# Lefschetz decomposition and the $c d$-index of fans 

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#### Abstract

The goal of this article is to give a Lefschetz type decomposition for the $c d$-index of a complete fan.

To a complete simplicial fan one can associate a toric variety $X$, the even Betti numbers $h_{i}$ of $X$ and the numbers $g_{i}=h_{i}-h_{i-1}$. If the fan is projective, then non-negativity of $g_{i}$ follows from the Lefschetz decomposition of the cohomology.

In the case of a nonsimplicial complete fan, one can analogously compute the flag h-numbers $h_{S}$ and, by a change of variable formula, the cd-index. We give an analogue of the Lefschetz operation for the cd-index. This gives another proof of the non-negativity of the cd-index for complete fans.


## 1. Introduction

Let $\Delta$ be a complete simplicial $n$-dimensional fan. Let $f_{i}$ be the number of $i$-dimensional cones in $\Delta$ and let $h_{k}$ be defined by the formula

$$
\sum_{i} f_{n-i}(t-1)^{i}=\sum_{k} h_{n-k} t^{k}
$$

The numbers $h_{k}$ for $k=0, \ldots, n$ are the even Betti numbers $h_{k}=\operatorname{dim} H^{2 k}\left(X_{\Delta}, \mathbb{C}\right)$ of a toric variety $X_{\Delta}$ if the fan $\Delta$ is rational. If $\Delta$ is also projective, then there exists a Lefschetz operation:

$$
L: H^{2 k}\left(X_{\Delta}, \mathbb{C}\right) \rightarrow H^{2 k+2}\left(X_{\Delta}, \mathbb{C}\right), \quad L^{k}: H^{n-k}\left(X_{\Delta}, \mathbb{C}\right) \stackrel{\simeq}{\rightarrow} H^{n+k}\left(X_{\Delta}, \mathbb{C}\right)
$$

giving rise to the Lefschetz decomposition of the cohomology. The existence of a Lefschetz operation implies that the numbers $g_{k}=h_{k}-h_{k-1}$ are non-negative for $0 \leq k \leq n / 2$.

For a complete but not necessarily simplicial fan one can consider the barycentric subdivision $B \Delta$ of $\Delta$, construct cohomology spaces $H^{S}(B \Delta)$ of dimension $h_{S}$ for $S \in \mathbb{N}^{n}$, and from the numbers $h_{S}$ compute the $c d$-index $\Psi_{\Delta}(c, d)$ of $\Delta$ (see below). Our goal is to find linear maps on $H^{S}(\Delta)$, the analogs of the Lefschetz operation, that guarantee non-negativity of the $c d$-index. Unlike the simplicial case, it is not clear how such maps should be defined. We will give in Definition 1.1 a rather weak notion of a Lefschetz operation which, nevertheless, is sufficient to imply non-negativity of the $c d$-index. We also

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## K. Karu

conjecture a stronger version in which the maps are defined by conewise linear functions on the fan, just as in the simplicial case. The rest of the introduction is spent on constructing the $c d$-index and motivating the definition of a Lefschetz operation. The precise statements and proofs are given in Sections 2 and 3 below.

Let us start by recalling the construction of the cohomology $H^{2 *}\left(X_{\Delta}, \mathbb{C}\right)$ (which we will denote simply $H^{*}(\Delta)$ ) in the case of a simplicial fan $\Delta$ (the references $[1,3]$ contain more details and generalization to the intersection cohomology). Let $\mathcal{A}(\Delta)$ be the vector space of complex-valued conewise polynomial functions on the fan $\Delta$. In other words, an element of $\mathcal{A}(\Delta)$ is a continuous function on the support of the fan $\Delta$ that restricts to a polynomial on each cone $\sigma \in \Delta$. We can multiply a conewise polynomial function with a globally polynomial function. In fact, this makes the space $\mathcal{A}(\Delta)$ into a free module under the action of the ring $A=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ of global polynomial functions, graded by degree. The graded vector space $\mathcal{A}(\Delta) / m \mathcal{A}(\Delta)$, where $m \subset A$ is the maximal homogeneous ideal, is the cohomology space $H^{*}(\Delta)$ with Poincaré polynomial

$$
P_{\Delta}(t)=\sum_{k} h_{k} t^{k}, \quad h_{k}=\operatorname{dim} H^{k}(\Delta)
$$

The fan $\Delta$ is projective if and only if there exists a strictly convex conewise linear function $L \in \mathcal{A}(\Delta)$. Multiplication with $L$ induces a Lefschetz operation in cohomology.

In case when the fan $\Delta$ is complete, but not necessarily simplicial, we proceed as follows (see [6] or Section 2 below for more details). Let $B \Delta$ be a first barycentric subdivision of $\Delta$ and consider the space $\mathcal{A}(B \Delta)$ of conewise polynomial functions on this subdivision, which again is a free $A$-module. It is possible to modify the $A$-module structure, so that $\mathcal{A}(B \Delta)$ has a grading by $\mathbb{N}^{n}$ and the module structure is compatible with this grading (where $A=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ has the standard grading by $\mathbb{N}^{n}$ ). To do this, note that the generating rays of a maximal cone $\sigma \in B \Delta$ are labeled by $1, \ldots, n$ : the $i$ 'th ray is the barycenter of a cone of dimension $i$. Let's map the cone $\sigma$ linearly onto the positive orthant of $\mathbb{R}^{n}$ so that the $i$ 'th ray goes to the $i$ 'th coordinate axis. These maps can be chosen compatibly for all maximal cones, defining a piecewise linear map from $B \Delta$ to $\mathbb{R}^{n}$ that "folds" the fan onto the positive orthant. The $A$-module structure on $\mathcal{A}(B \Delta)$ is defined via pullback by this map. Since this map identifies polynomial functions on a maximal cone with polynomials on the positive orthant, i.e., with $A$, we get a grading by $\mathbb{N}^{n}$ on polynomials on each cone $\sigma$, hence on $\mathcal{A}(B \Delta)$.

With the module structure and grading on $\mathcal{A}(B \Delta)$ defined above, the quotient $H^{*}(B \Delta):=\mathcal{A}(B \Delta) / m \mathcal{A}(B \Delta)$ inherits a similar grading. Consider the corresponding Poincaré polynomial

$$
P_{B \Delta}\left(t_{1}, \ldots, t_{n}\right)=\sum_{S \in \mathbb{N}^{n}} h_{S} t_{1}^{S_{1}} \cdots t_{n}^{S_{n}}, \quad h_{S}=\operatorname{dim} H^{S}(B \Delta)
$$

The cohomology $H^{*}(B \Delta)$ satisfies Poincaré duality $h_{S}=h_{(1, \ldots, 1)-S}$. In fact, this duality is defined by a non-degenerate Poincaré pairing (see Section 2.4). This in particular
implies that the nonzero coefficients in the Poincaré polynomial can be indexed by subsets $S \subset\{1, \ldots, n\}$. The numbers $h_{S}$ are called the flag $h$-numbers of the fan $\Delta$.

Let $\mathbb{Q}\langle c, d\rangle$ be the polynomial ring in non-commuting variables $c$ and $d$ of degree 1 and 2 , respectively. There is an embedding of vector spaces

$$
\phi: \mathbb{Q}\langle c, d\rangle \hookrightarrow \mathbb{Q}\left[t_{1}, t_{2}, \ldots\right]
$$

defined as follows. $\phi$ maps constants to constants and if $f(c, d) c+g(c, d) d$ is a homogeneous $c d$-polynomial of degree $m>0$, define inductively

$$
\phi(f(c, d) c+g(c, d) d)=\phi(f(c, d))\left(t_{m}+1\right)+\phi(g(c, d))\left(t_{m-1}+t_{m}\right)
$$

For example, there are $3 c d$-monomials of degree 3 :

$$
\begin{aligned}
c^{3} & =\left(t_{1}+1\right)\left(t_{2}+1\right)\left(t_{3}+1\right), \\
c d & =\left(t_{1}+1\right)\left(t_{2}+t_{3}\right), \\
d c & =\left(t_{1}+t_{2}\right)\left(t_{3}+1\right)
\end{aligned}
$$

It is shown in [2] that the Poincaré polynomial $P_{B \Delta}\left(t_{1}, \ldots, t_{n}\right)$ of a complete fan $\Delta$ can be expressed as a homogeneous $c d$-polynomial of degree $n$, called the $c d$-index $\Psi_{\Delta}(c, d)$ of $\Delta$. The coefficients of the polynomial are integers [2] and non-negative $[7,6]$.

One approach to proving non-negativity of the coefficients of the $c d$-index is to decompose the cohomology $H^{*}(B \Delta)$ into summands corresponding to different $c d$-monomials, so that the coefficients of $\Psi_{\Delta}(c, d)$ are the dimensions of the corresponding components. If we know the non-negativity of the $c d$-index, then the existence of such a decomposition follows trivially. Figure 1 shows the dimensions of the pieces corresponding to different $c d$-monomials in the 3 -dimensional case. The bold dots indicate the $t_{i}$-monomial being a summand of the $c d$-monomial.

In analogy with the Lefschetz decomposition in the singly-graded case, we expect the decomposition to be defined by linear maps. More precisely, we look for endomorphisms $L_{i}: H^{*}(B \Delta) \rightarrow H^{*}(B \Delta)$ of degree $e_{i}=(0, \ldots, 1, \ldots, 0)$. If a $c d$-monomial $m$ can be written as $m=\ldots\left(t_{i}+1\right) \ldots$, then $L_{i}$ should map in the corresponding piece $H_{m}^{*}$ of the cohomology decomposition:

$$
L_{i}: H_{m}^{(*, \ldots, *, 0, *, \ldots, *)} \xrightarrow{\simeq} H_{m}^{(*, \ldots, *, 1, *, \ldots, *)} .
$$

For example, $L_{1}$ should define an isomorphism from the back face of the cube to the front face in Figure 1 for the monomials $c^{3}$ and $c d$; the component corresponding to the monomial $d c$ should lie in the kernel of $L_{1}$.

### 1.1. The main construction

The definition of a Lefschetz operation is given inductively using a construction that we call "the main construction". It essentially describes the action of $L_{1}$ on the $A$-module $\mathcal{A}(B \Delta)$ as described in the previous paragraph.


Figure 1. $c d$-monomials in terms of $t_{i}$-monomials.

Let $A_{l, m}$ be the polynomial ring $\mathbb{C}\left[x_{l}, \ldots, x_{m}\right]$, graded by $\mathbb{N}^{m-l}$, with $x_{i}$ having degree $e_{i}$. Let the dualizing module of $A_{l, m}$ be $\omega_{l, m}$, the principal ideal in $A_{l, m}$ generated by $x_{l} \cdots x_{m}$.

Let $M$ be a finitely generated free graded $A_{l, m}$-module. A Poincaré pairing on $M$ is an $A_{l, m}$-bilinear symmetric map

$$
<\cdot, \cdot>: M \times M \rightarrow \omega_{l, m},
$$

inducing a non-degenerate pairing on $\bar{M}=M /\left(x_{l}, \ldots, x_{m}\right) M$. We always assume that $M$ is graded in non-negative degrees. Then the existence of a Poincaré pairing implies that $\bar{M}$ is graded by subsets of $\{l, \ldots, m\}$.

Let $M$ be a free $A_{l, m}$-module with a Poincaré pairing and let $L: M \rightarrow M$ be an endomorphism of degree $e_{l}$ which is self-adjoint with respect to the pairing:

$$
<L m_{1}, m_{2}>=<m_{1}, L m_{2}>
$$

We can write

$$
M /\left(x_{l}\right) M=M^{0}+M^{1}
$$

where $M^{i}$ consists of elements of degree $(i, *, \ldots, *)$. Then $L$ induces a map $M^{0} \rightarrow M^{1}$.
Assume that the map $L: M^{0} \rightarrow M^{1}$ is injective and the quotient is annihilated by $x_{l+1}$ :

$$
\begin{equation*}
0 \rightarrow M^{0} \xrightarrow{L} M^{1} \rightarrow Q \rightarrow 0, \quad x_{l+1} Q=0 \tag{1}
\end{equation*}
$$

It is elementary (see Lemma 3.1 in Section 3 ) that $Q$ is a free $A_{l+2, m}=\mathbb{C}\left[x_{l+2}, \ldots, x_{m}\right]$ module and we get a long-exact Tor sequence:

$$
\begin{equation*}
0 \rightarrow Q\left[e_{l}-e_{l+1}\right] \rightarrow M^{0} /\left(x_{l+1}\right) M^{0} \xrightarrow{L} M^{1} /\left(x_{l+1}\right) M^{1} \rightarrow Q \rightarrow 0 \tag{2}
\end{equation*}
$$

Let $C$ be the cokernel of the embedding $Q\left[e_{l}-e_{l+1}\right] \rightarrow M^{0} /\left(x_{l+1}\right) M^{0}$ :

$$
\begin{equation*}
0 \rightarrow Q\left[e_{l}-e_{l+1}\right] \rightarrow M^{0} /\left(x_{l+1}\right) M^{0} \rightarrow C \rightarrow 0 \tag{3}
\end{equation*}
$$

## Lefschetz decomposition and the $c d$-index of fans

Then $C$ is also a free $A_{l+2, m}$-module (see Section 3 below). We will show that $Q$ and $C$ both inherit Poincaré pairings from $M$. The construction of $Q$ and $C$ from $M$ and $L$ is what we call the main construction.

Let us explain how the construction of $C$ and $Q$ corresponds to the $c d$-variables $c$ and $d$, respectively. First, since $Q$ lies in degrees $(1,0, *, \ldots, *)$, let us replace it with the shifted module $Q^{\prime}=Q\left[e_{l}\right]$, which is a free $A_{l+2, m}$-module in degrees $(0,0, *, \ldots, *)$. Also, since $c$ and $d$ have degrees 1 and 2 , respectively, we replace $C$ with the free $A_{l+1, m^{-}}$ module $C^{\prime}=C \otimes_{A_{l+2, m}} A_{l+1, m}$. Now let $P_{M}\left(t_{1}, \ldots, t_{n}\right)$ (respectively $P_{Q^{\prime}}\left(t_{l+2}, \ldots, t_{m}\right)$, $P_{C^{\prime}}\left(t_{l+1}, \ldots, t_{m}\right)$ ) be the Hilbert polynomial of $\bar{M}$ (respectively $\overline{Q^{\prime}}, \overline{C^{\prime}}$ ). From the exact sequences (2) and (3), we get

$$
\begin{equation*}
P_{M}=\left(1+t_{l}\right) P_{C^{\prime}}+\left(t_{l}+t_{l+1}\right) P_{Q^{\prime}}=c P_{C^{\prime}}+d P_{Q^{\prime}} \tag{4}
\end{equation*}
$$

Thus, if $P_{C^{\prime}}$ and $P_{Q^{\prime}}$ are both $c d$-polynomials with non-negative coefficients, then the same is true for $P_{M}$. (Here we changed slightly the $\operatorname{map} \phi$ on $\mathbb{Q}\langle c, d\rangle$ by letting its image be in $\mathbb{Q}\left[t_{l}, \ldots\right]$ for $P_{M}, \mathbb{Q}\left[t_{l+1}, \ldots\right]$ for $P_{C^{\prime}}$, and similarly for $P_{Q^{\prime}}$.) From this formula, we see that the main construction of $C^{\prime}$ and $Q^{\prime}$ from $M$ corresponds to contracting the polynomial $P_{M}$ from the left with $c$ and $d$, respectively.
Definition 1.1. Let $M$ be a finitely generated free $A_{l, m}$-module with a Poincaré pairing. We say that $M$ has a Lefschetz operation if there exists an endomorphism $L: M \rightarrow M$ of degree $e_{l}$, satisfying the assumptions of the main construction, such that the modules $C^{\prime}$ and $Q^{\prime}$ also have Lefschetz operations. More precisely:

- $L$ is self-adjoint with respect to the pairing on $M$.
- $L: M^{0} \rightarrow M^{1}$ is injective with cokernel annihilated by $x_{l+1}$.
- Inductively, the $A_{l+1, m}$-module $C^{\prime}=C \otimes_{A_{l+2, m}} A_{l+1, m}$ and the $A_{l+2, m}$-module $Q^{\prime}=Q\left[e_{l}\right]$ have Lefschetz operations.
To start the induction, if $l=m+1$ and $M$ is a finite dimensional vector space, then it trivially has a Lefschetz operation.

From the computation (4) above, it is clear that if $M$ has a Lefschetz operation, then the Hilbert function of $\bar{M}$ can be written as a homogeneous $c d$-polynomial of degree $m-l$ with non-negative integer coefficients.

The main result of this article is:
Theorem 1.2. Let $\Delta$ be a complete fan of dimension $n$. Then the $A_{1, n}$-module $\mathcal{A}(B \Delta)$ has a Lefschetz operation. In particular, the cd-index of $\Delta$ has non-negative integer coefficients.

Recall that $\mathcal{A}(B \Delta)$ is a ring. If $L_{i} \in \mathcal{A}(B \Delta)$ is an element of degree $e_{i}$, then multiplication with $L_{i}$ defines an endomorphism of $\mathcal{A}(B \Delta)$ of degree $e_{i}$, self-adjoint with respect to the natural Poincaré pairing. Thus, $L_{1}$ is a good candidate for the Lefschetz operation on $\mathcal{A}(B \Delta)$, and inductively, $L_{i}$ for $i>1$ could be used to define the endomorphisms of $Q$ and $C$.

Conjecture 1.3. Let $L_{i} \in \mathcal{A}(B \Delta)$ be a general element of degree $e_{i}$ for $i=1, \ldots, n$. Then $L_{i}$ define a Lefschetz operation on $\mathcal{A}(B \Delta)$.

We remark that a Lefschetz operation on $M$ does not define a canonical decomposition of $\bar{M}$ into components corresponding to the $c d$-monomials. To decompose $\bar{M}$, we need to choose a splitting of the sequence (3), so that

$$
\bar{M} \simeq \bar{C}^{\prime} \oplus \bar{C}^{\prime}\left[-e_{l}\right] \oplus \bar{Q}^{\prime}\left[-e_{l}\right] \oplus \bar{Q}^{\prime}\left[-e_{l+1}\right]
$$

corresponding to the formula (4). Inductive decomposition of $\bar{C}^{\prime}$ and $\bar{Q}^{\prime}$ then give a complete decomposition of $\bar{M}$.

To prove Theorem 1.2, we express $\mathcal{A}(B \Delta)$ as the space of global sections of a sheaf $\mathcal{L}$ on $\Delta$. The main construction can be sheafified, i.e., performed on the stalks of the sheaf $\mathcal{L}$ simultaneously. We show that a Lefschetz operation on the space of global sections comes from a sheaf homomorphism.

We also consider Conjecture 1.3 in the context of sheaves and reduce it to a KleimanBertini type problem of torus actions on a vector space. Let an algebraic torus $T$ act on a finite dimensional vector space $V$ with possibly infinitely many orbits. When does the general translate of a subspace $K \subset V$ intersect another subspace transversely? Conjecture 3.13 claims sufficient conditions for this, implying Conjecture 1.3.

Theorem 1.2 gives another proof of non-negativity of the $c d$-index for a complete fan. In [6] non-negativity was proved more generally for Gorenstein* posets. The current proof does not extend to that more general situation. The two proofs are based on the same idea. However, the proof we give here is simpler because we work with modules only, avoiding derived categories.

## 2. Sheaves on fans

All our vector spaces are over the field of complex numbers $\mathbb{C}$. Let $A_{l, m}=$ $\mathbb{C}\left[x_{l}, x_{l+1}, \ldots, x_{m}\right]$, graded so that $x_{i}$ has degree $e_{i}$. For a graded $A_{l, m}$-module $M$ we denote the shift in grading by $M[\cdot]$. We also write $\bar{M}=M /\left(x_{l}, \ldots, x_{m}\right) M$.

For a graded set, the superscript refers to degree. If $\Delta$ is a fan, then $\Delta \geq m$ consists of all cones of dimension at least $m$. Similarly, $\Delta^{[l, m]}$ is the subset of cones of dimension $d \in[l, m]$.

### 2.1. Fan spaces

Let us recall the notion of sheaves on fans. The main reference for the general theory is $[1,3]$ and for the specific sheaves used here [6].

We fix a complete $n$-dimensional fan $\Delta$ (see [5] for terminology). Consider $\Delta$ as a finite partially ordered set of cones, graded in degrees $0, \ldots, n$. It is sometimes convenient to add a maximal element $\hat{1}$ of degree $n+1$ to $\Delta$.

The fan $\Delta$ is given the topology in which open sets are the (closed) subfans of $\Delta$. Then a sheaf $F$ of vector spaces on $\Delta$ consists of the data:

- A vector space $F_{\sigma}$ for each $\sigma \in \Delta$.
- Linear maps $r e s_{\tau}^{\sigma}: F_{\sigma} \rightarrow F_{\tau}$ for $\sigma>\tau$, satisfying the compatibility condition $r e s_{\rho}^{\tau} \circ r e s_{\tau}^{\sigma}=r e s_{\rho}^{\sigma}$ for $\sigma>\tau>\rho$.

On sheaves we can perform the usual sheaf operations. For example, a global section $f \in \Gamma(F, \Delta)$ consists of the data $f_{\sigma} \in F_{\sigma}$ for each $\sigma \in \Delta$, such that $r e s_{\tau}^{\sigma} f_{\sigma}=f_{\tau}$. Equivalently, we only need to give $f_{\sigma} \in F_{\sigma}$ for maximal cones $\sigma$, such that their restrictions to smaller dimensional cones agree.

Define a sheaf of rings $\mathcal{A}$ on $\Delta$ as follows:

- $\mathcal{A}_{\sigma}=A_{1, d}=\mathbb{C}\left[x_{1}, \ldots, x_{d}\right]$ if $\operatorname{dim} \sigma=d$.
- $\operatorname{res} \tau_{\tau}^{\sigma}: \mathbb{C}\left[x_{1}, \ldots, x_{d}\right] \rightarrow \mathbb{C}\left[x_{1}, \ldots, x_{l}\right]$ is the standard projection $x_{i} \mapsto x_{i}$ for $1 \leq i \leq l$ and $x_{i} \mapsto 0$ for $i>l$.

Given the sheaf of rings $\mathcal{A}$ on $\Delta$, we consider sheaves of $\mathcal{A}$-modules $\mathcal{F}$. This means that the stalks $\mathcal{F}_{\sigma}$ are $\mathcal{A}_{\sigma}$-modules and the restriction maps are module homomorphisms. Note that the sheaf $\mathcal{A}$ is graded by $\mathbb{N}^{n}$. We assume that all sheaves of $\mathcal{A}$-modules are similarly graded.

There exists an indecomposable sheaf $\mathcal{L}$ of $\mathcal{A}$-modules satisfying the following conditions:

- Locally free: $\mathcal{L}_{\sigma}$ is a graded free $\mathcal{A}_{\sigma}$-module.
- Minimally flabby: for $\sigma=0, \mathcal{L}_{0}=\mathbb{C}$ in degree 0 ; for $\sigma>0$, the restriction maps induce an isomorphism

$$
\overline{\mathcal{L}}_{\sigma} \rightarrow \overline{\Gamma(\mathcal{L}, \partial \sigma)}
$$

where $\partial \sigma$ is the boundary fan of $\sigma$.
These two conditions define $\mathcal{L}$ up to an isomorphism. In fact, $\Gamma(\mathcal{L}, \partial \sigma)$ is a free $A_{1, d-1^{-}}$ module if $\operatorname{dim} \sigma=d$, and we can inductively define

$$
\mathcal{L}_{\sigma}=\Gamma(\mathcal{L}, \partial \sigma) \otimes_{A_{1, d-1}} A_{1, d}
$$

### 2.2. Barycentric subdivisions

Let $B \Delta$ be a barycentric subdivision of $\Delta$. As a poset, it consists of chains $x=(0<$ $\left.\sigma_{1}<\ldots<\sigma_{m}\right)$ in $\Delta$. Define a sheaf of rings $\mathcal{B}$ on $B \Delta$ as follows:

- $\mathcal{B}_{x}=\mathbb{C}\left[x_{i}\right]_{i \in S}$, where $x=\left(0<\sigma_{1}<\ldots<\sigma_{m}\right), S=\left\{\operatorname{dim} \sigma_{1}, \ldots, \operatorname{dim} \sigma_{m}\right\}$.
- res $y_{y}^{x}$ is the standard projection.

One can construct as above a sheaf $\mathcal{L}$ with respect to $\mathcal{B}$, but this sheaf is isomorphic to $\mathcal{B}$.

Lemma 2.1 ([6]). We have

$$
\pi_{*} \mathcal{B} \simeq \mathcal{L}
$$

where $\pi: B \Delta \rightarrow \Delta$ is the subdivision map sending $x=\left(0<\sigma_{1}<\ldots<\sigma_{m}\right)$ to $\sigma_{m}$.
It is often more convenient to work with the sheaf $\mathcal{B}$ because it is a sheaf of rings. The space of global sections $\Gamma(\mathcal{B}, B \Delta)$ (which is isomorphic to $\Gamma(\mathcal{L}, \Delta)$ by the previous lemma) is what we called $\mathcal{A}(B \Delta)$ in the introduction. Since $\mathcal{B}$ and $\mathcal{L}$ are sheaves of $A_{1, n}$-modules, so are the spaces of global sections.

### 2.3. The cellular complex

Let us fix an orientation for each cone $\sigma \in \Delta$ and for $\sigma>\tau, \operatorname{dim} \sigma=\operatorname{dim} \tau+1$, let

$$
o r_{\tau}^{\sigma}= \pm 1
$$

depending on whether the orientations of $\sigma$ and $\tau$ agree or not.
The cellular complex of a sheaf $F$ on $\Delta$ is

$$
C_{n}^{\bullet}(F, \Delta)=0 \rightarrow C^{0} \rightarrow C^{1} \rightarrow \ldots \rightarrow C^{n} \rightarrow 0
$$

where

$$
C^{i}=\bigoplus_{\operatorname{dim} \sigma=n-i} F_{\sigma}
$$

and the differentials are defined as sums of $o r_{\tau}^{\sigma} r e s_{\tau}^{\sigma}: F_{\sigma} \rightarrow F_{\tau}$.
For a complete fan $\Delta$, the cellular complex $C_{n}^{\bullet}(F, \Delta)$ computes the cohomology of $F$. Applying this to the flabby sheaf $\mathcal{L}$, we get

$$
H^{i}\left(C_{n}^{\bullet}(\mathcal{L}, \Delta)\right)= \begin{cases}\Gamma(\mathcal{L}, \Delta) & \text { if } i=0 \\ 0 & \text { otherwise }\end{cases}
$$

Moreover, $\Gamma(\mathcal{L}, \Delta)$ is a graded free $A_{1, n}$-module.
If $\sigma \in \Delta, \operatorname{dim} \sigma=d$, then $\partial \sigma$ is combinatorially equivalent to a complete fan of dimension $d-1$, hence we may use $C_{d-1}^{\bullet}(\mathcal{L}, \partial \sigma)$ to compute $\Gamma(\mathcal{L}, \partial \sigma)$. This gives an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma} \rightarrow \bigoplus_{\tau<\sigma, \operatorname{dim} \tau=d-1} \mathcal{L}_{\tau} \rightarrow \bigoplus_{\rho<\sigma, \operatorname{dim} \rho=d-2} \mathcal{L}_{\rho} \rightarrow \ldots \rightarrow \mathcal{L}_{0} \rightarrow 0 \tag{5}
\end{equation*}
$$

### 2.4. Poincaré pairing

Define the dualizing module $\omega_{1, n}=\left(x_{1} \cdots x_{n}\right) A_{1, n}$. I.e., $\omega_{1, n}$ is the principal ideal generated by $x_{1} \cdots x_{n}$. There exists an $A_{1, n}$-bilinear non-degenerate pairing

$$
\Gamma(\mathcal{L}, \Delta) \times \Gamma(\mathcal{L}, \Delta) \rightarrow \omega_{1, n}
$$

The pairing is best constructed using the isomorphism $\Gamma(\mathcal{L}, \Delta) \simeq \Gamma(\mathcal{B}, B \Delta)$. On $\Gamma(\mathcal{B}, B \Delta)$ the pairing is defined by multiplication ( $\mathcal{B}$ is a sheaf of rings), followed by an evaluation map into $\omega_{1, n}$.

One can give a simple description of the evaluation map as in [4], depending on the orientations or $r_{\tau}^{\sigma}$. For $x=\left(0<\sigma_{1}<\ldots<\sigma_{n}\right)$ a maximal element of $B \Delta$ of dimension $n$, define

$$
\varepsilon_{x}=o r_{\sigma_{n}}^{\hat{1}} o r_{\sigma_{n-1}}^{\sigma_{n}} \cdots o r_{0}^{\sigma_{1}}= \pm 1
$$

Now if $f \in \Gamma(\mathcal{B}, B \Delta)$, then it can be shown that

$$
\sum_{\operatorname{dim} x=n} \varepsilon_{x} f_{x}
$$

is an element of $A_{1, n}$ that is divisible by $x_{1} x_{2} \cdots x_{n}$, hence lies in $\omega_{1, n}$. This defines the $A_{1, n}$-linear evaluation map $\Gamma(\mathcal{B}, B \Delta) \rightarrow \omega_{1, n}$ and the Poincaré pairing on $\Gamma(\mathcal{B}, B \Delta)$.

If $\sigma \in \Delta$ is a $d$-dimensional cone, then $\partial \sigma$ is combinatorially equivalent to a complete fan of dimension $d-1$. By the same construction as above we get a pairing on $\Gamma(\mathcal{L}, \partial \sigma) \simeq$ $\mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma}$.

In summary, for each cone $\sigma \in \Delta$, $\operatorname{dim} \sigma=d$, we have a non-degenerate symmetric bilinear pairing

$$
<\cdot, \cdot>_{\sigma}: \mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma} \times \mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma} \rightarrow \omega_{1, d-1}
$$

These pairings are related as follows. For $f, g \in \mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma}$,

$$
<f, g>_{\sigma}=\sum_{\operatorname{dim} \tau=d-1} o r_{\tau}^{\sigma}<f_{\tau}, g_{\tau}>_{\tau}
$$

where $f_{\tau}$ and $g_{\tau}$ are the restrictions of $f$ and $g$ to $\tau$ and the pairing on the right hand side is the $A_{1, d-1}$-bilinear extension of the $A_{1, d-2}$-bilinear pairing $<\cdot, \cdot>_{\tau}$.

## 3. The main construction on sheaves

Let us return to the situation of Section 1.1 and prove the claims made there.
We have a finitely generated free $A_{l, m}$-module $M$ with Poincaré pairing

$$
<\cdot, \cdot>_{M}: M \times M \rightarrow \omega_{l, m}
$$

Write

$$
M / x_{l} M=M^{0} \oplus M^{1}
$$

where $M^{i}$ consists of elements of degree $(i, *, \ldots, *)$. Assume that $L: M^{0} \rightarrow M^{1}$ is a $A_{l+1, m}$-module homomorphism of degree $e_{l}$, self-adjoint with respect to the pairing, and such that $L$ is injective with quotient $Q$ annihilated by $x_{l+1}$ :

$$
0 \rightarrow M^{0} \xrightarrow{L} M^{1} \rightarrow Q \rightarrow 0, \quad x_{l+1} Q=0
$$

Lemma 3.1. $Q$ is a free $A_{l+2, m}$-module.

Proof. Since $M^{0}$ and $M^{1}$ are free $A_{l+1, m}$-modules, we get from the exact sequence above that

$$
\operatorname{Tor}_{i}^{A_{l+1, m}}(Q, \mathbb{C})=0, \quad i \geq 2
$$

Because $Q$ is a $A_{l+2, m}$-module, annihilated by $x_{l+1}$, this implies that

$$
\operatorname{Tor}_{1}^{A_{l+2, m}}(Q, \mathbb{C})=0
$$

hence $Q$ is free.
Now assuming that $Q$ is free, we get an exact sequence

$$
0 \rightarrow Q\left[e_{l}-e_{l+1}\right] \rightarrow M^{0} /\left(x_{l+1}\right) M^{0} \xrightarrow{L} M^{1} /\left(x_{l+1}\right) M^{1} \rightarrow Q \rightarrow 0
$$

where all terms are free $A_{l+2, m}$-modules. Define $C$ by the exact sequence

$$
0 \rightarrow Q\left[e_{l}-e_{l+1}\right] \rightarrow M^{0} /\left(x_{l+1}\right) M^{0} \rightarrow C \rightarrow 0
$$

Then $\operatorname{Tor}_{1}^{A_{l+2, m}}(C, \mathbb{C})=0$ and $C$ is also a free $A_{l+2, m}$-module.
Let us construct bilinear pairings on $C$ and $Q$. On $C$ the pairing is

$$
<x, y>_{C}=<x, L y>_{M}
$$

This is well-defined and gives an $A_{l+2, m}$-linear map of degree $e_{l}$

$$
C \otimes_{A_{l+2, m}} C \rightarrow \omega_{l, m} \otimes_{A_{l, m}} A_{l+2, m} .
$$

Dividing by $x_{l}$ we get a degree 0 map into $\omega_{l+1, m} \otimes_{A_{l+1, m}} A_{l+2, m}$. Finally, replacing $C$ by $C^{\prime}=C \otimes_{A_{l+2, m}} A_{l+1, m}$ and extending the pairing linearly, we have a $A_{l+1, m}$-bilinear map

$$
<\cdot, \cdot>_{C^{\prime}}: C^{\prime} \times C^{\prime} \rightarrow \omega_{l+1, m}
$$

To define the pairing on $Q$, let $\alpha$ be the composition

$$
\alpha: Q \stackrel{\simeq}{\leftrightharpoons} Q\left[e_{l}-e_{l+1}\right] \hookrightarrow M^{0} / x_{l+1} M^{0} .
$$

On the elements $[q] \in Q$ this map is given by

$$
\alpha([q])=L^{-1}\left(x_{l+1} q\right)
$$

Now define the pairing

$$
<x, y>_{Q}=<\alpha(x), y>_{M}
$$

One can check that this pairing is well-defined. Taking into account that $\alpha$ has degree $e_{l+1}-e_{l}$, we get a degree $0 A_{l+2, m}$-bilinear map on $Q^{\prime}=Q\left[e_{l}\right]$

$$
<\cdot, \cdot>_{Q^{\prime}}: Q^{\prime} \times Q^{\prime} \rightarrow \omega_{l+2, m}
$$

It is easy to see that the bilinear maps on $C^{\prime}$ and $Q^{\prime}$ are symmetric.
Lemma 3.2. The pairings $<\cdot, \cdot>_{Q^{\prime}}$ and $<\cdot, \cdot>_{C^{\prime}}$ are non-degenerate.

Proof. One checks the non-degeneracy of the pairing on $C$ using the definition and self-adjointness of $L$. Then it follows that the pairing between $Q$ and $Q\left[e_{l}-e_{l+1}\right]$ is non-degenerate.

We next want to sheafify the main construction. Recall that $\mathcal{L}$ is a sheaf on $\Delta$ with stalks $\mathcal{L}_{\sigma}$ free $\mathcal{A}_{\sigma}$-modules with Poincaré pairings. To perform the main construction simultaneously on all stalks of $\mathcal{L}$, the first step is to split

$$
\mathcal{L} / x_{1} \mathcal{L}=\mathcal{L}^{0} \oplus \mathcal{L}^{1}
$$

and then find a map of sheaves of degree $e_{1}$

$$
L: \mathcal{L}^{0} \rightarrow \mathcal{L}^{1}
$$

If one looks at the stalks, it becomes clear that $\mathcal{L}^{i}$ should be considered as sheaves on $\Delta^{\geq 2}$ (i.e., on the poset of cones of dimension at least 2), and the cokernel $Q$ of the map $L$ should be a sheaf on $\Delta^{\geq 3}$. Therefore we will consider sheaves on $\Delta^{\geq m}$ for $m \geq 1$.

### 3.1. Sheaves on $\Delta \geq m$

We let $\Delta^{\geq m}$ have the the topology induced from $\Delta$. To give a sheaf on $\Delta^{\geq m}$ is equivalent to giving a sheaf on $\Delta$ with all stalks zero on cones of dimension less than $m$.

Define the structure sheaf $\mathcal{A}$ on $\Delta^{\geq m}$ as follows. For $\sigma \in \Delta, \operatorname{dim} \sigma=d \geq m$, let

$$
\mathcal{A}_{\sigma}=A_{m, d}=\mathbb{C}\left[x_{m}, \ldots, x_{d}\right],
$$

with restriction maps $r e s_{\tau}^{\sigma}$ the standard projections.
Definition 3.3. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{A}$-modules on $\Delta \geq m$. We say that $\mathcal{F}$ is minimally flabby if all the restriction maps $\operatorname{res}_{\beta}^{\alpha}$ are surjective and for every $\sigma \in \Delta$, $\operatorname{dim} \sigma=d \geq m$, we have an exact sequence, the "augmented cellular complex" (compare with (5))

$$
\begin{equation*}
0 \rightarrow \mathcal{F}_{\sigma} / x_{d} \mathcal{F}_{\sigma} \rightarrow \bigoplus_{\tau<\sigma, \operatorname{dim} \tau=d-1} \mathcal{F}_{\tau} \rightarrow \ldots \rightarrow \bigoplus_{\rho<\sigma, \operatorname{dim} \rho=m} \mathcal{F}_{\rho} \rightarrow G_{\sigma} \rightarrow 0 \tag{6}
\end{equation*}
$$

where

- The augmentation $G_{\sigma}$ is a vector space (i.e., an $A_{1, n}$-module annihilated by $\left.x_{1}, \ldots, x_{n}\right)$.
- All differentials, except the maps to $G_{\sigma}$, are defined by or ${ }_{\beta}^{\alpha} r e s_{\beta}^{\alpha}$ as in the usual cellular complex.

Remark 3.4. (1) It should be noted that a minimally flabby sheaf is not flabby in the topology of $\Delta \geq m$.
(2) We do not need the surjectivity of the restriction maps res ${ }_{\beta}^{\alpha}$ for the proof of Theorem 1.2. These conditions are only necessary to state Conjectures 1.3 and 3.13. However, surjectivity of the restriction maps follows easily for all sheaves we consider.

Example 3.5. (1) Let $\mathcal{L}$ be the indecomposable sheaf on $\Delta$. Then $\left.\mathcal{L}\right|_{\Delta \geq 1}$ is a minimally flabby sheaf on $\Delta^{\geq 1}$. In this case we have $G_{\sigma}=\mathcal{L}_{0}=\mathbb{C}$ for all $\sigma$.
(2) In general, the vector spaces $G_{\sigma}$ depend on the cone $\sigma$. Let $\pi_{1}, \pi_{2} \in \Delta$ be two cones of dimension $m-1$, and let $\mathcal{L}^{\pi_{i}}$ be the indecomposable sheaf constructed on the poset $\operatorname{Star} \pi_{i}$. Then $\mathcal{F}=\left.\mathcal{L}^{\pi_{1}} \oplus \mathcal{L}^{\pi_{2}}\right|_{\Delta \geq m}$ is a minimally flabby sheaf and we have

$$
G_{\sigma}= \begin{cases}\mathbb{C} \oplus \mathbb{C} & \text { if } \pi_{1}, \pi_{2}<\sigma \\ \mathbb{C} & \text { if } \pi_{1}<\sigma \text { or } \pi_{2}<\sigma, \text { but not both } \\ 0 & \text { otherwise }\end{cases}
$$

Note that a minimally flabby sheaf on $\Delta^{\geq m}$ is determined by its restriction to $\Delta^{[m, m+1]}$. Indeed, the exact sequence (6) can be used to recover $\mathcal{F}_{\sigma}$ for $\operatorname{dim} \sigma>m+1$. Similarly, given two minimally flabby shaves $\mathcal{F}$ and $\mathcal{E}$, a morphism defined between the restrictions of these sheaves to $\Delta^{[m, m+1]}$ can be lifted to a morphism on $\Delta^{\geq m}$.
Lemma 3.6. Let $\mathcal{E}$ and $\mathcal{F}$ be minimally flabby sheaves on $\Delta \geq m$, and $L: \mathcal{E} \rightarrow \mathcal{F} a$ homomorphism of $\mathcal{A}$-modules.
(1) If $L$ is injective on cones $\sigma \in \Delta, \operatorname{dim} \sigma=m$, then $L$ is injective on all cones.
(2) If $L$ is an isomorphism on cones $\sigma \in \Delta, \operatorname{dim} \sigma=m$, then the cokernel $\mathcal{Q}$ of $L$ :

$$
0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0
$$

is a minimally flabby sheaf on $\Delta^{\geq m+1}$.
Proof. The first statement follows by induction on $\operatorname{dim} \sigma$ from the exact sequence (6).
To prove the second statement, first note that the surjectivity of the restriction maps $\operatorname{res}_{\beta}^{\alpha}$ for $\mathcal{Q}$ is clear. The morphism $L$ defines a map between the augmented cellular complexes of $\mathcal{E}$ and $\mathcal{F}$ which is injective except possibly in the $G_{\sigma}$ terms. The quotient gives the cellular complex for $Q$. By induction on $\operatorname{dim} \sigma$ it follows that $Q_{\sigma}$ is annihilated by $x_{m}$, hence is a free $A_{m+1, d}$-module by Lemma 3.1. We get the augmentation for $\mathcal{Q}$ by removing tha augmentations of $\mathcal{E}$ and $\mathcal{F}$ and considering the long-exact cohomology sequence of the short-exact sequence of complexes.

Definition 3.7. Let $\mathcal{F}$ be a minimally flabby sheaf on $\Delta \geq m$. We say that $\mathcal{F}$ is a Poincaré sheaf if for every $\sigma \in \Delta, \operatorname{dim} \sigma=d \geq m$, we have an $A_{m, d-1}$-bilinear non-degenerate symmetric pairing

$$
<\cdot, \cdot>_{\sigma}: F_{\sigma} / x_{d} F_{\sigma} \times F_{\sigma} / x_{d} F_{\sigma} \rightarrow \omega_{m, d-1}
$$

satisfying the compatibility condition:

$$
\begin{equation*}
<f, g>_{\sigma}=\sum_{\tau<\sigma} o r_{\tau}^{\sigma}<\operatorname{res}_{\tau}^{\sigma} f, \operatorname{res}_{\tau}^{\sigma} g>_{\tau}, \quad f, g \in F_{\sigma} / x_{d} F_{\sigma} \tag{7}
\end{equation*}
$$

Here on the right hand side $<\cdot, \cdot>_{\tau}$ denotes the $A_{m, d-1}$-bilinear extension of the $A_{m, d-2^{-}}$ bilinear pairing $<\cdot, \cdot>_{\tau}$.

Example 3.8. The sheaf $\left.\mathcal{L}\right|_{\Delta \geq 1}$ is a Poincaré sheaf on $\Delta \geq 1$.
Let $\mathcal{F}$ be a Poincaré sheaf on $\Delta \geq m$. Then $\overline{\mathcal{F}}_{\sigma}$ for $\operatorname{dim} \sigma=d \geq m$ is a vector space graded by subsets of $\{m, \ldots, d-1\}$. Write $\mathcal{F} / x_{m} \mathcal{F}$ for the sheaf with stalks

$$
\left(\mathcal{F} / x_{m} \mathcal{F}\right)_{\sigma}=\mathcal{F}_{\sigma} / x_{m} \mathcal{F}_{\sigma}
$$

This is a locally free sheaf on $\Delta^{\geq m+1}$, and we can split it as

$$
\mathcal{F} / x_{m} \mathcal{F}=\mathcal{F}^{0} \oplus \mathcal{F}^{1}
$$

where $\mathcal{F}_{\sigma}^{i}$ consists of elements of degree $(i, *, \ldots, *)$.
Lemma 3.9. Let $\mathcal{F}$ be a Poincaré sheaf on $\Delta \geq m$. Then $\mathcal{F}^{0}$ and $\mathcal{F}^{1}$ are minimally flabby sheaves on $\Delta^{\geq m+1}$.

Proof. Let us cut the sequence (6) into two exact sequences

$$
\begin{gathered}
0 \rightarrow \mathcal{F}_{\sigma} / x_{d} \mathcal{F}_{\sigma} \rightarrow \bigoplus_{\tau<\sigma, \operatorname{dim} \tau=d-1} \mathcal{F}_{\tau} \rightarrow \ldots \rightarrow S \rightarrow 0 \\
0 \rightarrow S \rightarrow \bigoplus_{\rho<\sigma, \operatorname{dim} \rho=m} \mathcal{F}_{\rho} \rightarrow G_{\sigma} \rightarrow 0
\end{gathered}
$$

From the second sequence we get that $S$ is a free $\mathbb{C}\left[x_{m}\right]$-module, hence the first sequence remains exact after taking quotient by the ideal $\left(x_{m}\right)$ and splitting into two according to degree. The two sequences are the augmented cellular complexes for $\mathcal{F}^{0}$ and $\mathcal{F}^{1}$.

Now we are ready to define the sheafified version of the main construction. Let $\mathcal{F}$ be a Poincaré sheaf on $\Delta^{\geq m}$ and $L: \mathcal{F} \rightarrow \mathcal{F}$ an endomorphism of $\mathcal{A}$-modules of degree $e_{m}$, such that $L_{\sigma}: \mathcal{F}_{\sigma} \rightarrow \mathcal{F}_{\sigma}$ is self-adjoint with respect to the pairing for each $\sigma$. (More precisely, $L_{\sigma}: \mathcal{F}_{\sigma} \rightarrow \mathcal{F}_{\sigma}$ has to be self-adjoint with respect to the $A_{m, d}$-linear extension of the pairing $<\cdot, \cdot>_{\sigma}$.) Assume that the induced morphism $L: \mathcal{F}^{0} \rightarrow \mathcal{F}^{1}$ is injective on cones $\sigma \in \Delta, \operatorname{dim} \sigma=m+1$; then it is an isomorphism on these cones by Poincaré duality. Lemma 3.6 gives an exact sequence

$$
0 \rightarrow \mathcal{F}^{0} \rightarrow \mathcal{F}^{1} \rightarrow \mathcal{Q} \rightarrow 0
$$

where $\mathcal{Q}$ is a minimally flabby sheaf on $\Delta^{\geq m+2}$. In order to have $\mathcal{Q}$ in correct degrees, we have to replace it with $\mathcal{Q}^{\prime}=\mathcal{Q}\left[e_{m}\right]$.

We also construct the sheaf $\mathcal{C}$ as follows. First, we have an exact sequence of minimally flabby sheaves on $\Delta^{\geq m+2}$ :

$$
0 \rightarrow \mathcal{Q}\left[e_{m}-e_{m+1}\right] \rightarrow \mathcal{F}^{0} / x_{m+1} \mathcal{F}^{0} \rightarrow \mathcal{F}^{1} / x_{m+1} \mathcal{F}^{1} \rightarrow \mathcal{Q} \rightarrow 0
$$

Define $\mathcal{C}$ by the exact sequence

$$
0 \rightarrow \mathcal{Q}\left[e_{m}-e_{m+1}\right] \rightarrow \mathcal{F}^{0} / x_{m+1} \mathcal{F}^{0} \rightarrow \mathcal{C} \rightarrow 0
$$

Then one easily sees that $\mathcal{C}$ is also minimally flabby on $\Delta \geq m+2$ (to get the augmented cellular complex for $\mathcal{C}$, it is more convenient to consider the short exact sequence

$$
\left.0 \rightarrow \mathcal{C} \xrightarrow{L} \mathcal{F}^{1} / x_{m+1} \mathcal{F}^{1} \rightarrow \mathcal{Q} \rightarrow 0\right)
$$

We should again replace $\mathcal{C}$ with an almost flabby sheaf $\mathcal{C}^{\prime}$ on $\Delta^{\geq m+1}$, such that $\mathcal{C}=$ $\mathcal{C}^{\prime} / x_{m+1} \mathcal{C}^{\prime}$. We will not do this because inductively, the next step to construct a Lefschetz operation is to go from $\mathcal{C}^{\prime}$ to $\mathcal{C}$ and split it according to degree. The fact that we don't have $\mathcal{C}^{\prime}$ that induces $\mathcal{C}$ will cause us some trouble later when we look for an endomorphism of $\mathcal{C}$.

Summarizing, we have defined the sheafified version of the main construction. Starting with a Poincaré sheaf $\mathcal{F}$ on $\Delta \geq m$ and a morphism $L$, we constructed minimally flabby sheaves $\mathcal{Q}$ and $\mathcal{C}$ on $\Delta^{\geq m+2}$. The construction on stalks agrees with the main construction on modules. The stalks of the shaves $\mathcal{Q}$ and $\mathcal{C}$ inherit Poincaré pairings from the pairing on $\mathcal{F}$, which is clearly compatible with the restriction morphisms. Hence the two new sheaves are also Poincaré sheaves.

It remains to see when can we find an appropriate endomorphism $L$ of $\mathcal{F}$.
Lemma 3.10. Let $\mathcal{F}$ be a Poincaré sheaf on $\Delta^{\geq m}$ and $L: \mathcal{F} \rightarrow \mathcal{F}$ a homomorphism of degree $e_{m}$. Then $L$ is self-adjoint with respect to the pairings on $\sigma \in \Delta \geq m$ if and only if it is self-adjoint on cones $\rho$ of dimension $m$.

Proof. This follows by induction on the dimension of a cone from the formula (7).
Lemma 3.11. Let $\mathcal{F}$ be a Poincaré sheaf on $\Delta^{\geq m}$. Then there exists a homomorphism $L: \mathcal{F} \rightarrow \mathcal{F}$ of degree $e_{m}$ that is self-adjoint with respect to the pairings on the stalks $\mathcal{F}_{\sigma}$ and such that the induced homomorphism $L: \mathcal{F}^{0} \rightarrow \mathcal{F}^{1}$ is injective.

Proof. For $\operatorname{dim} \rho=m$, let $L_{\rho}: \mathcal{F}_{\rho} \rightarrow \mathcal{F}_{\rho}$ be a self-adjoint homomorphism of degree $e_{m}$. (Note that $\mathcal{F}_{\rho} \simeq \mathbb{C}\left[x_{m}\right]^{\oplus a_{\rho}}$ for some $a_{\rho} \geq 0$.) We claim that a suitable collection of $L_{\rho}$ induces the required $L$. For this we need to check that $L_{\rho}$ can be extended to cones $\tau$ of dimension $m+1$ (hence can be extended to all cones), and that on such $\tau$ it defines an injection $\mathcal{F}_{\tau}^{0} \rightarrow \mathcal{F}_{\tau}^{1}$.

Let $\operatorname{dim} \tau=m+1$ and consider the augmented cellular complex of $\tau$ :

$$
0 \rightarrow \mathcal{F}_{\tau} / x_{m+1} \mathcal{F}_{\tau} \rightarrow \bigoplus_{\rho<\tau} \mathcal{F}_{\rho} \rightarrow G_{\tau} \rightarrow 0
$$

Here $G_{\tau} \simeq \mathbb{C}^{a}$ for some $a \geq 0, \bigoplus_{\rho<\tau} \mathcal{F}_{\rho} \simeq \mathbb{C}\left[x_{m}\right]^{\oplus 2 a}$ and $\mathcal{F}_{\tau} / x_{m+1} \mathcal{F}_{\tau} \simeq \mathbb{C}\left[x_{m}\right]^{\oplus a} \oplus$ $\mathbb{C}\left[x_{m}\right]\left[-e_{m}\right]^{\oplus a}$.

The maps $L_{\rho}$ are compatible with the zero $\operatorname{map} G_{\tau} \rightarrow G_{\tau}$ of the augmentation. It follows that $L_{\rho}$ induce a map $L_{\tau}: \mathcal{F}_{\tau} \rightarrow \mathcal{F}_{\tau}$, compatible with restriction maps, hence there is an extension to a morphism $L: \mathcal{F} \rightarrow \mathcal{F}$.

Let $V=\bigoplus_{\rho<\tau} \overline{\mathcal{F}}_{\rho} \simeq \mathbb{C}^{2 a}$ and let $K \simeq \mathbb{C}^{a}$ be the kernel of $V \rightarrow G_{\tau}$. Then $K=\mathcal{F}_{\tau}^{0}$. The map $L_{\rho}$ comes from a linear map $\bar{L}_{\rho}=\frac{1}{x} L_{\rho}: \overline{\mathcal{F}}_{\rho} \rightarrow \overline{\mathcal{F}}_{\rho}$. The maps $\bar{L}_{\rho}$ together define a linear map $L_{V}: V \rightarrow V$. Now the condition that $L$ is injective is equivalent to $L_{\tau}: \mathcal{F}_{\tau}^{0} \rightarrow \mathcal{F}_{\tau}^{1}$ being injective, which is equivalent to the condition that the intersection of $K$ and $L_{V}(K)$ is zero.

## Lefschetz decomposition and the $c d$-index of fans

Let us also bring the Poincaré pairing into the picture. We have a non-degenerate symmetric pairing on each $\overline{\mathcal{F}}_{\rho}$, combined to a pairing on $V$. The pairing on $\mathcal{F}_{\tau}$ induces a non-degenerate pairing between $\mathcal{F}_{\tau}^{0}$ and $\mathcal{F}_{\tau}^{1}$, which restricts to the zero pairing on $\mathcal{F}_{\tau}^{0}$, hence the compatibility condition implies that the pairing on $V$ restricted to $K$ is zero. In other words, $K=K^{\perp}$. The proof that a suitable set of $L_{\rho}$ gives a required $L$ is given in the lemma below.

Finally, let us consider the case when $L$ is defined by a multiplication with an element in $L \in \Gamma(\mathcal{A}, \Delta)$ of degree $e_{m}$. In this case the linear maps $\bar{L}_{\rho}$ are given my multiplication with a constant $c_{\rho}$ (where $\left.L\right|_{\rho}=c_{\rho} x_{m}$ ). Note also that since the restriction maps res $_{\rho}^{\tau}$ are surjective, the projection $V \rightarrow \overline{\mathcal{F}}_{\rho}$ maps $K$ onto $\overline{\mathcal{F}}_{\rho}$. Thus, if the conjecture below is true then $L$ defines an injective morphism.

Lemma 3.12. Let $V=\oplus V_{i}$ be a finite dimensional vector space. Suppose that each $V_{i}$ has a non-degenerate symmetric bilinear pairing, giving a pairing on $V$. Let $K \subset V$ be a subspace such that $K \subset K^{\perp}$. Then there exist self-adjoint linear maps $L_{i}: V_{i} \rightarrow V_{i}$, combined to $L: V \rightarrow V$, satisfying $K^{\perp} \cap L(K)=0$.

Proof. Let $v_{1}, \ldots, v_{2 a}$ be an orthogonal basis of $V$ consisting of elements from $V_{i}$ and let $y_{1}, \ldots, y_{2 a}$ be the dual basis giving coordinates on $V$. Let $T$ be the algebraic torus of dimension $\operatorname{dim} V$ acting on $V$ by:

$$
\left(t_{1}, \ldots, t_{2 a}\right) \cdot\left(y_{1}, \ldots, y_{2 a}\right)=\left(t_{1} y_{1}, \ldots, t_{2 a} y_{2 a}\right)
$$

An element $t \in T$ defines a linear map $V \rightarrow V$ of the required type. We claim that for a general $t$ we have $K^{\perp} \cap t(K)=0$.

Now $V$ has finitely many $T$-orbits. By Kleiman-Bertini theorem, for a general $t$, the restrictions of $K^{\perp}$ and $K$ to any orbit $O$ intersect transversely. Thus, it suffices to show that the expected dimension of this intersection is zero.

Let $W \subset V$ be a subspace spanned by a subset of the $v_{j}$. Then the pairing on $V$ restricts to a non-degenerate pairing on $W$. Since $K \subset K^{\perp}$, it follows that

$$
\operatorname{dim}\left(K^{\perp} \cap W\right)+\operatorname{dim}(K \cap W) \leq \operatorname{dim}(W)
$$

Conjecture 3.13. Let the notation be as in the previous lemma. Additionally assume that the projections $V \rightarrow V_{i}$ map $K$ onto $V_{i}$ for each $i$.Then the statement of the lemma remains true if we let $L_{i}$ be multiplication by some constant $c_{i}$.
Remark 3.14. Starting with a Poincaré sheaf $\mathcal{F}$ on $\Delta^{\geq m}$, we apply the previous lemmas to perform the main construction on $\mathcal{F}$ and produce new sheaves $\mathcal{Q}$ and $\mathcal{C}$. Then inductively we apply the same construction on $\mathcal{C}$ and $\mathcal{Q}$. As explained above, we should consider $\mathcal{C}$ as coming from a sheaf $\mathcal{C}^{\prime}$ on $\Delta^{\geq m+1}$, so that the main construction should be applied to $\mathcal{C}^{\prime}$ rather than $\mathcal{C}$. Let us show that we don't need $\mathcal{C}^{\prime}$ for the existence of the required $L: \mathcal{C} \rightarrow \mathcal{C}$.

Recall that $\mathcal{C}$ was defined by the exact sequence of minimally flabby sheaves on $\Delta^{\geq m+2}$ :

$$
0 \rightarrow \mathcal{Q}\left[e_{m}-e_{m+1}\right] \rightarrow \mathcal{F}^{0} / x_{m+1} \mathcal{F}^{0} \rightarrow \mathcal{C} \rightarrow 0
$$

## K. Karu

On the sheaf $\mathcal{F}^{0}$ we can define a bilinear pairing by the same formula as on $\mathcal{C}$. This pairing is degenerate, but it induces the pairing on $\mathcal{C}$. Now as in Lemma 3.11 we construct a homomorphism $L: \mathcal{F}^{0} \rightarrow \mathcal{F}^{0}$ of degree $e_{m+1}$. We claim that this homomorphism induces the injective homomorphism $\mathcal{C}^{0} \rightarrow \mathcal{C}^{1}$. Indeed, we are reduced to the same Lemma 3.12. The difference now is that we may have a strict inclusion $K \subset K^{\perp}$, while the two spaces were equal in the proof of Lemma 3.11.

Let us now put everything together and finish the proof of Theorem 1.2. We start with the Poincaré sheaf $\left.\mathcal{L}\right|_{\Delta \geq 1}$ and apply the main construction to produce new Poincaré sheaves $\mathcal{C}$ and $\mathcal{Q}$. Then inductively we apply the main construction to $\mathcal{C}$ and $\mathcal{Q}$. These constructions give a Lefschetz operation on each stalk $\mathcal{L}_{\sigma} / x_{d} \mathcal{L}_{\sigma}, \operatorname{dim} \sigma=d$. Considering

$$
\mathcal{L}_{\hat{1}} / x_{n+1} \mathcal{L}_{\hat{1}} \simeq \Gamma(\mathcal{L}, \Delta),
$$

we get a Lefschetz operation on $\Gamma(\mathcal{L}, \Delta)$ as stated in the theorem.

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