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Virtual links, orientations of chord diagrams and Khovanov homology

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Dedicated to the memory of Raoul Bott.

Abstract. By adding or removing appropriate structures to Gauss diagrams, one can create useful objects related to virtual links. In this paper few objects of this kind are studied: twisted virtual links generalizing virtual links; signed chord diagrams staying halfway between twisted virtual links and Kauffman bracket/Khovanov homology; alternatable virtual links intermediate between virtual and classical links. The most profound role here belongs to a structure that we dare to call orientation of chord diagrams. Khovanov homology is generalized to oriented signed chord diagrams and links in an oriented thickened surface such that the link projection realizes the first Stiefel-Whitney class of the surface.

1. Introduction

In the middle of nineties Lou Kauffman suggested a natural extension of classical knot theory by replacing classical links with *virtual links.* From the point of view of traditional 3-dimensional topology, as it was shown by Greg Kuperberg [13], virtual links are nothing but links in oriented thickennings of orientable closed surfaces, considered up to homeomorphisms of thickennings. Virtual links provide a way to extend the range of applications of combinatorial diagrammatic technique. It became especially interesting due to recent development of link homology theories built on link diagrams.

This paper was written in an attempt to analyze the difficulties which emerge when the construction of Khovanov homology is extended to virtual links. I analyzed few geometric objects closely related to a virtual link and found that a key role is played by an additional structure: orientation of a chord diagram, that is a *signed chord diagram*. Existence of this structure is solely responsible for the possibility to construct a Khovanov complex literally as in the classical case.

Virtual links with orientable chord diagrams appeared in various occasions: virtual links that admit checkerboard colorable diagrams, virtual links which can be made alternating by crossing changes, virtual links with zero homologous modulo 2 irreducible model, virtual links with orientable atom, etc. I will call virtual links of this kind *alternatable*.

Construction of Khovanov homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ for any virtual link is quite straightforward. It was presented by V.O.Manturov in [14], [15] and [16]. In the same papers Manturov presented also two constructions of Khovanov homology with integer coefficients, but they do not generalize the original Khovanov homology of classical links and rely on preliminary geometric constructions.

Manturov's constructions of Khovanov complexes with integer coefficients split into geometric constructions cooking from an arbitrary virtual link alternatable ones, and the straightforward construction of Khovanov complex with integer coefficients which works only for alternatable links. The first construction is defined for a framed virtual link, and proceeds by doubling it. The second one relies on a 2-fold covering obtained as restriction to the link of an orientation covering of a surface, on which the link is naturally placed as a zero homologous modulo 2 curve. This surface with the link diagram and checkerboard coloring is called the *atom* of the link. For details see Manturov [14], or [15], or [16].

The construction of Khovanov complex for alternatable virtual link does not require all the information contained in the virtual link diagram. One can pass (without any loss) to a Gauss diagram and then forget orientations of its chords. A signed chord diagram obtained in this way contains everything needed for building the Khovanov complex. Alternability of a virtual link is orientability of its chord diagram.

Virtual links admit a generalization to *twisted virtual links*, which emerge in relation to links in oriented thickennings of non-orientable surfaces. Many link invariants, in particular Kauffman bracket and Jones polynomial, are extended to twisted virtual links and links in oriented thickennings of non-orientable surfaces.

A link in an oriented thickenning of a non-orientable surface which gives rise to an oriented signed chord diagram realizes homology class dual to the first Stiefel-Whitney class of the surface. This is the widest class of links in oriented thickennings of surfaces for which the classical construction of Khovanov complex works without any modification over the integers.

This creates a peculiar situation: the construction of Khovanov complex with integer coefficients works, say, for non-zero homologous links in the real projective space, but does not work for zero homologous links in the same space. It is difficult to believe that this is not just due to a lack of technique, especially since in the theory of Heegaard-Floer homology developed by Ozsváth and Szabó, which so far was quite similar to the Khovanov homology, these two classes of links in the projective space both have homology with integer coefficients. Khovanov, Rasmussen and Manturov conjectured that there should be a twisted version of Khovanov complex which works for all virtual links.

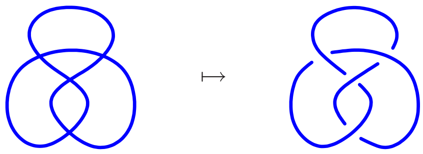
The main part of this paper starts with yet another introduction to virtual knot theory incorporating twisted virtual links. Then we review Kauffman bracket and Jones polynomial constructions, both for classical and virtual links, and show that they are defined for any signed chord diagram. The original part of the paper is devoted to the notion of orientation of chord diagrams and construction of Khovanov homology for orientable signed chord diagrams.

I am grateful to V.O.Manturov and A.N.Shumakovich for valuable information and interesting discussions.

2. Three faces of virtual knot theory

2.1. Link diagrams and Gauss diagrams

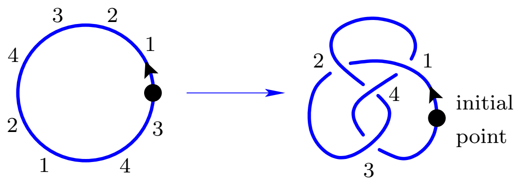
To describe graphically a classical link (that is, a closed smooth 1-dimensional submanifold of $\mathbb{R}^3$), one takes its generic projection to a plane and decorates the image at double points to show over and underpasses. This gives rise to a *link diagram:*



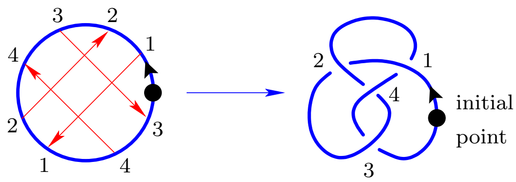
Genericity of projection means here that this is an immersion with multiple points of multiplicity at most 2 and transversality at double points.

A link diagram is a 2-dimensional picture of a link. In many cases 1-dimensional picture serves better. In particular, it is easier to convert to a combinatorial description, used as input data in computer programs.

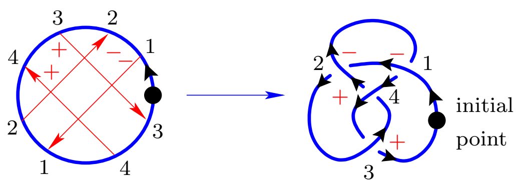
A 1-dimensional picture comes from a parametrization of the link:



The source of the parametrization is decorated. First, it is oriented. Second, each overpass is connected to the corresponding underpass with an arrow:



Third, each arrow is equipped with a sign:



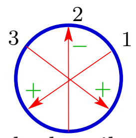
The sign is the *local writhe* of the crossing. It is + at a crossing which looks like: , and − at a crossing which looks like: . The signs depend on the orientation.

The result is called a *Gauss diagram* of the link. Gauss diagrams were introduced in our joint paper with Polyak [17]. The corresponding combinatorial objects called *Gauss codes* can be traced back to Gauss' notebooks. Transition from Gauss codes to their geometric counterparts, Gauss diagrams, is encouraged by geometric structures and operations, such as orientations and surgery, traditional for geometric objects, but difficult to recognize in a combinatorial context.

A Gauss diagram consists of an oriented 1-manifold (not necessarily connected) and chords connecting disjoint pairs of points on the 1-manifold. Each of the chords are oriented and equipped with a sign. The 1-manifold is called the *base* of the Gauss diagram.

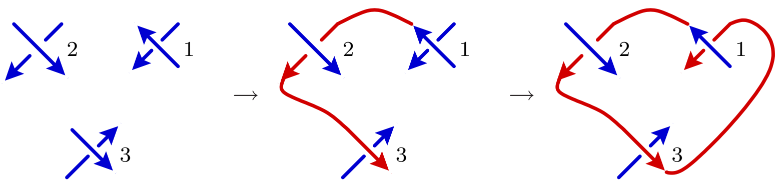
2.2. From Gauss diagrams to virtual links

Not any Gauss diagram can be obtained from a link diagram, but for any Gauss diagram one can try.

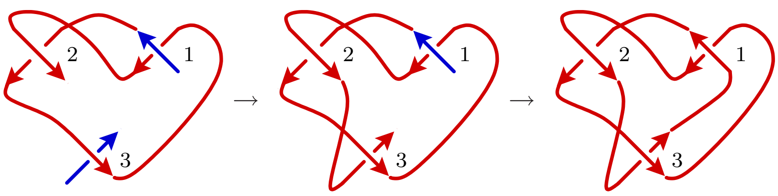
Take, for example, the Gauss diagram 

, and try to reconstruct the knot.

Let us start with crossings, as they are clearly described up to plane isotopy, then connect them step by step according to the Gauss diagram:

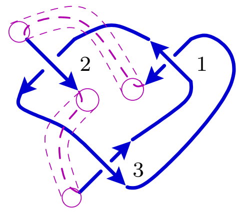


The next step meets an obstruction, as we need to penetrate through arcs that have been drawn. But let us continue neglecting the obstructions!



What is obtained looks like a knot diagram, but, besides usual crossings, it has double points which are not decorated. Such diagrams are called *virtual link diagrams*. They were introduced by Kauffman in the middle of nineties. Undecorated double points are called *virtual crossings*.

In the construction above, virtual crossings emerged inevitably. The only feasible way to avoid them is to attach handles to the plane and use them as bridges.



2.3. Link diagrams on orientable surfaces

A link diagram drawn on an orientable surface S, instead of the plane, defines a link in a thickened surface $S \times I$. It defines a Gauss diagram as well.

Any Gauss diagram appears in this way.

This is proven by the construction above, with handles added when needed. □

For each Gauss diagram, there is the smallest orientable closed surface with a link diagram defining this Gauss diagram.

Here by *smallest* I mean a surface with the greatest Euler characteristic, but without components disjoint from the link diagram. To eliminate a cheap possibility of making the Euler characteristic arbitrarily large by adding disjoint empty spheres, let us require that each connected component of the surface contains a piece of the link diagram.

To construct an orientable closed surface with greatest Euler characteristic accommodating a link diagram with a given Gauss diagram, one can first construct a germ of a link diagram on an oriented compact surface, which would contain the diagram as a deformation retract, and then cap each boundary component of the surface with a disk. For details, see [10]. □

A virtual link diagram may emerge as a projection to a plane of a link diagram on an orientable surface embedded in $\mathbb{R}^3$.

2.4. Moves

What happens to a link diagram when the link moves? It moves, too. A generic isotopy of a link can be decomposed into a sequence of isotopies each of which changes the diagram either as an isotopy of the plane or as one of the three Reidemeister moves (see Figure 1). It does not matter, if the link lies in $\mathbb{R}^3$ and its diagram lies on $\mathbb{R}^2$, or the link lies in a thickened oriented surface and its diagram lies on the surface.

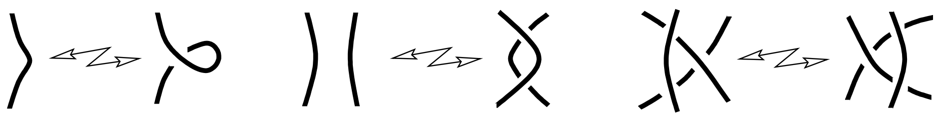


FIGURE 1. Reidemeister moves.

A virtual link diagram, which appears as a plane projection of a link diagram on a surface, moves also as shown in Figure 2 when the link moves generically in the thickened surface.

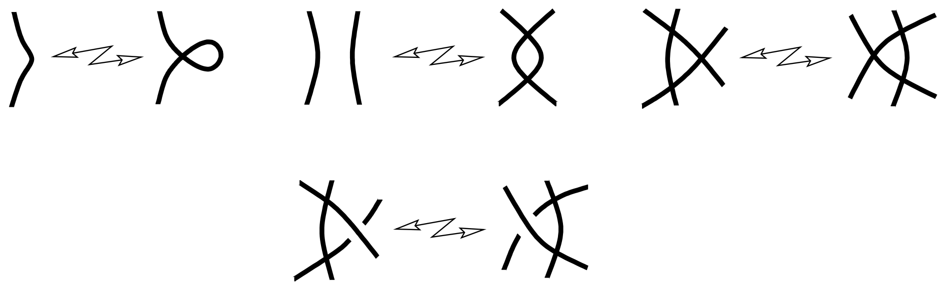


Figure 2. Virtual moves.

All virtual moves can be replaced by detour moves:

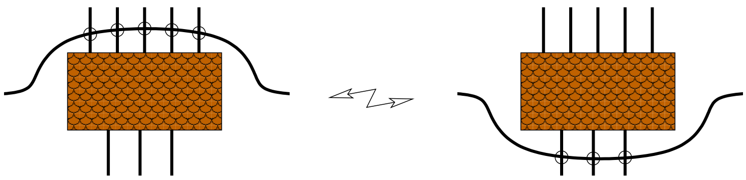


Figure 3. Detour move.

Gauss diagram does not change under virtual moves. Reidemeister moves act on Gauss diagram:

2.5. Three incarnations of virtual link theory

On the first stages of its development, the classical link theory received a completely combinatorial setup. Links were represented by link diagrams and isotopies were represented by Reidemeister moves.

We see that virtual link theory has even **two** similar **combinatorial incarnations.** Virtual links (whatever they are) are represented, on one hand, by *virtual link diagrams,* on the other hand, by *Gauss diagrams.* Furthermore, virtual isotopies (whatever they are) are represented, on one hand, by *Reidemeister and detour moves,* on the other hand, by the moves of Gauss diagrams corresponding to Reidemeister moves.

Third incarnation, truly topological one, is provided by Greg Kuperberg [13]. He proved that

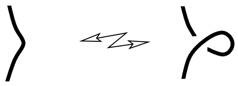
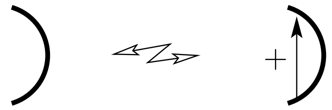
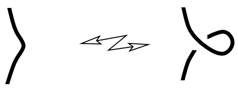
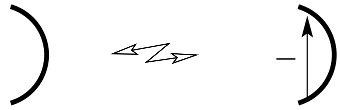
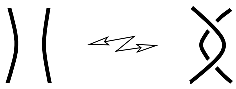
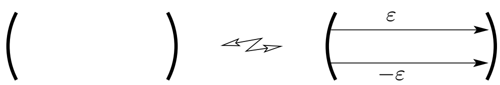
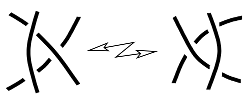
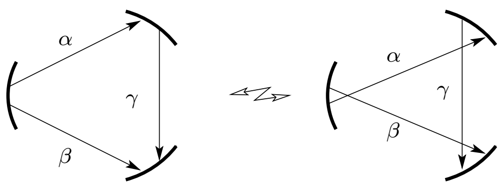
Move's name	Reidemeister move	Its action on Gauss diagram
Positive first move		
Negative first move		
Second move		
Third move		

Table 1. Action of Reidemeister moves on Gauss diagram.

Virtual links up to virtual isotopy are the same as irreducible links in thickened orientable surfaces up to orientation preserving homeomorphisms.

Here ***irreducible*** means that each connected component of the thickened surface contains some part of the link and it is impossible to find a simple curve C on the surface, which is

- disjoint from the projection of the link and
- either non-zero homologous on the surface, or separating two parts of the link projection from each other,

and it will be still impossible after any isotopy of the link.

In fact, Kuperberg [13] described more specifically how a virtual link diagram can be turned into an irreducible link in a thickened surface. He proved that a link diagram on an oriented surface can be ***destabilized*** to a link diagram of irreducible link on an oriented surface. A destabilization consists of embedded Reidemeister moves and Morse modifications of index 2 of the surface along a circle disjoint from the diagram.

Kuperberg's results bridge combinatorics (= 1-dimensional topology) with the 3-dimensional topology. The bridge can be used in both directions: both for extending combinatorial techniques like quantum link polynomials and link homology to links in 3-manifolds different from S^3, and for using traditional topological techniques, like signatures, in the combinatorial environment.

For instance, an oriented link in a thickened surface realizes a homology class. A homeomorphism maps a link homologous to zero to a link homologous to zero. Therefore the property of being homologous to zero is a property of the virtual link. The same holds for many other properties such as being homologous to zero modulo any number.

Furthermore, for a link homologous to zero modulo 2 in a thickened oriented surface one can define a *link signature.* Hence one can expect that there is a purely combinatorial construction of signature for virtual links of this kind.

2.6. Twisted virtual links

A non-orientable surface can also be thickened to an oriented 3-manifold. For example, take a Möbius band M embedded in $\mathbb{R}^3$ and thicken it, that is, take its regular neighborhood. See Figure 4. A neighborhood of M in $\mathbb{R}^3$ is orientable and fibers over M.

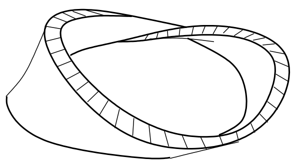
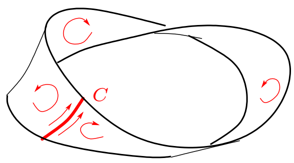
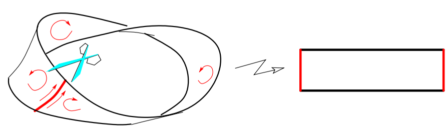


FIGURE 4.

To construct such a thickening, there is no need to embed a non-orientable surface to $\mathbb{R}^3$. Moreover, many of the non-orientable surfaces cannot be embedded into $\mathbb{R}^3$, but each of them has an orientable thickening. To thicken a non-orientable surface S:

(1) Find an *orientation change curve*[1] C (like *International date line*) on S.



(2) Cut S along C: $S \mapsto S$ ✂ C 

(3) Thicken: $(S$ ✂ $C) \times \mathbb{R}$.

(4) Paste over the sides of the cut $(x_+, t) \sim (x_-, -t)$.

[1]A curve realizing homology class Poincaré dual to the first Stiefel-Whitney class of S.

Although this construction seems to depend on the choice of the orientation change curve, the thickening does not: it is unique up to a homeomorphism and can be described as the total space of line bundles over the surface with the first Stiefel-Whitney class equal to the Stiefel-Whitney class of the surface.

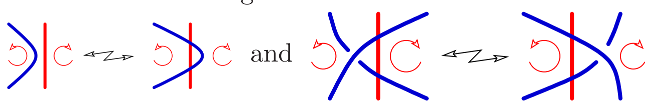
A link in an orientable thickening of a non-orientable surface has a diagram on the surface. Since the fibration of the thickened surface is not orientable one should take special care on the overpasses and underpasses. To distinguish them, one should orient the fiber over the crossing. Since it is impossible to orient all fibers coherently, the most natural solution is to use the orientation of the fibration over the complement of an orientation change curve and keep the curve as shown in the diagram.

To encode the orientation of the thickening in a diagram, one has to orient the complement of an orientation change curve on the base surface. The local orientation of the base and the global orientation of the total space determine the orientation of a fiber. This orientation is used to distinguish overpasses and underpasses.

The orientation of the complement of an orientation change curve on the base surface is not determined by the link in an oriented thickening of a non-oriented surface. It can be reversed on any component of the surface. The reversing switches over and underpassings on this component, but preserves local writhes which are defined as follows.

local writhe $= +1$ local writhe $= -1$ local writhe $= -1$ local writhe $= +1$

A generic isotopy of a link in an orientable thickening of a non-orientable surface decomposes into a sequence of isotopies each of which acts on the diagram either as an isotopy of the surface, or a Reidemeister move, or one of the following two additional moves, in which the orientation change curve is involved:



The first of these moves happens when a piece of link penetrates through the preimage of the orientation change curve. The second happens when a crossing moves through the orientation change curve. Since the orientation of the fiber changes at the moment, the overpass and underpass exchange.

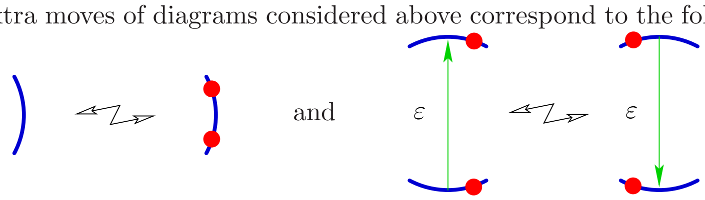
In practice, it is convenient to cut the diagram along the orientation change curve, but keep in mind the identification which would allow to recover the cut. Say, in the case of thickened projective plane, for an orientation change curve one can take the projective line. The cut along it gives a disk, which is much more comfortable to draw on than the projective plane. In this case, the theory sketched above turns into the theory of diagrams for links in the projective space developed by Julia Drobotukhina [6].

In general, links in an oriented thickening of a non-orientable surface were considered by Mario O.Bourgoin in his Research Statement [3]. He also suggested that the corresponding generalization of virtual links, twisted virtual links, and announced a generalization of Kuperberg's theorem to the twisted setup.

We consider the corresponding generalization of Gauss diagrams (rather than generalization of virtual link diagrams outlined by Bourgoin [3]).

Twisted Gauss diagram is a Gauss diagram with a finite set of points marked on the base curve. No marked point coincide with an end-point of an arrow.

The two extra moves of diagrams considered above correspond to the following moves:



2.7. Stripping of the third dimension

Recall that a classical link diagram is a decorated generic projection of the link. The generic projection is a generic immersion. A great piece of what was said above about link diagrams can be repeated, with appropriate simplifications, about generic immersions of a 1-manifold. For short, we call the image of a generic immersion of a closed 1-manifold to a surface S a *generic curve* on S. The immersion gives rise to a parametrization of the curve.

A closed 1-manifold with a finite number of chords connecting pairwise distinct points of the 1-manifold is called a *chord diagram.* The 1-manifold is called the *base* of the chord diagram. A chord diagram, each chord of which is equipped with orientation, is called an *arrow diagram.* A chord diagram, each chord of which is equipped with a sign, is called a *signed chord diagram.* Gauss diagrams considered above are signed arrow diagrams.

A generic immersion of a 1-manifold to a surface defines a chord diagram, in which the base is the source 1-manifold and each chord connects points having the same image. If the source 1-manifold and target surface of the immersion are oriented, the chords get natural orientations: direct a chord from branch A to branch B such that the orientation at the target is defined by the basis formed of vectors which are the images of tangent vectors to A and B defining the orientation of the source and taken in this order.

In the case of a link diagram, these orientations of chords can be obtained from the orientations and local writhes involved in the Gauss diagram by multiplying them: if the sign of the arrow in Gauss diagram is positive, take its orientation intact, if the sign is negative, reverse the orientation.

Not any arrow diagram or even a chord diagram can be generated by a generic immersion to a plane. This was the problem which led Gauss to think how one can recognize which diagrams appear in this way. This problem received solutions in a number of ways, but we refrain from going into this vast matter.

A step from generic curves on an oriented surface parallel to the step from link diagrams on an oriented surface to virtual link diagrams gives rise to *flat virtual knots*, see

D. Hrencecin, L. Kauffman [8]. The counterpart of Reidemeister moves are *flat Reidemeister moves*, see [8].

I am not aware about any counter-part of the Kuperberg Theorem, which would relate flat virtual knots considered up to flat Reidemeister moves to irreducible generic immersions of a 1-manifold to an orientable surface considered up to homeomorphism and homotopy.

3. Kauffman bracket of virtual links

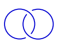
3.1. Digression on Kauffman bracket of classical links

Kauffman bracket of a classical link diagram is a Laurent polynomial in A with integer coefficients

$$\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]$$

For example,

$\langle \text{unknot} \rangle = \langle \bigcirc \rangle = -A^2 - A^{-2}$

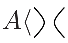
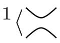
$\langle \text{Hopf link} \rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

$\langle \text{empty link} \rangle = \langle \quad \rangle = 1$

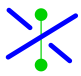
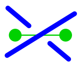
$\langle \text{trefoil} \rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

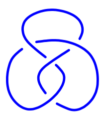
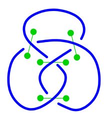
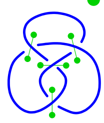
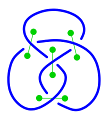
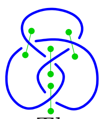
$\langle \text{figure-eight knot} \rangle = \langle$  $\rangle = -A^{10} - A^{-10}$

Kauffman bracket is defined by the following properties:

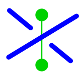
(1) $\langle \bigcirc \rangle = -A^2 - A^{-2}$,
(2) $\langle D \amalg \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle$, (here $\amalg$ means disjoint union)
(3) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$ (*Kauffman Skein Relation*) .

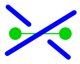
Indeed, applying the last property to each crossing of a link diagram, one reduces the diagram to a collection of embedded circles. Then the first two properties complete the job. This calculation can be summarized in the following *state sum model*.

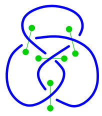
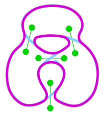
A *state* of diagram is a distribution of *markers* over all crossings. At each crossing of the diagram there should be a marker specifying a pair of vertical angles: , .

For example, the knot diagram:  has states: , , , , ... All together it has 2^c states, where c is the number of crossings.

Three numbers are associated to a state s of a diagram D:

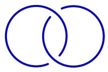
- the number $a(s)$ of *positive* markers ,

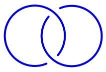
- the number $b(s)$ of ***negative*** markers  ,
- the number $|s|$ of components of the curve D_s obtained by smoothing of D along the markers of s.

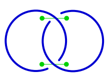
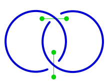
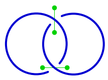
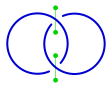
For example, state $s =$  has $a(s) = 1$, $b(s) = 3$, smoothing$(s) =$ , and $|s| = 2$.

The contribution of a state s to Kauffman bracket along with the calculation sketched above is $A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|}$. Finally, the whole Kauffman bracket is equal to the following ***state sum:***

$$\langle D \rangle = \sum_{s \text{ state of } D} A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|}$$

Example: Consider the Hopf link, 

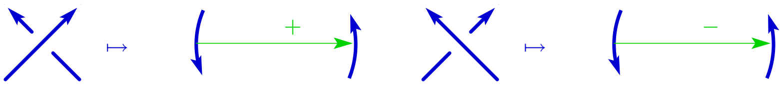
$\Big\langle$  $\Big\rangle =$

$\Big\langle$  $\Big\rangle + \Big\langle$  $\Big\rangle + \Big\langle$  $\Big\rangle + \Big\langle$  $\Big\rangle =$

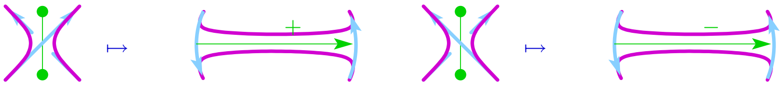
$A^2(-A^2 - A^{-2})^2 + 2(-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2 =$
$(A^6 + 2A^2 + A^{-2}) - 2A^2 - 2A^{-2} + (A^2 + 2A^{-2} + A^{-6}) =$
$A^6 + A^2 + A^{-2} + A^{-6}$.

3.2. Kauffman state sum in terms of Gauss Diagrams

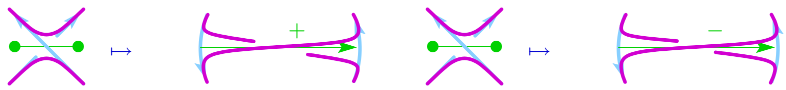
Let us rewrite the state sum above in terms of Gauss diagrams. Recall that to a crossing of a classical link diagram there corresponds an arrow equipped with a sign in the corresponding Gauss diagram.



Clearly, to a smoothing of a crossing there corresponds a surgery along the corresponding arrow:



positive marker, positive crossing   negative marker, negative crossing



negative marker, positive crossing   positive marker, negative crossing

We see that the type of smoothing depends only on two signs, the sign of the marker and the sign of the crossing. Namely, the surgery preserves orientation, if these signs coincide; if the signs are opposite, the surgery reverses orientation.

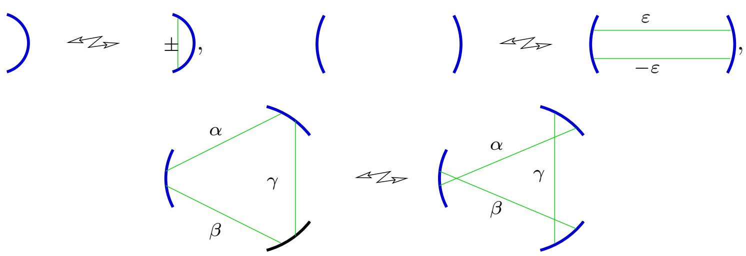
Thus, a state and all the numerical characteristic of a state involved in the Kauffman state sum can be read from the Gauss diagram. A state of Gauss diagram is a distribution of signs (signs of markers) over the set of all of its arrows. The number $|s|$ of components of the curve obtained by the smoothing along markers can be figured out from the Gauss diagram and signs of markers.

Moreover, to recover the Kauffman state sum we do not even need an important ingredient of the Gauss diagram: directions of its arrows. Therefore we can forget about them. Let us check if this is a useful possibility.

3.3. Blunted Gauss diagrams

A Gauss diagram with directions of arrows forgotten is a signed chord diagram. The forgetting of directions is called *blunting*, the result is called a *blunted Gauss diagram*.

Moves of Gauss diagrams defined by Reidemeister moves under blunting turns to the following moves of signed chord diagrams:



We shall call these moves *Reidemeister moves of signed chord diagrams*.

As shown above, Kauffman state sum is defined in terms of the corresponding blunted Gauss diagram. Such a state sum is defined for any signed chord diagram. The classical proof of invariance of the Kauffman bracket under the second and third Reidemeister moves works perfectly in the setup of signed chord diagrams. Moreover, under the first Reidemeister move the Kauffman bracket of a signed diagram behaves exactly as in the classical setup: the positive first move causes multiplication of the Kauffman bracket by $-A^3$, the negative one by $-A^{-3}$.

Thus, the Laurent polynomial

$$f_D(A) = (-A)^{-3w(D)}\langle D\rangle,$$

where $w(D)$ is the sum of signs of all chords of a signed chord diagram D, is invariant under all Reidemeister moves. As well-known, it is closely related to the Jones polynomial in the classical case. However, as everything here works just fine for blunted Gauss diagrams, it can be (and it was) taken as a definition of the Jones polynomial for virtual links.

3.4. Twisted versus blunted

Observe that under each of the additional two moves of twisted Gauss diagrams the signs of arrows do not change.

Therefore if we forget both directions of arrows and points in a twisted Gauss diagram, we get a signed chord diagram and moves of twisted Gauss diagrams turn into Reidemeister moves of signed chord diagrams.

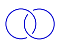
This gives Kauffman bracket and Jones polynomial for twisted virtual links. This has been done in literature, as well. For links in oriented thickened projective plane (which is equivalent to the 3-dimensional projective space $\mathbb{R}P^3$, as the oriented thickened projective plane is the complement of a point in $\mathbb{R}P^3$), the Kauffman bracket and Jones polynomial were defined and studied by Y. V. Drobotukhina [5] in 1990. For general twisted virtual links these polynomials and their refinement was outlined by Bourgoin [3].

At first glance, everything is preserved when we pass from the classical links to virtual and even twisted virtual ones. However, this impression is a little bit misleading. Some properties of the Kauffman bracket and Jones polynomial change drastically.

3.5. Exponents in the Kauffman bracket

As we have seen in the examples

$\langle\text{unknot}\rangle = \langle\bigcirc\rangle = -A^2 - A^{-2}$

$\langle\text{Hopf link}\rangle = \langle$  $\rangle = A^6 + A^2 + A^{-2} + A^{-6}$

$\langle\text{empty link}\rangle = \langle\ \rangle = 1$

$\langle\text{trefoil}\rangle = \langle$  $\rangle = A^7 + A^3 + A^{-1} - A^{-9}$

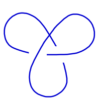
$\langle\text{figure-eight knot}\rangle = \langle$  $\rangle = -A^{10} - A^{-10}$,

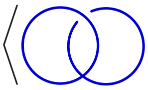
exponents in Kauffman bracket of a classical link are congruent to each other modulo 4.

This happens because:

- the contribution $A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|}$ of each state has this property,
- from each state we can get to any other state changing one marker a time,
- and changing a single marker in a state changes $a(s) - b(s)$ by 2 and $|s|$ by 1.

However, this is not the case for virtual links. For example,

$\Big\langle$  $\Big\rangle = A^{-4} + A^{-6} - A^{-10}$

$\Big\langle$  $\Big\rangle = A + A^{-1}$

What is wrong in the proof above if the link is virtual?

- Still, in the contribution $A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|}$ of each state all the exponents are congruent to each other modulo 4.
- Still, from each state we can get to any other state changing one marker a time.
- Still, changing a single marker changes $a(s) - b(s)$ by 2. However, $|s|$ may be preserved.

Change of a single marker causes a Morse modification of the result of smoothing. In the classical case this *Morse modification is embedded in plane, and therefore preserves orientation.*

As well-known, *an orientation preserving Morse modification of a 1-manifold changes the number of components by one. A Morse modification, which does not preserve orientation of a connected 1-manifold preserves the number of connected components.* A Morse modification which does not preserve orientation would cause a shift of exponents by 2.

4. Orientations of chord diagrams

4.1. Orientation of a chord diagram

Which structure on a blunted Gauss diagram would guarantee that, for any of its states, the corresponding smoothing has an orientation such that the change of any marker causes a Morse modification of the result of smoothing *preserving* the orientation?

Such a structure is a collection of orientations of arcs of the base 1-manifold between end points of chords such that these orientations define an orientation of each smoothing. For this, the orientations should have the following two properties:

- The orientations cannot be extended over an end point of a chord,
- For each chord, one of its end points is attractive, while the other one is repulsive.

See Figure 5.

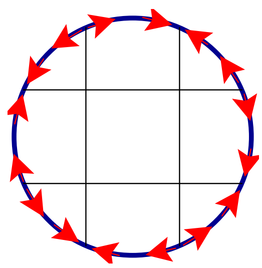


Figure 5. Orientation of a chord diagram.

The first property means that the orientations of arcs induce anonymously an orientation of each end point of each chord. The second property means that for each chord the induced orientations of its end points are opposite to each other.

Orientations of arcs of a chord diagram satisfying these two properties is called an *orientation* of a chord diagram.

Orient chords so that the induced orientations of their end points are opposite to the orientations induced by orientations of adjacent arcs. See Figure 6.

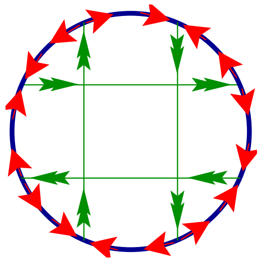


FIGURE 6. Orientations of chords in an oriented chord diagram.

Chords become arrows. Orientations of the arrows alternate in the sense that along the base of the diagram each arrowhead is followed by an arrowtail and each arrowtail is followed by an arrowhead.

Observe that an orientation of a chord diagram defines an orientation on the result of any of its smoothings. Indeed, the result of a smoothing is composed from arcs of the base and arcs which go along chords. The orientations of these pieces agree with each other.

4.2. Alternatable virtual link diagrams

Observe that directions of arrows in a Gauss diagram of a classical alternating link diagram alternate in the same way as the orientations of chords considered above.

Recall that a classical link diagram is called *alternating*, if along a branch of the link overcrossings would always follow undercrossings and undercrossings would always follow overcrossings.

Similarly, a virtual link is called *alternating*, if along a branch of the link over-crossing always follow after under-crossing and under-crossing always follow after over-crossing. By switching some overpasses and underpasses, a virtual link diagram such that its blunted Gauss diagram is orientable, can be made alternating. Vice versa, if, by switching some overpasses and underpasses, a virtual link diagram can be made alternating, its blunted Gauss diagram is orientable.

Therefore a virtual link diagram with orientable blunted Gauss diagram is called *alternatable*.

4.3. Moves of chord diagrams

Transformations of a chord diagram shown in Figure 7 are called *Reidemeister moves*.

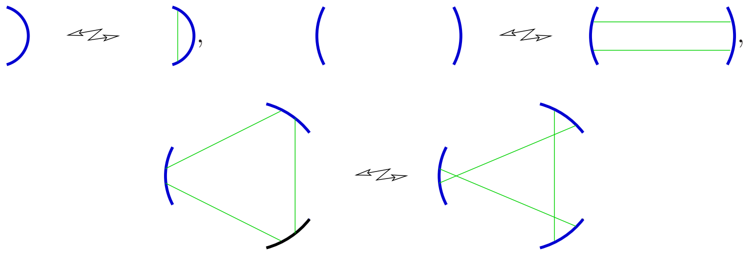
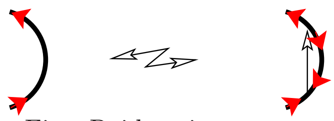


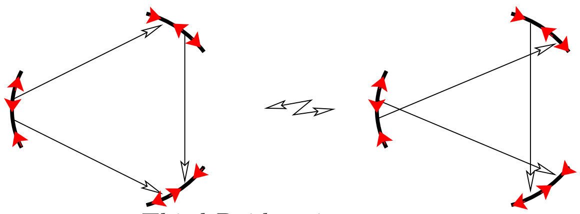
FIGURE 7. Reidemeister moves of chord diagrams

Theorem 4.A. *The result of any first or third Reidemeister move applied to an orientable chord diagram is orientable.*

Proof. An orientation of a chord diagram prior to the move in the fragment which is about to change is unique up to reversing. Its replacement admits an orientation coinciding with the original one near the boundary. See Figure 8. □



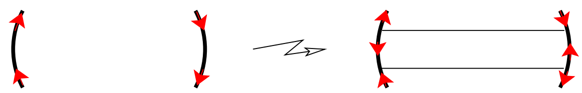
First Reidemeister move.



Third Reidemeister move.

FIGURE 8. Behavior of orientation of a signed chord diagram under first and third Reidemeister moves.

The second Reidemeister move can destroy orientability. Of course, a second Reidemeister move can transform an orientable chord diagram to an orientable one. All second Reidemeister moves decreasing the number of chords and about half of all second Reidemeister moves increasing it which can be applied to a chord diagram preserve orientability. One can easily recognize if a second Reidemeister move preserves orientability, see Figure 9.



An orientation surviving under a second Reidemeister move

Second Reidemeister move distroys an orientation

FIGURE 9. Behavior of orientations of a chord diagram under second Reidemeister moves.

4.4. Checkerboard coloring of a classical link diagram

Let us analyze where the orientation of a chord diagram underlying a Gauss diagram of a classical link comes from.

A generic curve on a plane can be considered as a 1-cycle with coefficients in $\mathbb{Z}_2$ on the plane. As a plane has trivial homology, this 1-cycle bounds a 2-chain. The latter is described as the union of all black domains in the checkerboard coloring of the diagram.

Each connected component of the complement of a generic curve inherits an orientation from the whole plane. These orientations of the black domains induce orientations on their boundaries. The boundary of a black domain consists of pieces of the curve. Thus an arc of the curve between any two consecutive double points gets a natural orientation. See Figure 10. It is called a *checkerboard orientation.*

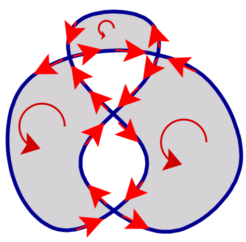


FIGURE 10. Checkerboard coloring and orientations.

Obviously, the checkerboard orientation gives orientation of the corresponding chord diagram. The orientation induced by the orientation of a chord diagram on the result of a smoothing of the curve can also be obtained as a checkerboard orientation.

Indeed, the result of a smoothing also admits a checkerboard coloring which coincides with the checkerboard coloring of the diagram outside small neighborhoods of its double points. The orientation induced on the smoothing by the orientations of the black domains agree with the checkerboard orientation of the diagram. See Figure 11.

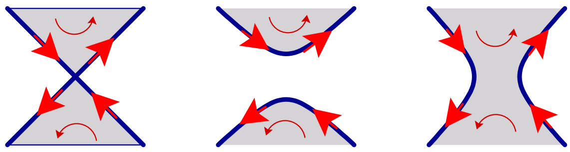


FIGURE 11. Checkerboard orientation of smoothings.

4.5. On orientable surfaces

The same arguments work for a link diagram zero homologous modulo 2 on an oriented surface. An orientable surface with a zero homologous generically immersed curve admits a checkerboard coloring. We see that:

Theorem 4.B. *The chord diagram of a generic curve on an orientable surface admitting a checkerboard coloring is orientable.*

Corollary (N. Kamada [9])***.*** *A link diagram on an orientable surface admitting a checkerboard coloring is alternatable.* □

Corollary (N. Kamada [9])***.*** *Exponents of monomials in the Kauffman bracket of a checkerboard colorable link diagram on an oriented surface are congruent to each other modulo 4.* □

It may happen that an alternatable link diagram on an orientable surface does not admit a checkerboard coloring. Indeed, any diagram admitting a checkerboard coloring can be spoiled by a Morse modification of index 1, that is, removing two disjoint open disks and attaching a tube connecting the boundary circles. For this, take the disks in the complementary domains colored with different colors. Of course, this stabilization of a virtual link diagram does not destroy alternatability, which is a property of Gauss diagrams.

Nonetheless, according to the following theorem, which is also basically due to Naoko Kamada [9], this cannot happen to irreducible diagram.

Theorem 4.C. *A generic curve on an orientable surface such that each connected component of its complement is a disk and its chord diagram is orientable admits a checkerboard coloring.*

Corollary (N. Kamada [9])***.*** *An alternatable link diagram on an orientable surface such that each connected component of its complement is a disk admits a checkerboard coloring.*

Proof of Theorem 4.C. An orientation of the chord diagram gives rise to an orientation of the boundary of each connected component of the complement of the curve. Since each such component is a disk, the orientation of its boundary orients the component itself. On the other hand, the orientation of the whole surface induces an orientation on each of these components. For some of the components these two orientations coincide, for the others they are opposite to each other. The components for which the orientations coincide form a chain modulo 2 bounded by the curve. By coloring these components in black and the others in white, we get the desired checkerboard coloring of the surface. □

The condition about components of the complement can be weakened: instead of requiring that they are homeomorphic to a disk, it suffices to require that the intersection of the curve with the closure of each of the components is connected.

4.6. Alternatable virtual links

Recall that a virtual link diagram which gives rise to an orientable blunted Gauss diagram is called *alternatable*.

Theorem 4.D (Corollary of Theorem *4.A*)**.** *The result of any first or third Reidemeister move applied to an alternatable virtual link diagram is alternatable.* □

A second Reidemeister move can turn an alternatable virtual link diagram to a non-alternatable diagram, see Figure 12.

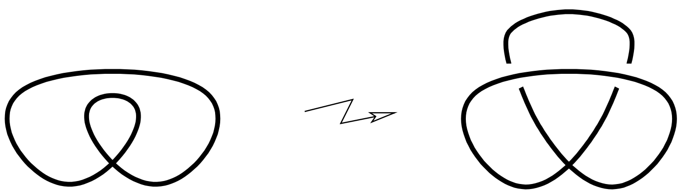


Figure 12. Creating a non-alternatable virtual knot diagram from a virtual diagram of unknot with single virtual crossing by a second Reidemeister move.

Moreover, each alternatable virtual diagram can be made non-alternatable by a single second Reidemeister move. Therefore alternatability is not a property shared by all virtual diagrams of a given virtual link.

Alternatable isotopy is a sequence of Reidemeister moves preserving alternatability and virtual moves of an alternatable virtual link diagram.

Theorem 4.E. *Alternatable virtual link diagrams are virtualy isotopic, if and only if they can be related by an alternatable isotopy.*

Proof. Let D_1 and D_2 be virtually isotopic alternatable link diagrams. Realize each of them as a diagram admitting a checkerboard coloring on an oriented closed surface.

According to the Kuperberg Theorem [13], virtually isotopic link diagrams can be destabilized to link diagrams on an oriented surface S, where they can be related by embedded moves. A destabilization consists of embedded Reidemeister moves and Morse modifications of index 2 of the ambient surface along a circle disjoint with the diagram.

Neither an embedded Reidemeister move, nor a Morse modification of index 2 can destroy a checkerboard coloring. A Morse modification of index 2 does not change the Gauss diagram. Therefore the sequence of moves connecting the Gauss diagrams corresponding to D_1 and D_2 which correspond to the Reidemeister moves existing by the Kuperberg Theorem constitute an alternatable isotopy. $\square$

Thus alternatable virtual links do not form a new category of objects similar to virtual links, but are virtual links of special kind. They have special properties. For example, the exponents of Kauffman bracket are congruent to each other modulo 4, see the second Corollary of Theorem *4.B* above. From the point of view of 3-dimensional topology, they can be characterized as irreducible links in thickened oriented surfaces which realize trivial homology class modulo 2.

4.7. On non-orientable surfaces

Theorem *4.B* and its corollary cannot be extended literally to twisted link diagrams on non-orientable surfaces . For example, link 2_1 on the projective plane has checkerboard colorable diagram, but its Kauffman bracket is $A^{-4} + A^{-6} - A^{-10}$, see [5].

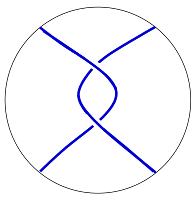
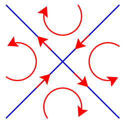


Figure 13. Projective link 2_1

However, a generalization recovers, if the property of being zero homologous modulo 2 is generalized in an other way.

Theorem 4.F. *The chord diagram of a generic curve realizing the homology class dual to the first Stiefel-Whitney class of the surface is orientable.*

Proof. Let C be a generic curve on a surface S realizing a homology class dual to $w_1(S) \in H^1(S;\mathbb{Z}_2)$. The complement $S \smallsetminus C$ admits an orientation which cannot be extended over C at any point. Each arc of C is involved in C with multiplicity 2 in the boundary of the corresponding fundamental class $[S \smallsetminus C]$. At a double point the orientations of $S \smallsetminus C$ and arcs of C look as follows:

.

Thus the orientations of arcs of C gives rise to an orientation of the corresponding chord diagram. $\square$

Corollary. *Exponents of monomials in the Kauffman bracket of a link which is non-zero homologous in the real projective space are congruent to each other modulo 4.* $\square$

Theorem 4.G (Generalization of Theorem *4.C*)***.*** *If the chord diagram of a generic curve on a surface is orientable and each connected component of its complement on the surface is a disk then the curve realizes the homology class dual to the first Stiefel-Whitney class of the surface.*

Proof. An orientation of the chord diagram gives rise to an orientation of the boundary of each connected component of the complement of the curve. Since each such component is a disk, the orientation of its boundary orients the component itself.

The orientations of the components of the complement form an orientation of the whole complement, which does not extend over the curve, because from both sides of the curve it induces the same orientation on each piece of the curve. Therefore the curve realizes the obstruction to orientability of the surface. $\square$

4.8. Obstruction to orientability of a chord diagram

Near each chord of a chord diagram an orientation of the diagram looks standard, up to simultaneous reversing. For any choice of the orientation, on each arc of the base either the orientations at the end points extend to the whole arc, or not. Thus for any choice of orientations of chords there is a well-defined $\mathbb{Z}/2\mathbb{Z}$-valued function of the set of arcs of the base. It can be considered as a 1-cochain with values in $\mathbb{Z}/2\mathbb{Z}$ on the chord diagram. Reversing of a single chord's orientation causes change of values on each of the four adjacent arcs.

This means that there is a well-defined 1-dimensional cohomology class with values in $\mathbb{Z}/2\mathbb{Z}$ of the underlying space of the chord diagram (the union of the base and chords). This class is zero, if and only if the diagram is orientable. Therefore we call it the *obstruction to orientability.*

The class is defined by topology of the pair consisting of the underlying space and the union of all chords. It is not defined by the homotopy type of the underlying space, although it belongs to a cohomology group depending only on the homotopy type. It is not defined by the topology of the 4-valent graph obtained by contracting each of the chords, either. Indeed, it depends on division of arcs adjacent to a vertex to pairs of opposite arcs, that is, arcs adjacent to one end point of the chord in the underlying space of the chord diagram.

The simplest example is a circle with two chords intersecting each other, on one hand, and two circles connected with two chords, on the other hand. The quotient 4-valent graphs are homeomorphic.

5. Khovanov homology of oriented signed chord diagrams

5.1. Khovanov homology of classical links

In 1998 Khovanov [12] categorified Jones polynomial. His construction is a refinement of the Kauffman bracket construction. To any classical link diagram D it associates a bi-graded chain complex of abelian groups $C^{i,j}(D)$ with differential $d: C^{i,j}(D) \to C^{i+1,j}(D)$ and homology groups $\mathcal{H}^{i,j}(D)$ such that the Reidemeister moves of the diagram induce homotopy equivalences of the complex. The Khovanov homology $\mathcal{H}^{i,j}(D)$ categorifies the Jones polynomial $f_D(A)$ in the sense that

$$f_D(A) = \sum_{i,j} (-1)^{i+j} A^{-2j} \operatorname{rk} \mathcal{H}^{i,j}(D).$$

This slightly strange formula is due to the fact that Khovanov used different normalization of the Jones polynomial, namely $K(D)(q) = f_D(\frac{1}{\sqrt{-q}})$. Ranks of the Khovanov homology groups are related to K as follows:

$$K(D)(q) = \sum_{i,j} (-1)^i q^j \operatorname{rk} \mathcal{H}^{i,j}(D).$$

For any state s of D, the construction associates to each connected component C of D_s a copy $\mathcal{A}_C$ of a graded free abelian group $\mathcal{A}$ with two generators, 1 of grading 1 and x of grading -1. To the whole s the construction associates a graded group $C(s)$ which is the tensor product of all copies $\mathcal{A}_c$ of $\mathcal{A}$ associated to the connected components of D_s with grading shifted by $\frac{3w(D)-a(s)+b(s)}{2}$, where $w(D)$ is the writhe of D.

For a state s of diagram D denote $\frac{w(D)-a(s)+b(s)}{2}$ by $i(s)$ and consider $C(s)$ as a bigraded group with the first grading to be identically equal to $i(s)$ and the second grading as defined above (that is, the grading of the tensor product shifted by $\frac{3w(D)-a(s)+b(s)}{2}$). Denote by $\mathcal{C}(D)$ the bigraded group $\sum_s C(s)$. This is the total group of the Khovanov chain complex.

To define a differential, we need to fix an order of all crossings of the diagram. We also need to fix the multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defined by the formulas $1 \otimes 1 \mapsto 1$, $1 \otimes x \mapsto x$, $x \otimes 1 \mapsto x$ and $x \otimes x \mapsto 0$, and co-multiplication $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by formulas $1 \mapsto 1 \otimes x + x \otimes 1$ and $x \mapsto x \otimes x$.

The differential $C^{i,j} \to C^{i+1,j}$ is defined as the sum of partial differentials $C^j(s) \to C^j(t)$, where s and t are states with $i(s) = i$, $i(t) = i+1$ such that t differs from s only by a marker at one crossing. Denote this crossing by x and its number by k. Denote by r the number of negative markers of s at crossings with numbers greater than k. At x the marker of s is positive, while the marker of t is negative.

Let two connected components of D_s pass near x. Denote them by C_1 and C_2. Then there is only one connected component of D_t passing near x. Denote this component by C. Then $C(s) = \mathcal{A}_{C_1} \otimes \mathcal{A}_{C_2} \otimes B$ and $C(t) = \mathcal{A}_C \otimes B$ and the partial differential $C(s) \to C(t)$ is defined by the formula $(-1)^r m \otimes id_B$.

Let only one component of D_s pass near x. Denote it by C. Then two components of D_t pass near x. Denote them by D_{C_1} and D_{C_2}. Then $C(s) = \mathcal{A}_C \otimes B$ and $C(t) = \mathcal{A}_{C_1} \otimes \mathcal{A}_{C_2} \otimes B$ and the partial differential $C(s) \to C(t)$ is defined by the formula $(-1)^r \Delta \otimes id_B$.

5.2. Khovanov complex of an oriented signed chord diagram

It is clear that the construction of Khovanov complex sketched above can be expressed completely in terms of signed chord diagram, as it was done with the Kauffman bracket in Section 3.2.

If a signed chord diagram is orientable, all properties of its Khovanov complexes proven in [12] or [2] can be repeated without any change. In particular, this is a complex. The only property of classical link diagrams that is used in the proof of this is that change of a single marker in a state s gives rise to a change of the number of connected components in D_s, and this is a corollary of orientability of the chord diagram.

Furthermore, the Reidemeister moves of signed chord diagrams induce homotopy equivalences of the Khovanov complex, provided they preserve orientation. The proof is also borrowed from the classical case without any change.

According to Theorem *4.E*, if alternatable virtual diagrams are virtualy isotopic then they can be related by an alternatable isotopy. Therefore, the homotopy type of the Khovanov complex and, in particular, Khovanov homology groups are invariants of alternatable virtual links.

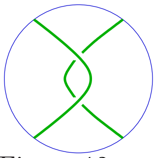
5.3. Failure in the non-orientable case

If the signed chord diagram is not orientable, there is no problem to extend the definition of the differential, because the partial differentials along a change of marker destroys orientation of D_s and hence preserving the number of components vanish for grading reasons.

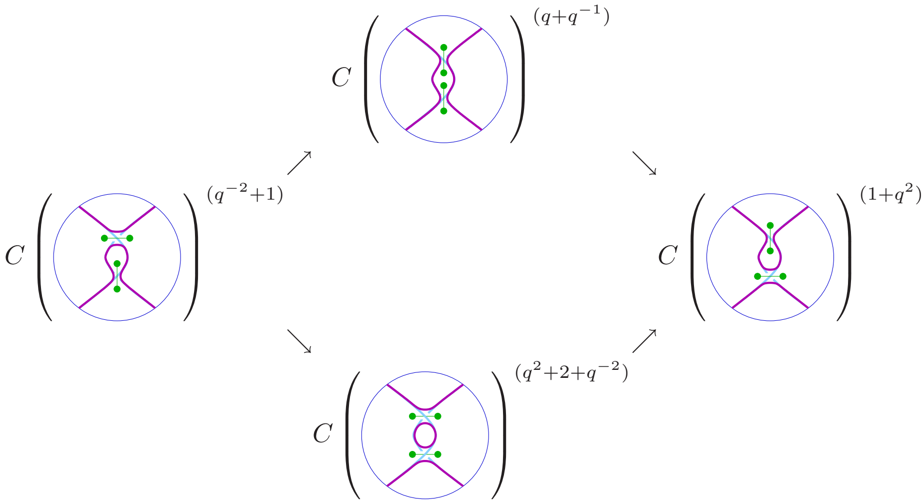
Indeed, $C(s)$ is obtained from $\mathcal{A}^{|s|}$ by a shift of the second grading by $\frac{3w(D)-a(s)+b(s)}{2}$. The graded rank [2] of $C(s)$ is $q^{\frac{3w(D)-a(s)+b(s)}{2}}(q+q^{-1})^{|s|}$. All exponents of this graded rank are congruent to each other modulo 2. When $a(s)$ and $b(s)$ change by one and $|s|$ does not change, the graded rank is multiplied by q, and parity of all exponents changes by 1. Thus any homomorphism preserving the grading is trivial.

However, the homomorphism $\mathcal{C}(D) \to \mathcal{C}(D)$ obtained in this way is not a differential: its square is not zero. This can be easily seen in the very simple example.

[2] Recall that the graded rank of a finitely generated graded group $W = \oplus_j W_j$ is the Laurent polynomial $\Sigma_j q^j \operatorname{rk} W_j$.

The unknot in $\mathbb{R}P^3$ is isotopic to . The corresponding virtual link diagram is shown on the right hand side of Figure 12.

Consider the Khovanov chain groups. There are 4 states giving rise to Khovanov chain groups which looks as follows:



Here the graded ranks are shown as exponents. For the grading reasons (the partial differentials are of degree 0), the upper arrows should be zero. Therefore the composition of the bottom two arrows should be zero. Otherwise the square of the differential would not be zero.

But it is non-trivial in the component with grading 0. Indeed, the first of them maps 1 to $\Delta(1) = 1 \otimes x + x \otimes 1$, then the second one maps $1 \otimes x$ to $m(1 \otimes x) = x$ and $x \otimes 1$ to x. Hence the composition sends 1 to $2x$.

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