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Elliptic Gromov-Witten Invariants of Del-Pezzo surfaces

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Abstract. We obtain a formula for the number of genus one curves with a variable complex structure of a given degree on a del-Pezzo surface that pass through an appropriate number of generic points of the surface. This is done using Getzler's relationship among cohomology classes of certain codimension 2 cycles in $\overline{M}_{1,4}$ and recursively computing the genus one Gromov-Witten invariants of del-Pezzo surfaces. Using completely different methods, this problem has been solved earlier by Bertram and Abramovich ([3]), Ravi Vakil ([23]), Dubrovin and Zhang ([8]) and more recently using Tropical geometric methods by M. Shoval and E. Shustin ([22]). We also subject our formula to several low degree checks and compare them to the numbers obtained by the earlier authors. Our numbers agree with the numbers obtained by Ravi Vakil, except for one number where we get something different. We give geometric reasons to explain why our answer is likely to be correct and hence conclude that the number written by Ravi Vakil is likely to be a minor typo (since our numbers are consistent with the other numbers he has obtained).

1. Introduction

One of the most fundamental problems in enumerative algebraic geometry is:

Question 1.1. *What is $E_d^{(g)}$, the number of genus g degree d curves in $\mathbb{CP}^2$ (with a variable complex structure) that pass through $3d-1+g$ generic points?*

Although the computation of $E_d^{(g)}$ is a classical question, a complete solution to the above problem (even for genus zero) was unknown until the early $90^{'s}$ when Ruan–Tian ([21]) and Kontsevich–Manin ([16]) obtained a formula for $E_d^{(0)}$.
The computation of $E_d^{(g)}$ is now very well understood from several different perspectives. The formula by Caporasso–Harris [6], computes $E_d^{(g)}$ for all g and d. Since then, the computation of $E_d^{(g)}$ has been studied from many different perspectives; these include (among others), the algorithm by Gathman ([10], [11]) and the method of virtual localization by Graber and Pandharipande ([15]) to compute the genus g Gromov-Witten invariants of $\mathbb{CP}^n$ (although for $n>2$ and $g>0$, the Gromov-Witten invariants are not enumerative).

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More recently, the problem of computing $E_d^{(g)}$ has been studied using the method of tropical geometry by Mikhalkin in [18] (using the results of that paper, one can in principle compute $E_d^{(g)}$ for all g and d).
A more general situation is as follows: let X be a projective manifold and $\beta \in H_2(X;\mathbb{Z})$ a given homology class. Given cohomology classes $\mu_1, \ldots, \mu_k \in H^*(X,\mathbb{Q})$, the k-pointed genus g, Gromov-Witten invariant of X is defined to be

$$N_{\beta,X}^{(g)}(\mu_1,\ldots,\mu_k) := \int\limits_{\overline{M}_{g,k}(X,\beta)} \mathrm{ev}_1^*(\mu_1) \smile \ldots \smile \mathrm{ev}_k^*(\mu_k) \smile \ \big[\overline{M}_{g,k}(X,\beta)\big]^{\mathrm{vir}}, \qquad (1)$$

where $\overline{M}_{g,k}(X,\beta)$ denotes the moduli space of genus g stable maps into X with k marked points representing β and ev_i denotes the i^{th} evaluation map. For $g = 0$, this is a smooth, irreducible and proper Deligne-Mumford stack and has a fundamental class. However, for $g > 0$, $\overline{M}_{g,k}(X,\beta)$ is not smooth or irreducible, hence it does not posses a fundamental class. Behrend, Behrend-Fantechi and Li-Tian, have however defined the virtual fundamental class

$$\big[\overline{M}_{g,k}(X,\beta)\big]^{\mathrm{vir}} \in H^{2\Theta}(\overline{M}_{g,k}(X,\beta)), \qquad \Theta := c_1(TX)\cdot\beta + (\dim X - 3)(1-g) + k;$$

which is used to define the Gromov-Witten invariants (see [4],[5] and [17]). When all the $\mu_1, \ldots, \mu_k$ represent the class Poincare dual to a point (and the degree of the cohomology class that is being paired in (1), is equal to the virtual dimension of the moduli space), then we abbreviate $N_{\beta,X}^{(g)}(\mu_1,\ldots,\mu_k)$ as $N_\beta^{(g)}$. The number of genus g curves of degree β in X, that pass through $c_1(TX)\cdot\beta + (\dim X - 3)(1-g)$ generic points is denoted by $E_\beta^{(g)}$. In general, $E_\beta^{(g)}$ is not necessarily equal to $N_\beta^{(g)}$, i.e. the Gromov-Witten invariant is not necessarily enumerative (this happens for example when $X := \mathbb{CP}^3$ and $g = 1$).
An important class of surfaces for which the enumerative geometry is particularly important are Fano surfaces, which are also called del-Pezzo surfaces (see section 4 for the definition of a del-Pezzo surface). When $g = 0$, it is proved in ([14], Theorem 4.1, Lemma 4.10) that for del-Pezzo surfaces $N_\beta^{(0)} = E_\beta^{(0)}$.
In [23], Vakil generalizes the approach of Caporasso-Harris in [6] to compute the numbers $E_\beta^{(g)}$ for all g and β for del-Pezzo surfaces. It is also shown in ([23], Section 4.2) that all the genus g Gromov-Witten invariants of del-Pezzo surfaces are enumerative (i.e. $N_\beta^{(g)} = E_\beta^{(g)}$). The enumerative geometry of del-Pezzo surfaces has also been studied extensively by Abramovich and Bertram (in [3]). More recently, this question has been approached using methods of tropical geometry. In [22], M. Shoval and E. Shustin give a formula to compute all the genus g Gromov-Witten invariants of del-Pezzo surfaces using methods of tropical geometry.
The genus one Gromov-Witten invariants of $\mathbb{CP}^n$ can also be computed from a completely different method from the ones developed in [10], [11] and [15]. In [12], Getzler finds a relationship among certain codimension two cycles in $\overline{M}_{1,4}$ and uses that to compute the genus one Gromov-Witten invariants of $\mathbb{CP}^2$ and $\mathbb{CP}^3$. In [9], using ideas from

Physics, Eguchi, Hori and Xiong made a remarkable conjecture concerning the genus g Gromov-Witten invariants of projective manifolds; this is known as the Virasoro conjecture. The conjecture in particular produces an explicit formula for $N_d^{(1)}$ (for $\mathbb{CP}^2$), which aprori looks very different from the formula obtained by Getzler (in [12]). It is shown by Pandharipande (in [20]), that the formula obtained by Getzler for $\mathbb{CP}^2$ is equivalent to a completely different looking formula predicted in [9].
In this paper, we extend the approach of Getzler to compute the genus one Gromov-Witten invariants of del-Pezzo surfaces. The formula we obtain has a completely different appearance from the one obtained by Vakil in [23]. We verify that our final numbers are consistent with the numbers he obtains, except for one number (see section 8 for details). The Virasoro conjecture for projective manifolds (which is conjectured in [9]) has been a topic of active research in mathematics for the last twenty years. In [8], Dubrovin and Zhang compute the genus one Gromov-Witten invariants of $\mathbb{CP}^1 \times \mathbb{CP}^1$ by showing that it follows from the Virasoro conjecture. We have verified that our numbers agree with all the numbers computed by them ([8], Page 463). They prove that the genus zero and genus one Virasoro Conjecture is true for all projective manifolds having semi-simple quantum cohomology. It is proved in [1] that the quantum cohomology of del-Pezzo surfaces is semi simple. It would be interesting to see if one can use the result of this paper and apply the method of [20] to obtain a formula for the genus one Gromov-Witten invariants of del-Pezzo surfaces, analogous to the one predicted for $\mathbb{CP}^2$ by Eguchi Hori and Xiong (in [9]). That would give a *direct* confirmation of the Virasoro conjecture in genus one for del-Pezzo surfaces. A detailed survey of the Virasosro conjecture is given in [13].

2. Main Result

The main result of this paper is the following:

Main Result. *Let X be a del-Pezzo surface and $\beta \in H_2(X, \mathbb{Z})$ be a given effective homology class. We obtain a formula for $N_\beta^{(1)}$ (equation* (2)*) using Getzler's relation*

Remark. We note that by ([23], Section 4.2), we conclude that $N_\beta^{(1)} = E_\beta^{(1)}$. Alternatively, we note that $N_\beta^{(1)} = E_\beta^{(1)}$ follows from ([24], Theorem 1.1).

Our formula for $N_\beta^{(1)}$ is a recursive formula, involving $N_\beta^{(0)}$. The latter can be computed via the algorithm given in [16] and [14]. The base case of our recursive formula are given by equations (3) and (4). We have written a C++ program that implements (2); it is available on our web page:

`http://www.iiserpune.ac.in/~chitrabhanu/`.

3. Recursive formula

We will now give the recursive formula to compute $N^{(1)}_\beta$. First, we will develop some notation that is used throughout this paper. Let

$$\begin{array}{lll} \xi_X & := c_1(TX), & \text{for both the cohomology class and the divisor,} \\ \kappa_\beta & := \xi_X \cdot \beta, & \text{where } \beta \in H_2(X,\mathbb{Z}), \\ b_2(X) & := \dim H_2(X,\mathbb{Q}), & \text{the second betti number of } X, \\ d_X & := \xi_X \cdot \xi_X, & \text{the degree of } X. \end{array}$$

Moreover, $\cdot$ is used for both the cup product in cohomology as well as cap product between a homology and a cohomology class.

We are now ready to state the formula. First, let us define the following four quantities:

$$\begin{aligned} T_1 := \sum_{\beta_1+\beta_2+\beta_3=\beta} & \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} 2\kappa_{\beta_2}\kappa^2_{\beta_3}(\beta_1\cdot\beta_2) \\ & \Big((4\kappa_{\beta_1} + \kappa_{\beta_2} - 2\kappa_{\beta_3})(\beta_2\cdot\beta_3) - 3\kappa_{\beta_2}(\beta_1\cdot\beta_3) \Big) N^{(1)}_{\beta_1} N^{(0)}_{\beta_2} N^{(0)}_{\beta_3}, \end{aligned}$$

$$\begin{aligned} T_2 := \sum_{\beta_1+\beta_2=\beta} & \Bigg[\binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} 4\kappa^2_{\beta_2}\Big(2\kappa_{\beta_1}\kappa_{\beta_2} - \kappa^2_{\beta_2} - 3d_X(\beta_1\cdot\beta_2)\Big) + \\ & \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} 2\kappa_{\beta_2}\Big(d_X(\beta_1\cdot\beta_2)(4\kappa_{\beta_1}+\kappa_{\beta_2}) + 2\kappa_{\beta_1}\kappa_{\beta_2}(2\kappa_{\beta_1}-\kappa_{\beta_2})\Big) \Bigg] N^{(1)}_{\beta_1} N^{(0)}_{\beta_2}, \end{aligned}$$

$$\begin{aligned} T_3 := -\frac{1}{12} \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} \kappa^2_{\beta_2}(\beta_1\cdot\beta_2) \Bigg[& \kappa^2_{\beta_1}\Big(\kappa_{\beta_1} - 2\kappa_{\beta_2} - 6(\beta_1\cdot\beta_2)\Big) \\ & + \kappa_{\beta_2}(\beta_1\cdot\beta_1)\Big(4\kappa_{\beta_1}+\kappa_{\beta_2}\Big)\Bigg] N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}, \end{aligned}$$

$$T_4 := -\frac{1}{12}\kappa^3_\beta\Big((2+b_2(X))\kappa_\beta - d_X\Big) N^{(0)}_\beta.$$

The number $N^{(1)}_\beta$ satisfies the following recursive relation:

$$6d_X^2 N^{(1)}_\beta = T_1 + T_2 + T_3 + T_4. \tag{2}$$

We will now give the initial conditions for the recursion (2). Let X be $\mathbb{P}^2$ blown up at upto $k = 8$ points. Then the initial condition of the recursion is

$$N^{(1)}_L = 0 \qquad \text{and} \qquad N^{(1)}_{E_i} = 0 \qquad \forall i = 1 \text{ to } k. \tag{3}$$

Here L denotes the class of a line and E_i denotes the exceptional divisors. If $X := \mathbb{P}^1 \times \mathbb{P}^1$, then

$$N^{(1)}_{e_1} = 0 \qquad \text{and} \qquad N^{(1)}_{e_2} = 0. \tag{4}$$

Here e_1 and e_2 denote the class of $[\mathrm{pt}\times\mathbb{P}^1]$ and $[\mathbb{P}^1\times\mathrm{pt}]$ respectively. The initial conditions (3) and (4), combined with the values of $N_\beta^{(0)}$ obtained from [16] and [14], give us the values of $N_\beta^{(1)}$ for any β.

Remark. We would like to mention that the formula (2) yields Getzler's recursion relation, equation (0.1) of [12], after some symmetrization of the summation indices of T_1 and T_3.

4. Del-Pezzo surfaces

A del-Pezzo surface X is a smooth projective algebraic surface with an ample anti-canonical divisor ξ_X. The degree of the surface is defined to be the self-intersection number

$$d_X = \xi_X \cdot \xi_X.$$

This degree d_X varies between 1 and 9. X can be obtained as a blow-up of $\mathbb{P}^2$ at $k = 9-d_X$ general points, except, when $d_X = 8$ the surface can also be $\mathbb{P}^1 \times \mathbb{P}^1$.

If X has degree $9-k$ and is not $\mathbb{P}^1 \times \mathbb{P}^1$, then we have the blow up morphism $Bl : X \to \mathbb{P}^2$. We denote by $E_1, \dots, E_k$ the exceptional divisors of Bl and by L the pull-back of the class of a hyperplane in $\mathbb{P}^2$. We have

$$H^2(X,\mathbb{Z}) = \mathbb{Z}\langle L, E_1, \dots, E_k\rangle,$$

and $L\cdot L = 1$, $E_i \cdot E_i = -1$, $L \cdot E_i = E_i \cdot E_j = 0$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$. The anti-canonical divisor is given by $\xi_X = 3L - E_1 - \dots - E_k$.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $e_1 = \mathrm{pr}_1^*[\mathrm{pt}]$ and $e_2 = \mathrm{pr}_2^*[\mathrm{pt}]$, then $\xi_X = 2e_1 + 2e_2$, $e_1 \cdot e_2 = 1$ and $e_2 \cdot e_1 = 1$ whereas $e_i \cdot e_i = 0$ for $i = 1, 2$.

5. Basic Strategy

We will now recall the basic setup of [12], where Getzler computes the number $N_d^{(1)}$ when X is $\mathbb{CP}^2$. First, let us consider the space $\overline{M}_{1,4}$, the moduli space of genus one curves with four marked points. We shall be interested in certain S_4 invariant codimension 2 boundary strata in $\overline{M}_{1,4}$ which we list in Figure 1. In the figure we draw the topological type and the marked point distribution of the generic curve in each strata. We use the same nomenclature as [12] except for $\delta_{0,0}$ which was denoted by δ_β in [12], (to avoid confusion between notations). See section 1 of [12] for a list of all the codimension 2 strata. There the strata are denoted by the dual graph of the generic curve.

These strata define cycles in $H^4(\overline{M}_{1,4}, \mathbb{Q})$. Let us now define the following cycle in $H^4(\overline{M}_{1,4}, \mathbb{Q})$, given by

$$\mathcal{R} := -2\delta_{2,2} + \frac{2}{3}\delta_{2,3} + \frac{1}{3}\delta_{2,4} - \delta_{3,4} - \frac{1}{6}\delta_{0,3} - \frac{1}{6}\delta_{0,4} + \frac{1}{3}\delta_{0,0}.$$

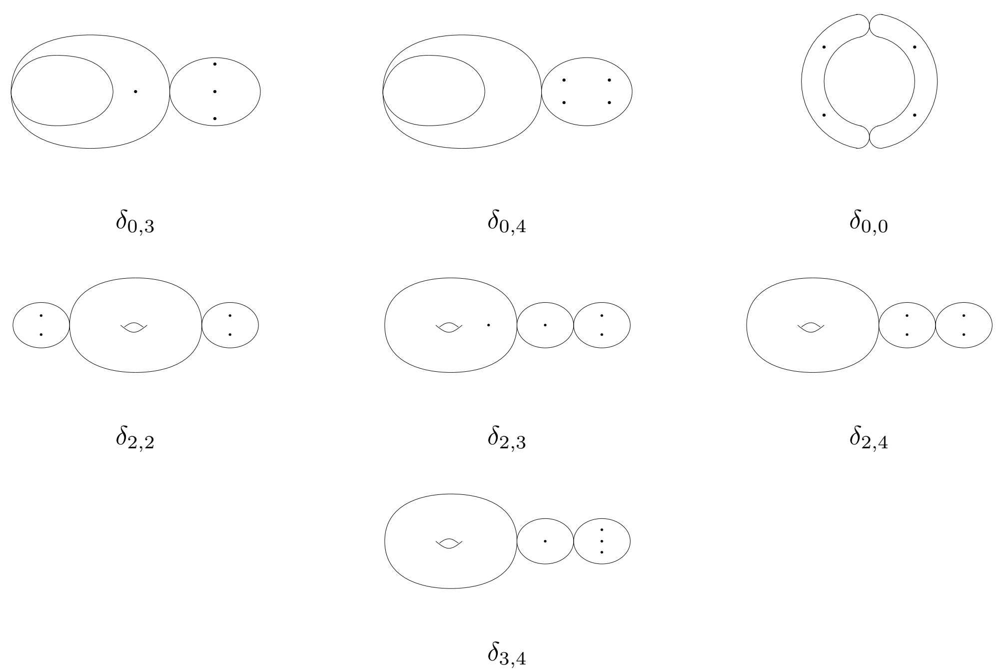

FIGURE 1. Codimension 2 strata in $\overline{M}_{1,4}$.

The main result of [12] is that $\mathcal{R} = 0$. This will subsequently be referred to as Getzler's relation. In [20], Pandharipande has shown that this relation, in fact, comes from a rational equivalence.

Now we explain how to obtain our formula. Consider the natural forgetful morphism

$$\pi : \overline{M}_{1,\kappa_\beta+2}(X,\beta) \longrightarrow \overline{M}_{1,4}.$$

We shall pull-back the cycle $\mathcal{R}$ to $H^*(\overline{M}_{1,\kappa_\beta+2}(X,\beta),\mathbb{Q})$ and intersect it with a cycle of a complementary dimension; that will give us an equality of numbers and subsequently the formula. Let $\mu \in H^4(X,\mathbb{Q})$ be the class of a point. Define

$$\mathcal{Z} := \mathrm{ev}_1^*(\xi_X) \cdot \ldots \cdot \mathrm{ev}_4^*(\xi_X) \cdot \mathrm{ev}_5^*(\mu) \cdot \ldots \cdot \mathrm{ev}_{\kappa_\beta+2}^*(\mu).$$

The class ξ_X is used since it is ample and hence numerically effective. Since $\mathcal{R} = 0$ by Getzler's relation, we conclude that

$$\int_{\overline{M}_{1,\kappa_\beta+2}(X,\beta)} (\pi^*\mathcal{R} \cdot \mathcal{Z}) \cdot \left[\overline{M}_{1,\kappa_\beta+2}(X,\beta)\right]^{\mathrm{vir}} = 0. \tag{5}$$

We can also compute the left hand side of (5) using the composition axiom for Gromov-Witten invariants which will give us the recursive formula.

6. Axioms for Gromov-Witten Invariants

We shall make use of certain axioms for Gromov-Witten invariants. These are quite standard, see for example [7], however for completeness we list them here. We assume X is a smooth projective variety.

Degree axiom: If $\deg \mu_1 + \ldots + \deg \mu_n \neq 2n + 2\kappa_\beta + 2(3 - \dim X)(g-1)$ then

$$N^{(g)}_{\beta,X}(\mu_1, \ldots, \mu_n) = 0.$$

Fundamental class axiom: If $[X]$ is the fundamental class of X and $2g + n \geq 4$ or $\beta \neq 0$, then

$$N^{(g)}_{\beta,X}([X], \mu_1, \ldots, \mu_{n-1}) = 0.$$

Divisor axiom: If D is a divisor of X and $2g + n \geq 4$. then

$$N^{(g)}_{\beta,X}(D, \mu_1, \ldots, \mu_{n-1}) = (D \cdot \beta) N^{(g)}_{\beta,X}(\mu_1, \ldots, \mu_{n-1}).$$

Composition axiom: This is a bit complicated to write down, so we refer to [12], section 2.11. It is a combination of the splitting and reduction axioms of [16] section 2.

We also need the following results which do not follow from the above axioms:

$$N^{(0)}_{0,X}(\mu_1, \mu_2, \mu_3) = \int_X \mu_1 \smile \mu_2 \smile \mu_3,$$

and

$$N^{(1)}_{0,X}(\mu) = -\frac{1}{24} c_1(TX) \cdot \mu.$$

7. Intersection of cycles

Now we are in a position to compute the left hand side of (5). Fix a homogeneous basis $\{\gamma_1, \ldots, \gamma_{b(X)}\}$ of $H^*(X, \mathbb{Q})$. Let $g_{ij} = \int_X \gamma_i \smile \gamma_j$ and $((g^{ij})) = ((g_{ij}))^{-1}$. For a cycle δ in $H^*(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$, we introduce the following notation

$$N^{\delta}_{\beta,X}(\mu_1, \ldots, \mu_n) = \int\limits_{\overline{M}_{g,n}(X,\beta)} \delta \cdot \mathrm{ev}_1^*(\mu_1) \cdots \mathrm{ev}_n^*(\mu_n) \cdot \left[\overline{M}_{g,n}(X, \beta)\right]^{\mathrm{vir}}.$$

Let $\mu_1 = \ldots = \mu_4 = \xi_X$, and $\mu_5 = \ldots = \mu_{\kappa_\beta+2} = [pt]$ be the class of a point. If $\delta = \pi^*\delta_{2,2}$, by the composition axiom

$$\begin{aligned} N^{\delta}_{\beta,X} =& N^{\delta}_{\beta,X}(\mu_1,\ldots,\mu_{\kappa_\beta+2}) \\ =& \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C}} \sum_{i,j,k,l} g^{ij}g^{kl} N^{(1)}_{\beta_1,X}(\gamma_i,\gamma_k,\mu_\alpha|\alpha\in A) \\ & \times N^{(0)}_{\beta_2,X}(\gamma_j,\mu_\alpha|\alpha\in B) \times N^{(0)}_{\beta_3,X}(\gamma_l,\mu_\alpha|\alpha\in C), \end{aligned}$$

where the second sum is over i,j,k,l ranging from 1 to $b(X)$ and the first sum is over disjoint sets A,B,C satisfying

$$A \sqcup B \sqcup C = \{1,\ldots,\kappa_\beta+2\}, \quad |B\cap\{1,2,3,4\}| = |C\cap\{1,2,3,4\}| = 2.$$

Note that if $\beta_1,\beta_2,\beta_3 > 0$, by the degree axiom the only non-trivial terms occur when $|A| = \kappa_{\beta_1}, |B| = \kappa_{\beta_2}+1, |C| = \kappa_{\beta_3}+1$. The limiting case $\beta_1 = 0$ does not yield anything, however $\beta_2 = 0$ or $\beta_3 = 0$ have non-trivial contributions to the sum. When $\beta_3 = 0, \beta_1, \beta_2 > 0$, the non-trivial contribution occurs precisely when $|C| = 2, \gamma_l = [X]$, $|A| = \kappa_{\beta_1} - 1, \gamma_k = [pt]$, and $|B| = \kappa_{\beta_2}+1$. Finally when $\beta_2 = \beta_3 = 0$, the only non-zero term occurs when $|B| = |C| = 2$, $\gamma_l = \gamma_j = [X]$ and $\gamma_k = \gamma_i = [pt]$. Making use of the fact that for any $\sigma,\tau \in H^*(X,\mathbb{Q})$

$$\sum_{i=1}^{b(X)}\sum_{j=1}^{b(X)} g^{ij}(\sigma\cdot\gamma_i)(\gamma_j\cdot\tau) = (\sigma\cdot\tau),$$

we obtain the following expression

$$\begin{aligned} N^{\pi^*\delta_{2,2}}_{\beta,X} =& 3(\xi_X\cdot\xi_X)^2 N^{(1)}_{\beta} \\ &+ 3\sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} (\beta_2\cdot\xi_X)^2(\beta_3\cdot\xi_X)^2(\beta_1\cdot\beta_2)(\beta_1\cdot\beta_3) N^{(1)}_{\beta_1}N^{(0)}_{\beta_2}N^{(0)}_{\beta_3} \\ &+ 6\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}(\xi_X\cdot\xi_X)(\beta_1\cdot\beta_2)(\beta_2\cdot\xi_X)^2 N^{(1)}_{\beta_1}N^{(0)}_{\beta_2}. \end{aligned} \tag{6}$$

Next, let us consider the cycle $\delta_{2,3}$. We then have

$$\begin{aligned} N^{\pi^*\delta_{2,3}}_{\beta,X} =& \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C}} \sum_{i,j,k,l} g^{ij}g^{kl} N^{(1)}_{\beta_1,X}(\gamma_i,\mu_\alpha|\alpha\in A) \\ & \times N^{(0)}_{\beta_2,X}(\gamma_j,\gamma_k,\mu_\alpha|\alpha\in B) \times N^{(0)}_{\beta_3,X}(\gamma_l,\mu_\alpha|\alpha\in C), \end{aligned}$$

where the sum is over sets A,B,C satisfying

$$A \sqcup B \sqcup C = \{1,\ldots,\kappa_\beta+2\}, \quad |A\cap\{1,2,3,4\}| = |B\cap\{1,2,3,4\}| = 1.$$

All the cases are similar to the previous calculation except, when $\beta_2 = 0$. In this case we can either have $|B| = 1, |A| = \kappa_{\beta_1}$, $\gamma_i = [pt]$ and $\gamma_j = [X]$; or $|B| = 1, |C| = \kappa_{\beta_3}$, $\gamma_k = [X]$ and $\gamma_l = [pt]$. We get

$$
\begin{aligned}
&N_{\beta,X}^{\pi^*\delta_{2,3}} \\
&= 12 \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} (\beta_1\cdot\xi_X)(\beta_2\cdot\xi_X)(\beta_3\cdot\xi_X)^2(\beta_1\cdot\beta_2)(\beta_2\cdot\beta_3) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} N_{\beta_3}^{(0)} \\
&\quad + 12 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} (\beta_1\cdot\xi_X)(\beta_2\cdot\xi_X)\Big((\xi_X\cdot\xi_X)(\beta_1\cdot\beta_2) + (\beta_1\cdot\xi_X)(\beta_2\cdot\xi_X)\Big) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} \\
&\quad + 12 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} (\beta_1\cdot\xi_X)(\beta_2\cdot\xi_X)^3 N_{\beta_1}^{(1)} N_{\beta_2}^{(0)}. \qquad (7)
\end{aligned}
$$

Moving on to $\delta_{2,4}$ we have

$$
\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{2,4}} = & \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C}} \sum_{i,j,k,l} g^{ij} g^{kl} N_{\beta_1,X}^{(1)}(\gamma_i, \mu_\alpha | \alpha \in A) \\
& \times N_{\beta_2,X}^{(0)}(\gamma_j,\gamma_k,\mu_\alpha|\alpha\in B) \times N_{\beta_3,X}^{(0)}(\gamma_l,\mu_\alpha|\alpha\in C),
\end{aligned}
$$

where the sum is over sets A, B, C satisfying

$$
A \sqcup B \sqcup C = \{1, \ldots, \kappa_\beta + 2\}, \quad |B \cap \{1,2,3,4\}| = |C \cap \{1,2,3,4\}| = 2.
$$

Now there is no contribution when $\beta_2 = 0$, however we have a non-trivial contribution when $\beta_1 = 0$. We can use (6) to calculate this

$$
\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{2,4}} = & \; 6 \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} (\beta_2\cdot\xi_X)^2(\beta_3\cdot\xi_X)^2(\beta_1\cdot\beta_2)(\beta_2\cdot\beta_3) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} N_{\beta_3}^{(0)} \\
& + 6 \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1}} (\beta_2\cdot\xi_X)^2(\xi_X\cdot\xi_X)(\beta_1\cdot\beta_2) N_{\beta_1}^{(1)} N_{\beta_2}^{(0)} \\
& + 6 \sum_{\beta_1+\beta_2=\beta} \left(-\frac{1}{24}\right) \binom{\kappa_\beta - 2}{\kappa_{\beta_1}-1} (\xi_X\cdot\beta_1)^3(\beta_2\cdot\xi_X)^2(\beta_1\cdot\beta_2) N_{\beta_1}^{(0)} N_{\beta_2}^{(0)} \\
& + 6\left(-\frac{1}{24}\right)(\xi_X\cdot\beta)^3(\xi_X\cdot\xi_X) N_\beta^{(0)}. \qquad (8)
\end{aligned}
$$

For $\delta_{3,4}$ we have

$$
\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{3,4}} = & \sum_{\substack{\beta_1+\beta_2+\beta_3=\beta \\ A,B,C}} \sum_{i,j,k,l} g^{ij} g^{kl} N_{\beta_1,X}^{(1)}(\gamma_i, \mu_\alpha | \alpha \in A) \\
& \times N_{\beta_2,X}^{(0)}(\gamma_j,\gamma_k,\mu_\alpha|\alpha\in B) \times N_{\beta_3,X}^{(0)}(\gamma_l,\mu_\alpha|\alpha\in C),
\end{aligned}
$$

where the first sum is over sets A, B, C satisfying

$$
A \sqcup B \sqcup C = \{1, \ldots, \kappa_\beta + 2\}, \quad |B \cap \{1,2,3,4\}| = 1, |C \cap \{1,2,3,4\}| = 3.
$$

The calculation is similar to the previous cases, so we omit the details. We obtain

$$\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{3,4}} = 4 &\sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_2}-1, \kappa_{\beta_3}-1} (\beta_2\cdot\xi_X)(\beta_3\cdot\xi_X)^3(\beta_1\cdot\beta_2)(\beta_2\cdot\beta_3)N_{\beta_1}^{(1)}N_{\beta_2}^{(0)}N_{\beta_3}^{(0)} \\
&+ 4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}}(\beta_2\cdot\xi_X)^3(\beta_1\cdot\xi_X)N_{\beta_1}^{(1)}N_{\beta_2}^{(0)} \\
&+ 4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}(\beta_2\cdot\xi_X)^4N_{\beta_1}^{(1)}N_{\beta_2}^{(0)} \\
&+ 4\sum_{\beta_1+\beta_2=\beta}\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}\left(-\frac{1}{24}\right)(\xi_X\cdot\beta_1)^2(\beta_2\cdot\xi_X)^3(\beta_1\cdot\beta_2)N_{\beta_1}^{(0)}N_{\beta_2}^{(0)} \\
&+ 4\left(-\frac{1}{24}\right)(\xi_X\cdot\xi_X)(\beta\cdot\xi_X)^3N_\beta^{(0)}. \qquad (9)
\end{aligned}$$

The remaining cycles all have 2 genus zero components so the calculations are simpler. We will first consider $\delta_{0,3}$:

$$\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{0,3}} = \frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta\\ A,B}}\sum_{i,j,k,l} g^{ij}g^{kl}N_{\beta_1,X}^{(0)}(\gamma_i,\gamma_j,\gamma_k,\mu_\alpha|\alpha\in A) \\
\times N_{\beta_2,X}^{(0)}(\gamma_l,\mu_\alpha|\alpha\in B),
\end{aligned}$$

where the first sum is over sets A, B satisfying

$$A\sqcup B = \{1,\dots,\kappa_\beta+2\},\quad |A\cap\{1,2,3,4\}| = 1.$$

The factor of $\frac{1}{2}$ appears since the dual graph of a generic curve in $\delta_{0,3}$ has an automorphism of order 2. Neither $\beta_1 = 0$, nor $\beta_2 = 0$ has any non-trivial contribution so it is straight forward to see that

$$N_{\beta,X}^{\pi^*\delta_{0,3}} = \sum_{\beta_1+\beta_2=\beta} 2\binom{\kappa_\beta-2}{\kappa_{\beta_1}-1}(\beta_1\cdot\xi_X)(\beta_2\cdot\xi_X)^3(\beta_1\cdot\beta_2)(\beta_1\cdot\beta_1)N_{\beta_1}^{(0)}N_{\beta_2}^{(0)}. \qquad (10)$$

The calculation for $\delta_{0,4}$ is a bit more subtle:

$$\begin{aligned}
N_{\beta,X}^{\pi^*\delta_{0,4}} = \frac{1}{2}\sum_{\substack{\beta_1+\beta_2=\beta\\ A,B}}\sum_{i,j,k,l} g^{ij}g^{kl}N_{\beta_1,X}^{(0)}(\gamma_i,\gamma_j,\gamma_k,\mu_\alpha|\alpha\in A) \\
\times N_{\beta_2,X}^{(0)}(\gamma_l,\mu_\alpha|\alpha\in B),
\end{aligned}$$

where the first sum is over sets A, B satisfying

$$A\sqcup B = \{1,\dots,\kappa_\beta+2\},\quad A\cap\{1,2,3,4\} = \emptyset.$$

Contribution from $\beta_2 = 0$ is 0. When $\beta_1 = 0$, we must have $A = \emptyset$ which leads to

$$\begin{aligned} N^{\pi^*\delta_{0,4}}_{\beta,X} =& \frac{1}{2} \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\beta_2 \cdot \xi_X)^4 (\beta_1 \cdot \beta_2)(\beta_1 \cdot \beta_1) N^{(0)}_{\beta_1} N^{(0)}_{\beta_2} \\ &+ \frac{1}{2}(2 + b_2(X))(\beta \cdot \xi_X)^4 N^{(0)}_{\beta}. \end{aligned} \tag{11}$$

Finally, let us consider the cycle $\delta_{0,0}$:

$$\begin{aligned} N^{\pi^*\delta_{0,0}}_{\beta,X} = \frac{1}{2} \sum_{\substack{\beta_1+\beta_2=\beta \\ A,B}} \sum_{i,j,k,l} & g^{ij} g^{kl} N^{(0)}_{\beta_1,X}(\gamma_i, \gamma_k, \mu_\alpha | \alpha \in A) \\ & \times N^{(0)}_{\beta_2,X}(\gamma_j, \gamma_l, \mu_\alpha | \alpha \in B), \end{aligned}$$

where the first sum is over sets A, B satisfying

$$A \sqcup B = \{1, \ldots, \kappa_\beta + 2\}, \quad |A \cap \{1, 2, 3, 4\}| = 2.$$

By an analogous calculation as the previous situations we have

$$N^{\pi^*\delta_{0,0}}_{\beta,X} = \frac{3}{2} \sum_{\beta_1+\beta_2=\beta} \binom{\kappa_\beta - 2}{\kappa_{\beta_1} - 1} (\beta_1 \cdot \xi_X)^2 (\beta_2 \cdot \xi_X)^2 (\beta_1 \cdot \beta_2)^2 N^{(0)}_{\beta_1} N^{(0)}_{\beta_2}. \tag{12}$$

Now collecting all these terms and using relation (5) we obtain the desired formula (2).

8. Low degree checks

We will now describe some concrete low degree checks that we have performed. Let X_k be a del-Pezzo surface obtained by blowing up $\mathbb{P}^2$ at $k \leq 8$ points. We claim that

$$N^{(1)}_{dL+\sigma_1 E_1+\ldots+\sigma_r E_r, X_k} = N^{(1)}_{dL+\sigma_1 E_1+\ldots+\sigma_{r-1} E_{r-1}, X_{k-1}},$$

if σ_r is -1 or 0. Let us explain why this is so. Consider X_1 which is $\mathbb{P}^2$ blown up at the point p. Let us consider the number $N^{(1)}_{dL-E_1,X_1}$; this is the number of genus one curves in X_1 representing the class $dL - E_1$ and passing through $3d-1$ generic points. Let $\mathcal{C}$ be one of the curves counted by the above number. The curve $\mathcal{C}$ intersects the exceptional divisor exactly at one point. Furthermore, since the $3d - 1$ points are generic, they can be chosen not to lie in the exceptional divisor; let us call the points $p_1, p_2, \ldots, p_{3d-1}$. Hence, when we consider the blow down from X_1 to $\mathbb{P}^2$, the curve $\mathcal{C}$ becomes a curve in $\mathbb{P}^2$ passing through $p_1, p_2, \ldots, p_{3d-1}$ and the blow up point p. We thus get a genus one, degree d curve in $\mathbb{P}^2$ passing through $3d$ points. There is a one to one correspondence between curves representing the class $dL - E_1$ in X_1 passing through $3d - 1$ points and degree d curves in $\mathbb{P}^2$ passing through $3d$ points. Hence $N^{(1)}_{dL-E_1,X_1} = N^{(1)}_{dL,\mathbb{P}^2}$. A similar argument holds when there are more than one blowup points. The same argument also shows that $N^{(1)}_{dL+0E_1,X_1} = N^{(1)}_{dL,\mathbb{P}^2}$; the same reasoning holds by taking a curve in the blowup and then considering its image under the blow down. The blow down gives a one to one correspondence between the two sets and hence, the corresponding numbers are the same.

We have verified this assertion in many cases. For instance we have verified that

$$N^{(1)}_{5L-E_1-E_2,X_2} = N^{(1)}_{5L-E_1,X_1}, \;=\; N^{(1)}_{5L+0E_1,X_1} \;=\; N^{(1)}_{5L,\mathbb{P}^2}\,.$$

The reader is invited to use our program and verify these assertions. Hence without ambiguity we write $N^{(1)}_{dL+\sigma_1 E_1+\ldots+\sigma_r E_r}$ for $N^{(1)}_{dL+\sigma_1 E_1+\ldots+\sigma_r E_r,X_r}$.

Next, we note that in [8], Dubrovin has computed the genus one Gromov-Witten Invariants of $\mathbb{P}^1\times\mathbb{P}^1$; our numbers agree with the numbers he has listed in his paper (Page 463).

Finally, in [23], Ravi Vakil has explicitly computed some $N^{(g)}_{\beta}$ for del-Pezzo surfaces (Page 78). Our numbers agree with the following numbers he has listed:

$$N^{(1)}_{5L-2E_1} = 13775,\; N^{(1)}_{5L-2E_1-2E_2-2E_3} = 225,\; N^{(1)}_{5L-2E_1-2E_2-2E_3-2E_4} = 20,$$
$$N^{(1)}_{5L-3E_1} = 240,\; N^{(1)}_{5L-3E_1-2E_2} = 20 \quad\text{and}\quad N^{(1)}_{5L-3E_1-2E_2-2E_3} = 1.$$

However, there is one number he has listed which does not agree with what we obtain: our formula predicts that

$$N^{(1)}_{5L-2E_1-2E_2} = 1920.$$

However, in Ravi Vakil's paper ([23], Page 78), it is given that

$$N^{(1)}_{5L-2E_1-2E_2} = 1887.$$

The value of 1920 is in agreement with the value communicated to us by Ritiwik Mukherjee, which we now briefly explain. Let us first consider the following question: what is $f(d)$, the number of degree d curves (not necessarily irreducible) in $\mathbb{P}^2$, passing through $\frac{d(d+3)}{2}-9$ generic points, having 3 nodes (unordered) and having two more nodes at two fixed points? By modifying the result of his thesis [19], Ritwik Mukherjee has obtained the value of $f(d)$ to be

$$f(d) = -1026 - \frac{4207}{2}d + 248d^2 + \frac{927}{2}d^3 - \frac{117}{2}d^4 - 27d^5 + \frac{9}{2}d^6.$$

We first note that $f(6) = 19581$. This is precisely equal to the number $R^{6,5}_{2^2}$ obtained by Ravi Vakil in [23] (Page 79); it is the number of degree 6 maps (not necessarily reducible) from a genus 5 surface into X_2, representing the class $6L-2E_1-2E_2$.

Next, we note that $f(5) = 1969$. We will now determine the reducible configurations counted by $f(5)$; recall that $f(5)$ is the number of quintics in $\mathbb{P}^2$ passing through 11 generic points, having 3 unordered nodes and having two more nodes at two fixed points. The following reducible configurations can take place: we can place a line through the two fixed point and place a nodal quartic through the remaining 11 points (and also through the two fixed points). There are 27 nodal quartics through 13 generic points (it is a well known fact that the number of degree d curves in $\mathbb{P}^2$ through $\frac{d(d+3)}{2}-1$ points, having one node is $3(d-1)^2$; there are several references for this fact such as [2]). Next, we could choose 10 out of the 11 points and place a quartic through one of the fixed points with a node at the other fixed point and place a line through the first fixed point and the remaining 11^{th} point. The number of quartics through 11 points having a node at a fixed given point is 1. Hence, there are a total of 11 such configurations (22 by interchanging the role of the fixed points). Hence, the total number of reducible configurations counted by $f(5)$ is $27+11+11 = 49$. Hence, the number of irreducible quintics through 11 generic points, having 3 unordered nodes and having two more nodes at two fixed points is

$$1969 - (27 + 11 + 11) = 1920.$$

This number is precisely equal to $N^1_{5L-2E_1-2E_2}$. Hence, we believe that the number obtained by our formula is correct as it satisfies this nontrivial geometric consistency check. We believe that the number 1887 written in [23] is likely to be a minor typo made by the author (since our numbers are consistent with the other numbers the author has written and our formula staisfies several other non trivial low degree checks).

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