^t dp_2 \circ \kappa^{\tau_2} \circ dp_2(n) | n \in \Pi_T\} \\ = &\{\lambda + \kappa^\tau(n) | n \in \Pi_T\}. \end{aligned}$$

Since ${}^t di_1 \circ {}^t dp_2 = 0$, ${}^t di_1$ is just a projection onto first factor. □

Lemma 1.22. *Let $q : S \to T$ be a finite covering. Then,*

$$char(q)([\lambda]_T) = \sum_{m \in \Pi_T / dq(\Pi_S)} [{}^t dq(\lambda + \kappa^\tau(m))]_S.$$

Remark 1.23. *$m \in \Pi_T/dq(\Pi_S)$ is a chosen representative element. Let us notice that the orbit $[{}^t dq(\lambda + \kappa^\tau(m))]_S$ is independent of the choice of the representative element.*

Proof. It follows from the following.

$$\begin{aligned} &\{{}^t dq(\lambda + \kappa^\tau(n)) | n \in \Pi_T\} = \{{}^t dq(\lambda) + {}^t dq(\kappa^\tau(n)) | n \in \Pi_T\} \\ = &\{{}^t dq(\lambda) + {}^t dq(\kappa^\tau(dq(n') + m)) | n' \in \Pi_S, m \in \Pi_T/dq(\Pi_S)\} \\ = &\coprod_{m \in \Pi_T/dq(\Pi_S)} \{{}^t dq(\lambda + \kappa^\tau(m)) + \kappa^{q^*\tau}(n') | n' \in \Pi_S\}. \end{aligned}$$

□

1.2.2. The decomposition of a local injection associated with a positive central extension

While *char* is not a functor, it satisfies the "functoriality" we need. To state it, we need to rewrite a local injection as a composition of finite coverings and an inclusion into the first factor of a direct product canonically.

Let T and S be tori, $f : S \to T$ be a local injection and τ be a positive central extension of LT. At first sight, our formulae Lemma 1.21, 1.22 are very special. However, we can rewrite any local injection f as a composition of finite coverings and an inclusion into the first factor of a direct product canonically. We start from an easy observation.

Proposition 1.24. *We can canonically decompose a local injection f as follows*

$$S \xrightarrow{q} S/\ker f \xrightarrow{i} T$$

where q is a finite covering, and i is an injection.

The above tells us that we can assume that f is an injection.

Since we assume that $f^*\tau$ is positive, $df(\mathfrak{s}) \cap df(\mathfrak{s})^\perp = 0$, where $df(\mathfrak{s})^\perp$ is the orthogonal complement under $\langle \cdot, \cdot \rangle_\tau$ in $\mathfrak{t}$. Let us verify the key lemma to the decomposition of f.

Lemma 1.25. *$df(\mathfrak{s})^\perp \cap \Pi_T$ is a $\mathbb{Z}$ free module of rank $n - n'$, where $n = \dim(T)$ and $n' = \dim(S)$.*

Proof. Since Π_T is a free module, $df(\mathfrak{s}) \cap \Pi_T \subseteq \Pi_T$ is also free. Let us recall that κ^τ determines a $\mathbb{Z}$-valued bilinear form on Π_T and it extends to $\Pi_T \otimes \mathbb{Q}$. So we can use linear algebra over $\mathbb{Q}$.

Fix a $\mathbb{Z}$-basis $\{v_1, v_2, \cdots, v_{n'}\}$ of Π_S, and let us notice that $df(\Pi_S) \subseteq \Pi_T$ and $df(\Pi_S) \otimes \mathbb{Q} = df(\Pi_S \otimes \mathbb{Q}) \subseteq \Pi_T \otimes \mathbb{Q}$. Let us notice that

$$(df(\mathfrak{s}))^\perp \cap (\Pi_T \otimes \mathbb{Q}) = (df(\Pi_S \otimes \mathbb{Q}))^\perp = \ker \begin{pmatrix} \kappa^\tau(df(v_1)) \\ \kappa^\tau(df(v_2)) \\ \vdots \\ \kappa^\tau(df(v_{n'})) \end{pmatrix}$$

Needless to say, $\kappa^\tau(df(v_j))$ can be regarded as a linear functional on $\Pi_T \otimes \mathbb{Q}$. If we notice that the rank of the matrix ${}^t\begin{pmatrix}\kappa^\tau(df(v_1)) & \kappa^\tau(df(v_2)) & \cdots & \kappa^\tau(df(v_{n'}))\end{pmatrix}$ is n' (recall that κ^τ is injective), the dimension of the LHS is $n - n'$. So we can take a $\mathbb{Q}$ basis $\{w_j\}_{j=1}^{n-n'}$. Notice that for any $w \in \Pi_T \otimes \mathbb{Q}$, there exists $N \in \mathbb{Z}$ such that $Nw \in \Pi_T$, therefore, we may assume that $w_j \in \Pi_T$ for any j. Since $\{w_j\}$ are linearly independent over $\mathbb{Z}$, $\mathrm{rank}(df(\mathfrak{s})^\perp \cap \Pi_T) \geq n - n'$. On the other hand, $\mathrm{rank}(df(\mathfrak{s})^\perp \cap \Pi_T) \leq n - n' = \dim(df(\mathfrak{s})^\perp)$. Therefore $\{w_j\}_{j=1}^{n-n'}$ is a basis of the free module $df(\mathfrak{s})^\perp \cap \Pi_T$. $\square$

The above implies the followings.

Corollary 1.26.

$$S^\perp := df(\mathfrak{s})^\perp / (df(\mathfrak{s})^\perp \cap \Pi_T) \subseteq T$$

is a torus. We call it the "orthogonal complement torus". Let $j : S^{\perp} \hookrightarrow T$ be the natural inclusion.

From the above construction, we obtain the following decomposition.

Proposition 1.27. *If $f : S \to T$ is an injection, we can define the new torus $S \times S^{\perp}$ and decompose f as follows.*

$$S \xrightarrow{i_1} S \times S^{\perp} \xrightarrow{f \cdot j} T.$$

Moreover, $f \cdot j$ is a finite covering, where $(f \cdot j)(t_1, t_2)$ is defined by $f(t_1) \cdot j(t_2)$.

Example 1.28. *Let $f : U(1) \hookrightarrow [U(1)]^2$ be $f(z) := (z, z^{-1})$. We suppose that the central extension of $L[U(1)]^2$ is invariant under the action $(z_1, z_2) \mapsto (z_2, z_1)$.*

Then, the orthogonal complement torus is

$$\{(z, z) | z \in U(1)\}.$$

Clearly $\ker(f \cdot j) = \{\pm(1, 1)\}$ *and $f \cdot j$ is truly a non-trivial covering map.*

Combining Proposition 1.24 with Proposition 1.27, we obtain the following theorem.

Theorem 1.29. *We can decompose f as follows*

$$S \xrightarrow{q} S/\ker(f) \xrightarrow{i_1} S/\ker(f) \times (S/\ker(f))^{\perp} \xrightarrow{f \cdot j} T,$$

where $(S/\ker(f))^{\perp}$ is the orthogonal complement torus associated with f and τ, q is the natural finite covering, i_1 is the natural inclusion into the first factor, and j is the natural inclusion.

Let us notice that $df(\mathfrak{s}) \perp df(\mathfrak{s})^{\perp}$ under the bilinear form $\langle \cdot, \cdot \rangle_{\tau}$ by the definition. Combining it with Lemma 1.18, we obtain the following.

Lemma 1.30.

$$(f \cdot j)^* \tau = p_1^* i_1^* (f \cdot j)^* \tau + p_2^* i_2^* (f \cdot j)^* \tau.$$

That is, the map $i_1 : S/\ker(f) \to S/\ker(f) \times (S/\ker(f))^{\perp}$ satisfies the assumption of Lemma 1.21.

1.2.3. The functoriality we need

We use the same notations in the previous section. The following is the functoriality we need mentioned above. The most important point is the definition of $S^{\perp}$.

Theorem 1.31.

$$char(f) = char(q) \circ char(i_1) \circ char(f \cdot j)$$

Proof. We divide the proof into two parts. Let us start from the easier one.

Lemma 1.32. *If we have the following commutative diagram*

$$\begin{tikzcd} S' \arrow[rr, "f"] \arrow[dr, "q"'] & & T \\ & S \arrow[ur, "g"'] & \end{tikzcd}$$

where f and g are local injections and q is a finite covering, the equality

$$char(f) = char(q) \circ char(g)$$

holds.

Proof. Let us take an orbit $[\lambda]_T \in char(T, \tau)$ and compute $char(f)([\lambda]_T)$ and $char(q) \circ char(g)([\lambda]_T)$. Since ${}^t df = {}^t dq \circ {}^t dg$, the image of the orbit $[\lambda]_T$ under ${}^t df$ coincides with the one under ${}^t dq \circ {}^t dg$. Therefore, when $char(f)([\lambda]_T) = \sum_{i=1}^{N} [\mu_i]_{S'}$,

$$char(q) \circ char(g)([\lambda]_T) = \sum_{i=1}^{N} a_i [\mu_i]_{S'}$$

for some $a_i \in \mathbb{Z}_{>0}$. It is sufficient to verify that $a_i = 1$ for any i.

If $char(g)([\lambda]_T) = \sum_{j=1}^{M} [\nu_j]_S$,

$$char(q) \circ char(g)([\lambda]_T) = \sum_{j=1}^{M} char(q)([\nu_j]_S) = \sum_{i=1}^{N} a_i [\mu_i]_{S'},$$

where $a_i = \#\{j \in \{1, 2, \cdots, M\} | {}^t dq(\nu_j + \kappa^{g^*\tau}(\Pi_S)) \supseteq \mu_i + \kappa^{f^*\tau}(\Pi_{S'})\}$.

However, if $[\nu]_S \neq [\nu']_S$, the images of orbits $\{\nu + \kappa^{g^*\tau}(n) | n \in \Pi_S\}$ and $\{\nu' + \kappa^{g^*\tau}(n) | n \in \Pi_S\}$ under ${}^t dq$ are disjoint because ${}^t dq$ is injective. Therefore, $a_i = 1$ for any i. $\square$

The above tells us that we may assume that f is an injection. Let us consider the following commutative diagram.

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "i_1"'] & & T \\ & S \times S^{\perp} \arrow[ur, "f \cdot j"'] & \end{tikzcd}$$

Let $[\lambda]_T \in char(T, \tau)$ be an orbit, $char(f)([\lambda]_T) = \sum_{i=1}^{N} [\mu_i]_S$, and $char(i_1) \circ char(f \cdot j)([\lambda]_T) = \sum_{i=1}^{N} a_i [\mu_i]_S$. Let us verify that $a_i = 1$ for any i. Since

$$char(f \cdot j)([\lambda]_T)$$
$$= \sum_{m \in \Pi_T / d(f \cdot j)(\Pi_{S \times S^{\perp}})} [{}^t d(f \cdot j)(\lambda + \kappa^{\tau}(m))]_{S \times S^{\perp}},$$

that $a_i = 1$ for any i is equivalent to that

$$\left[{}^t di_1({}^t d(f\cdot j)(\lambda+\kappa^\tau(m))) + \kappa^{f^*\tau}(\Pi_S)\right] \cap \left[{}^t di_1({}^t d(f\cdot j)(\lambda+\kappa^\tau(m'))) + \kappa^{f^*\tau}(\Pi_S)\right] = \emptyset$$

if $m \neq m'$. From the linearity of these maps, it is sufficient to verify the equality under the assumption that $m' \in d(f\cdot j)(\Pi_{S\times S^\perp})$.

Let us assume that

$$\left[{}^t di_1({}^t d(f\cdot j)(\lambda+\kappa^\tau(m))) + \kappa^{f^*\tau}(\Pi_S)\right] \cap \left[{}^t di_1({}^t d(f\cdot j)(\lambda)) + \kappa^{f^*\tau}(\Pi_S)\right] \neq \emptyset.$$

We verify that $m \in d(f\cdot j)(\Pi_{S\times S^\perp})$. The assumption is equivalent to that for some $n, n' \in \Pi_S$,

$$ {}^t di_1({}^t d(f\cdot j)\circ\kappa^\tau(m)) + \kappa^{f^*\tau}(n) = \kappa^{f^*\tau}(n')$$
$$ {}^t df\circ\kappa^\tau(m) = \kappa^{f^*\tau}(n'-n)$$

. That is,

$$ {}^t df(\kappa^\tau(m)) \in \kappa^{f^*\tau}(\Pi_S).$$

Therefore, there exists $k \in \Pi_S$ such that

$$ {}^t di_1 \circ {}^t d(f\cdot j)\circ\kappa^\tau(m) = {}^t di_1\circ {}^t d(f\cdot j)\circ\kappa^\tau\circ df(k).$$

So, ${}^t d(f\cdot j)(\kappa^\tau(m) - \kappa^\tau(df(k))) \in \ker({}^t di_1)$.

Since $df(\mathfrak{s}) \perp df(\mathfrak{s})^\perp$ under $\langle\cdot,\cdot\rangle_\tau$,

$$\ker({}^t di_1) = \kappa^{(f\cdot j)^*\tau}(di_1(\mathfrak{s})^\perp).$$

Therefore, there exists $v \in di_1(\mathfrak{s})^\perp = \mathfrak{s}^\perp$ such that

$$ {}^t d(f\cdot j)\circ(\kappa^\tau(m) - \kappa^\tau(df(k))) = \kappa^{(f\cdot j)^*\tau}(v) = {}^t d(f\cdot j)\circ\kappa^\tau\circ d(f\cdot j)(v).$$

Since ${}^t d(f\cdot j)\circ\kappa^\tau$ is injective, $m - df(k) = d(f\cdot j)(v)$. Moreover, because m and $df(k)$ are elements of Π_T, $m - df(k) = d(f\cdot j)(v) \in \Pi_T \cap df(\mathfrak{s})^\perp$.

By the definition of the orthogonal complement torus, $j = (f\cdot j)|_{S^\perp}$ is injective and

$$dj|_{\Pi_{S^\perp}} = d(f\cdot j)|_{\Pi_{S^\perp}} : \Pi_{S^\perp} \to \Pi_T \cap df(\mathfrak{s})^\perp$$

is bijective. Therefore, $v \in \Pi_{S^\perp} \subseteq \Pi_{S\times S^\perp}$. Therefore,

$$m = d(f\cdot j)(di_1(k)+v) \in d(f\cdot j)(\Pi_{S\times S^\perp}).$$

□

1.3. Twisted character for general groups

We can extend the above constructions and functoriality to more general cases using reduction to maximal tori. Let G be a compact connected Lie group with torsion-free π_1 and T be a maximal torus of G. Let $i : T \to G$ be the natural inclusion. Let us suppose that $\lambda \in \Lambda_T^\tau$ determines a regular orbit in Λ_T^τ, that is, the stabilizer of λ in $W_{aff}^e(G)$ is trivial. Then the $W_{aff}^e(G)$-orbit $W_{aff}^e(G).\lambda$ can be regarded as the union of Π_T-orbits $\coprod_{w\in W(G)}(w.\lambda + \kappa^\tau(\Pi_T))$. So we define $char(i)([\lambda]_G)$ as follows.

Definition 1.33.
$$char(i)([\lambda]_G) := \sum_{w \in W(G)} [w.\lambda]_T$$

Clearly, it is an injection.

1.3.1. The definition of the induced homomorphism of $char(f)$

Let us consider the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ {\scriptstyle i}\uparrow & & {\scriptstyle k}\uparrow \\ S & \xrightarrow{f|_S} & T \end{array}$$

where H and G are compact connected Lie groups with torsion-free π_1, S and T are maximal tori of H and G respectively satisfying the decomposable condition and that $f(S) \subseteq T$. Let τ be a positive central extension of LG.

Definition 1.34. *$char(f) : char(G,\tau) \to char(H, f^*\tau)$ is defined so that the following diagram commutes.*

$$\begin{array}{ccc} char(G,\tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \\ {\scriptstyle char(k)}\downarrow & & {\scriptstyle char(i)}\downarrow \\ char(T, k^*\tau) & \xrightarrow{char(f|_S)} & char(S, i^*f^*\tau). \end{array}$$

We have to verify that the above is well-defined. Firstly, we verify the regularity of image of regular orbits. For any $[\lambda]_G \in char(G,\tau)$, $char(f|_S) \circ char(k)([\lambda]_G)$ determines a subset in $\Lambda_S^{i^*f^*\tau}$. We verify that the subset is a union of $W_{aff}^e(H)$-regular orbits. Since this subset coincides with the image of $W_{aff}^e(G).\lambda$ under ${}^t df$, it is sufficient to verify the following lemma.

Lemma 1.35. *If f satisfies the decomposable condition, the image of a regular orbit under ${}^t df$ is $W_{aff}^e(H)$-invariant and has trivial stabilizer.*

Proof. Let $\lambda \in \Lambda_T^\tau$ determine a $W_{aff}^e(G)$-regular orbit. Let us consider the $W_{aff}^e(G)$-orbit $W_{aff}^e(G).\lambda$ and the image of it under ${}^t df$. Firstly, we verify that ${}^t df(W_{aff}^e(G).\lambda)$ is $W_{aff}^e(H)$-invariant, then we verify that it has the trivial stabilizer.

For any $\mu \in {}^t df(W_{aff}^e(G).\lambda)$, there exists $\nu \in W_{aff}^e(G).\lambda$ such that $\mu = {}^t df(\nu)$. $W_{aff}^e(H)$-invariance of ${}^t df(W_{aff}^e(G).\lambda)$ follows from the followings.

$$w.\mu = w.{}^t df(\nu) = {}^t df(f_*(w).\nu) \in {}^t df(W_{aff}^e(G).\lambda)$$

$$\mu + \kappa^{f^*\tau}(n) = {}^t df(\nu + \kappa^\tau(df(n))) \in {}^t df(W_{aff}^e(G).\lambda)$$

where $w \in W(H)$ and $n \in \Pi_S$. These imply the invariance of ${}^t df(W_{aff}^e(G).\lambda)$.

Now, let us verify the triviality of the stabilizer of ${}^t df(W^e_{aff}(G).\lambda)$. Suppose that $(w,n) \in W^e_{aff}(H)$ stabilizes $\mu \in {}^t df(W^e_{aff}(G).\lambda)$. Then

$$\begin{aligned} w.\mu + \kappa^{f^*\tau}(n) &= w.{}^t df(\nu) + {}^t df \circ \kappa^\tau(df(n)) \\ &= {}^t df(f_*(w).\nu + \kappa^\tau(df(n))) = \mu = {}^t df(\nu). \end{aligned}$$

If we recall that we can define the difference between two twisted characters as an ordinary character (Λ_T acts on Λ_T^τ transitively and faithfully), we obtain that

$$f_*(w).\nu + \kappa^\tau(df(n)) - \nu \in \ker({}^t df) = \kappa^\tau(df(\mathfrak{s})^\perp).$$

On the other hand, that

$$f_*(w).\nu + \kappa^\tau(df(n)) - \nu \in \kappa^\tau(df(\mathfrak{s}))$$

holds. It can be verified as follows.

Let us recall that we can identify two objects Λ_T and Λ_T^τ as $W(G)$ spaces (Lemma 4.20 in [FHT1]). So we can identify Λ_T^τ as a subset of $\mathfrak{t}^*$. Let us notice that ν can be orthogonally decomposed as $\nu = \nu_1 + \nu_2$, where $\nu_1 \in \kappa^\tau(df(\mathfrak{s}))$ and $\nu_2 \in \kappa^\tau(df(\mathfrak{s}))^\perp$. $f_*(W(H))$ preserves $df(\mathfrak{s})$ and acts on $\mathfrak{s}^\perp$ trivially from the condition (3) of Definition 1.12. Therefore, $f_*(w).\nu - \nu = f_*(w).\nu_1 - \nu_1 \in \kappa^\tau(df(\mathfrak{s}))$.

Since $df(\mathfrak{s}) \cap df(\mathfrak{s})^\perp = 0$,

$$f_*(w).\nu + \kappa^\tau(df(n)) - \nu = 0.$$

Since ν has trivial stabilizer, $(f_*(w), \kappa^\tau(df(n))) = (e_{W(G)}, 0)$. Since f_*, df and κ^τ are injective, we obtain the conclusion.

□

Secondly, we deal with two special cases. We have to verify that $char(f)([\lambda]_G) \in char(S, i^*f^*\tau)$ is really an element of $char(H, f^*\tau)$.

Lemma 1.36. *If the restriction of f to S is a finite covering, the above is well-defined.*

Proof. We can write $char(f|_S)$ explicitly as

$$char(f|_S)([\lambda]_T) = \sum_{m \in \Pi_T/df(\Pi_S)} [{}^t df(\lambda + \kappa^\tau(m))]_S.$$

Since ${}^t df$ is injective and the orbit $W^e_{aff}(G).\lambda$ is regular, for any $w, w' \in W(H)$ and $m, m' \in \Pi_T/df(\Pi_S)$ such that $(w,m) \neq (w', m')$, the following holds.

$$\left[w.{}^t df(\lambda + \kappa^\tau(m)) + \kappa^{i^*f^*\tau}(\Pi_S)\right] \cap \left[w'.{}^t df(\lambda + \kappa^\tau(m')) + \kappa^{i^*f^*\tau}(\Pi_S)\right] = \emptyset,$$

that is, $[{}^t df(w.\lambda + \kappa^\tau(m))]_S \neq [{}^t df(w'.\lambda + \kappa^\tau(m'))]_S$.

$$char(f|_S)(\sum_{w \in W(G)} [w.\lambda]_T) = \sum_{w \in W(G)} \sum_{m \in \Pi_T/df(\Pi_S)} [{}^t df(w.\lambda + \kappa^\tau(m))]_S.$$

Since Π_T is $W(G)$-invariant and κ^τ is $W(G)$-equivariant ($\langle \cdot, \cdot \rangle_\tau$ is $W(G)$-invariant), for any $w \in W(G)$ and $m \in \Pi_T$, $w^{-1}.m \in \Pi_T$ and

$$w.\lambda + \kappa^\tau(m) = w.(\lambda + \kappa^\tau(w^{-1}.m)).$$

Let us consider the quotient space $W(G)/f_*(W(H)) = \{[W_1], [W_2], \cdots, [W_k]\}$. That is, for any w, there is a unique solution $w' \in W(H)$ and $l \in \{1, 2, \cdots, k\}$ to the equation $w = f_*(w')W_l$, where W_l is a fixed representative element of $[W_l]$. Since ${}^t df \circ f_*(w') = w'.{}^t df$,

$$\sum_{m \in \Pi_T/df(\Pi_S)} \sum_{w \in W(G)} [{}^t df(w.\lambda + \kappa^\tau(m))]_S$$

$$= \sum_{l=1}^{k} \sum_{m \in \Pi_T/df(\Pi_S)} \sum_{w' \in W(H)} [w'.{}^t df(W_l.\lambda + \kappa^\tau(f_*(w')^{-1}m))]_S.$$

By summing over $W(H)$, ambiguity of the choice of representatives $W_1, W_2, \cdots, W_k$ does not appear. $W(H)$-equivariance of κ^τ implies that for any $m' \in \Pi_T/df(\Pi_S)$, there exists $m \in \Pi_T/df(\Pi_S)$ such that $f(w')^{-1}.\kappa^\tau(m) = \kappa^\tau(m')$. Since

$$\sum_{w' \in W(H)} [w'.{}^t df(W_l.\lambda + \kappa^\tau(m))]_S$$

is an element of $char(H, f^*\tau)$, so is $char(f|_S)(\sum_{w \in W(G)} [w.\lambda]_T)$. $\square$

Lemma 1.37. *If $T = S \times S^\perp$, the restriction of f to S is the inclusion into the first factor and $k^*\tau = p_1^* i^* f^* \tau + p_2^* j^* k^* \tau$, where $p_1 : S \times S^\perp \to S$ and $p_2 : S \times S^\perp \to S^\perp$ are the natural projection, the above is well-defined.*

Proof. We can write $char(f|_S)$ explicitly as $char(f|_S)([\lambda]_T) = [{}^t df(\lambda)]_S$. So

$$char(f|_S)(\sum_{w \in W(G)} [w.\lambda]_T) = \sum_{w \in W(G)} [{}^t df(w.\lambda)]_S$$

$$= \sum_{[w] \in W(G)/W(H)} \sum_{w' \in W(H)} [w'.{}^t df([w].\lambda)]_S$$

Since $\sum_{w' \in W(H)} [w'.{}^t df(W_l.\lambda)]_S$ is an element of $char(H, f^*\tau)$, so is the right hand side. $\square$

Let us state the functoriality we need for this case. Thanks to the decomposable condition, we have already known the following commutative diagram.

$$\begin{tikzcd}
H \arrow[r, "q"] & H/\ker(f) \arrow[r, "i_1"] & H/\ker(f) \times S^\perp \arrow[r, "f \cdot j"] & G \\
S \arrow[u] \arrow[r, "q|_S"'] & S/\ker(f|_S) \arrow[u] \arrow[r, "i_1|_S"'] & S/\ker(f|_S) \times S^\perp \arrow[u] \arrow[r, "f|_S \cdot j"'] & T \arrow[u]
\end{tikzcd}$$

Let us recall that $f = f \cdot j \circ i_1 \circ q$. We obtain the following from the definition of $char(f)$ and the above three lemmas.

Theorem 1.38. *$char(f) : char(G, \tau) \to char(H, f^*\tau)$ is well-defined and*

$$char(f) = char(f \cdot j) \circ char(i_1) \circ char(q).$$

2. Twisted equivariant K-theory

In this chapter, we deal with twisted equivariant K-theory. Our main tool is a topological Mackey decomposition.

2.1. Mackey decomposition and its functoriality

We use a Mackey decomposition to compute $K_T^{\tau+\dim(T)}(T)$ for a torus T, which is an analogue of the isomorphism in un-twisted equivariant K-theory $K_G(X) \cong K(X) \otimes R(G)$ for trivial G-space X. Let us recall that

$$R(G) \cong K(\{\text{isomorphism classes of irreducible representations of } G\}),$$

$$K_G(X) \cong K(X \times \{\text{isomorphism classes of irreducible representations of } G\}).$$

The space $X \times \{\text{isomorphism classes of irreducible representations of } G\}$ is a trivial fiber bundle with discrete fibers. The following construction is a twisted version of it. We state it for only trivial T-spaces without the proof. See [FHT3] for detail.

Let T be a torus, X be a smooth manifold, T act on X trivially, P be a T-equivariant projective bundle over X, and τ be the twisting represented by P. Covering space Y and the associated twisting τ' over Y are defined by the following.

(i) a T-equivariant family, parametrized by X, of central extension T^τ of T by $U(1)$;

(ii) a covering space $p : Y \to X$, whose fibers label the isomorphism classes of irreducible, τ-twisted representations Λ_T^τ;

(iii) a tautological projective bundle $\mathbb{P}R \to Y$ whose fiber $\mathbb{P}R_\lambda$ at $\lambda \in Y$ is the projectification of the τ-twisted representation of T labelled by λ;

(iv) a formal $\mathbb{P}R$-twisted K-theory class $[R]$, represented by R;

(v) a non-equivariant twisting τ' over Y and an isomorphism as T-equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.

Construction: The composition of the following sequence is an isomorphism and our Mackey decomposition is the inverse of it.

$$K^{\tau'}(Y) \to K_T^{\tau'}(Y) \cong K_T^{p^*\tau-\mathbb{P}R}(Y) \xrightarrow{\otimes[R]} K_T^{p^*\tau}(Y) \xrightarrow{p_!} K_T^\tau(X),$$

where the first map is defined by regarding a family of Fredholm operators as an equivariant one via the trivial action, the second isomorphism is defined by the isomorphism of twistings, the third map is defined by the tensor product with the formal K-theory class $[R]$, and the last map is the push-forward along $p : Y \to X$. $[R]$ is the line bundle over Y, whose fiber at $\lambda \in \Lambda_{T,x}^\tau$ is acted on by T^τ via λ.

Since T^τ is an abelian group ([FHT1] Lemma 4.1), τ-twisted irreducible representations of T are 1 dimensional. Therefore, the fiber $\mathbb{P}R_\lambda$ of $\mathbb{P}R$ at λ is a point. In other words, the twisting $\mathbb{P}R \cong 0$.

Theorem 2.1 ([FHT3])**.** *The inverse of the composition of the above sequence*

$$\mathrm{Mac}^\tau_{X,T} : K_T^{\tau+k}(X) \to K^{\tau'+k}(Y)$$

is an isomorphism associated with an isomorphism $\tau' \cong p^*\tau - \mathbb{P}R$ in (v).

We verify naturality with respect to pull back along a continuous map between two spaces and pull back of group action along a local injection between two tori. Let us construct a Mackey decomposition, and verify naturality with respect to F^* and $f^\natural$, where F is a proper smooth map between two manifolds and f is a locally injective group homomorphism. $f^\natural$ is the pull back of group action along f.

Let us verify naturality of a Mackey decomposition.

Theorem 2.2. *Let X and X' be manifolds, T be a torus acting on X and X' trivially, τ be a T-equivariant twisting over X, and $F : X' \to X$ be a smooth proper map. $p : Y_T \to X$ and $p' : Y'_T \to X'$ are the covering spaces constructed above.*

The following diagram commutes.

$$\begin{tikzcd}
K^{\tau'+k}(Y_T) \arrow[r, "\widetilde{F}^*"] & K^{\widetilde{F}^*\tau'+k}(Y'_T) \\
K_T^{\tau+k}(X) \arrow[r, "F^*"] \arrow[u, "\mathrm{Mac}^\tau_{X,T}"] & K_T^{F^*\tau+k}(X') \arrow[u, "\mathrm{Mac}^{F^*\tau}_{X',T}"']
\end{tikzcd}$$

where $\widetilde{F}$ is defined below.

Proof. Let us consider the following diagram

$$\begin{tikzcd}
K^{\tau'+k}(Y_T) \arrow[r] \arrow[d, "\widetilde{F}^*"] & K_T^{\tau'+k}(Y_T) \arrow[r, "\cong"] \arrow[d, "\widetilde{F}^*"] & K_T^{p^*\tau-\mathbb{P}R_T+k}(Y_T) \arrow[r, "\otimes[R_T]"] \arrow[d, "\widetilde{F}^*"] & {} \\
K^{\widetilde{F}^*\tau'+k}(Y'_T) \arrow[r] & K_T^{\widetilde{F}^*\tau'+k}(Y'_T) \arrow[r, "\cong"] & K_T^{\widetilde{F}^*p^*\tau-\mathbb{P}\widetilde{F}^*R_T+k}(Y'_T) \arrow[r, "\otimes[R'_T]"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\otimes[R_T]"] & K_T^{p^*\tau+k}(Y_T) \arrow[r, "p_!"] \arrow[d, "\widetilde{F}^*"] & K_T^{\tau+k}(X) \arrow[d, "F^*"] \\
{} \arrow[r, "\otimes[R'_T]"] & K_T^{\widetilde{F}^*p^*\tau+k}(Y'_T) \arrow[r, "p'_!"] & K_T^{F^*\tau+k}(X').
\end{tikzcd}$$

The compositions of horizontal sequences are the inverse of $\mathrm{Mac}^\tau_{X,T}$ and $\mathrm{Mac}^{F^*\tau}_{X',T}$ respectively. If we verify the commutativity of four squares, we obtain the conclusion.

The first is clear.

The second is verified by pull back of isomorphism between two twistings along $\widetilde{F}$.

The third is clear if we notice that $\widetilde{F}^*R_T \cong R'_T$ as T-equivariant bundles.

The fourth follows from functoriality of push-forward maps if we notice that $Y'_T \cong F^*Y_T$. It can be verified by the definition of $F^*\tau$. $\square$

Let us verify naturality with respect to pull back of group action along $f : T' \to T$. We can define $f^\natural$ by the restriction of group actions. For example $f^\natural\tau$ is a T'-equivariant twisting over X and we can define a homomorphism $f^\natural : K_T^\tau(X) \to K_{T'}^{f^\natural\tau}(X)$. However, the sense of "naturality with respect to $f^\natural$" is not clear, because $K^{\tau'+k}(Y_T)$ is a non-equivariant twisted K-group. In this paper, we regard the following theorem as naturality with respect to $f^\natural$.

Theorem 2.3. *Let $p : Y_T \to X$ and $p' : Y_{T'} \to X$ be the covering spaces associated with (T, τ) and $(T', f^\natural\tau)$ respectively. The following diagram commutes.*

$$\begin{array}{ccc} K^{\tau'+k}(Y_T) & \xrightarrow{\cong} & K_T^{\tau+k}(X) \\ {\scriptstyle (\widetilde{{}^t df})_!}\downarrow & & {\scriptstyle f^\natural}\downarrow \\ K^{\tau''+k}(Y_{T'}) & \xrightarrow{\cong} & K_{T'}^{f^\natural\tau+k}(X) \end{array}$$

where τ', τ'' and $\widetilde{{}^t df}$ are defined as in the following.

Remark 2.4. (1) *The map $\widetilde{{}^t df}$ is defined by*

$$\widetilde{{}^t df}(x, \lambda) := (x, {}^t df(\lambda)) \in Y_{T'}$$

for $(x, \lambda) \in Y_T$. $\widetilde{{}^t df}$ is clearly a continuous fiber map.

(2) *τ' is the associated twisting with τ and the Mackey decomposition $\mathrm{Mac}_{X,T}^\tau$ in Theorem 2.1. τ'' is defined so that $\widetilde{{}^t df}^*\tau'' = f^\natural\tau'$ (we regard τ' and τ'' as T- and T'-equivariant twistings with the trivial action respectively) and therefore we can define the push-forward map.*

(3) *The theorem is verified by functoriality of each objects. But the main idea is quite simple. The theorem is a family version of the following commutative diagram.*

$$\begin{array}{ccc} K(\Lambda_T) & \xrightarrow{\cong} & R(T) = K_T(\{pt\}) \\ {\scriptstyle ({}^t df)_!}\downarrow & & {\scriptstyle f^\natural}\downarrow \\ K(\Lambda_{T'}) & \xrightarrow{\cong} & R(T') = K_{T'}(\{pt\}) \end{array}$$

Usually, $f^\natural$ above is written as f^* which is the pull back of representations along f.

Proof. Let us consider the following diagram associated with the Mackey decompositions and the pull backs along f.

$$\begin{tikzcd}
K^{\tau'+k}(Y_T) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "(1)"] & K^{\tau'+k}(Y_T) \arrow[r, "(\widetilde{{}^t df})_!"] \arrow[d] \arrow[dr, phantom, "(2)"] & K^{\tau''+k}(Y_{T'}) \arrow[d] \\
K_T^{\tau'+k}(Y_T) \arrow[r, "f^\natural"] \arrow[d, "\cong"] \arrow[dr, phantom, "(3)"] & K_{T'}^{f^\natural\tau'+k}(Y_T) \arrow[r, "(\widetilde{{}^t df})_!"] \arrow[d, "\cong"] \arrow[dr, phantom, "(4)"] & K_{T'}^{\tau''+k}(Y_{T'}) \arrow[d, "\cong"] \\
K_T^{p^*\tau-\mathbb{P}R_T+k}(Y_T) \arrow[r, "f^\natural"] \arrow[d, "\otimes[R_T]"] \arrow[dr, phantom, "(5)"] & K_{T'}^{f^\natural(p^*\tau-\mathbb{P}R_T)+k}(Y_T) \arrow[r, "(\widetilde{{}^t df})_!"] \arrow[d, "\otimes(\widetilde{{}^t df})^*{[R_{T'}]}"] \arrow[dr, phantom, "(6)"] & K_{T'}^{p'^*f^\natural\tau-\mathbb{P}R_{T'}+k}(Y_{T'}) \arrow[d, "\otimes[R_{T'}]"] \\
K_T^{p^*\tau+k}(Y_T) \arrow[r, "f^\natural"] \arrow[d, "p_!"] & K_{T'}^{f^\natural p^*\tau+k}(Y_T) \arrow[r, "(\widetilde{{}^t df})_!"] \arrow[d, phantom, "(7)"] & K_{T'}^{p'^*f^\natural\tau+k}(Y_{T'}) \arrow[d, "p'_!"] \\
K_T^{\tau+k}(X) \arrow[rr, "f^\natural"] & & K_{T'}^{f^\natural\tau+k}(X)
\end{tikzcd}$$

If we can prove all of the commutativities from (1) to (7), we obtain the required commutativity. Let us notice that $f^\natural p^*\tau = \widetilde{{}^t df}^* p'^* f^\natural\tau$ and $f^\natural(p^*\tau - \mathbb{P}R_T) = \widetilde{{}^t df}^*(p'^* f^\natural\tau - \mathbb{P}R_{T'})$. Therefore, the above push-forward maps are well-defined.

(1), (2) are clear from the definitions of each maps.

To verify that (3), (4) commute, we pull back the isomorphism between two twistings. τ' and τ'' are defined so that $\widetilde{{}^t df}^*\tau'' = f^\natural\tau'$.

(5) commutes since $(\widetilde{{}^t df})^*[R_{T'}] \cong f^\natural[R_T]$.

(6) commutes from Remark 4.2 in [CW].

(7) commutes from the functoriality of push-forward maps with respect to pull back along group action if we notice that $p = p' \circ \widetilde{{}^t df}$.

$\square$

2.2. Twisted equivariant K-theory for tori

We can describe $K_T^{\tau+n}(T)$ by use of the above construction. Let T be a torus and τ be a positive central extension of LT and the associated T-equivariant twisting over T. Fix an orientation of T. The following is just a corollary of a Mackey decomposition.

Corollary 2.5 ([FHT3])**.**

$$K_T^{\tau+k}(T) \cong \begin{cases} char(T,\tau) & (k \equiv \dim(T)) \\ 0 & (k \equiv \dim(T)+1) \end{cases}$$

We write the isomorphism $K_T^{\tau+\dim(T)}(T) \xrightarrow{\cong} char(T,\tau)$ as $M.d._T$, where $M.d.$ comes from a Mackey decomposition.

Proof. Associated covering space Y_T is $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$, Π_T acts on Λ_T^τ through κ^τ (Definition 1.7) and on $\mathfrak{t}$ via the translation. $\kappa^\tau(n)$ represents how the character changes when one travels in T along the geodesic loop n. Since τ is positive, κ^τ is injective. Therefore, $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ has a structure of a trivial vector bundle over $\Lambda_T^\tau/\kappa^\tau(\Pi_T)$ whose fiber is $\mathfrak{t}$. We write this vector bundle as

$$\pi : \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} \to \Lambda_T^\tau/\kappa^\tau(\Pi_T).$$

Moreover, since any connected component of $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ is contractible, a twisting over $\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ is automatically trivial. So we have the following series of isomorphisms

$$K_T^{\tau+\dim(T)}(T) \xrightarrow{\mathrm{Mac}_{T,T}^\tau} K^{\dim(T)}(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) \xrightarrow{\pi_!} K(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) \cong char(T,\tau).$$

The first isomorphism is the Mackey decomposition and $\pi_!$ is Thom isomorphism, so this isomorphism depends on the orientation of T.

Since $K^1(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) \cong 0$, we obtain that$K_T^{\tau+\dim(T)+1}(T) \cong 0$. □

We omit the proof of the following theorem. One can find it in [FHT3].

Theorem 2.6. *Let $k : T \hookrightarrow G$ be a chosen maximal torus. We can define the isomorphism $M.d._G : K_G^{\tau+\mathrm{rank}(G)}(G) \to char(G,\tau)$ so that the following diagram commutes.*

$$\begin{tikzcd}
K_G^{\tau+\mathrm{rank}(G)}(G) \arrow[r, "k^*\circ k^\natural"] \arrow[d, "M.d._G"'] & K_T^{k^*\tau+\dim(T)}(T) \arrow[d, "M.d._T"'] \\
char(G,\tau) \arrow[r, "char(k)"] & char(T,k^*\tau)
\end{tikzcd}$$

In this section, we construct $f^\#$ for a local injection $f : S \to T$ so that it has a compatibility with $char(f)$ by use of the isomorphism $char(T,\tau) \cong K(\Lambda_T^\tau/\kappa^\tau(\Pi_T))$.

2.2.1. Direct product

Let T_1, T_2 be tori, $T = T_1 \times T_2$, n_j be the dimension of T_j, $n := n_1 + n_2$ be the dimension of T, τ_j be a positive central extension of LT_j, $i_j : T_j \hookrightarrow T$ be the natural inclusion into the j'th factor, $p_j : T \twoheadrightarrow T_j$ be the natural projection onto the j'th factor ($j = 1, 2$). We obtain a positive central extension $\tau = p_1^*\tau_1 + p_2^*\tau_2$ of LT from these data. The orientation of T is naturally obtained from ones of T_1 and T_2.

We verify the following theorem.

Theorem 2.7. *We define $i_1^\#$ as the composition of the following sequence.*

$$K_T^{\tau+n}(T) \xrightarrow{i_1^\natural} K_{T_1}^{i_1^\natural\tau+n}(T) \xrightarrow{(p_1)_!} K_{T_1}^{\tau_1+n_1}(T_1),$$

where the second arrow is the push-forward map along the natural projection $p_1 : T \to T_1$. Then the following diagram commutes.

$$\begin{tikzcd}
K_T^{\tau+\dim(T)}(T) \arrow[r, "i_1^{\#}"] \arrow[d, "M.d._T"'] & K_{T_1}^{\tau_1+\dim(T_1)}(T_1) \arrow[d, "M.d._{T_1}"'] \\
char(T,\tau) \arrow[r, "char(i_1)"] & char(T_1,\tau_1)
\end{tikzcd}$$

Proof. Let us consider the following diagram.

$$\begin{tikzcd}
K_T^{\tau+n}(T) \arrow[r, "i_1^{\natural}"] \arrow[d, "\mathrm{Mac}_{T,T}^{\tau}"'] \arrow[dr, phantom, "(1)"] & K_{T_1}^{p_1^*\tau_1+n}(T) \arrow[r, "(p_1)_!"] \arrow[d, dashed, "\mathrm{Mac}_{T,T_1}^{p_1^*\tau_1}"'] \arrow[dr, phantom, "(2)"] & K_{T_1}^{\tau_1+n_1}(T_1) \arrow[d, "\mathrm{Mac}_{T_1,T_1}^{\tau_1}"] \\
K^n(Y_T) \arrow[r, "(\widetilde{{}^t di_1})_!"] \arrow[d, "\pi_!"] & K^n(Y_{T_1} \times T_2) \arrow[r, "(p_1)_!"] \arrow[d, phantom, "(3)"] & K^{n_1}(Y_{T_1}) \arrow[d, "\pi_{1!}"] \\
K(\Lambda_T^{\tau}/\kappa^{\tau}(\Pi_T)) \arrow[rr, "(({}^t di_1)/\Pi_T)_!"] \arrow[d, "\cong"] & \arrow[d, phantom, "(4)"] & K(\Lambda_{T_1}^{\tau_1}/\kappa^{\tau_1}(\Pi_{T_1})) \arrow[d, "\cong"] \\
char(T,\tau) \arrow[rr, "char(i_1)"] & & char(T_1,\tau_1)
\end{tikzcd}$$

If we verify that each square commutes, we obtain the result. In short, our idea is rewriting $i_1^{\#}$ and $char(i_1)$ using Mackey decompositions.

Let us start from the easiest part, the rewriting of $char(i_1)$ in terms of K-theory. The following can be verified if one knows the description of push-forward maps along a projection with discrete fibers, and we obtain the commutativity of (4).

Lemma 2.8. *$char(i_1)$ corresponds to $[({}^t di_1)/\Pi_T]_!$ which is the map between two orbit spaces induced by ${}^t di_1 : \Lambda_T^{\tau} \to \Lambda_{T_1}^{\tau_1}$.*

We deal with so many maps like $({}^t di_1)/\Pi_T$, which is defined in the same way and we do not write all the definition from now on.

We can lift this construction to the level of covering spaces. One can verify that the maps constructed below is well-defined by an easy computation.

Lemma 2.9. *The following diagram commutes.*

$$\begin{tikzcd}
\Lambda_T^{\tau} \times_{\Pi_T} \mathfrak{t} \arrow[r, "({}^t di_1 \times dp_1)/\Pi_T"] \arrow[d, "\pi"'] & \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1 \arrow[d, "\pi_1"'] \\
\Lambda_T^{\tau}/\kappa^{\tau}(\Pi_T) \arrow[r, "({}^t di_1)/\Pi_T"] & \Lambda_{T_1}^{\tau_1}/\kappa^{\tau_1}(\Pi_{T_1})
\end{tikzcd}$$

where the vertical arrows π and π_1 are the trivial vector bundles mentioned in the proof of Corollary 2.5. Moreover, $({}^t di_1 \times dp_1)/\Pi_T$ can be written as the composition of the following sequence

$$\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} \xrightarrow{({}^t di_1 \times \mathrm{id})/\Pi_T} \Lambda_{T_1}^{\tau_1} \times_{\Pi_T} \mathfrak{t} \xrightarrow{(\mathrm{id} \times dp_1)/\Pi_T} \Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1,$$

where Π_T acts on $\Lambda_{T_1}^{\tau_1}$ through $dp_1 : \Pi_T \to \Pi_{T_1}$ and κ^{τ_1}.

$({}^t di_1 \times \mathrm{id})/\Pi_T$ can be written as $\widetilde{{}^t di_1}$ in the notation in Theorem 2.3. Moreover, through the diffeomorphism $\Lambda_{T_1}^{\tau_1} \times_{\Pi_T} \mathfrak{t} \cong (\Lambda_{T_1}^{\tau_1} \times_{\Pi_{T_1}} \mathfrak{t}_1) \times T_2$, $(\mathrm{id} \times dp_1)/\Pi_T$ can be thought as the projection onto the first factor p_1.

With functoriality of push-forward maps, we obtain the following and therefore the commutativity of (3).

Lemma 2.10. *The following diagram commutes.*

$$\begin{tikzcd}
K^n(Y_T) \arrow[r, "(\widetilde{{}^t di_1})_!"] \arrow[d, "\pi_!"] & K^n(Y_{T_1} \times T_2) \arrow[r, "(p_1)_!"] & K^{n_1}(Y_{T_1}) \arrow[d, "\pi_{1!}"] \\
K(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) \arrow[rr, "(({}^t di_1)/\Pi_T)_!"] & & K(\Lambda_{T_1}^{\tau_1}/\kappa^{\tau_1}(\Pi_{T_1}))
\end{tikzcd},$$

The following is the proof of the commutativity of (1) and (2).

Lemma 2.11. *The following diagram commutes.*

$$\begin{tikzcd}
K_T^{\tau+n}(T) \arrow[r, "i_1^\natural"] \arrow[d, "\mathrm{Mac}_{T,T}^\tau"'] & K_{T_1}^{p_1^*\tau_1+n}(T_1 \times T_2) \arrow[r, "(p_1)_!"] \arrow[d, "\mathrm{Mac}_{T_1 \times T_2, T_1}^{p_1^*\tau}"'] & K_{T_1}^{\tau_1+n_1}(T_1) \arrow[d, "\mathrm{Mac}_{T_1,T_1}^{\tau_1}"'] \\
K^n(Y_T) \arrow[r, "(\widetilde{{}^t di_1})_!"] & K^n(Y_{T_1} \times T_2) \arrow[r, "(p_1)_!"] & K^{n_1}(Y_{T_1})
\end{tikzcd}$$

where $Y_T := \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}$ and $Y_{T_1} := \Lambda_{T_1}^\tau \times_{\Pi_{T_1}} \mathfrak{t}_1$.

Let us explain the outline of the proof of this lemma. The left square commutes from Theorem 2.3. The right one commutes from push-forward maps version of Theorem 2.2. We leave the details of it for the readers.

Combining all of them, we obtain the commutativity of the biggest square and therefore the theorem. $\square$

2.2.2. Finite covering

Let $q : T' \to T$ be a finite covering of an n dimensional torus and τ be a positive central extension of LT and the associated T-equivariant twisting over T. In this section, we verify the following theorem.

Theorem 2.12. *If we define*

$$q^\# := q^* \circ q^\natural$$

the following diagram commutes.

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{q^\#} & K_{T'}^{q^*\tau+\dim(T')}(T') \\ {\scriptstyle M.d._T}\downarrow & & {\scriptstyle M.d._{T'}}\downarrow \\ char(T,\tau) & \xrightarrow{char(q)} & char(T', q^*\tau) \end{array}$$

Remark 2.13. *$q^\#$ is the composition of*

$$K_T^{\tau+\dim(T)}(T) \xrightarrow{q^\natural} K_{T'}^{q^\natural\tau+\dim(T)}(T) \xrightarrow{q^*} K_{T'}^{q^*\tau+\dim(T')}(T').$$

It is written as q^* simply in [FHT1]*. In the terminology of groupoids, q determines a functor $(q,q) : T'//T' \to T//T$ and q^* is the pull back along the functor (q,q).*

Proof. Let us consider the following diagram. $n := \dim(T)$.

$$\begin{tikzcd} K_T^{\tau+n}(T) \arrow[rr, "q^\natural"] \arrow[d, "\mathrm{Mac}^{\tau}_{T,T}"] & & K_{T'}^{q^\natural\tau+n}(T) \arrow[rr, "q^*"] \arrow[d, "\mathrm{Mac}^{q^\natural\tau}_{T,T'}"] & & K_{T'}^{q^*\tau+n}(T') \arrow[d, "\mathrm{Mac}^{q^*\tau}_{T',T'}"] \\ K^n(\Lambda_T^\tau \times_{\Pi_T} \mathfrak{t}) \arrow[rr, "((^t dq\times \mathrm{id})/\Pi_T)_!"] \arrow[d, "\pi_!"] \arrow[rr, phantom, shift left=8, "(1)"] & & K^n(\Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t}) \arrow[rr, "((\mathrm{id}\times dq)/\Pi_{T'})^*"] \arrow[d, "'\pi_!"] \arrow[rr, phantom, shift left=8, "(2)"] & & K^n(\Lambda_{T'}^{q^*\tau} \times_{\Pi_{T'}} \mathfrak{t}') \arrow[d, "\pi'_!"] \\ K(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) \arrow[rr, "((^t dq)/\Pi_T)_!"] \arrow[d] \arrow[rr, phantom, shift left=8, "(3)"] & & K(\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))) \arrow[rr, "r^*"] \arrow[rr, phantom, shift left=8, "(4)"] \arrow[d, phantom, "(5)"] & & K(\Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'})) \arrow[d] \\ char(T,\tau) \arrow[rrrr, "char(q)"] & & & & char(T', q^*\tau) \end{tikzcd}$$

As we explain below, $\Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t}$ is the covering space of T associated with T'-equivariant twisting $q^\natural\tau$, and $'\pi : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \to \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))$ is the associated trivial vector bundle (notice that Π_T acts on $\Lambda_{T'}^{q^*\tau}$ through the homomorphism $\Pi_T \xrightarrow{\kappa^\tau} \Lambda_T \xrightarrow{^t dq} \Lambda_{T'}$). Moreover, r is the natural projection between two orbit spaces.

Since $(^t dq \times \mathrm{id})/\Pi_T$ and $(\mathrm{id} \times dq)/\Pi_{T'}$ correspond to $\widetilde{^t dq}$ and $\widetilde{q}$ respectively, we have known the commutativity of (1) and (2) already frome Theorem 2.2 and 2.3. Our idea is rewriting of $(^t dq \times \mathrm{id})/\Pi_T$ and $(\mathrm{id} \times dq)/\Pi_{T'}$ in terms of K-theory of orbit spaces.

Let us start from the computation $K_{T'}^{q^\natural\tau+\dim(T)}(T)$ and the associated covering space by use of a Mackey decomposition.

Proposition 2.14.

$$K_{T'}^{q^\natural\tau+k}(T) \cong \begin{cases} \mathbb{Z}[\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))] & k = \dim(T) \mod 2 \\ 0 & k = \dim(T)+1 \mod 2 \end{cases}$$

Proof. By the definition of $q^\natural\tau$, if one travels on T along a geodesic $n \in \Pi_T$, the character $\lambda \in \Lambda_{T'}^{q^\natural\tau}$ changes to $\lambda + {}^t dq(\kappa^\tau(n))$. Therefore, the covering space is

$$'p : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \to T,$$

where Π_T acts on $\Lambda_{T'}^{q^*\tau}$ via ${}^t dq \circ \kappa^\tau : \Pi_T \to \Lambda_{T'}$ and $\Lambda_{T'} \curvearrowright \Lambda_{T'}^{q^*\tau}$. It has a structure of a trivial vector bundle

$$'\pi : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \to \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)).$$

Through Thom isomorphism $'\pi_!$, we obtain the conclusion. □

In order to obtain the commutativity of (5), we verify the following.

Lemma 2.15. *If we consider the maps*

$$\Lambda_T^\tau/\kappa^\tau(\Pi_T) \xrightarrow{({}^t dq)/\Pi_T} \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)) \xleftarrow{r} \Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'}),$$

where

$$r([\mu]_{T'}) := [\mu].$$

$[\mu]$ *is the* Π_T*-orbit of* μ *in* $\Lambda_{T'}^{q^*\tau}$, $char(q)$ *corresponds to* $r^* \circ (({}^t dq)/\Pi_T)_!$. *That is, the following diagram commutes.*

$$\begin{tikzcd}
K(\Lambda_T^\tau/\kappa^\tau(\Pi_T)) \arrow[r, "(({}^t dq)/\Pi_T)_!"] \arrow[d, "\cong"] & K(\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))) \arrow[r, "r^*"] & K(\Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'})) \arrow[d, "\cong"] \\
char(T,\tau) \arrow[rr, "char(q)"] & & char(T', q^*\tau)
\end{tikzcd}$$

Proof. Let us compute the value of $(({}^t dq)/\Pi_T)_!$ at the trivial bundle $\delta_{[\lambda]_T}$ supported at $[\lambda]_T$. Since $({}^t dq)/\Pi_T$ is an injection, for $\lambda \in \Lambda_T^\tau$,

$$(({}^t dq)/\Pi_T)_!(\delta_{[\lambda]_T}) = \delta_{[{}^t dq(\lambda)]}.$$

On the other hand, for $\lambda' \in \Lambda_{T'}^{q^*\tau}$,

$$r^*(\delta_{[\lambda']}) = \sum_{m \in \Pi_T/dq(\Pi_{T'})} \delta_{[\lambda' + {}^t dq(\kappa^\tau(m))]_{T'}}.$$

According to Lemma 1.22, the composition above corresponds to $char(q)$. □

Let us lift r to the covering spaces. One can verify the following lemma in the same way in Lemma 2.9.

Lemma 2.16. *The following diagram commutes.*

$$\begin{tikzcd}
\Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \arrow[d, "'\pi"'] & \Lambda_{T'}^{q^*\tau} \times_{\Pi_{T'}} \mathfrak{t}' \arrow[l, "(\mathrm{id}\times dq)/\Pi_{T'}"'] \arrow[d, "\pi'"'] \\
\Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)) & \Lambda_{T'}^{q^*\tau}/\kappa^{q^*\tau}(\Pi_{T'}) \arrow[l, "r"']
\end{tikzcd}$$

It tells us that the vector bundle $\pi' : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \to \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T))$ is isomorphic to the induced bundle $r^*({}'\pi : \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \to \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)))$. Thanks to functoriality of push-forward maps, we obtain the commutativity of (4).

In order to verify the commutativity of (3), let us lift $({}^t dq)/\Pi_T$ to the covering spaces. The followings can be obtained in the same way.

Lemma 2.17. *The following diagram commutes.*

$$\begin{array}{ccc} \Lambda_T^\tau \times_{\Pi_T} \mathfrak{t} & \xrightarrow{({}^t dq \times \mathrm{id})/\Pi_T} & \Lambda_{T'}^{q^*\tau} \times_{\Pi_T} \mathfrak{t} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle '\pi}\downarrow \\ \Lambda_T^\tau/\kappa^\tau(\Pi_T) & \xrightarrow{({}^t dq)/\Pi_T} & \Lambda_{T'}^{q^*\tau}/{}^t dq(\kappa^\tau(\Pi_T)) \end{array}$$

Thanks to the functoriality of push-forward maps, we obtain the commutativity of (3).

Combining all of them, we obtain the commutativity of the biggest square and therefore the theorem has been verified.

□

2.2.3. Local injection

Let S and T be tori, $f : S \to T$ be a local injection and τ be a positive central extension.

Let us recall that f can be canonically decomposed as follows.

$$S \xrightarrow{q} S/\ker(f) \xrightarrow{i_1} S/\ker(f) \times S^\perp \xrightarrow{f \cdot j} T,$$

where q is the natural finite covering, i_1 is the natural inclusion into the first factor, and $j : S^\perp \to T$ is the natural inclusion. Theorems 1.31, 2.7 and 2.12 imply the following theorem.

Theorem 2.18. *If we define as*

$$f^\# := q^\# \circ i_1^\# \circ (f \cdot j)^\#,$$

the following diagram commutes.

$$\begin{array}{ccc} K_T^{\tau+\dim(T)}(T) & \xrightarrow{f^\#} & K_S^{f^*\tau+\dim(S)}(S) \\ {\scriptstyle M.d._T}\downarrow & & {\scriptstyle M.d._S}\downarrow \\ char(T,\tau) & \xrightarrow{char(f)} & char(S, f^*\tau) \end{array}$$

2.3. Twisted equivariant K-theory for general groups

In this section, we extend the above constructions to more general groups. Let G and H be compact connected Lie groups with torsion-free π_1, τ be a positive central extension of LG and the associated G-equivariant twisting over G, f satisfy the decomposable condition in Definition 1.12, and S and T be chosen maximal tori of H and G respectively such that $f(S) \subseteq T$. That is, the following diagram commutes.

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ {\scriptstyle i}\uparrow & & {\scriptstyle k}\uparrow \\ S & \xrightarrow{f} & T \end{array}$$

In this section, we use the same symbol to represent the restriction of homomorphisms to subgroups or induced maps to the quotient groups. For example, $f : H \to G$ induces group homomorphisms $f : H/\ker(f) \to G$ and $f : S \to T$.

Let us recall the result in [FHT1].

Theorem 2.19 ([FHT1] Theorem 4.27)**.** *If f is an injection and* $\mathrm{rank}(G) = \mathrm{rank}(H)$, *the following diagram commutes.*

$$\begin{array}{ccc} K_G^{\tau+\mathrm{rank}(G)}(G) & \xrightarrow{f^*\circ f^\natural} & K_H^{f^*\tau+\mathrm{rank}(H)}(H) \\ {\scriptstyle M.d._G}\downarrow & & {\scriptstyle M.d._H}\downarrow \\ char(G,\tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \end{array}$$

This theorem tells us that $f^\#$ has been defined and holds naturality with $char(f)$ in special cases. Let us extend this results to more general cases, that is, we verify the following theorem.

Theorem 2.20. *We can define $f^\#$ for f satisfying the decomposable condition and the following diagram commutes.*

$$\begin{array}{ccc} K_G^{\tau+\mathrm{rank}(G)}(G) & \xrightarrow{f^\#} & K_H^{f^*\tau+\mathrm{rank}(H)}(H) \\ {\scriptstyle M.d.G}\downarrow & & {\scriptstyle M.d._H}\downarrow \\ char(G,\tau) & \xrightarrow{char(f)} & char(H, f^*\tau) \end{array}$$

Just like commutative case, we deal with $f^\#$ in two special cases, and define $f^\#$ as the composition of special cases later.

2.3.1. The decomposition of a homomorphism satisfying the decomposable condition

Let us recall that $f : H \to G$ can be written as the composition of the following series

$$H \xrightarrow{q} H/\ker(f) \xrightarrow{i_1} H/\ker(f) \times S^\perp \xrightarrow{f\cdot j} G$$

if f satisfies the decomposable condition. By restricting this sequence to maximal tori, we obtain the following commutative diagram

$$\begin{tikzcd} H \arrow[r, "q"] & H/\ker(f) \arrow[r, "i_1"] & H/\ker(f)\times S^{\perp} \arrow[r, "f\cdot j"] & G & \\ S \arrow[u, "i"] \arrow[r, "q"] & S/\ker(f) \arrow[u, "i"] \arrow[r, "i_1"] & S/\ker(f)\times S^{\perp} \arrow[u, "i\times \mathrm{id}"] \arrow[r, "f\cdot j"] & T \arrow[u, "k"] & S^{\perp} \arrow[l, "j"] \arrow[ul, "j"'] \end{tikzcd}$$

Moreover, by the definition of $S^{\perp}$, $(f\cdot j)^*\tau$ can be written as

$$p_1^*\tau_{H/\ker(f)} + p_2^*\tau_{S^{\perp}},$$

where $\tau_{H/\ker(f)} = ((f\cdot j)\circ i_1)^*\tau$ and $\tau_{S^{\perp}} = j^*\tau$.

We can verify Theorem 2.20 in the same way in Theorem 2.18 if we verify naturality in two special cases, finite covering maps and inclusions into the first factor of direct products with a torus.

2.3.2. Direct product

In this subsection, we deal with the cases of

a (possibly non-commutative) Lie group $\times$ a torus.

Let H be a compact connected Lie group with torsion-free π_1, S be a maximal torus of H, $S^{\perp}$ be a torus, τ_H and $\tau_{S^{\perp}}$ be positive central extensions of LH and $LS^{\perp}$ respectively. To be consistent, we use the symbol $S^{\perp}$ to represent the second factor. $i_1 : H \to H\times S^{\perp}$, $p_1 : H\times S^{\perp} \to H$ and $p_2 : H\times S^{\perp} \to S^{\perp}$ are as usual. Then, $\tau := p_1^*\tau_H + p_2^*\tau_{S^{\perp}}$ is a positive central extension of $L(H\times S^{\perp})$ and the associated $H\times S^{\perp}$-equivariant twisting over $H\times S^{\perp}$. n and m are the dimension of S and $S^{\perp}$ respectively.

Theorem 2.21. *If we define $i_1^{\#}$ as the composition of the following sequence*

$$K^{\tau+n+m}_{H\times S^{\perp}}(H\times S^{\perp}) \xrightarrow{i_1^{\natural}} K^{i_1^{\natural}\tau+n+m}_{H}(H\times S^{\perp}) \xrightarrow{(p_1)_!} K^{\tau|_H+n}_{H}(H)$$

the following diagram commutes.

$$\begin{tikzcd} K^{\tau+n+m}_{H\times S^{\perp}}(H\times S^{\perp}) \arrow[r, "i_1^{\#}"] \arrow[d, "M.d._{H\times S^{\perp}}"'] & K^{\tau_H+n}_{H}(H) \arrow[d, "M.d._H"'] \\ char(H\times S^{\perp}, \tau) \arrow[r, "char(i_1)"] & char(H, \tau_H) \end{tikzcd}$$

Proof. Let us consider the following diagram.

$$\begin{tikzcd}
K^{\tau+n+m}_{H\times S^{\perp}}(H\times S^{\perp}) \arrow[rrr, "i_1^{\#}"] \arrow[dr, "(i\times\mathrm{id})^{\#}"] \arrow[ddd, "M.d._{H\times S^{\perp}}"'] & \arrow[r, phantom, "(1)"] & & K_H^{\tau_H+n}(H) \arrow[dl, "i^{\#}"'] \arrow[ddd, "M.d._H"] \\
\arrow[r, phantom, "(2)"] & K^{(i\times\mathrm{id})^*\tau+n+m}_{S\times S^{\perp}}(S\times S^{\perp}) \arrow[r, "i_1^{\#}"] \arrow[d, "M.d._{S\times S^{\perp}}"] \arrow[dr, phantom, "(3)"] & K_S^{i^*\tau_H+n}(S) \arrow[d, "M.d._S"] \arrow[r, phantom, "(4)"] & \\
& char(S\times S^{\perp},(i\times\mathrm{id})^*\tau) \arrow[r, "char(i_1)"] & char(S,i^*\tau_H) & \\
char(H\times S^{\perp},\tau) \arrow[rrr, "char(i_1)"] \arrow[ur, "char(i\times\mathrm{id})"] & \arrow[r, phantom, "(5)"] & & char(H,\tau_H) \arrow[ul, "char(i)"']
\end{tikzcd}$$

What we want to verify is the commutativity of the biggest square. Since $char(i)$ is injective, it is sufficient to verify that

$$char(i)\circ M.d._H\circ i_1^{\#} = char(i)\circ char(i_1)\circ M.d._{H\times S^{\perp}}.$$

So if each square commutes, we obtain the result by a diagram chasing.

The followings have been verified.

(2) and (4) commute from Theorem 2.19.

(3) commutes from Theorem 2.7.

(5) commutes from the definition of $char(i_1) : char(H\times S^{\perp},\tau)\to char(H,\tau|_H)$.

(1) commutes from the following. It can be verified from functoriality of push-forward maps and commutativity between pull back of group action and push-forward maps.

Proposition 2.22. *The following diagram commutes.*

$$\begin{tikzcd}
K^{\tau+n+m}_{H\times S^{\perp}}(H\times S^{\perp}) \arrow[r, "(i\times\mathrm{id})^{\#}"] \arrow[d, "i_1^{\#}"'] & K^{(i\times\mathrm{id})^*\tau+n+m}_{S\times S^{\perp}}(S\times S^{\perp}) \arrow[d, "i_1^{\#}"'] \\
K_H^{\tau_H+n}(H) \arrow[r, "i^{\#}"] & K_S^{i^*\tau_H+n}(S)
\end{tikzcd}$$

where $(i\times\mathrm{id})^{\#}$ *and* $i^{\#}$ *have been defined in Theorem 2.19.*

□

2.3.3. The same rank, possibly not injective (the restriction to the torus is a finite covering)

Let H, G be compact connected Lie groups with the same rank and with torsion-free π_1, τ be a positive central extension of LG and the associated G-equivariant twisting over G, and $f : H\to G$ be a smooth group homomorphism satisfying the decomposable condition such that the restriction of f to maximal torus is a finite covering. Let S and

T be maximal tori of H and G respectively such that $f(S) \subseteq T$. That is, the following diagram commutes.

$$
\begin{tikzcd}
H \arrow[r, "f"] & G \\
S \arrow[u, "i"] \arrow[r, "f"] & T \arrow[u, "k"']
\end{tikzcd}
$$

We verify the following theorem.

Theorem 2.23. *If we define as*

$$f^{\#} := f^* \circ f^{\natural},$$

the following diagram commutes.

$$
\begin{tikzcd}
K_G^{\tau+\mathrm{rank}(G)}(G) \arrow[r, "f^{\#}"] \arrow[d, "M.d._G"'] & K_H^{f^*\tau+\mathrm{rank}(H)}(H) \arrow[d, "M.d._H"'] \\
char(G,\tau) \arrow[r, "char(f)"] & char(H, f^*\tau)
\end{tikzcd}
$$

Remark 2.24. *In Theorem 2.19, the restriction of f to maximal torus is supposed to be injective.*

Proof. Let us consider the following diagram just like the proof of Theorem 2.21. $n := \mathrm{rank}(H) = \mathrm{rank}(G)$.

$$
\begin{tikzcd}
K_G^{\tau+n}(G) \arrow[rrr, "f^{\#}"] \arrow[dr, "k^{\#}"] \arrow[ddd, "M.d._G"'] & & & K_H^{f^*\tau+n}(H) \arrow[dl, "i^{\#}"'] \arrow[ddd, "M.d._H"] \\
& K_T^{k^*\tau+n}(T) \arrow[r, "f^{\#}"] \arrow[d, "M.d._T"] & K_S^{f^*k^*\tau+n}(S) \arrow[d, "M.d._S"] & \\
& char(T, k^*\tau) \arrow[r, "char(f)"'] & char(S, f^*k^*\tau) & \\
char(G,\tau) \arrow[ur, "char(k)"] \arrow[rrr, "char(f)"'] & & & char(H, f^*\tau) \arrow[ul, "char(i)"']
\arrow[from=1-2, to=1-3, phantom, "(*)"]
\arrow[from=2-1, to=3-2, phantom, "\circlearrowleft"]
\arrow[from=2-2, to=3-3, phantom, "\circlearrowleft"]
\arrow[from=2-3, to=3-4, phantom, "\circlearrowleft"]
\arrow[from=4-2, to=4-3, phantom, "\circlearrowleft"]
\end{tikzcd}
$$

Verified commutativity is represented by "$\circlearrowleft$". What we want to know is that the biggest square commutes. If we verify that $(*)$ commutes, we obtain the result in the same way in the proof of Theorem 2.21.

If we notice that $f \circ i = k \circ f|_S$, the following can be verified from functoriality of K-theory in terms of groupoids.

Proposition 2.25. *The following diagram commutes.*

$$\begin{array}{ccc} K_G^{\tau+n}(G) & \xrightarrow{f^\#} & K_H^{f^*\tau+n}(H) \\ {\scriptstyle k^\#}\big\downarrow & & {\scriptstyle i^\#}\big\downarrow \\ K_T^{k^*\tau+n}(T) & \xrightarrow{f^\#} & K_S^{i^*f^*\tau+n}(S) \end{array}$$

□

3. Positive energy representation group

In this section, we deal with representation theory of loop groups of tori to verify Theorem 0.7. Our main tool is an explicit description of positive energy representations.

3.1. An explicit construction of positive energy representation of LT (review)

Representation theory for loop groups is well known ([PS]). Especially we have known an explicit description of irreducible positive energy representations of loop groups of tori already.

Let τ be a positive central extension of LT. Let us recall that we can decompose LT^τ as $U^\tau \boxtimes (T \times \Pi_T)^\tau$. U^τ is the infinite dimensional version of the Heisenberg group and has the unique irreducible positive energy representation $\rho_{\mathcal{H}} : U^\tau \to U(V_{\mathcal{H}}(U^\tau))$ at level τ up to equivalence. More precisely, $V_{\mathcal{H}}(U^\tau) := \widehat{S}((L\mathfrak{t}_{\mathbb{C}})_+)$ is a completion of the symmetric tensor algebra $S((L\mathfrak{t}_{\mathbb{C}})_+)$ of $(L\mathfrak{t}_{\mathbb{C}})_+$, which we call the Heisenberg representation.

When we choose a character $\lambda \in \Lambda_T^\tau$, $U^\tau \boxtimes T^\tau$ acts on $V_{\mathcal{H}}(U^\tau) \otimes \mathbb{C}_\lambda$ by

$$\rho_{\mathcal{H}} \boxtimes \lambda(u \boxtimes t)(v \otimes z) := \rho_{\mathcal{H}}(u)(v) \otimes \lambda(t)z,$$

where $u \in U^\tau$, $t \in T^\tau$, $v \in V_{\mathcal{H}}(U^\tau)$ and $z \in \mathbb{C}_\lambda$.

Then $LT^\tau = U^\tau \boxtimes (T \times \Pi_T)^\tau$ acts on

$$V_{[\lambda]} := \sum_{n \in \Pi_T} V_{\mathcal{H}}(U^\tau) \otimes \mathbb{C}_{\lambda + \kappa^\tau(n)}$$

where Π_T permutes the components. The isomorphism classes of irreducible positive energy representations are in 1:1 correspondence with the points of $\Lambda_T^\tau/\kappa^\tau(\Pi_T)$ ([PS] Proposition 9.5.11). So we can define an isomorphism

$$l.w._T : R^\tau(LT) \to char(T, \tau)$$

by the formula $l.w._T(V_{[\lambda]}) := [\lambda]_T$. The symbol $l.w.$ comes from the lowest weight.

3.2. Induced homomorphism

In this section, we construct the induced homomorphism $f^! : R^\tau(LT) \to R^{f^*\tau}(LS)$ for a locally injective Lie group homomorphism $f : S \to T$ so that the following theorem holds.

Theorem 3.1. *The following diagram commutes.*

$$\begin{tikzcd} R^\tau(LT) \arrow[r, "f^!"] \arrow[d, "l.w._T"'] & R^{f^*\tau}(LS) \arrow[d, "l.w._S"] \\ char(T,\tau) \arrow[r, "char(f)"] & char(S, f^*\tau) \end{tikzcd}$$

As usual, it is sufficient to deal with two special cases, inclusion into the first factor of direct products and finite coverings.

3.2.1. Direct product

Let T_1 and T_2 be tori, τ_1 and τ_2 be positive central extensions of LT_1 and LT_2 respectively, i_1 be the natural inclusion into the first factor of $T_1 \times T_2$, and $p_j : T_1 \times T_2 \to T_j$ be the natural projection onto tha j' th factor $(j = 1, 2)$. Then $T := T_1 \times T_2$ is a torus and $\tau := p_1^*\tau_1 + p_2^*\tau_2$ is a positive central extension of $LT \cong LT_1 \times LT_2$.

Using the above description, we construct $i_1^! : R^\tau(LT) \to R^{i_1^*\tau}(LT_1)$ and verify that the following diagram commutes.

$$\begin{tikzcd} R^\tau(LT) \arrow[r, "i_1^!"] \arrow[d, "l.w._T"'] & R^{i_1^*\tau}(T_1) \arrow[d, "l.w._{T_1}"] \\ char(T,\tau) \arrow[r, "char(i_1)"] & char(T_1, i_1^*\tau) \end{tikzcd}$$

In this situation, we have a natural isomorphisms $LT \cong LT_1 \times LT_2$ and $LT^\tau \cong LT_1^{\tau_1} \boxtimes LT_2^{\tau_2}$, that is, for any $l_1, l_1' \in LT_1^{\tau_1}$ and $l_2, l_2' \in LT_2^{\tau_2}$, $l_1 \boxtimes l_2 \cdot l_1' \boxtimes l_2' = l_1 l_1' \boxtimes l_2 l_2'$. Moreover, $LT^\tau \cong U_1^{\tau_1} \boxtimes U_2^{\tau_2} \boxtimes (T_1 \times \Pi_1)^{\tau_1} \boxtimes (T_2 \times \Pi_2)^{\tau_2}$. From the commutativity between $LT_1^{\tau_1}$ and $LT_2^{\tau_2}$, we can verify the followings.

Lemma 3.2. *Let $V_{\mathcal{H}}(U^\tau)$ be the Heisenberg representation space of U^τ. Then*

$$V_{\mathcal{H}}(U^\tau) \cong V_{\mathcal{H}}(U_1^{\tau_1}) \otimes V_{\mathcal{H}}(U_2^{\tau_2})$$

as representation spaces.

Lemma 3.3. *Let $\lambda \in \Lambda_T^\tau$ be a τ-twisted character of $T^\tau \cong T_1^{\tau_1} \boxtimes T_2^{\tau_2}$, then*

$$\mathbb{C}_\lambda \cong \mathbb{C}_{{}^t di_1(\lambda)} \otimes \mathbb{C}_{{}^t di_2(\lambda)}$$

as representation spaces.

The above lemmas imply the following theorem.

Theorem 3.4. *We have an isomorphism*

$$V_{[\lambda]} \cong V_{[{}^t di_1(\lambda)]} \otimes V_{[{}^t di_2(\lambda)]}$$

as representation spaces.

Proof.

$$
\begin{aligned}
V_{[\lambda]} &= \sum_{n\in\Pi_T} V_{\mathcal{H}}(U^\tau)\otimes\mathbb{C}_{\lambda+\kappa^\tau(n)} \\
&\cong \sum_{n_1\in\Pi_{T_1},n_2\in\Pi_{T_2}} V_{\mathcal{H}}(U_1^{\tau_1})\otimes V_{\mathcal{H}}(U_2^{\tau_2})\otimes\mathbb{C}_{{}^t di_1(\lambda)+\kappa^{\tau_1}(n_1)}\otimes\mathbb{C}_{{}^t di_2(\lambda)+\kappa^{\tau_2}(n_2)} \\
&\cong \sum_{n_1\in\Pi_{T_1},n_2\in\Pi_{T_2}} \big[V_{\mathcal{H}}(U_1^{\tau_1})\otimes\mathbb{C}_{{}^t di_1(\lambda)+\kappa^{\tau_1}(n_1)}\big]\otimes\big[V_{\mathcal{H}}(U_2^{\tau_2})\otimes\mathbb{C}_{{}^t di_2(\lambda)+\kappa^{\tau_2}(n_2)}\big] \\
&\cong \Big[\sum_{n_1\in\Pi_{T_1}} V_{\mathcal{H}}(U_1^{\tau_1})\otimes\mathbb{C}_{{}^t di_1(\lambda)+\kappa^{\tau_1}(n_1)}\Big]\otimes\Big[\sum_{n_2\in\Pi_{T_2}} V_{\mathcal{H}}(U_2^{\tau_2})\otimes\mathbb{C}_{{}^t di_2(\lambda)+\kappa^{\tau_2}(n_2)}\Big] \\
&= V_{[{}^t di_1(\lambda)]}\otimes V_{[{}^t di_2(\lambda)]}
\end{aligned}
$$

Factorization as representation spaces follows from Lemma 3.2, 3.3 and the bijection in the proof of Lemma 1.21. □

Since the dimension of $V_{[{}^t di_2(\lambda)]}$ is infinite, we obtain the following.

Corollary 3.5. *When we regard an irreducible representation of LT^τ as a representation of $LT_1^{\tau_1}$, it is never finitely reducible if* $\dim T_2\geq 1$.

It tells us that ordinary pull back of representations should not be the induced homomorphism between representation groups.

Motivated by this observation, we define the new "induced" representation $i_1^! V_{[\lambda]}$ so that it is compatible with $char(i_1)$.

Definition 3.6. *Let V be a finitely reducible representation space of LT^τ.*

$$
\begin{aligned}
i_1^! V &:= \sum_{[\lambda_2]\in\Lambda_{T_2}^{\tau_2}/\kappa^{\tau_2}(\Pi_{T_2})} \mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V) \\
&\cong \mathrm{Hom}_{LT_2^{\tau_2}}\Big(\sum_{[\lambda_2]\in\Lambda_{T_2}^{\tau_2}/\kappa^{\tau_2}(\Pi_{T_2})} V_{[\lambda_2]}, i_2^* V\Big)
\end{aligned}
$$

where, $\mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V)$ *is the set of bounded intertwining operators. The norm of the left hand side is defined in the proof of Theorem 3.8.*

Proposition 3.7. *$i_1^! V$ is a representation space of $LT_1^{\tau_1}$. The action is defined by*

$$(l.F)(v) := l.(F(v)),$$

where $F\in i_1^! V$, $l\in LT_1^{\tau_1}$ and $v\in\sum_{[\lambda_2]\in\Lambda_{T_2}^{\tau_2}/\kappa^{\tau_2}(\Pi_{T_2})} V_{[\lambda_2]}$.

Proof. It is sufficient to verify that $l.F$ is an intertwining operator. We may assume that $v\in V_{[\lambda_2]}$ for some $[\lambda_2]\in\Lambda_{T_2}^{\tau_2}/\kappa^{\tau_2}(\Pi_{T_2})$. Let $l\in LT_1^{\tau_1}$, $l'\in LT_2^{\tau_2}$ and $F\in \mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V)$.

$$
\begin{aligned}
(l.F)(l'.v) &= l.(F(l'.v)) = l.(l'.(F(v))) \\
&= (ll').F(v) = (l'l).F(v)
\end{aligned}
$$

$$= l'.(l.(F(v))) = l'.(l.F)(v).$$

The first and the last equalities follow from the definition. The second one follows from the assumption that F is an intertwining operator. The third and the fifth ones hold since a representation is a group homomorphism. The fourth one follows from the commutativity between $LT_1^{\tau_1}$ and $LT_2^{\tau_2}$.

□

Clearly, for any finitely reducible positive energy representation space V_1 and V_2 of LT^τ at level τ, $i_1^!(V_1 \oplus V_2) \cong i_1^! V_1 \oplus i_1^! V_2$. Therefore, we can assume that V is irreducible.

Theorem 3.8.

$$i_1^! V_{[\lambda]} \cong V_{[{}^t di_1(\lambda)]},$$

therefore, $i_1^!$ is compatible with $char(i_1)$.

If we define an inner product of the left hand side by $(F_1, F_2) := (F_1(v), F_2(v))_{V_{[\lambda]}}$, where v is a chosen unit vector in $V_{[{}^t di_2(\lambda)]}$, this inner product does not depend on the choice of v, and the above isomorphism is isometric.

Proof. Fix a completely orthonormal system $\{w_j\}_{j\in\mathbb{N}}$ of $V_{[{}^t di_1(\lambda)]}$. From Theorem 3.4, $V_{[\lambda]} \cong \sum_j \mathbb{C}w_j \otimes V_{[{}^t di_2(\lambda)]}$ as representation spaces of $LT_2^{\tau_2}$. So we can decompose $F \in \mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]})$ as

$$F = \sum_j F_j,$$

where $F_j : V_{[{}^t di_2(\lambda)]} \to \mathbb{C}w_j \otimes V_{[{}^t di_2(\lambda)]} \cong V_{[{}^t di_2(\lambda)]}$.

By Schur's lemma, $F_j = c_j \mathrm{id}$ for some $c_j \in \mathbb{C}$. Since F determines an operator, for any $v \in V_{[{}^t di_2(\lambda)]}$, $\sum_j F_j(v) = v \otimes \sum_j c_j w_j \in V_{[\lambda]}$, therefore, $\sum_j |c_j|^2 < \infty$.

Moreover, by Schur's lemma, if $[\lambda_2] \neq [{}^t di_2(\lambda)]$, $\mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]}) = 0$. Therefore, we can define an isomorphisms

$$\sum_{[\lambda]\in \Lambda_{T_2}^{\tau_2}/\kappa^{\tau_2}(\Pi_{T_2})} \mathrm{Hom}_{LT_2^{\tau_2}}(V_{[\lambda_2]}, i_2^* V_{[\lambda]}) \cong \mathrm{Hom}_{LT_2^{\tau_2}}(V_{[{}^t di_2(\lambda)]}, i_2^* V_{[\lambda]}) \to V_{[i_1^* \lambda]}$$

by

$$F \mapsto \sum_j c_j w_j.$$

Since $\sum_j |c_j|^2 < \infty$, the infinite sum $\sum_j c_j w_j$ converges.

Moreover, since $\sum_j c_j w_j$ is determined by $F(v) = v \otimes \sum_j c_j w_j$, this isomorphism is independent of the choice of v and completely orthonormal system. □

From this theorem, we obtain the following.

Corollary 3.9. *The following diagram commutes.*

$$\begin{array}{ccc} R^{\tau}(LT) & \xrightarrow{i_1^!} & R^{i_1^*\tau}(LT_1) \\ {\scriptstyle l.w._T}\downarrow & & {\scriptstyle l.w._{T_1}}\downarrow \\ char(T,\tau) & \xrightarrow{char(i_1)} & char(T_1, i_1^*\tau) \end{array}$$

3.2.2. Finite covering

In this section, we deal with finite coverings. Let $q : T' \to T$ be a finite covering. Fortunately, the definition of $q^!$ looks very natural.

Definition 3.10. *If V is a finitely reducible representation space of LT^{τ},*

$$q^! V := (Lq)^* V.$$

Let us notice that for any finitely reducible positive energy representation spaces V_1 and V_2 of LT^{τ} at level τ, $q^!(V_1 \oplus V_2) \cong q^! V_1 \oplus q^! V_2$. That is, we can assume that V is irreducible.

Theorem 3.11.

$$q^!(V_{[\lambda]}) \cong \sum_{m \in \Pi_T / dq(\Pi_{T'})} V_{[{}^t dq(\lambda + \kappa^{\tau}(m))]}.$$

Moreover, the following diagram commutes.

$$\begin{array}{ccc} R^{\tau}(LT) & \xrightarrow{q^!} & R^{q^*\tau}(LT') \\ {\scriptstyle l.w._T}\downarrow & & {\scriptstyle l.w._{T'}}\downarrow \\ char(T,\tau) & \xrightarrow{char(q)} & char(T', q^*\tau) \end{array}$$

Proof. Since T and T' are locally isomorphic, the set of the derivatives of contractible loops whose initial values are 0 of T corresponds to one of T' by the natural way. That is, $LT^{\tau} \cong (T \times \Pi_T)^{\tau} \boxtimes U^{\tau}$, $LT'^{q^*\tau} \cong (T' \times \Pi_{T'})^{q^*\tau} \boxtimes U^{q^*\tau}$ and $U^{\tau} \cong U^{q^*\tau}$. Therefore, $U^{\tau} \subseteq LT^{\tau}$ and $U^{q^*\tau} \subseteq LT'^{q^*\tau}$ has the isomorphic Heisenberg representations.

$$q^!(V_{[\lambda]}) = q^!\Big(\sum_{n \in \Pi_T} V_{\mathcal{H}}(U^{\tau}) \otimes \mathbb{C}_{\lambda + \kappa^{\tau}(n)}\Big) = \sum_{n \in \Pi_T} V_{\mathcal{H}}(U^{q^*\tau}) \otimes \mathbb{C}_{{}^t dq(\lambda + \kappa^{\tau}(n))}$$

$$= \sum_{m \in \Pi_T / dq(\Pi_{T'})} \sum_{n' \in \Pi_{T'}} V_{\mathcal{H}} \otimes \mathbb{C}_{{}^t dq(\lambda + \kappa^{\tau}(m)) + \kappa^{q^*\tau}(n')}$$

$$= \sum_{m \in \Pi_T / dq(\Pi_{T'})} V_{[{}^t dq(\lambda + \kappa^{\tau}(m))]}$$

Commutativity of the above diagram is clear from the above and Lemma 1.22. ☐

3.2.3. Local injection

Let $f : S \to T$ be a local injection. From Theorem 1.29, we can decompose f as the following sequence

$$S \xrightarrow{q} S/\ker(f) \xrightarrow{i_1} S/\ker(f) \times S^{\perp} \xrightarrow{f\cdot j} T,$$

where q and $f \cdot j$ are finite coverings, i_1 is the natural inclusion into the first factor.

Definition 3.12.

$$f^! := q^! \circ i_1^! \circ (f \cdot j)^!$$

The following theorem follows from Corollary 3.9, Theorem 3.11, Theorem 1.29 and the same argument in the proof of Theorem 2.18.

Theorem 3.13. *The following diagram commutes.*

$$\begin{tikzcd} R^{\tau}(LT) \arrow[r, "f^!"] \arrow[d, "l.w._T"'] & R^{f^*\tau}(LS) \arrow[d, "l.w._S"'] \\ char(T,\tau) \arrow[r, "char(f)"] & char(S, f^*\tau) \end{tikzcd}$$

4. Proof of the main theorem

Freed, Hopkins and Teleman constructed isomorphism between twisted equivariant K-theory and positive energy representation group by use of a family of "Dirac operators" parametrized by the set of connections over the trivial bundle $S^1 \times G$ in [FHT2]. We write FHT isomorphism for G as FHT_G.

Let us describe the essence of the proof of FHT isomorphism by Freed, Hopkins and Telemann in [FHT2] for tori using our terminology.

Theorem 4.1 ([FHT2] Proposition 4.8)**.** *Let T be a torus and τ be a positive central extension of LT. The following commutative diagram holds.*

$$\begin{tikzcd} R^{\tau}(LT) \arrow[rr, "FHT_T"] \arrow[dr, "l.w._T"'] & & K_T^{\tau+\dim(T)}(T) \arrow[dl, "M.d._T"] \\ & char(T,\tau) & \end{tikzcd}$$

Let us verify our main theorem using the above. The following diagram commutes.

$$\begin{tikzcd}
R^{\tau}(LT) \arrow[rrr, "f^{!}"] \arrow[dr, "l.w._T"] \arrow[dd, "FHT_T"'] & & & R^{f^*\tau}(LS) \arrow[dl, "l.w._S"'] \arrow[dd, "FHT_S"] \\
\arrow[r, phantom, "(2)"] & char(T,\tau) \arrow[r, "char(f)"] \arrow[ur, phantom, "(1)"] \arrow[dr, phantom, "(4)"] & char(S,f^*\tau) \arrow[r, phantom, "(3)"] & \\
K_T^{\tau+\dim(T)}(T) \arrow[rrr, "f^{\#}"'] \arrow[ur, "M.d._T"'] & & & K_S^{f^*\tau+\dim(S)}(S) \arrow[ul, "M.d._S"]
\end{tikzcd}$$

Commutativity of (1) is Theorem 3.13, and commutativity of (4) is Theorem 2.18. Commutativity of (2) and (3) are the above theorem. Since $M.d._T$ and $M.d._S$ are isomorphisms, we obtain the conclusion.

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References

[AS] M. Atiyah, G. Segal, *Twisted K-theory*, Ukr. Mat. Visn. **1**, (2004), no. 3, 287–330; translation in Ukr. Math. Bull. **1** (2004), no. 3, 291–334.

[CW] A. L. Carey and B. L. Wang, *Thom isomorphism and push-forward map in twisted K-theory*, Journal of K-theory **1** (2008), 357–393.

[FHT1] D. S. Freed, M. J. Hopkins, and C. Teleman, *Loop groups and twisted K-theory I*, Journal of Topology **4** (2011), 737–798.

[FHT2] D. S. Freed, M. J. Hopkins, and C. Teleman, *Loop groups and twisted K-theory II*, Journal of the American Mathematical Society **26** (2013), 595–644.

[FHT3] D. S. Freed, M. J. Hopkins, and C. Teleman, *Loop groups and twisted K-theory III*, Annals of Mathematics **174** (2011), 947–1007.

[FHT4] D. S. Freed, M. J. Hopkins, and C. Teleman, *Twisted equivariant K-theory with complex coefficients*, Journal of Topology **1** (2008), 16–44.

[Fu] M. Furuta, *Index theorem* (Japanese), Iwanami Shoten, 2008.

[Ka] M. Karoubi, *K-theory - An Introduction*, Springer, 1978.

[PS] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, New York, 1986.

[Se] G. Segal, *Equivariant K-theory*, Publ. Math. I.H.E.S. **34** (1968), 129–151.

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