Proceedings of 12^{th} Gökova
Geometry-Topology Conference
pp. 157 – 164

Published online at
GokovaGT.org

Deformations of special Lagrangian submanifolds; An approach via Fredholm alternative

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Dedicated to the memory of Raoul Bott.

Abstract. In an earlier paper, [9], we showed that the moduli space of deformations of a smooth, compact, orientable special Lagrangian submanifold L in a symplectic manifold X with a non-integrable almost complex structure is a smooth manifold of dimension $H^1(L)$, the space of harmonic 1-forms on L. We proved this first by showing that the linearized operator for the deformation map is surjective and then applying the Banach space implicit function theorem. In this paper, we obtain the same surjectivity result by using a different method, the Fredholm Alternative, which is a powerful tool for compact operators in linear functional analysis.

1. Introduction

In [8], McLean showed that the moduli space of nearby submanifolds of a smooth, compact special Lagrangian submanifold L in a Calabi-Yau manifold X is a smooth manifold and its dimension is equal to the dimension of $H^1(L)$, the space of harmonic 1-forms on L. Special Lagrangian submanifolds have attracted much attention after Strominger, Yau and Zaslow proposed a mirror Calabi-Yau construction using special Lagrangian fibration [11]. For more information about special Lagrangian submanifolds and examples, see [3], [4], [6].

One can also define special Lagrangian submanifolds of symplectic manifolds equipped with a nowhere vanishing complex valued $(n, 0)$-form, [9]. Such symplectic manifolds were studied recently by Smith, Thomas and Yau in [10].

In [9], we showed that the moduli space of special Lagrangian deformations of L in a symplectic manifold with non-integrable almost complex structure is also a smooth manifold of dimension $b_1(L)$, the first Betti number of L. In order to prove this result we first modified the definition of a special Lagrangian submanifold for symplectic manifolds, extended the parameter space of deformations and showed that the linearization of the deformation map is onto and finally applied the infinite dimensional Banach space implicit function theorem.

In this paper, we obtain the same result by a different approach. In particular, we show that the linearized operator for the deformation map is invertible by using Fredholm Alternative, a technique from linear functional analysis.

2. Deformations of special Lagrangian submanifolds

Let $(M^{2n}, \omega, J, g, \Omega)$ be a Calabi-Yau manifold with a Kähler 2-form ω, a complex structure J, a compatible Riemannian metric g and a nowhere vanishing holomorphic $(n, 0)$-form Ω which is normalized with respect to ω. Then one can define a special Lagrangian submanifold of M.

Definition 2.1. An n-dimensional submanifold $L \subseteq M$ is *special Lagrangian* if L is Lagrangian (i.e. $\omega|_L \equiv 0$) and $Im(\Omega)$ restricts to zero on L. Equivalently, $\mathrm{Re}(\Omega)$ restricts to be the volume form on L with respect to the induced metric.

McLean studied the deformations of compact special Lagrangian submanifolds in Calabi-Yau manifolds and proved the following theorem, [8].

Theorem 2.2. *The moduli space of all deformations of a smooth, compact, orientable special Lagrangian submanifold L in a Calabi-Yau manifold M within the class of special Lagrangian submanifolds is a smooth manifold of dimension equal to dim$(H^1(L))$.*

One natural generalization of McLean's result is for symplectic manifolds. Now let (X, ω, J, g, ξ) denote a $2n$-dimensional symplectic manifold X with symplectic 2-form ω, an almost complex structure J which is tamed by ω, the compatible Riemannian metric g and a nowhere vanishing complex valued $(n, 0)$-form $\xi = \mu + i\beta$, where μ and β are real valued n-forms. Here we also take ξ to be normalized with respect to ω.

Note that the holomorphic form Ω is a closed form on Calabi-Yau manifolds. However, on a symplectic manifold which is equipped with a nowhere vanishing complex-valued $(n, 0)$ form ξ, it is not necessarily closed.

For more general special Lagrangian calibrations, one can introduce an additional term $e^{i\theta}$, where for each fixed angle θ we have a corresponding form $e^{i\theta}\xi$ and its associated geometry. Here θ is the phase factor of the calibration and using this as the new parameter one can define special Lagrangian submanifolds in a symplectic manifold and study their deformations, [9].

Definition 2.3. An n-dimensional submanifold $L \subseteq X$ is *special Lagrangian* if L is Lagrangian (i.e. $\omega|_L \equiv 0$) and $Im(e^{i\theta}\xi)$ restricts to zero on L, for some $\theta \in \mathbb{R}$. Equivalently, $Re(e^{i\theta}\xi)$ restricts to be the volume form on L with respect to the induced metric.

Now we recall the basics of a technique for compact operators in linear functional analysis, known as Fredholm Alternative. One can find more information about the subject in [5] and [7].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two real Banach spaces.

Definition 2.4. A bounded linear operator $\mathcal{K} : \mathcal{X} \to \mathcal{Y}$ is called *compact* provided for each bounded sequence $\{u_k\}_{k=1}^{\infty}$ is precompact in $\mathcal{Y}$.

Now let H denote a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$.

Theorem 2.5. *Let $\mathcal{K}: H \to H$ be a compact operator. Then*

(i) $\ker(I-\mathcal{K})$ *is finite dimensional,*
(ii) Range$(I-\mathcal{K})$ *is closed,*
(iii) Range$(I-\mathcal{K}) = \ker(I-\mathcal{K}^*)^{\perp}$*,*
(iv) $\ker(I-\mathcal{K}) = \{0\}$ *if and only if Range*$(I-\mathcal{K}) = H$
(v) dim $\ker(I-\mathcal{K}) =$ *dim* $\ker(I-\mathcal{K}^*)$.

Remark 2.6. Theorem 2.5 asserts in particular either

(a) for each $f \in H$, the equation $u - \mathcal{K}u = f$ has a unique solution

or else

(b) the homogeneous equation $u - \mathcal{K}u = 0$ has solutions $u \neq 0$.

In addition, should (a) obtain the space of solutions of the homogeneous problem is finite dimensional and the nonhomogeneous equation $u - \mathcal{K}u = f$ has a solution if and only if $f \in \ker(I-\mathcal{K}^*)^{\perp}$.

Now we prove the following theorem, [9], using the Fredholm Alternative:

Theorem 2.7. *Let L be a smooth, compact, orientable special Lagrangian submanifold of a symplectic manifold X. Then the moduli space of all deformations of L in X within the class of special Lagrangian submanifolds is a smooth manifold of dimension $H^1(L)$.*

Proof. Given a domain Ω, let $C^{k,\alpha}(\Omega)$ denote the Hölder norms defined as

$$C^{k,\alpha}(\Omega) = \{f \in C^k(\Omega) |\ \ [D^{\gamma} f]_{\alpha,\Omega} < \infty, |\gamma| \leqq k\}$$

where

$$[f]_{\alpha,\Omega} = \underset{x,y\in\Omega,\ x\neq y}{Sup} \frac{dist(f(x), f(y))}{(dist(x,y))^{\alpha}} \quad \text{in } \Omega.$$

Then for a small vector field V and a scalar $\theta \in \mathbb{R}$, we define the deformation map as follows,

$$F : C^{1,\alpha}(\Gamma(N(L))) \times \mathbb{R} \to C^{0,\alpha}(\Omega^2(L)) \oplus C^{0,\alpha}(\Omega^n(L))$$

$$F(V,\theta) = ((\exp_V)^*(-\omega), (\exp_V)^*(Im(e^{i\theta}\xi)).$$

Here $N(L)$ denotes the normal bundle of L, $\Gamma(N(L))$ the space of sections of the normal bundle, and $\Omega^2(L)$, $\Omega^n(L)$ denote the differential 2-forms and n-forms, respectively.

Since the symplectic form ω is closed on X and the restriction of $Im(e^{i\theta}\xi)$ is a top dimensional form on L the image of the deformation map F lies in the closed 2-forms and closed n-forms. So by Hodge decomposition we get

$$F : C^{1,\alpha}(\Gamma(N(L))) \times \mathbb{R} \to C^{0,\alpha}(d\Omega^1(L)) \oplus C^{0,\alpha}(d\Omega^{n-1}(L) \oplus \mathcal{H}^n(L))$$

where $d\Omega^{n-1}(L)$ denotes the space of exact n-forms and $\mathcal{H}^n(L)$ denotes the space of harmonic n-forms on L.

In [9], we computed the linearization of F at (0,0),

$$dF(0,0): C^{1,\alpha}(\Gamma(N(L)))\times\mathbb{R}\to C^{0,\alpha}(d\Omega^1(L))\oplus C^{0,\alpha}(d\Omega^{n-1}(L)\oplus\mathcal{H}^n(L))$$

where

$$dF(0,0)(V,\theta)=\frac{\partial}{\partial t}F(tV,s\theta)|_{t=0,s=0}+\frac{\partial}{\partial s}F(tV,s\theta)|_{t=0,s=0}$$

$$=(-d(i_V\omega)|_L,(i_Vd\beta+d(i_V\beta))|_L+\theta)$$

$$=(dv,\zeta+d*v+\theta),\quad\text{where }\ \zeta=i_V(d\beta)|_L.$$

Here i_V is the interior derivative and v is the dual 1-form to the vector field V with respect to the induced metric. For the details of local calculations see [8], [9].

Let $x_1,x_2,...,x_n$ and $x_1,x_2,...,x_{2n}$ be the local coordinates on L and X, respectively. Then for any given normal vector field $V=(V_1\frac{\partial}{\partial x_{n+1}},...,V_n\frac{\partial}{\partial x_{2n}})$ to L we can show that

$$\zeta=i_V(d\beta)|_L=-n(V_1\cdot g_1+...+V_n\cdot g_n)\text{dvol}$$

where g_i $(0<i\leq n)$ are combinations of coefficient functions in the connection-one forms.

One can decompose the n-form $\zeta=da+d^*b+h_2$ by using Hodge Theory and because ζ is a top dimensional form on L, ζ is closed and the equation becomes

$$dF(0,0)(V,\theta)=(dv,da+d*v+h_2+\theta)$$

for some $(n-1)$-form a and harmonic n-form h_2. Also the harmonic projection for $\zeta=-n(V_1.g_1+...+V_n.g_n)\text{dvol}$ is given by $(\int_L -n(V_1.g_1+...+V_n.g_n)\text{dvol})\text{dvol}$ and therefore one can show that

$$da=-n(V_1.g_1+...+V_n.g_n)\text{dvol}+(n\int_L(V_1.g_1+...+V_n.g_n)\text{dvol})\text{dvol}$$

and

$$h_2=(-n\int_L(V_1.g_1+...+V_n.g_n)\text{dvol})\text{dvol}.$$

One should note that the differential forms a and h_2 both depend on V.

The Implicit Function Theorem says that $F^{-1}(0,0)$ is a manifold and its tangent space at $(0,0)$ can be identified with the kernel of dF.

$$(dv)\oplus(\zeta+d*v+\theta)=(0,0)$$

which implies

$$dv=0\quad\text{and}\quad\zeta+d*v+\theta=da+d*v+h_2+\theta=0.$$

The space of harmonic n-forms $\mathcal{H}^n(L)$, and the space of exact n-forms $d\Omega^{n-1}(L)$, on L are orthogonal vector spaces by Hodge Theory. Therefore, $dv=0$ and $da+d*v+h_2+\theta=0$ is equivalent to $dv=0$ and $d*v+da=0$ and $h_2+\theta=0$.

One can see that the special Lagrangian deformations (the kernel of dF) can be identified with the 1-forms on L which satisfy the following equations:

$$\begin{aligned} (i) &\;\; dv = 0 \\ (ii) &\;\; d*(v+\kappa(v)) = 0 \\ (iii) &\;\; h_2 + \theta = 0. \end{aligned}$$

Here, $\kappa(v)$ is a linear functional that depends on v and h_2 is the harmonic part of ζ which also depends on v. These equations can be formulated in a slightly different way in terms of decompositions of v and $*a$.

If $v = dp + d^*q + h_1$ and $*a = dm + d^*n + h_3$ then we have

$$\begin{aligned} (i) &\;\; dd^*q = 0 \\ (ii) &\;\; \Delta(p \pm m) = 0 \\ (iii) &\;\; h_2 + \theta = 0. \end{aligned}$$

This formulation of the solutions will provide the proof of the surjectivity of the linearized operator without using $\kappa(v)$.

Now we show that the linearized operator is surjective at $(0,0)$. Recall that the deformation map is given as

$$F : C^{1,\alpha}(\Gamma(N(L))) \times \mathbb{R} \to C^{0,\alpha}(d\Omega^1(L)) \oplus C^{0,\alpha}(d\Omega^{n-1}(L) \oplus \mathcal{H}^n(L)).$$

Therefore, for any given exact 2-form x and closed n-form $y = u + z$ in the image of the deformation map (here u is the exact part and z is the harmonic part of y), we need to show that there exists a 1-form v and a constant θ that satisfy the equations,

$$\begin{aligned} (i) &\;\; dv = x \\ (ii) &\;\; d*(v+\kappa(v)) = u \\ (iii) &\;\; h_2 + \theta = z. \end{aligned}$$

Alternatively, we can solve the following equations for p, q and θ.

$$\begin{aligned} (i) &\;\; dd^*q = x \\ (ii) &\;\; \Delta(p \pm m) = *u \\ (iii) &\;\; h_2 + \theta = z, \end{aligned}$$

where the star operator $*$ in (ii) is defined on L.

For (i), since x is an exact 2-form we can write $x = d(dr + d^*s+\text{harmonic form})$ by Hodge Theory. Then one can solve (i) for q by setting $q = s$.

For (ii), since $\Delta m = d^* dm = d^* * a = *d**a = \pm * da$,

$$\Delta(p \pm m) = \Delta p \pm *da,$$

where a depends on p and we obtain

$$\Delta p \pm (-n(V_1.g_1 + ... + V_n.g_n) + (n \int_L (V_1.g_1 + ... + V_n.g_n)\text{dvol})) = *u \quad (1)$$

We can show the solvability as follows: Since $V = (V_1, ..., V_n)$ is the dual vector field of the one form $v = dp + d^*q + h_1$ we can write the equation (1) as

$$\Delta p \pm (-n(v \cdot g) + (n \int_L (v \cdot g)\text{dvol})) = *u \quad (2)$$

$$\Delta p \pm (-n(dp + d^*q + h_1) \cdot g + (n \int_L (dp + d^*q + h_1) \cdot g \text{ dvol})) = *u, \quad (3)$$

where $v \cdot g$ represents the action of the one form v on the vector field $g = (g_1, .., g_n)$ and $n \int_L (dp + d^*q + h_1 \cdot g)\text{dvol}$ is the harmonic projection of $-n(dp + d^*q + h_1) \cdot g$.

Then we get

$$\Delta p \pm n(-(dp \cdot g) + \int_L dp \cdot g \text{ dvol}) = *u \mp n[-(d^*q + h_1) \cdot g + \int_L (d^*q + h_1) \cdot g \text{ dvol}].$$

For simplicity we put

$$*u \mp n[-(d^*q + h_1) \cdot g + \int_L (d^*q + h_1) \cdot g \text{ dvol}] = h.$$

Since $\int_L * u = 0$ and $\int_L (d^*q + h_1) \cdot g$ dvol is equal to the harmonic projection of $(d^*q + h_1) \cdot g$, we get $\int_L h = 0$.

Since L is a compact manifold without boundary, by Stoke's Theorem,

$$\int_L dp \cdot g \text{ dvol} = -\int_L p \cdot \text{div} g \text{ dvol}$$

and the equation becomes

$$\Delta p \pm n(-(dp \cdot g) - \int_L p \cdot \text{div} g \text{ dvol}) = h.$$

Then by adding and subtracting p from the equation

$$(\Delta - Id)p = [\pm n(-(dp \cdot g) - \int_L p \cdot \text{div} g \text{ dvol}) - p + h]$$

and

$$p = (\Delta - Id)^{-1}[.....]p + \overline{h} = \mathcal{K}(p) + \overline{h}$$

where $\overline{h} = (\Delta - Id)^{-1}h$.
Since

$$||(\Delta - Id)^{-1}\int_L p \cdot \mathrm{div} g||_{L^2_1} \leq C|\int_L p \cdot \mathrm{div} g| \leq C||p||_{L^2}$$

$\mathcal{K}(p)$ is a compact operator which takes bounded sets in L^2 to bounded sets in L^2_1. Also note that we assumed here $1 \notin \mathrm{spec}(\Delta)$, and if this is not the case then we can modify the above argument by adding and subtracting λp, $\lambda \notin \mathrm{spec}(\Delta)$ from the equation.

Next we show that the set of solutions of the equation

$$\Delta p \pm n(-(dp \cdot g) - \int_L p \cdot \mathrm{div} g \;\mathrm{dvol}) = 0 \tag{4}$$

is constant functions and therefore of dimension 1. Note that this set of solutions also satisfy the equation $(Id - \mathcal{K})(p) = 0$.

Also note that $\int_L p \cdot \mathrm{div} g$ dvol is a constant which depends on p. We denote this as $C(p)$. At maximum values of p, Δp will be negative which imply that $C(p) \leq 0$ and at minimum values of p, Δp will be positive which imply that $C(p) \geq 0$ so $C(p)$ should be zero. Then the maximum principle holds for the equation $\Delta p \pm n(-(dp \cdot g)) = 0$ and since L is a compact manifold without boundary the solutions of this equation are constant functions. Hence the dimension of the kernel of $(Id - \mathcal{K})$ is one.

Next we find the kernel of $(Id - \mathcal{K}^*)$.

$$\int_L (\Delta p \pm n(-(dp \cdot g) - \int_L p \cdot \mathrm{div} g))q(y)dy \tag{5}$$

$$= \int_L p\Delta q(y) \pm n\int_L - (dp \cdot g)q(y)dy - n\int_L (\int_L p \cdot \mathrm{div} g)q(y)dy.$$

$$= \int_L p(y)\Delta q \pm n\int_L + (p\mathrm{div}(g \cdot q)(y)dy - n\int_L p(x) \cdot \mathrm{div} g(x) \int_L q(y)dydx$$

$$= \int_L p(y)\Delta q \pm n\int_L + (p\mathrm{div}(g \cdot q)(y)dy - n\int_L p(y) \cdot \mathrm{div} g(y) \int_L q(x)dxdy$$

$$= \int_L p(y)(\Delta q \pm n(+\mathrm{div}(g \cdot q) - \mathrm{div} g \int_L q(x)dx)dy.$$

Since we assumed that $1 \notin \mathrm{spec}(\Delta)$, dim ker(Id$-\mathcal{K}^*$)($\Delta-$Id) =dim ker(Id$-\mathcal{K}^*$) and the kernel of (Id$-\mathcal{K}^*$) is equivalent to the solution space of the equation

$$\Delta q \pm n(+\mathrm{div}(g \cdot q) - \mathrm{div} g \int_L q(x)dx) = 0. \tag{6}$$

By Fredholm Alternative, the dimension of this kernel is 1 and one can check that a constant function $q = 1$ satisfies this equation, therefore the kernel consists of constant functions. Moreover these functions satisfy the compatibility condition $\int h.q = 0$.

Then by Fredholm Alternative, Theorem 2.5, we can conclude the existence of solutions of the equation

$$\Delta p \pm (-n(V_1.g_1 + ... + V_n.g_n) + (n\int_L (V_1.g_1 + ... + V_n.g_n)\ \mathrm{dvol})) = *u.$$

(iii) is straightforward.

It follows from [9] that the image of the deformation map F_1 lies in $d\Omega^1(L)$ and the image of F_2 lies in $d\Omega^{n-1}(L) \oplus \mathcal{H}^n(L)$. So we conclude that dF is surjective at $(0,0)$. Also since both the index of $d + *d^*(v)$ and $d + *d^*(v + \kappa(v))$ are equal to $b_1(L)$ the dimension of tangent space of special Lagrangian deformations in a symplectic manifold is also $b_1(L)$, the first Betti number of L. By infinite dimensional version of the implicit function theorem and elliptic regularity, the moduli space of all deformations of L within the class of special Lagrangian submanifolds is a smooth manifold and has dimension $b_1(L)$.

□

Remark 2.8. One can study the deformations of special Lagrangian submanifolds in much more general settings. In a forthcoming paper we plan to study these deformations using the techniques which we developed recently for associative submanifolds of G_2 manifolds, [1], [2].

Acknowledgements. The author is grateful to Peng Lu for many useful conversations and valuable comments.

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