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On moduli of pointed real curves of genus zero

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Abstract. We introduce the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ of pointed real curves of genus zero and give its natural stratification. The strata of $\mathbb{R}\overline{M}_{(2k,l)}$ correspond to real curves of genus zero with different degeneration types and are encoded by trees with certain decorations. By using this stratification, we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ and construct the orientation double cover $\mathbb{R}\widetilde{\overline{M}}_{(2k,l)}$ of $\mathbb{R}\overline{M}_{(2k,l)}$.

1. Introduction

The moduli space $\overline{M}_n$ of stable n-pointed (complex) curves of genus zero has been extensively studied as one of the fundamental models of moduli problems in algebraic geometry (see [12, 15, 13, 14, 18]). The moduli space $\overline{M}_n$ carries a set of anti-holomorphic involutions, whose fixed point sets are the moduli spaces of pointed real curves of genus zero. These moduli spaces parameterize the isomorphism classes of pointed curves of genus zero with a real structure. For each of these spaces, a certain set of labeled points stays in the real parts of the curves while other pairs of labeled points are conjugated by the real structures of the curves.

The moduli spaces of pointed real curves have recently attracted attention in various contexts such as multiple ζ-motives [7], representations of quantum groups [4, 9] and Welschinger invariants [22, 23].

The aim of this work is to explore the topological properties of the moduli spaces of pointed real curves of genus zero. Hence, we first introduce a natural combinatorial stratification of the moduli spaces of pointed real curves of genus zero through the stratification of $\overline{M}_n$. Each stratum is determined by the degeneration type of the real curve. They are identified with the product of the spaces of real point configurations in the projective line $\mathbb{CP}^1$ and the moduli spaces $\overline{M}_m$. The degeneration types of the pointed real curves are encoded by trees with corresponding decorations. Secondly, we calculate the first Stiefel-Whitney classes of the moduli spaces in terms of their stratifications. The moduli spaces of pointed real curves are not orientable for $n \geq 5$ and the set of labeled real points of real curves is not empty. We construct the orientation double covers of the moduli spaces for the non-orientable cases. The double covering in this work significantly differs from the 'double covering' in the recent literature on open Gromov-Witten invariants and moduli spaces of pseudoholomorphic discs (see [5, 17]): Our double covering has no boundaries which suits better for the use of intersection theory.

This paper is organized as follows. Section 2 contains a brief overview of facts about the moduli space $\overline{M}_n$. In Section 3, we introduce real structures on $\overline{M}_n$ and real parts $\mathbb{R}\overline{M}_{(2k,l)}$ as moduli spaces of $(2k,l)$-pointed real curves. The following section, the stratification of $\mathbb{R}\overline{M}_{(2k,l)}$ is given according to the degeneration types of pointed real curves of genus zero. In Section 5, the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is calculated by using the stratification given in Section 4. Then in Section 6, we construct the orientation double coverings $\mathbb{R}\widetilde{M}_{(2k,l)} \to \mathbb{R}\overline{M}_{(2k,l)}$.

In this paper, the genus of the curves is zero except when the contrary is stated explicitly. Therefore, we omit mentioning the genus of the curves.

2. Pointed complex curves and their moduli

This section reviews the basic facts on pointed complex curves of genus zero and their moduli space.

2.1. Pointed curves and their trees

Definition 2.1. An *n-pointed curve* $(\Sigma; \mathbf{p})$ is a connected complex algebraic curve Σ with distinct, smooth, *labeled points* $\mathbf{p} = (p_1, \cdots, p_n) \subset \Sigma$, satisfying the following conditions:

- Σ has only nodal singularities.
- The arithmetic genus of Σ is equal to zero.

A *family of n-pointed curves* over a complex manifold S is a proper, holomorphic map $\pi_S : \mathcal{U}_S \to S$ with n sections $p_1, \cdots, p_n$ such that each geometric fiber $(\Sigma(s); \mathbf{p}(s))$ is an n-pointed curve.

Two such curves, $(\Sigma; \mathbf{p})$ and $(\Sigma'; \mathbf{p}')$, are *isomorphic* if there exists a bi-holomorphic equivalence $\Phi : \Sigma \to \Sigma'$ mapping p_i to p'_i.

An n-pointed curve is *stable* if its automorphism group is trivial (i.e., on each irreducible component, the number of singular points plus the number of labeled points is at least three).

2.1.1. Graphs

Definition 2.2. A *graph* Γ is a collection of finite sets of *vertices* V_Γ and *flags (or half edges)* F_Γ with a boundary map $\partial_\Gamma : F_\Gamma \to V_\Gamma$ and an involution $j_\Gamma : F_\Gamma \to F_\Gamma$ ($j_\Gamma^2 = id$). We call $E_\Gamma = \{(f_1, f_2) \in F_\Gamma^2 \mid f_1 = j_\Gamma f_2 \ \&\ f_1 \neq f_2\}$ the set of *edges*, and $T_\Gamma = \{f \in F_\Gamma \mid f = j_\Gamma f\}$ the set of *tails*. For a vertex $v \in V_\Gamma$, let $F_\Gamma(v) = \partial_\Gamma^{-1}(v)$ and $|v| = |F_\Gamma(v)|$ be the *valency* of v.

We think of a graph Γ in terms of its following *geometric realization* $||\Gamma||$: Consider the disjoint union of closed intervals $\bigsqcup_{f_i \in F_\Gamma} [0,1] \times f_i$ and identify $(0, f_i)$ with $(0, f_j)$ if $\partial_\Gamma f_i = \partial_\Gamma f_j$, and identify (t, f_i) with $(1-t, j_\Gamma f_i)$ for $t \in]0,1[$ and $f_i \neq j_\Gamma f_i$. The geometric realization of Γ has a piecewise linear structure.

Definition 2.3. A *tree* γ is a graph whose geometric realization is connected and simply-connected. If $|v| > 2$ for all $v \in V_\gamma$, then such a tree is called *stable*.

We associate a subtree γ_v for each vertex $v \in V_\gamma$ which is given by $V_{\gamma_v} = \{v\}, F_{\gamma_v} = F_\gamma(v)$, $j_{\gamma_v} = id$, and $\partial_{\gamma_v} = \partial_\gamma$.

Definition 2.4. Let γ and τ be trees with n tails. A *morphism* between these trees $\phi : \gamma \to \tau$ is a pair of maps $\phi_F : F_\tau \to F_\gamma$ and $\phi_V : V_\gamma \to V_\tau$ satisfying the following conditions:

- ϕ_F is injective and ϕ_V is surjective.
- The following diagram commutes

$$\begin{array}{ccc} F_\gamma & \xrightarrow{\partial_\gamma} & V_\gamma \\ {\scriptstyle \phi_F}\uparrow & & \downarrow{\scriptstyle \phi_V} \\ F_\tau & \xrightarrow{\partial_\tau} & V_\tau . \end{array}$$

- $\phi_F \circ j_\tau = j_\gamma \circ \phi_F$.
- $\phi_T := \phi_F|_T$ is a bijection.

An *isomorphism* $\phi : \gamma \to \tau$ is a morphism where ϕ_F and ϕ_V are bijections. We denote the isomorphic trees by $\gamma \approx \tau$.

Each morphism induces a piecewise linear map on geometric realizations.

Lemma 2.1. *Let γ and τ be stable trees with n tails. Any isomorphism $\phi : \gamma \to \tau$ is uniquely defined by its restriction on tails $\phi_T : T_\tau \to T_\gamma$.*

Proof. Let $\phi, \varphi : \gamma \to \tau$ be two isomorphisms such that their restriction on tails are the same. Consider the path ${}_{f_1}P_{f_2}$ in $||\gamma||$ that connects a pair of tails f_1, f_2. The automorphism $\varphi^{-1} \circ \phi$ of γ maps ${}_{f_1}P_{f_2}$ to itself; otherwise, the union of the ${}_{f_1}P_{f_2}$ and its image $\varphi^{-1} \circ \phi({}_{f_1}P_{f_2})$ gives a loop in $||\gamma||$, which contradicts simply-connectedness. Moreover, the restriction of $\varphi^{-1} \circ \phi$ to the path ${}_{f_1}P_{f_2}$ is the identity map since it preserves distances of vertices to tails f_1, f_2. This follows from the compatibility of the automorphism $\varphi^{-1} \circ \phi$ with ∂_γ and j_γ.

The geometric realization $||\gamma||$ of γ can be covered by paths that connects pairs of tails of γ. We conclude that the automorphism $\varphi^{-1} \circ \phi$ is the identity since it is the identity on every such path. □

There are only finitely many isomorphisms classes of stable trees whose set of tails is $T_\gamma = \{1, \cdots, n\}$. We call the isomorphism classes of such trees *n-trees*. We denote the set of all n-trees by $\mathcal{T}ree$.

2.1.2. Dual trees of pointed curves

Let $(\Sigma; \mathbf{p})$ be an n-pointed curve and $\eta : \hat{\Sigma} \to \Sigma$ be its normalization. Let $(\hat{\Sigma}_v; \hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve: $\hat{\Sigma}_v$ is a component of $\hat{\Sigma}$, and $\hat{\mathbf{p}}_v$ is the set of points consisting of the preimages of *special* (i.e. labeled and nodal) points on $\Sigma_v := \eta(\hat{\Sigma}_v)$. The points $\hat{\mathbf{p}}_v = (p_{f_1}, \cdots, p_{f_{|v|}})$ on $\hat{\Sigma}_v$ are ordered by the flags $f_* \in F_\tau(v)$.

Definition 2.5. The *dual tree* γ of an n-pointed curve $(\Sigma; \mathbf{p})$ is the tree consisting of following data:

- V_γ is the set of components of $\hat{\Sigma}$.
- F_γ is the set consisting of the preimages of special points.
- $\partial_\gamma : f \mapsto v$ if and only if $p_f \in \hat{\Sigma}_v$.
- $j_\gamma : f \mapsto f$ if and only if p_f is a labeled point, and $j_\gamma : f_1 \mapsto f_2$ if and only if $p_{f_1} \in \hat{\Sigma}_{v_1}$ and $p_{f_2} \in \hat{\Sigma}_{v_2}$ are the preimages of the nodal point $\Sigma_{v_1} \cap \Sigma_{v_2}$.

Lemma 2.2. *Let* Φ *be an isomorphism between the* n*-pointed stable curves* $(\Sigma; \mathbf{p})$ *and* $(\Sigma'; \mathbf{p}')$*.*
(i) Φ *induces an isomorphism* ϕ *between their dual trees* γ, τ*.*
(ii) Φ *is uniquely defined by its restriction on labeled points.*

Proof. (i) The result follows from the decomposition of Φ into its restriction to each irreducible component and the Def. 2.4.

(ii) Due to Lemma 2.1, the isomorphism $\phi : \gamma \to \tau$ is determined by the restriction of Φ to the labeled points. The isomorphism ϕ determines which component of Σ is mapped to which component of Σ' as well as the restriction of Φ to the special points. Each component of Σ is rational and has at least three special points. Therefore, the restriction of Φ to a component is uniquely determined by the images of the three special points. □

2.2. Deformations of pointed curves

Let γ be the dual tree of $(\Sigma; \mathbf{p})$ and $\hat{\Sigma} \to \Sigma$ be the normalization. Let $(\hat{\Sigma}_v; \hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve corresponding to the irreducible component Σ_v of $(\Sigma; \mathbf{p})$. Let Ω^1_Σ be the sheaf of Kähler differentials.

The infinitesimal deformations of a nodal curve Σ with divisor $D_{\mathbf{p}} = p_1 + \cdots + p_n$ is canonically identified with the complex vector space

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma), \tag{1}$$

and the obstruction lies in

$$Ext^2_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}), \mathcal{O}_\Sigma).$$

In this case, it is known that there are no obstructions (see, for example [17] or [8]).

The space of infinitesimal deformations is the tangent space of the space of deformations at $(\Sigma; \mathbf{p})$. It can be written explicitly in the following form:

$$\bigoplus_{v \in V_\gamma} H^1(\hat{\Sigma}_v, T_{\hat{\Sigma}_v}(-D_{\hat{\mathbf{p}}_v})) \oplus \bigoplus_{(f_e, f^e) \in E_\gamma} T_{p_{f_e}} \hat{\Sigma} \otimes T_{p_{f^e}} \hat{\Sigma}. \tag{2}$$

The first part corresponds to the equisingular deformations of Σ with the divisor $D_{\hat{\mathbf{p}}_v} = \sum_{f_i \in F_\gamma(v)} p_{f_i}$, and the second part corresponds to the smoothing of nodal points p_e of the edges $e = (f_e, f^e)$ (see [8]).

2.2.1. Combinatorics of degenerations

Let $(\Sigma;\mathbf{p})$ be an n-pointed curve with the dual tree γ. Consider the deformation of a nodal point of $(\Sigma;\mathbf{p})$. Such a deformation of $(\Sigma;\mathbf{p})$ gives a *contraction* of an edge of γ: Let $e=(f_e,f^e)\in E_\gamma$ be the edge corresponding to the nodal point and $\partial_\gamma(e)=\{v_e,v^e\}$, and consider the equivalence relation $\sim$ on the set of vertices, defined by: $v\sim v$ for all $v\in V_\gamma\setminus\{v_e,v^e\}$, and $v_e\sim v^e$. Then, there is an n-tree γ/e whose vertices are $V_\gamma/\sim$ and whose flags are $F_\gamma\setminus\{f_e,f^e\}$. The boundary map and involution of γ/e are the restrictions of ∂_γ and j_γ.

We use the notation $\gamma<\tau$ to indicate that τ is obtained by contracting some edges of γ.

2.3. Stratification of the moduli space $\overline{M}_n$

The moduli space $\overline{M}_n$ is the space of isomorphism classes of n-pointed stable curves. This space is stratified according to degeneration types of n-pointed stable curves which are given by n-trees. The principal stratum M_n corresponds to the one-vertex n-tree and is the quotient of the product $(\mathbb{CP}^1)^n$ minus the diagonals $\Delta=\bigcup_{k<l}\{(p_1,\cdots,p_n)|p_k=p_l\}$ by $Aut(\mathbb{CP}^1)=PSL_2(\mathbb{C})$.

Theorem 2.3 (Knudsen & Keel, [15, 12])**.** *(i) For any $n\geq 3$, $\overline{M}_n$ is a smooth projective algebraic variety of (real) dimension $2n-6$.*

(ii) Any family of n-pointed stable curves over S is induced by a unique morphism $\kappa:S\to\overline{M}_n$. The universal family of curves $\overline{\mathcal{U}}_n$ of $\overline{M}_n$ is isomorphic to $\overline{M}_{n+1}$.

(iii) For any n-tree γ, there exists a quasi-projective subvariety $D_\gamma\subset\overline{M}_n$ parameterizing the curves whose dual tree is given by γ. D_γ is isomorphic to $\prod_{v\in V_\gamma}M_{|v|}$. Its (real) codimension is $2|E_\gamma|$.

(iv) $\overline{M}_n$ is stratified by pairwise disjoint subvarieties D_γ. The closure of any stratum D_γ is stratified by $\{D_{\gamma'}\mid\gamma'\leq\gamma\}$.

2.3.1. Examples

(i) For $n<3$, $\overline{M}_n$ is empty due to the definition of n-pointed stable curves. $\overline{M}_3$ is simply a point, and its universal curve $\overline{\mathcal{U}}_3$ is $\mathbb{CP}^1$ endowed with three points.

(ii) The moduli space $\overline{M}_4$ is $\mathbb{CP}^1$ with three points. These points D_{τ_1},D_{τ_2} and D_{τ_3} correspond to the curves with two irreducible components, and M_4 is the complement of these three points (see Fig. 1). The universal family $\overline{\mathcal{U}}_4$ is a del Pezzo surface of degree five which is obtained by blowing up three points of $\mathbb{CP}^1\times\mathbb{CP}^1$.

(iii) The moduli space $\overline{M}_5$ is isomorphic to $\overline{\mathcal{U}}_4$. It has ten divisors and each of these divisors contains three codimension two strata. The corresponding 5-trees are shown in Fig. 1.

2.4. Forgetful morphism

We say that $(\Sigma;p_1,\cdots,p_{n-1})$ is obtained by forgetting the labeled point p_n of the n-pointed stable curve $(\Sigma;p_1,\cdots,p_n)$. However, the resulting pointed curve may well

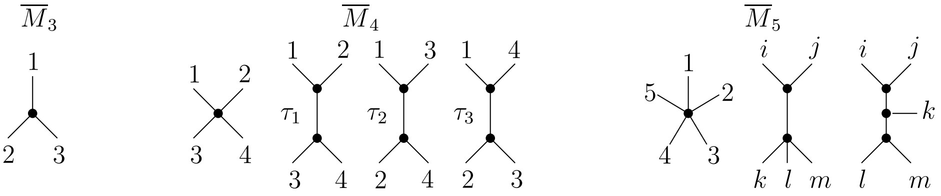

FIGURE 1. Dual trees encoding the strata of $\overline{M}_n$ for $n = 3, 4$, and 5.

be unstable. This happens when the component Σ_v of Σ supporting p_n has only two additional special points. In this case, we contract this component to its intersection point(s) with the components adjacent to Σ_v. With this *stabilization* we extend this map to whole space and obtain $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$. There exists a canonical isomorphism $\overline{M}_n \to \overline{\mathcal{U}}_{n-1}$ commuting with the projections to $\overline{M}_{n-1}$. In other words, $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$ can be identified with the universal family of curves. For details, see [12, 15].

2.5. Automorphisms of $\overline{M}_n$

The open stratum M_n of the moduli space $\overline{M}_n$ can be identified with the orbit space $((\mathbb{CP}^1)^n \setminus \Delta)/PSL_2(\mathbb{C})$. The latter orbit space may be viewed as the configuration space of $(n-3)$ ordered distinct points of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$M_n \cong \{\mathbf{p} = (z_1, \cdots, z_n) \in \mathbb{C}^{n-3} \mid z_i \neq z_j \ \forall i \neq j,\ z_{n-2} = 0, z_{n-1} = 1, z_n = \infty\},$$

where $z_i := [z_i : 1]$ are the coordinates of labeled points p_i in an affine chart of $\mathbb{CP}^1$.

Let $\psi = (\psi_1, \ldots, \psi_{n-3}) : M_n \to M_n$ be a non-constant holomorphic map. In [10], Kaliman discovered the following fact:

Theorem 2.4 (Kaliman, [10]). *For $n \geq 4$, every non-constant holomorphic endomorphism $\psi = (\psi_1, \cdots, \psi_{n-3})$ of M_n is an automorphism and its components ψ_r are of the form*

$$\psi_r(\mathbf{p}) = \frac{z_{\varrho(r)} - z_{\varrho(n-2)}}{z_{\varrho(r)} - z_{\varrho(n)}} \Bigg/ \frac{z_{\varrho(n-2)} - z_{\varrho(n-1)}}{z_{\varrho(n)} - z_{\varrho(n-1)}},\ 1 \leq r \leq n-3$$

where $\varrho \in S_n$ is a permutation not depending on r.

Kaliman's theorem implies the following corollary.

Theorem 2.5 (Kaliman & Lin, [10, 16]). *Every holomorphic automorphism of M_n is produced by a certain permutation $\varrho \in S_n$. Hence, $Aut(M_n) \cong S_n$.*

On the other hand, the permutation group S_n acts on the compactification $\overline{M}_n$ of M_n via relabeling: for $\varrho \in S_n$, there is a holomorphic map ψ_ϱ which is given by

$$\psi_\varrho : (\Sigma; \mathbf{p}) \mapsto (\Sigma; \varrho(\mathbf{p})) := (\Sigma; p_{\varrho(1)}, \cdots, p_{\varrho(n)}). \tag{3}$$

Therefore, the permutation action given in (3) forms a subgroup of holomorphic automorphisms $Aut(\overline{M}_n)$.

Let $Aut_\sharp(\overline{M}_n)$ be the group of holomorphic automorphisms of $\overline{M}_n$ that respect the stratification: $\psi \in Aut_\sharp(\overline{M}_n)$ maps D_τ onto D_γ where $\dim D_\tau = \dim D_\gamma$. Kaliman's theorem leads us to the following immediate corollary.

Theorem 2.6. *The group $Aut_\sharp(\overline{M}_n)$ is S_n.*

Proof. Let $\psi \in Aut_\sharp(\overline{M}_n)$. The restriction of ψ to the open stratum gives the permutation action on M_n since the automorphism group of the open stratum M_n contains only permutations ψ_ϱ. The unicity theorem of holomorphic maps implies that $\psi = \psi_\varrho$ since they coincide on the open stratum $\psi|_{M_n} = \psi_\varrho|_{M_n}$. □

Remark 2.1. Note that the whole group of holomorphic automorphisms $Aut(\overline{M}_n)$ is not necessarily isomorphic to S_n. For example, the automorphism group of $\overline{M}_4$ is $PSL_2(\mathbb{C})$. To the best of our knowledge, there is no systematic exposition of $Aut(\overline{M}_n)$ for $n > 5$.

3. Moduli of pointed real curves of genus zero

A *real structure* on complex variety X is an anti-holomorphic involution $c_X : X \to X$. The fixed point set $\mathbb{R}X = \mathrm{Fix}(c_X)$ of the involution is called the *real part* of the variety (or of the real structure).

In this section, we introduce the moduli spaces of pointed real curves of genus zero as the fixed point sets of real structures on $\overline{M}_n$.

3.1. Real structures on $\overline{M}_n$

The moduli space $\overline{M}_n$ comes equipped with a natural real structure. The involution $c : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$ acts on $\overline{M}_n$. Here a complex curve Σ is regarded as a pair $\Sigma = (C, J)$, where C is the underlying 2-dimensional manifold and J is a complex structure on it, and $\overline{\Sigma} = (C, -J)$ is its complex conjugated pair.[1]

Lemma 3.1. *The map c is a real structure on $\overline{M}_n$.*

Proof. The differentiability of c follows form the Kodaira-Spencer construction of infinitesimal deformations. We need to show that the differential of c is anti-linear at each $(\Sigma; \mathbf{p}) \in \overline{M}_n$. It is sufficient to show that it is anti-linear without taking the quotient with respect to $PSL_2(\mathbb{C})$.

The infinitesimal deformations of a nodal curve Σ with divisor $D_\mathbf{p} = p_1 + \cdots + p_n$ is canonically identified with the complex vector space $Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$, (see Section 2.2). By reversing the complex structure on Σ, we reverse the complex structure on the tangent space $Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$ at $(\Sigma; \mathbf{p})$. The differential of the map $(\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma) \to Ext^1_{\mathcal{O}_{\overline{\Sigma}}}(\Omega_{\overline{\Sigma}}(D_\mathbf{p}), \mathcal{O}_{\overline{\Sigma}})$$

is clearly anti-linear. □

[1] There is some notational ambiguity: The bar over $\overline{M}_n$ and that over $\overline{\Sigma}$ refer to different structures on underlying manifolds: the first one refers to the compactification of $\overline{M}_n$ and second refers to the manifold with reverse complex structure. Both of these notations are widely used, we use the bar for both cases. The context should make it clear which structure is referred to.

The subgroup $Aut_\sharp(\overline{M}_n) \cong S_n$ of holomorphic automorphisms acts on $\overline{M}_n$ via relabeling as given in (3). For each involution $\sigma \in S_n$, we have an additional real structure on $\overline{M}_n$:

$$c_\sigma := c \circ \psi_\sigma : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \sigma(\mathbf{p})). \tag{4}$$

Lemma 3.2. *Every real structure of $\overline{M}_n$ preserving the stratification is of the form (4) where $\sigma \in S_n$ is an involution.*

Proof. By their definition, anti-holomorphic automorphisms of $\overline{M}_n$ are obtained by composing the principal real structure $c : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$ with elements of $Aut(\overline{M}_n)$. The real structure c maps each stratum of $\overline{M}_n$ onto itself. Therefore, an anti-holomorphic automorphism $c \circ \psi$ respects the stratification of $\overline{M}_n$ if and only if $\psi \in Aut(\overline{M}_n)$ respects the stratification of $\overline{M}_n$ i.e., each real structure preserving the stratification of $\overline{M}_n$ is given by a certain involution $\sigma \in S_n$ and is of the form (4). □

3.2. σ-invariant curves and σ-equivariant families

Definition 3.1. An n-pointed stable curve $(\Sigma; \mathbf{p})$ is called *σ-invariant* if it admits a real structure $conj : \Sigma \to \Sigma$ such that $conj(p_i) = p_{\sigma(i)}$ for all $i \in \{1, \cdots, n\}$.

A family of n-pointed stable curves $\pi_S : \mathcal{U}_S \to S$ is called *σ-equivariant* if there exist a pair of real structures

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{c_\mathcal{U}} & \mathcal{U}_S \\ \pi_S \downarrow & & \downarrow \pi_S \\ S & \xrightarrow{c_S} & S. \end{array}$$

such that the fibers $\pi^{-1}(s)$ and $\pi^{-1}(c_S(s))$ are Σ and $\overline{\Sigma}$ respectively, and $c_\mathcal{U}$ maps $z \in \pi^{-1}(s)$ to $z \in \pi^{-1}(c_S(s))$.

Remark 3.1. If $(\Sigma; \mathbf{p})$ is σ-invariant, then the real structure $conj : \Sigma \to \Sigma$ is uniquely determined by the permutation σ due to Lemma 2.2.

Lemma 3.3. *If $\pi : \mathcal{U}_S \to S$ is a σ-equivariant family, then each $(\Sigma; \mathbf{p}) \in \mathbb{R}S$ is σ-invariant.*

Proof. If $c_S((\Sigma; \mathbf{p})) = (\Sigma; \mathbf{p})$ then there exists a unique bi-holomorphic equivalence $conj : \Sigma \to \overline{\Sigma}$ (in other words, anti-holomorphic $conj : \Sigma \to \Sigma$) such that $conj(p_i) = p_{\sigma(i)}$. The restriction of $conj$ on labeled points is an involution. By applying Lemma 2.2 to $conj^2$, we determine that $conj$ is an involution on Σ. □

3.3. The moduli space of pointed real curves $\mathbb{R}\overline{M}_{(2k,l)}$

Let $\mathrm{Fix}(\sigma)$ be the fixed point set of the action of σ on the labeling set $\{1, \cdots, n\}$, and let $\mathrm{Perm}(\sigma)$ be its complement. Let $|\mathrm{Fix}(\sigma)| = l$ and $|\mathrm{Perm}(\sigma)| = 2k$.

Let $\varrho \in S_n$, and ψ_ϱ be the corresponding automorphism of $\overline{M}_n$. The conjugation of real structure c_σ with ψ_ϱ provides a conjugate real structure $c_{\sigma'} = \psi_\varrho \circ c_\sigma \circ \psi_{\varrho^{-1}}$.

The conjugacy classes of real structures are determined by the cardinalities $|\mathrm{Fix}(\sigma)|$ and $|\mathrm{Perm}(\sigma)|$. Therefore, from now on, we only consider c_σ where

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & 2k & 2k+1 & \cdots & 2k+l \\ k+1 & \cdots & 2k & 1 & \cdots & k & 2k+1 & \cdots & 2k+l \end{pmatrix}, \qquad (5)$$

and $n = 2k + l$.

Definition 3.2. For σ as above in (5), σ-invariant curves are called *$(2k, l)$-pointed real curves.*

The fixed point set $\mathrm{Fix}(c_\sigma)$ is called the *moduli space of $(2k, l)$-pointed real curves* and denoted by $\mathbb{R}\overline{M}_{(2k,l)}$.

Theorem 3.4. *(i) For any $n \geq 3$, $\mathbb{R}\overline{M}_{(2k,l)}$ is a smooth real projective manifold of dimension $n-3$.*

(ii) The universal family of curves $\pi : \overline{\mathcal{U}}_n \to \overline{M}_n$ is a σ-equivariant family.

(iii) Any σ-equivariant family of n-pointed stable curves over $\pi_S : \mathcal{U}_S \to S$ is induced by a unique pair of real morphisms

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{\hat{\kappa}} & \overline{\mathcal{U}}_n \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\kappa} & \overline{M}_n. \end{array}$$

(iv) Let $\mathfrak{M}_\sigma$ be the contravariant functor that sends real varieties (B, c_B) to the set of σ-equivariant families of curves over B. The moduli functor $\mathfrak{M}_\sigma$ is represented by the real variety $(\overline{M}_n, c_\sigma)$.

(v) Let $\mathbb{R}\mathfrak{M}_\sigma$ be the contravariant functor that sends real analytic manifolds R to the set of families of σ-invariant curves over R. The moduli functor $\mathbb{R}\mathfrak{M}_\sigma$ is represented by the real part $\mathbb{R}\overline{M}_{(2k,l)}$ of $(\overline{M}_n, c_\sigma)$.

Proof. (i) The smoothness of the real part of c_σ is a consequence of the implicit function theorem, and $dim_{\mathbb{R}}\ \mathbb{R}\overline{M}_{(2k,l)} = dim_{\mathbb{C}}\ \overline{M}_n = n-3$ since the real part $\mathbb{R}\overline{M}_{(2k,l)}$ is not empty.

(ii) The fiber over $(\Sigma; \mathbf{p}) \in \overline{M}_n$ is $\pi^{-1}((\Sigma; \mathbf{p})) = \Sigma$. We define real structures on $\overline{M}_n$ and $\overline{\mathcal{U}}_n$ as follows;

$$\begin{array}{llll} c_\sigma : & (\Sigma; \mathbf{p}) & \mapsto & (\overline{\Sigma}; \sigma(\mathbf{p})), \\ \hat{c}_\sigma : & z \in \pi^{-1}((\Sigma; \mathbf{p})) & \mapsto & z \in \pi^{-1}((\overline{\Sigma}; \sigma(\mathbf{p}))), \end{array}$$

The real structures $c_\sigma, \hat{c}_\sigma$ satisfy the conditions of σ-equivariant families in Definition 3.1.

(iii) Due to Knudsen's theorem (see Section 2.3), each of the morphisms $\kappa : S \to \overline{M}_n$ and $\hat{\kappa} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$ are unique. Therefore, they are the same as $c_\sigma \circ \kappa \circ c_S : S \to \overline{M}_n$ and $\hat{c}_\sigma \circ \hat{\kappa} \circ c_{\mathcal{U}} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$. Hence, the morphisms $\kappa, \hat{\kappa}$ are real.

(iv)-(v) The statements follows from (iii) and the definition of moduli functors. □

It was believed that the real locus $\mathbb{R}\overline{M}_{(2k,l)}$ does not represent any moduli functor for $k \neq 0$, but it has only a meaning in operadic setting. Theorem 3.4 shows the contrary.

4. Stratification of $\mathbb{R}\overline{M}_{(2k,l)}$

A stratification for $\mathbb{R}\overline{M}_{(2k,l)}$ can be obtained by the stratification of $\overline{M}_n$ given in Section 2.3.

Lemma 4.1. *Let γ and $\overline{\gamma}$ be the dual trees of $(\Sigma; \mathbf{p})$ and $(\overline{\Sigma}; \sigma(\mathbf{p}))$ respectively.*

(i) If γ and $\overline{\gamma}$ are not isomorphic, then the restriction of c_σ on the union of complex strata $D_\gamma \bigcup D_{\overline{\gamma}}$ gives a real structure with empty real part.

(ii) If γ and $\overline{\gamma}$ are isomorphic, then the restriction of c_σ on D_γ gives a real structure whose corresponding real part $\mathbb{R}D_\gamma$ is the intersection of $\mathbb{R}\overline{M}_{(2k,l)}$ with D_γ.

Proof. (i) Since γ and $\overline{\gamma}$ are not isomorphic, D_γ and $D_{\overline{\gamma}}$ are disjoint complex strata. The restriction of c_σ on $D_\gamma \bigcup D_{\overline{\gamma}}$ swaps the strata. Therefore, the real part of this real structure is empty.

(ii) Since γ and $\overline{\gamma}$ are isomorphic, the n-pointed curves $(\Sigma; \mathbf{p})$ and $(\overline{\Sigma}; \sigma(\mathbf{p}))$ are in the same stratum D_γ. Therefore, the restriction of c_σ on D_γ is a real structure. The real part $\mathbb{R}D_\gamma$ of the σ-equivariant family D_γ is $\mathbb{R}\overline{M}_{(2k,l)} \bigcap D_\gamma$ since $\mathbb{R}\overline{M}_{(2k,l)} = \mathrm{Fix}(c_\sigma)$. □

Definition 4.1. A tree γ is called *σ-invariant* if it is isomorphic to $\bar{\gamma}$. We denote the set of σ-invariant n-trees by $\mathcal{T}ree(\sigma)$.

Theorem 4.2. $\mathbb{R}\overline{M}_{(2k,l)}$ *is stratified by real analytic subsets $\mathbb{R}D_\gamma$ where $\gamma \in \mathcal{T}ree(\sigma)$.*

Although the notion of σ-invariant trees leads us to a combinatorial stratification of $\mathbb{R}\overline{M}_{(2k,l)}$ as given in Theorem 4.2, it does not give a stratification in terms of connected strata. For a σ-invariant γ, the real part of the stratum $\mathbb{R}D_\gamma$ has many connected components. In the next subsection, we refine this stratification by using the spaces of $\mathbb{Z}_2$-equivariant point configurations in the projective line $\mathbb{CP}^1$.

4.1. Spaces of $\mathbb{Z}_2$-equivariant point configurations in $\mathbb{CP}^1$

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$. Consider the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{CP}^1 \mid \Im(z) > 0\}$ (resp. lower half plane $\mathbb{H}^- = \{z \in \mathbb{CP}^1 \mid \Im(z) < 0\}$) as a half of the $\mathbb{CP}^1$ with respect to $z \mapsto \bar{z}$, and the real part $\mathbb{RP}^1$ as its boundary. Denote by $\mathbb{H}$ the compact disc $\mathbb{H}^+ \cup \mathbb{RP}^1$.

4.1.1. Irreducible $(2k, l)$-pointed real curves and configuration spaces

Let $(\Sigma; \mathbf{p})$ be an irreducible $(2k, l)$-pointed real curve. As a real curve, Σ is isomorphic to $\mathbb{CP}^1$ with real structure which is either $z \mapsto \bar{z}$ or $z \mapsto -1/\bar{z}$ (see, for example [19]). The real structure $z \mapsto -1/\bar{z}$ has empty real part. Notice that a $(2k, l)$-pointed curve with empty real part is possible only when $\mathrm{Fix}(\sigma) = \emptyset$ i.e., $l = 0$.

We will consider the spaces of real curves with non-empty and empty real parts as separate cases.

Case I. ***Configurations in*** $\mathbb{CP}^1$ ***with non-empty real part.*** Each finite subset $\mathbf{p}$ of $\mathbb{CP}^1$ which is invariant under the real structure $z \mapsto \bar{z}$ inherits additional structures:

I. *An oriented cyclic ordering on* $\mathrm{Fix}(\sigma)$: For any point $p \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ there is unique $q \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ which follows the point p in the positive direction of $\mathbb{RP}^1$ (the direction in which the coordinate $x := [x:1]$ on $\mathbb{RP}^1$ increases).
The elements of $\mathbf{p}$ are labeled, therefore the cyclic ordering can be seen as a linear ordering on $(\mathbf{p} \bigcap \mathbb{RP}^1) \backslash \{p_n\}$. This linear ordering gives an *oriented cyclic ordering* on $\mathrm{Fix}(\sigma) = \{2k+1, \cdots, n\}$ which we denote by $\{i_1\} < \cdots < \{i_{l-1}\} < \{i_l\}$ where $i_l = n$.

II. *A 2-partition on* $\mathrm{Perm}(\sigma)$: The subset $\mathbf{p} \bigcap (\mathbb{CP}^1 \setminus \mathbb{RP}^1)$ of $\mathbf{p}$ admits a partition into two disjoint subsets $\{p_i \in \mathbb{H}^\pm\}$. This partition gives an *ordered 2-partition* $\mathrm{Perm}^\pm := \{i \mid p_i \in \mathbb{H}^\pm\}$ of $\mathrm{Perm}(\sigma)$. The subsets $\mathrm{Perm}^\pm$ are swapped by the permutation σ.

The set of data

$$o := \{(\mathbb{CP}^1, z \mapsto \bar{z}); \mathrm{Perm}^\pm; \mathrm{Fix}(\sigma) = \{\{i_1\} < \cdots < \{i_l\}\}\}$$

is called the *oriented combinatorial type* of the $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on $(\mathbb{CP}^1, z \mapsto \bar{z})$.

The oriented combinatorial types of equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \bar{z})$ enumerate the connected components of the space $\widetilde{Conf}_{(2k,l)}$ of k distinct pairs of conjugate points on $\mathbb{H}^+ \bigcup \mathbb{H}^-$ and l distinct points on $\mathbb{RP}^1$:

$$\begin{aligned} \widetilde{Conf}_{(2k,l)} \;:=\; & \{(p_1, \cdots, p_{2k}; q_{2k+1}, \cdots, q_{2k+l}) \mid p_i \in \mathbb{CP}^1 \setminus \mathbb{RP}^1,\ p_i = p_j \Leftrightarrow i = j, \\ & p_i = \bar{p}_j \Leftrightarrow i = \sigma(j)\ \&\ q_i \in \mathbb{RP}^1,\ q_i = q_j \Leftrightarrow i = j\}. \end{aligned}$$

The number of connected components of $\widetilde{Conf}_{(2k,l)}$ is $2^k(l-1)!$.[2] They are all pairwise diffeomorphic; natural diffeomorphisms are given by σ-invariant relabeling.

Let $z := [z:1]$ be the affine coordinate on $\mathbb{CP}^1$ and $x := [x:1]$ be affine coordinate on $\mathbb{RP}^1$. The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$SL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \quad (\Lambda, z) \mapsto \Lambda(z) = \frac{az+b}{cz+d}, \quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

in affine coordinates. It induces an isomorphism $SL_2(\mathbb{R})/\pm I \to Aut(\mathbb{H})$. The automorphism group $Aut(\mathbb{H})$ acts on $\widetilde{Conf}_{(2k,l)}$

$$\Lambda : (z_1, \cdots, z_{2k}; x_{2k+1}, \cdots, x_{2k+l}) \mapsto (\Lambda(z_1), \cdots, \Lambda(z_{2k}); \Lambda(x_{2k+1}), \cdots, \Lambda(x_{2k+l})).$$

It preserves each of the connected components of $\widetilde{Conf}_{(2k,l)}$. This action is free when $2k + l \geq 3$, and it commutes with diffeomorphisms given by σ-invariant relabellings. Therefore, the quotient space $\widetilde{C}_{(2k,l)} := \widetilde{Conf}_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k + l - 3$ whose connected components are pairwise diffeomorphic.

[2]Here we use the convention $n! = 1$ whenever $n \leq 0$.

In addition to the automorphisms considered above, there is a diffeomorphism $-\mathbb{I}$ of $\widetilde{Conf}_{(2k,l)}$ which is given in affine coordinates as follows.

$$-\mathbb{I} : (z_1, \cdots, z_{2k}; x_{2k+1}, \cdots, x_{2k+l}) \mapsto (-z_1, \cdots, -z_{2k}; -x_{2k+1}, \cdots, -x_{2k+l}). \quad (6)$$

Consider the quotient space $Conf_{(2k,l)} = \widetilde{Conf}_{(2k,l)}/(-\mathbb{I})$. Note that, $-\mathbb{I}$ interchanges components with *reverse* combinatorial types. Namely, the combinatorial type $\overline{o}$ of $-\mathbb{I}(\mathbf{p})$ is obtained from the combinatorial type o of $\mathbf{p}$ by reversing the cyclic ordering on $\mathrm{Fix}(\sigma)$ and swapping Perm^+ and Perm^-. The equivalence classes of oriented combinatorial types with respect to the action of $-\mathbb{I}$ are called *un-oriented combinatorial types* of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \overline{z})$. The un-oriented combinatorial types enumerate the connected components of $Conf_{(2k,l)}$.

The diffeomorphism $-\mathbb{I}$ commutes with each σ-invariant relabeling and normalizing action of $Aut(\mathbb{H})$. Therefore, the quotient space $C_{(2k,l)} := Conf_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k + l - 3$, its connected components are diffeomorphic to the components of $\widetilde{C}_{(2k,l)}$, and, moreover, the quotient map $\widetilde{C}_{(2k,l)} \to C_{(2k,l)}$ is a trivial double covering.

Case II. ***Configurations in $\mathbb{CP}^1$ with empty real part.*** Let $(\Sigma; \mathbf{p})$ be an irreducible $(2k, 0)$-pointed real curve and let $\mathbb{R}\Sigma = \emptyset$. Such a pointed real curve is isomorphic to $(\mathbb{CP}^1, \mathbf{p})$ with real structure $conj : z \mapsto -1/\bar{z}$.

The group of automorphisms of $\mathbb{CP}^1$ which commutes with $conj$ is

$$Aut(\mathbb{CP}^1, conj) \cong SU(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}.$$

Thus, the group $Aut(\mathbb{CP}^1, conj)$ acts naturally on the space

$$Conf^{\emptyset}_{(2k,0)} := \{(z_1, \cdots, z_{2k}) \mid z_i = -1/\bar{z}_{i+k}\}$$

of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$. For $k \geq 2$, the action is free and the quotient $B_{(2k,0)} := Conf^{\emptyset}_{(2k,0)}/Aut(\mathbb{CP}^1, conj)$ is a $2k-3$ dimensional connected manifold.

The *combinatorial type* of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ is unique and given by the topological type of the real structure $z \mapsto -1/\bar{z}$.

4.1.2. A normal position of $\mathbb{Z}_2$-equivariant point configurations on $\mathbb{CP}^1$

By using the automorphisms we can make the following choices for the representatives of the points in $\widetilde{C}_{(2k,l)}$ and $B_{(2k,0)}$.

Case I. ***Configurations in $\mathbb{CP}^1$ with non-empty real part.*** Every element in $\widetilde{C}_{(2k,l)}$ is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in \widetilde{Conf}_{(2k,l)}$. In order to calibrate the choice by $Aut(\mathbb{H})$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labeled points in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(A) In the case $l \geq 3$, the three consecutive labeled points $(p'_{i_{l-1}}, p'_n, p'_{i_1})$ in $\mathbb{RP}^1$ are put in the position $x'_{i_{l-1}} = 1, x'_n = \infty, x'_{i_1} = 0$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_k, \overline{z}_1, \cdots, \overline{z}_k, x_{2k+1}, \cdots, x_{2k+l-1}, \infty).$$

(B) In the case $l = 1, 2$, the three labeled points $\{p_k, p_{2k}, p_n\}$ are put in the position $\{\pm\sqrt{-1},\ \infty\}$. Then,

$$\mathbf{p}' = \begin{cases} (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \infty) & \text{if } l = 2, \\ (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, \infty) & \text{if } l = 1 \end{cases}$$

where $\epsilon = \pm$.

(C) In the case $l = 0$, the labeled points $\{p_k, p_{2k}\}$ are fixed at $\{\pm\sqrt{-1}\}$ and p_i where $\{i\} = \{k-1, 2k-1\} \bigcap \mathrm{Perm}^+$ is placed on the interval $]0, \sqrt{-1}[\subset \mathbb{H}^+$. Then,

$$\mathbf{p}' = (z_1, \cdots, z_{k-2}, \epsilon_1\lambda\sqrt{-1}, \epsilon_2\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-2}, -\epsilon_1\lambda\sqrt{-1}, -\epsilon_2\sqrt{-1}).$$

where $\lambda \in]0, 1[$ and $\epsilon_i = \pm, i = 1, 2$.

Remark 4.1 $(\mathbf{A}')$**.** In case of $k > 1$ and $l > 3$, we can consider the alternative map which puts the labeled points $\mathbf{p}'$ into the following normal positions. In this case, three labeled points $\{p_k, p_{2k}, p_n\}$ can be put at $\{z_k, z_{2k}, x_n\} = \{\pm\sqrt{-1}, \infty\}$ by the action of $Aut(\mathbb{H})$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_1, \cdots, \overline{z}_{k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \cdots, x_{2k+l-1}, \infty)$$

where $\epsilon = \pm$.

Case II. ***Configurations in*** $\mathbb{CP}^1$ ***with empty real part.*** Let $k \geq 2$. Every element of $B_{(2k,0)}$ is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in Conf^{\emptyset}_{(2k,0)}$. Calibrating the choice by $Aut(\mathbb{CP}^1, conj)$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labeled points of $(\Sigma; \mathbf{p})$ in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

(D)

$$\mathbf{p} = (z_1, \cdots, z_{k-2}, \lambda\sqrt{-1}, \sqrt{-1}, \frac{-1}{\overline{z}_1}, \cdots, \frac{-1}{\overline{z}_{k-2}}, -\frac{\sqrt{-1}}{\lambda}, -\sqrt{-1})$$

where $\lambda \in]-1, 1[$.

4.2. O/U-planar trees: one-vertex case.

An *oriented planar (o-planar) structure* on the one-vertex n-tree τ is one of the two possible sets of data

$$o \ := \ \begin{cases} \{\mathbb{R}\Sigma \neq \emptyset; \text{a } \sigma-\text{equivariant two}-\text{partition } \mathrm{Perm}^{\pm} \text{ of } \mathrm{Perm}(\sigma); \\ \text{an oriented cyclic ordering on } \mathrm{Fix}(\sigma)\}, \\ \{\mathbb{R}\Sigma = \emptyset\}. \end{cases}$$

We denote the o-planar trees by (τ, o).

An *un-oriented planar (u-planar) structure* u on the one-vertex n-tree τ is a pair of reverse o-planar structures $\{o,\bar{o}\}$ when $\mathbb{R}\Sigma \neq \emptyset$, and equal to the o-planar structure when $o = \{\mathbb{R}\Sigma = \emptyset\}$. We denote the u-planar trees by (τ, u).

4.2.1. O/U-planar trees and connected components of configuration spaces

As shown in Section 4.1.1, each connected component of $C_{(2k,l)}$ for $l > 0$ (resp. $C_{(2k,0)} \bigcup B_{(2k,0)}$ for $l = 0$) is associated to a unique u-planar tree since the un-oriented combinatorial types of $\mathbb{Z}_2$-equivariant point configurations are encoded by the same set of data. We denote the connected components of $C_{(2k,l)}$ (and $C_{(2k,0)} \bigcup B_{(2k,0)}$) by $C_{(\tau,u)}$. Similarly, each connected component of $\widetilde{C}_{(2k,l)}$ is associated to a unique o-planar tree. We denote the connected components of $\widetilde{C}_{(2k,l)}$ by $C_{(\tau,o)}$.

4.2.2. Connected components of $\mathbb{R}M_{(2k,l)}$

Every $\mathbb{Z}_2$-equivariant point configuration defines a $(2k,l)$-pointed real curve. Hence, we define

$$\Xi : \bigsqcup_{(\tau,u)} C_{(\tau,u)} \to \mathbb{R}M_{(2k,l)} \tag{7}$$

which maps $\mathbb{Z}_2$-equivariant point configurations to the corresponding isomorphism classes of irreducible $(2k,l)$-pointed curves.

Lemma 4.3. *(i) The map Ξ is a diffeomorphism.*

(ii) The configuration space $C_{(\tau,u)}$ is diffeomorphic to

- $((\mathbb{H}^+)^k \setminus \Delta) \times \square^{l-3}$ *when* $l > 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}\})^{k-1} \setminus \Delta) \times \square^{l-1}$ *when* $l = 1, 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}, \sqrt{-1}/2\})^{k-2} \setminus \Delta) \times \square^1$ *when* $l = 0$ *and the type of real structure is* $(\mathbb{CP}^1, z \mapsto \bar{z})$,
- $((\mathbb{CP}^1 \setminus \{\sqrt{-1}, \sqrt{-1}/2, -\sqrt{-1}/2, -\sqrt{-1}\})^{k-2} \setminus (\Delta \cup \Delta^c)) \times \square^1$ *when* $l = 0$ *and the type of the real structure is* $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$.

Here, Δ is the union of all diagonals $z_i \neq z_j (i \neq j)$, Δ^c is the union of all cross-diagonals $z_i \neq -\frac{1}{\bar{z}_j} (i \neq j)$, and $\square^l$ is the l-dimensional open simplex.

Proof. (i) The map Ξ is clearly smooth. It is surjective since any $(2k,l)$-pointed irreducible curve is isomorphic either to $(\mathbb{CP}^1, z \mapsto \bar{z})$ or $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ with a $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on it. It is injective since the group of holomorphic automorphisms commuting with the real structure $z \mapsto \bar{z}$ is generated by $Aut(\mathbb{H})$ and $-\mathbb{I}$, and the group of holomorphic automorphisms commuting with the real structure $z \mapsto -1/\bar{z}$ is $Aut(\mathbb{CP}^1, conj)$. These automorphisms are taken into account during construction of the configuration spaces.

(ii) As it is shown in Section 4.1.1, $C_{(\tau,u)}$ is the quotient $C_{(\tau,o)} \bigsqcup C_{(\tau,\bar{o})}/(-\mathbb{I})$. The spaces $C_{(\tau,u)}$ and $C_{(\tau,o)}$ are clearly diffeomorphic. To replace $C_{(\tau,u)}$ by $C_{(\tau,o)}$, we choose an o-planar representative for each one-vertex u-planar tree (τ, u) among $(\tau, o), (\tau, \bar{o})$ as follows.

- $l \geq 3$ case: Let (τ, o) be the representative of (τ, u) for which $p_{2k+1} < p_{n-1} < p_n$ with respect to the cyclic ordering on $\mathrm{Fix}(\sigma)$.
- $l = 0, 1, 2$ case: Let (τ, o) be the representative of (τ, u) for which $k \in \mathrm{Perm}^+$.

We put the $\mathbb{Z}_2$-equivariant point configurations into a normal position as in 4.1.2. The parameterizations stated in (ii) for $l > 0$ cases follow from (**A**) and (**B**) of 4.1.2. In the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto \bar{z})$, according to (**C**) the configuration space $C_{(\tau,o)}$ is a locally trivial fibration over $\square^1 =]0, 1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{H}^+ \setminus \{\sqrt{-1}, \lambda\sqrt{-1}\})^{k-2} \setminus \Delta$. Similarly, in the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$, according to (**D**) the configuration space $C_{(\tau,u)}$ is a locally trivial fibration over $\square^1 =]-1, 1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{CP}^1 \setminus \{\sqrt{-1}, \lambda\sqrt{-1}, -\sqrt{-1}/\lambda, -\sqrt{-1}\})^{k-2} \setminus \Delta$. Since the bases of these locally trivial fibrations are contractible, they are trivial fibrations, and the result follows. $\square$

4.3. Pointed real curves and o-planar trees

This section extends the notions of o/u-planar structures to the all σ-invariant trees.

4.3.1. Notations

Let $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\gamma$ for some $\gamma \in \mathcal{T}ree(\sigma)$ and $conj : \Sigma \to \Sigma$ be the real structure on Σ. We denote the set of real components $\{v \in V_\gamma \mid conj(\Sigma_v) = \Sigma_v\}$ of Σ by $V_\gamma^{\mathbb{R}}$. If this set is empty, then $\mathbb{R}\Sigma$ is an isolated real node; we call the edge of γ representing the isolated real node the *special invariant edge*.

Two vertices $v, \bar{v} \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ are said to be *conjugate* if the real structure *conj* maps the components Σ_v and $\Sigma_{\bar{v}}$ onto each other. Similarly, we call the flags $f, \bar{f} \in F_\gamma \setminus \{f_* \in F_\gamma \mid p_{f_*} \in \mathbb{R}\Sigma\}$ conjugate if *conj* swaps the corresponding special points $p_f, p_{\bar{f}}$.

4.3.2. O/U-planar trees: general case.

Let $\gamma \in \mathcal{T}ree(\sigma)$. Each $(2k, l)$-pointed real curve $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\gamma$ with $\mathbb{R}\Sigma \neq \emptyset$ admits additional structures:

I. If $V_\gamma^{\mathbb{R}} \neq \emptyset$, one can fix an oriented combinatorial type of point configurations on each real component Σ_v. Namely, for $v \in V_\gamma^{\mathbb{R}}$ with $\mathbb{R}\Sigma \neq \emptyset$, an oriented combinatorial type o_v is given by an oriented cyclic ordering on the set of *invariant flags* $F_\gamma^{\mathbb{R}}(v) = \{f \mid p_f \in \mathbb{R}\Sigma_v\}$ and an ordered 2-partition $F_\gamma^{\pm}(v) = \{i \mid p_i \in \mathbb{H}^{\pm}\}$ of $F_\gamma(v) \setminus F_\gamma^{\mathbb{R}}(v)$. In the case of $V_\gamma^{\mathbb{R}} = \{v\}$ and $\mathbb{R}\Sigma = \emptyset$, the o-planar structure is simply given by the type of the real structure, i.e., $\mathbb{R}\Sigma_v = \emptyset$ (as in 4.1.1).

II. If $V_\gamma^{\mathbb{R}} = \emptyset$, then one can fix an ordering of the flags of the special invariant edge: $f_e \mapsto \pm$, $f^e \mapsto \mp$.

This additional structures motivate the following definition.

Definition 4.2. An *o-planar* structure on $\gamma \in \mathcal{T}ree(\sigma)$ is the following set of data,

$$o := \begin{cases} \{(\gamma_v, o_v) \mid v \in V_\gamma^{\mathbb{R}}\} & \text{when } V_\gamma^{\mathbb{R}} \neq \emptyset, \\ \{\text{ordering } \{f_e, f^e\} \to \{\pm\}\}, & \text{when } V_\gamma^{\mathbb{R}} = \emptyset, \end{cases}$$

where (γ_v, o_v) is an o-planar structure on the one-vertex tree γ_v for $v \in V_\gamma^{\mathbb{R}}$, and $e = (f_e, f^e)$ is the special invariant edge of γ when $V_\gamma^{\mathbb{R}} = \emptyset$.

Similarly, an *u-planar* structure on γ is the following set of data,

$$u := \begin{cases} \{(\gamma_v, u_v) \mid v \in V_\gamma^{\mathbb{R}}\} & \text{when } V_\gamma^{\mathbb{R}} \neq \emptyset, \\ \{\emptyset\} & \text{when } V_\tau^{\mathbb{R}} = \emptyset, \end{cases}$$

where (γ_v, u_v) is an u-planar structure on the one-vertex tree γ_v for $v \in V_\gamma^{\mathbb{R}}$. We denote u-planar trees by (γ, u).

4.3.3. Notations.

We associate the subsets of vertices $V_\gamma^{\pm}$ and flags $F_\gamma^{\pm}$ to every o-planar tree (γ, o) when $V_\gamma^{\mathbb{R}} \neq \emptyset$ as follows. Let $v_1 \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ and let $v_2 \in V_\gamma^{\mathbb{R}}$ be the closest invariant vertex to v_1 in $||\gamma||$. Let $f \in F_\gamma(v_2)$ be in the shortest path connecting the vertices v_1 and v_2. The set $V_\gamma^{\pm}$ is the subset of vertices $v_1 \in V_\gamma \setminus V_\gamma^{\mathbb{R}}$ such that the flag f (defined as above) is in $F_\gamma^{\pm}(v_2)$. The subset of flags $F_\gamma^{\pm}$ is defined as $\partial_\gamma^{-1}(V_\gamma^{\pm})$.

Similarly, we associate the subsets of vertices $V_\gamma^{\pm}$ and flags $F_\gamma^{\pm}$ to every o-planar tree (γ, o) when $V_\gamma^{\mathbb{R}} = \emptyset$: Let $f_e \mapsto +$, $\partial_\gamma(f_e) = v_e$ and $\partial_\gamma(f^e) = v^e$. The set V_γ^{+} is the subset of vertices v that are closer to v_e than v^e in $||\tau||$. The complement $V_\gamma \setminus V_\gamma^{+}$ is denoted by V_γ^{-}. The subset of flags $F_\gamma^{\pm}$ is defined as $\partial_\gamma^{-1}(V_\gamma^{\pm})$.

4.4. O-planar trees and their configuration spaces

We associate a product of configuration spaces of $\mathbb{Z}_2$-equivariant point configurations $C_{(\tau_v, o_v)}$ and moduli space of pointed complex curves $\overline{M}_{|v|}$ to each o-planar tree (τ, o):

$$C_{(\tau,o)} \quad := \quad \begin{cases} \prod_{v \in V_\tau^{\mathbb{R}}} C_{(\tau_v, o_v)} \times \prod_{v \in V_\tau^{+}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } \mathbb{R}\Sigma \neq \emptyset, \\ C_{(\tau_{v_r}, o_{v_r})} \times \prod_{\{v, \bar{v}\} \subset V_\tau \setminus V_\tau^{\mathbb{R}}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } \mathbb{R}\Sigma = \emptyset, \\ \prod_{v \in V_\tau^{+}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} = \emptyset. \end{cases}$$

For the case $\mathbb{R}\Sigma = \emptyset$, v_r is the vertex corresponding to the unique real component of point curves, and the product runs over the un-ordered pairs of conjugate vertices belonging to $V_\tau \setminus V_\tau^{\mathbb{R}}$ i.e., $\{v, \bar{v}\} = \{\bar{v}, v\}$.

For each u-planar tree (τ, u), we first choose an o-planar representative and then put $C_{(\tau,u)} = C_{(\tau,o)}$. Note that the so defined space $C_{(\tau,u)}$ does not depend on the o-planar representatives up to an isomorphism.

Lemma 4.4. *Let $\gamma \in \mathcal{T}ree(\sigma)$. The real part $\mathbb{R}D_\gamma$ is diffeomorphic to $\bigsqcup_{(\gamma,u)} C_{(\gamma,u)}$ where the disjoint union is taken over all possible u-planar structures of γ.*

Proof. The complex stratum D_γ is diffeomorphic to the product $\prod_{v \in V_\gamma} \overline{M}_{|v|}$. The real structure $c_\sigma : D_\gamma \to D_\gamma$ maps the factor $\overline{M}_{|v|}$ onto $\overline{M}_{|\bar{v}|}$ when v and $\bar{v}$ are conjugate vertices, and maps the factor $\overline{M}_{|v|}$ onto itself when $v \in V_\gamma^{\mathbb{R}}$. Therefore, the real part $\mathbb{R}D_\gamma$

of c_σ is given by

$$\prod_{v\in V_\gamma^{\mathbb{R}}} C_{(2k_v,l_v)} \times \prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when } |V_\gamma^{\mathbb{R}}|>1,$$
$$(C_{(|v_r|,0)} \bigsqcup B_{(|v_r|,0)}) \times \prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\gamma^{\mathbb{R}}|=1,$$
$$\prod_{\{v,\bar v\}\subset V_\gamma\setminus V_\gamma^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\gamma^{\mathbb{R}}|=0,$$

where $k_v = |F_\gamma^+(v)|$ and $l_v = |F_\gamma^{\mathbb{R}}(v)|$. The decompositions of the spaces $C_{(2k,l)}$ and $C_{(2k,0)} \bigsqcup B_{(2k,0)}$ into their connected components are given in Lemma 4.3. □

Theorem 4.5. *$\mathbb{R}\overline{M}_{(2k,l)}$ is stratified by $C_{(\gamma,u)}$.*

Proof. The moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ can be stratified by $\mathbb{R}D_\gamma$ due to Theorem 4.2. The claim directly follows from the decompositions of open strata of $\mathbb{R}\overline{M}_{(2k,l)}$ into their connected components given in Lemma 4.4. □

4.5. Boundaries of the strata

In this section we investigate the adjacency of the strata $C_{(\gamma,u)}$ in $\mathbb{R}\overline{M}_{(2k,l)}$. We start with considering the complex situation in order to introduce natural coordinates near each codimension one stratum $C_{(\gamma,u)}$.

4.5.1. Intermezzo: Coordinates around the codimension one strata

Let γ be a 2-vertex n-tree given by $V_\gamma = \{v_e, v^e\}$, $F_\gamma(v^e) = \{i_1,\cdots,i_s,f^e\}$ and $F_\gamma(v_e) = \{f_e, i_{s+1},\cdots,i_{n-1},n\}$. Let $(z,w) := [z:1]\times[w:1]$ be affine coordinates on $\mathbb{CP}^1\times\mathbb{CP}^1$. Here we introduce coordinates around D_γ.

Consider a neighborhood $V\subset D_\gamma$ of a nodal n-pointed curve $(\Sigma^o,\mathbf{p}^o)\in D_\gamma$. Any $(\Sigma;\mathbf{p})\in V$ can be identified with a nodal curve $\{(z-z_{f_e})\cdot(w-w_{f^e})=0\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$ with special points $\mathbf{p}_{v_e} = (a_{f_e}, a_{i_{s+1}},\cdots,a_{i_{n-1}},a_n)\subset\{w-w_{f^e}=0\}$ and $\mathbf{p}_{v^e} = (b_{f^e}, b_{i_1},\cdots,b_{i_s})\subset\{z-z_{f_e}=0\}$. In order to determine a nodal curve in $\mathbb{CP}^1\times\mathbb{CP}^1$ and the position of its special points uniquely defined by $(\Sigma;\mathbf{p})$, we make the following choice. Firstly, we fix three labeled points $a_{i_{s+1}}, a_{i_{n-1}}, a_n$ on the line $\{w-w_{f^e}=0\}$ whenever $|v_e|>3$, and three special points $a_{f_e}, a_{i_{n-1}}, a_n$ whenever $|v_e|=3$. Secondly, we fix three special points $b_{f^e}, b_{i_1}, b_{i_s}$ on $\{z-z_{f_e}=0\}$. Finally, we choose $a_{i_{s+1}}=(0,0)$, $a_{i_{n-1}}=(1,0)$, $a_n=(\infty,0)$ for $|v_e|>3$; $a_{f_e}=(0,0)$, $a_{i_{n-1}}=(1,0)$, $a_n=(\infty,0)$ for $|v_e|=3$; and $b_{i_1}=(z_{f_e},1)$, $b_{i_s}=(z_{f_e},\infty)$, $b_{f^e}=(z_{f_e},0)$. Then the components z and w of the special points provide a coordinate system in V; in particular, for $|v_e|>3$ such a coordinate system is formed by z_{f^e}, z_{i_*} with $i_* = i_{s+2},\cdots,i_{n-2}$, and w_{j_*} with $j_* = i_2,\cdots,i_{s-1}$.

We now consider a family of n-pointed curves over V times the ϵ-ball $B_\epsilon = \{|t|<\epsilon\}$. It is given by a family curves $\{(z-z_{f_e})\cdot w+t=0 \mid t\in B_\epsilon\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$. The labeled points

$(z_i, w_i), i = 1, \cdots n$ on these curves are chosen in the following way. If $|v_e| > 3$, we put $(z_{i_1}, w_{i_1}) = (z_{f_e} - t, 1)$, $(z_{i_s}, w_{i_s}) = (z_{f_e}, \infty)$, $(z_{i_{s+1}, w_{i_{s+1}}}) = (0, t/z_{f_e})$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t/(1 - z_{f_e}))$ and $(z_n, w_n) = (\infty, 0)$. Similarly, for $|v_e| = 3$, $(z_{i_1}, w_{i_1}) = (-t, 1)$, $(z_{i_{n-2}}, w_{i_{n-2}}) = (0, \infty)$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t)$ and $(z_n, w_n) = (\infty, 0)$. The other labeled points are taken in an arbitrary position. The component z of the special points and the parameter t provide a coordinate system in $V \times B_\epsilon$.

Due to Knudsen's theorem there exists a unique $\kappa : V \times B_\epsilon \to \overline{M}_n$ which gives the family of n-pointed curves given above.

Lemma 4.6. $\det(d\kappa) \neq 0$ *at* $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. *Hence,* κ *gives a local isomorphism.*

Proof. The parameter t gives a regular function on $\kappa(V \times B_\epsilon)$ which is vanishing along $D_\gamma \bigcap \kappa(V \times B_\epsilon)$. The differential $d\kappa(\vec{v}) = \vec{v}$ for $\vec{v} \in T_{(\Sigma^o, \mathbf{p}^o)}V$ since the restriction of κ on $V \times \{0\}$ is the identity map. We need to prove that $d\kappa(\partial_t) \neq 0$. In other words, the curves are non-isomorphic for different values of the parameter t. Let $(\Sigma(t_i), \mathbf{p}(t_i)) \in V \times B_\epsilon$ be two n-pointed curves for $t_1 \neq t_2$. A bi-holomorphic map $\Phi : \Sigma(t_1) \to \Sigma(t_2)$ is determined by the images of $p_{s+1}, p_{i_{n-1}}, p_n$ when $|v_e| > 3$, and by the images of $p_{i_{n-2}}, p_{i_{n-1}}, p_n$ when $|v_e| = 3$. However, the bi-holomorphic map Φ mapping $(p_{s+1}, p_{i_{n-1}}, p_n)(t_1) \mapsto (p_{s+1}, p_{i_{n-1}}, p_n)(t_2)$ (resp. $(p_{i_{n-2}}, p_{i_{n-1}}, p_n)(t_1) \mapsto (p_{i_{n-2}}, p_{i_{n-1}}, p_n)(t_2)$) maps $p_{i_1}(t_1) = (z_{f_e} - t_1, 1)$ to $(z_{f_e} - t_1, t_2/t_1) \neq p_{i_1}(t_2)$ (resp. $p_{i_1}(t_1) = (-t_1, 1)$ to $(-t_1, t_2/t_1) \neq p_{i_1}(t_2)$), i.e., Φ can not be an isomorphism. □

Remark 4.2. Due to Lemma 4.6, the coordinates on $V \times B_\epsilon$ provide a coordinate system at $(\Sigma^o; \mathbf{p}^o) \in D_\gamma$. There is a natural coordinate projection $\rho : V \times B_\epsilon \to V$.

For a σ-invariant γ and c_σ-invariant V, the above coordinates and the local isomorphism κ are equivariant with respect to a suitable real structure ($(z, w) \mapsto (\bar{z}, \bar{w})$ when $\mathbb{R}\Sigma \neq \emptyset$, and $(z, w) \mapsto (\bar{w}, \bar{z})$ when $\mathbb{R}\Sigma = \emptyset$) on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Therefore, the real part $\mathbb{R}V \times] - \epsilon, \epsilon[$ of $V \times B_\epsilon$ provides a neighborhood for a $(\Sigma^o; \mathbf{p}^o)$ in $\mathbb{R}D_\gamma$ with a set of coordinates on it.

4.5.2. Contraction morphisms for o-planar trees.

Let $(\gamma, \hat{o})$ be an o-planar tree and $\phi : \gamma \to \tau$ be a morphism of n-trees contracting an invariant set of edges $E_{con} = E_\gamma \setminus \phi_E(E_\tau)$. In such a situation, we associate a particular o-planar structure o on τ, as described below in separate cases **(a)** and **(b)**, and speak of a *contraction morphism* $\varphi : (\gamma, \hat{o}) \to (\tau, o)$. In all the cases, except **(a-2)**, the o-planar structure o is uniquely defined by $\hat{o}$.

- **(a)** Let $E_{con} = \{e = (f_e, f^e)\}$ and e be an invariant edge.
  - (1) If $\partial_\gamma(e) = \{v_e, v^e\} \subset V_\gamma^{\mathbb{R}}$, then we convert the o-planar structures

$$\begin{aligned} \hat{o}_{v_e} &= \{\mathbb{R}\Sigma_{v_e} \neq \emptyset; F_\gamma^{\pm}(v_e); F_\gamma^{\mathbb{R}}(v_e) = \{\{i_1\} < \cdots < \{i_m\} < \{f_e\}\}\} \\ \hat{o}_{v^e} &= \{\mathbb{R}\Sigma_{v^e} \neq \emptyset; F_\gamma^{\pm}(v^e); F_\gamma^{\mathbb{R}}(v^e) = \{\{i'_1\} < \cdots < \{i'_{m'}\} < \{f^e\}\}\}. \end{aligned}$$

at v_e and v^e to an o-planar structure at vertex $v = \phi_V(\{v_e, v^e\})$ of (τ, o) defining it by

$$\begin{aligned} o_v &= \{\mathbb{R}\Sigma_v \neq \emptyset; F_\tau^\pm(v) = F_\gamma^\pm(v_e) \bigcup F_\gamma^\pm(v^e); \\ & \quad F_\tau^\mathbb{R}(v) = \{\{i_1\} < \cdots < \{i_m\} < \{i'_1\} < \cdots < \{i'_{m'}\}\}\}. \end{aligned}$$

The o-planar structures are kept unchanged at all other invariant vertices.

(2) If e is a special invariant edge, then we convert the o-planar structure $\hat{o} = \{f_e \mapsto +, f^e \mapsto -\}$ of γ into an o-planar structure at the vertex $v = \phi_V(\{v_e, v^e\})$ of τ defining it by

$$o_v = \{\mathbb{R}\Sigma_v \neq \emptyset; F_\tau^+(v) = F_\gamma^+(v_e) \setminus \{f_e\}, F_\tau^-(v) = F_\gamma^-(v^e) \setminus \{f^e\}; F_\tau^\mathbb{R}(v) = \emptyset\}$$

or by

$$o_v = \{\mathbb{R}\Sigma_v = \emptyset\}.$$

(b) Let $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ where $f_{e_i}, i = 1, 2$ and $f^{e_i}, i = 1, 2$ are conjugate pairs of flags.

(1) If $\partial_\gamma(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in V_\gamma^\mathbb{R}$, $v^{e_i} \notin V_\gamma^\mathbb{R}$, then we convert the o-planar structure

$$o_{\hat{v}} = \{\mathbb{R}\Sigma_{\hat{v}} \neq \emptyset; F_\gamma^\pm(\hat{v}); F_\gamma^\mathbb{R}(\hat{v}) = \{\{i_1\} < \cdots < \{i_m\}\}\}.$$

at $\hat{v}$ to an o-planar structure at $v = \phi(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of τ defining it by

$$\begin{aligned} o_v &= \{\mathbb{R}\Sigma_v \neq \emptyset; F_\tau^+(v) = F_\gamma^+(\hat{v}) \bigcup F_\gamma^+(v^{e_1}) \setminus \{f_{e_1}, f^{e_1}\}, \\ & \quad F_\tau^-(v) = F_\gamma^-(\hat{v}) \bigcup F_\gamma^-(v^{e_2}) \setminus \{f_{e_2}, f^{e_2}\}; \\ & \quad F_\tau^\mathbb{R}(v) = \{\{i_1\} < \cdots < \{i_m\}\}\}. \end{aligned}$$

(2) If $\partial_\gamma(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in V_\gamma^\mathbb{R}$, $v^{e_i} \notin V_\gamma^\mathbb{R}$, then we convert the o-planar structure $o_{\hat{v}} = \{\mathbb{R}\Sigma_{\hat{v}} = \emptyset\}$ at the vertex $\hat{v}$ to an o-planar structure at $v = \phi(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of τ defining it by $o_v = \{\mathbb{R}\Sigma_v = \emptyset\}$.

(3) If $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ and $\partial_\gamma(e_i) \bigcap V_\gamma^\mathbb{R} = \emptyset$, then we define the o-planar structure at each v in τ to be the same as the o-planar structure at v of $(\gamma, \hat{o})$.

Remark 4.3. Let $(\gamma, \hat{o})$ be an o-planar tree with $\mathbb{R}\Sigma \neq \emptyset$, and let $\varphi_e : (\gamma, \hat{o}) \to (\tau, o)$ be the contraction of an edge $e \in E_\gamma$. If the o-planar tree (τ, o) and the u-planar tree $(\gamma, \hat{u})$ underlying $(\gamma, \hat{o})$ are given, then the o-planar structure $\hat{o}$ can be reconstructed. For this reason, when a stratum $C_{(\gamma,\hat{u})}$ contained in the boundary of $\overline{C}_{(\tau,u)}$ is given, we denote the corresponding o-planar structure $\hat{o}$ by $\delta(o)$.

Proposition 4.7. *A stratum $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,u)}$ if and only if the u-planar structures $u, \hat{u}$ can be lifted to o-planar structures $o, \hat{o}$ in such a way that (τ, o) is obtained by contracting an invariant set of edges of $(\gamma, \hat{o})$.*

Proof. We need to consider the statement only for the strata of codimension one and two. These cases correspond to the contraction morphisms from two/three-vertex o-planar (sub)trees to one-vertex o-planar (sub)trees given in **(a)** and **(b)**. For a stratum of higher codimension, the statement can be proved by applying the elementary contractions **(a)** and **(b)** inductively. Here, we consider only the case **(a-1)**. The proof for other cases is the same.

We first assume that (τ, o) is obtained by contracting the edge e of $(\gamma, \hat{o})$, where $(\gamma, \hat{o})$ is an o-planar two-vertex tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$. An element $(\Sigma; \mathbf{p}) \in C_{(\gamma,\hat{o})}$ can be represented by the nodal curve $\{(z - z_{f_e}) \cdot w = 0\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with special points $a_f = (z_f, 0)$ and $b_f = (z_{f_e}, w_f)$ such that

$$\begin{array}{ll} a_f \in \{w = 0 \ \& \ \Im(z) > 0\} & \text{for } f \in F_\gamma^+(v_e) \\ a_{\bar{f}} \in \{w = 0 \ \& \ \Im(z) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v_e) \\ \{a_{i_1} < \cdots < a_{i_m}\} \subset \{w = 0 \ \& \ \Im(z) = 0\} & \text{for } i_* \in F_\gamma^{\mathbb{R}}(v_e) \end{array}$$

on the axis $w = 0$, and $b_* = (z_{f_e}, w_*)$

$$\begin{array}{ll} b_f \in \{z - z_{f_e} = 0 \ \& \ \Im(w) > 0\} & \text{for } f \in F_\gamma^+(v^e) \\ b_{\bar{f}} \in \{z - z_{f_e} = 0 \ \& \ \Im(w) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v^e) \\ \{b_{i'_1} < \cdots < b_{i'_{m'}}\} \subset \{z - z_{f_e} = 0 \ \& \ \Im(w) = 0\} & \text{for } i'_* \in F_\gamma^{\mathbb{R}}(v^e) \end{array}$$

When we include the curve $\{(z - z_{f_e}) \cdot w = 0\}$ into the family $\{(z - z_{f_e}) \cdot w + t = 0\}$, the complex orientation defined on the irreducible components $w = 0$ and $z - z_{f_e} = 0$ by the halves $\Im(z) > 0$ and, respectively, $\Im(w) > 0$ extends continuously to a complex orientation of $\{(z - z_{f_e}) \cdot w + t = 0\}$ with $t \in [0, \epsilon[$ defined by, say, $\Im(z) > 0$. As a result, the curves $\{(z - z_{f_e}) \cdot w + t = 0\}$ with $t \in [0, \epsilon[$ acquire an o-planar structure given by

$$\begin{array}{ll} (z_f, w_f) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) > 0\} & \text{for } f \in F_\gamma^+(v_e) \bigcup F_\gamma^+(v^e) \\ (z_{\bar{f}}, w_{\bar{f}}) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) < 0\} & \text{for } \bar{f} \in F_\gamma^-(v_e) \bigcup \in F_\gamma^-(v^e) \\ (z_f, w_f) \in \{(z - z_{f_e}) \cdot w + t = 0 \ \& \ \Im(z) = 0\} & \text{for } f \in F_\gamma^{\mathbb{R}}(v_e) \bigcup F_\gamma^{\mathbb{R}}(v^e) \end{array}$$

where the points on the real part of the curves $\{(z - z_{f_e}) \cdot w + t = 0\}$ are cyclicly ordered by

$$z_{i_1} < \cdots < z_{i_m} < z_{i'_1} < \cdots < z_{i'_{m'}}.$$

This is exactly the o-planar structure (τ, o) defined in **(a-1)** of 4.5.2.

Now assume that $C_{(\gamma,\hat{u})}$, where $(\gamma, \hat{u})$ is an u-planar tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, is contained in the boundary of $\overline{C}_{(\tau,u)}$. There are four different o-planar representatives of $(\gamma, \hat{u})$, and any two non reverse to each other representatives $\hat{o}_1, \hat{o}_2$ provide by contraction two different o-planar structures $(\tau, o_i), i = 1, 2$. By the already proved part of the statement, $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,o_i)}$ for each $i = 1, 2$. It remains to notice that any codimension one stratum is adjacent to at most two main strata. $\square$

4.5.3. Examples

(i) The first nontrivial example is $\overline{M}_4$. There are three real structures: c_{σ_1}, c_{σ_2}, c_{σ_3}, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

These real structures then give $\mathbb{R}\overline{M}_{(2k,l)}$, where $(2k,l) = (0,4),(2,2)$, and $(4,0)$ respectively.

In the case $\sigma = id$, $\mathbb{R}M_{(0,4)}$ is the configuration space of four distinct points on $\mathbb{RP}^1$ up to the action of $PSL_2(\mathbb{R})$. The 4-pointed curves $(\Sigma;\mathbf{p}) \in \mathbb{R}M_{(0,4)}$ can be identified with $(0,x_2,1,\infty)$ where $x_2 \in \mathbb{RP}^1 \setminus \{0,1,\infty\}$. Hence, $\mathbb{R}M_{(0,4)} = \mathbb{RP}^1 \setminus \{0,1,\infty\}$ and its compactification is $\mathbb{R}\overline{M}_{(0,4)} = \mathbb{RP}^1$. The three intervals of $\mathbb{R}M_{(0,4)}$ are the three configuration spaces $C_{(\tau,u_i)}$ and the three points are the configuration spaces $C_{(\gamma_i,u_i)}$. The u-planar trees (τ,u_i) and (γ_i,u_i) are given in Fig. 2.

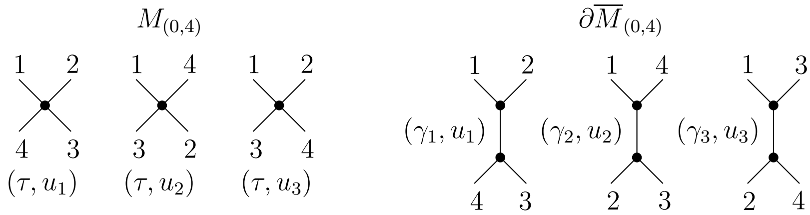

FIGURE 2. U-planar trees encoding the strata of $\overline{M}_{(0,4)}$.

In the case $\sigma = \sigma_1$, $\mathbb{R}M_{(2,2)}$ is the space of distinct configurations of two points in $\mathbb{RP}^1$ and a pair of complex conjugate points in $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. $(\Sigma;\mathbf{p}) \in \mathbb{R}M_{(2,2)}$ is identified with $(\sqrt{-1},-\sqrt{-1},x_3,\infty) \in C_{(\tau,u)}, -\infty < x_3 < \infty$. Hence, $\mathbb{R}M_{(2,2)} = \mathbb{RP}^1 \setminus \{\infty\}$ and its compactification is $\mathbb{R}\overline{M}_{(2,2)} = \mathbb{RP}^1$. The interval $\mathbb{R}M_{(2,2)}$ is $C_{(\tau,u)}$ and the point at its closure is $C_{(\gamma,u)}$.

In the case $\sigma = \sigma_2$, the space $\mathbb{R}M_{(4,0)}$ has different pieces parameterizing real curves with non-empty and empty real parts: The subspace of $\mathbb{R}M_{(4,0)}$ parameterizing the $(4,0)$-pointed real curve with $\mathbb{R}\Sigma \neq \emptyset$ is $(\lambda\sqrt{-1},\sqrt{-1},-\lambda\sqrt{-1},-\sqrt{-1})$ where $\lambda \in]-1,1[\setminus\{0\}$. The subspace of $\mathbb{R}M_{(4,0)}$ parameterizing the real curves with $\mathbb{R}\Sigma = \emptyset$ is $(\lambda\sqrt{-1},\sqrt{-1},-\sqrt{-1}/\lambda,-\sqrt{-1})$, where $\lambda \in]-1,1[$. Note that, the pieces parameterizing $\mathbb{R}\Sigma \neq \emptyset$ and $\mathbb{R}\Sigma = \emptyset$ are joined through the boundary points corresponding to curves with isolated real singular points. The compactification $\mathbb{R}\overline{M}_{(4,0)}$ is $\mathbb{RP}^1$.

(ii) The moduli space $\overline{M}_5$ has three different real structures $c_{\sigma_1}, c_{\sigma_2}$ and c_{σ_3} where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}.$$

The space $\mathbb{R}M_{(0,5)}$ is identified with the configuration space of five distinct points on $\mathbb{RP}^1$ modulo $PSL_2(\mathbb{R})$. It is $(\mathbb{RP}^1 \setminus \{0,1,\infty\})^2 \setminus \Delta$, where Δ is union of all diagonals. Each connected component of $\mathbb{R}M_{(0,5)}$ is isomorphic to a two dimensional simplex. The closure of each cell can be obtained by adding the boundaries given in Section 4.5; for an example see Fig. 3a. It gives the compactification of $\mathbb{R}\overline{M}_{(0,5)}$ which is a torus with 3 points blown up: the cells corresponding to u-planar trees (τ,u_1) and (τ,u_2) are glued along the face corresponding to (γ, u) which gives $(\tau, u_i), i = 1, 2$ by contracting some edges, see Fig. 3b.

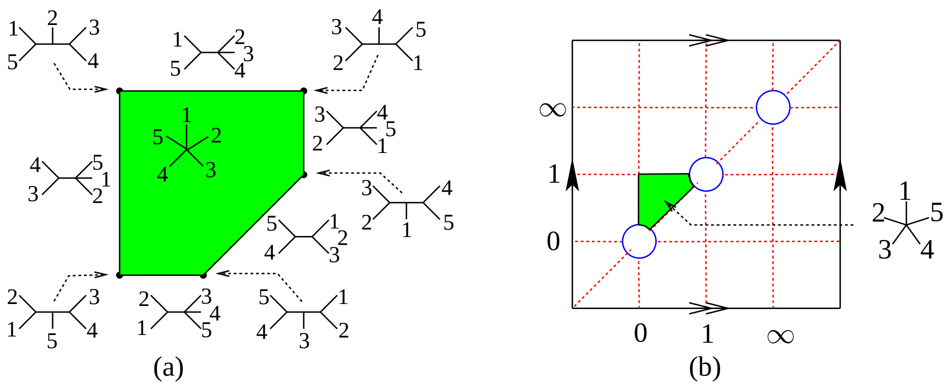

FIGURE 3. (a) Stratification of $\widetilde{C}_{(\tau,u)}$. (b) The stratification of $\mathbb{R}\overline{M}_{(0,5)}$.

The space $\mathbb{R}M_{(2,3)}$ is isomorphic to configurations of a conjugate pair of points on $\mathbb{CP}^1$. The automorphisms allows us to identify such configurations with $(z, \bar{z}, 0, 1, \infty)$ where $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, it can be given as $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. The $\mathbb{R}\overline{M}_{(2,3)}$ is obtained as a sphere with 3 points blown up according to the stratification given in Section 4.5.

Finally, elements of $\mathbb{R}M_{(4,1)}$ can be identified with $(z, \sqrt{-1}, \bar{z}, -\sqrt{-1}, \infty)$. Hence it can be identified with $\mathbb{CP}^1 \setminus (\mathbb{RP}^1 \cup \{\sqrt{-1}, -\sqrt{-1}\})$. Therefore, connected components are isomorphic to $\mathbb{H}^+ \setminus \{\sqrt{-1}\}$. The $\mathbb{R}\overline{M}_{(4,1)}$ is a sphere with a point blown up.

The moduli space $\overline{M}_5$ is a del Pezzo surface of degree 5 and these are all the possible real parts of this del Pezzo surface (see [1]).

5. The first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$

In this section we calculate the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ by using the stratification given in Theorem 4.5.

5.1. Orientations of top-dimensional strata

Let (τ, o) be a one-vertex o-planar tree. The coordinates on $C_{(\tau,o)}$ given in Section 4.1.2 determine an orientation of $C_{(\tau,o)}$. For instance, let $|\mathrm{Fix}(\sigma)| \geq 3$ and let the o-planar structure on (τ, o) be given by $\mathrm{Perm}^{\pm}$ and by a linear ordering $x_{i_1} = 0 < x_{i_2} < \cdots < x_{i_{l-1}} = 1 < x_{i_l} := x_n = \infty$ on $\mathrm{Fix}(\sigma)$. The coordinates in **(A)** of 4.1.2 generate the following top-dimensional differential form on $C_{(\tau,o)}$:

$$\omega_{(\tau,o)} := \left(\frac{\sqrt{-1}}{2}\right)^k \bigwedge_{\alpha_*\in\mathrm{Perm}^+} dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*} \bigwedge dx_{i_2}\wedge\cdots\wedge dx_{i_{l-2}}. \tag{8}$$

The multiplication of top-dimensional forms with a positive valued function $\Theta : C_{(\tau,o)} \to \mathbb{R}_{>0}$ defines an equivalence relation on sections of $\det(TC_{(\tau,o)})$. An *orientation* is an equivalence class of nowhere zero top-dimensional forms with respect to this equivalence relation. We denote the equivalence class of $\omega_{(\tau,o)}$ by $[\omega_{(\tau,o)}]$.

Similarly, using the coordinates given in **(A′)**, **(B)**, **(C)** and **(D)** in 4.1.2 and their ordering, we determine differential forms $\omega_{(\tau,o)}$ and orientations $[\omega_{(\tau,o)}]$ of $C_{(\tau,o)}$ for all (τ,o) with $|V_\tau| = 1$.

5.2. Orientations of codimension one strata

Let (γ,o) be a two-vertex o-planar tree. Let $V_\gamma = \{v_e, v^e\}$ and $e = (f_e, f^e)$ be the edge where $\partial_\gamma(n) = \partial_\gamma(f_e) = v_e$ and $\partial_\gamma(f^e) = v^e$.

By choosing three flags in $F_\gamma(v_e)$ and $F_\gamma(v^e)$, and using the calibrations as in 4.1.2 we obtain a coordinate system in $C_{(\gamma_v,o_v)}$ for each $v \in \{v_e, v^e\}$. More precisely, we use the following choice.

I. Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) \neq \emptyset$. If $|F^{\mathbb{R}}_\gamma(v_e)| \geq 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$), then we specify an isomorphism $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ (resp. $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$) by mapping three consecutive special points as follows: If $|F^{\mathbb{R}}_\gamma(v_e)| > 3$ and the special points p_{f_e} and p_n are not consecutive, then $\Phi_{v_e} : (p_{i_{l-1}}, p_n, p_{i_1}) \mapsto (1,\infty,0)$. If $|F^{\mathbb{R}}_\gamma(v_e)| \geq 3$ and the special points p_{f_e} and p_n are consecutive and

$$\begin{array}{llll} \{f_e\} < \{n\} < \{i_1\}, & \Longrightarrow & \Phi_{v_e} : (p_{f_e}, p_n, p_{i_1}) & \mapsto \ (1,\infty,0), \\ \{i_{l-1}\} < \{n\} < \{f_e\}, & \Longrightarrow & \Phi_{v_e} : (p_{i_{l-1}}, p_n, p_{f_e}) & \mapsto \ (1,\infty,0). \end{array}$$

For $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$, $\Phi_{v^e} : (p_{i_{q+1}}, p_{i_{q+r}}, p_{f^e}) \mapsto (1,\infty,0)$.

If $|F^{\mathbb{R}}_\gamma(v_e)| < 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| < 3$), then in addition to $p_n \mapsto \infty$ (resp. $p_{f^e} \mapsto 0$), we pick the maximal element α in $F^+_\gamma(v_e)$ (resp. $F^+_\gamma(v^e)$) and map the pair of conjugate labeled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$.

II. Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) = \emptyset$. We specify an isomorphism $\Phi_v : \Sigma_v \to \mathbb{CP}^1$ by mapping the pair of conjugate labeled points $(p_\alpha, p_{\overline{\alpha}})$ to $(\sqrt{-1}, -\sqrt{-1})$ for the maximal element α in $F^+_\gamma(v)$, and $p_{f_e} \mapsto 0$ (resp. $p_{f^e} \mapsto 0$).

III. Let $\mathbb{R}\Sigma$ be an isolated real node. We pick a maximal element α_{k-1} in $F^+_\gamma(v_e) \setminus \{n\}$ and specify isomorphisms $\Phi_{v_e} : \Sigma_{v_e} \to \mathbb{CP}^1$ and $\Phi_{v^e} : \Sigma_{v^e} \to \mathbb{CP}^1$ by mapping the special points $(p_{f_e}, p_{\alpha_{k-1}}, p_n)$ and, respectively, $(p_{f^e}, p_{\overline{\alpha}_{k-1}}, p_{\overline{n}})$ to $(0, \sqrt{-1}/2, \sqrt{-1})$.

By using the o-planar structure

$$o_v = \begin{cases} \{\mathbb{R}\Sigma_v \neq \emptyset; F^{\pm}_\gamma(v); F^{\mathbb{R}}_\gamma(v) = \{\{f_1\} < \cdots < \{f_{l_v}\}\}\} & \text{for case } \mathbf{I}, \\ \{\mathbb{R}\Sigma_v \neq \emptyset; F^{\pm}_\gamma(v); F^{\mathbb{R}}_\gamma(v) = \emptyset\} & \text{for case } \mathbf{II}, \end{cases}$$

of γ_v for each $v \in \{v_e, v^e\}$, arrange the coordinates of the special points in the following order

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}}, x_{i_1}, \cdots, x_{i_{l_v}}),$$

where $\alpha_* \in F_\gamma^+(v)$. We fix special points as in (**I**) and (**II**), and apply (8) to introduce top-dimensional differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})}$ and $\Omega_{(\gamma_{v_e}, o_{v_e})}$ on $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$ (note that the resulting forms do not depend on the order of z-coordinates). In the case (**III**), there are no real special points, so we may get a top-dimensional differential form $\Omega_{(\gamma,o)}$ on $C_{(\gamma,o)}$ via choosing the vertex $v \in V_\gamma^+$ with ordering arbitrarily the z-coordinates

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}})$$

where $F_\gamma^+ = \{\alpha_1, \cdots, \alpha_{k_v}\}$.

In such a way, we produce well-defined orientations $[\Omega_{(\gamma_{v^e}, o_{v^e})}]$ and $[\Omega_{(\gamma_{v_e}, o_{v_e})}]$ of, $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$ respectively, and finally get an orientation on $C_{(\gamma,o)}$ determined by

$$\begin{array}{ll} [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}] & \text{when } V_\gamma = V_\gamma^{\mathbb{R}} \\ [\Omega_{(\gamma,o)}] & \text{when } V_\gamma^{\mathbb{R}} = \emptyset \text{ and } v \in V_\gamma^+. \end{array}$$

5.3. Induced orientations on codimension one strata

Let $C_{(\tau,u)}$ be a top dimensional stratum, and let $C_{(\gamma,\hat{u})}$ be a codimension one stratum contained in the boundary of $\overline{C}_{(\tau,u)}$. We lift the u-planar structures $u, \hat{u}$ to o-planar representatives $o, \hat{o} = \delta(o)$ such that (τ, o) is obtained by contracting the edge of $(\gamma, \delta(o))$ (see Prop. 4.7). Then we pick a point $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$ and consider a tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^o, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ as in Section 4.5.

The orientation $[\omega_{(\tau,o)}]$, introduced in 5.1, induces some orientation on $C_{(\gamma,\delta(o))}$: The outward normal direction of $\overline{C}_{(\tau,o)}$ on $\mathbb{R}V \times \{0\} \subset C_{(\gamma,\delta(o))}$ is $-\partial_t$, where t is the standard coordinate on $[0, \epsilon[\subset \mathbb{R}$. Therefore a differential form $\omega_{(\gamma,\delta(o))}$ defines the induced orientation, if and only if

$$-dt \wedge \omega_{(\gamma,\delta(o))} = \Theta\, \omega_{(\tau,o)} \tag{9}$$

with $\Theta > 0$ at each point of $\mathbb{R}V \times]0, \epsilon[$.

In what follows, we compare the induced orientation $[\omega_{(\gamma,\delta(o))}]$ with $[\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$.

5.3.1. Case I: $\mathrm{Fix}(\sigma) \neq \emptyset$.

Lemma 5.1. *Let $(\gamma, \delta(o))$ be an o-planar tree as above, and let $|F_\gamma^{\mathbb{R}}| = l + 2 > 3$, and $|F_\gamma^{\mathbb{R}}(v^e)| = r + 1$. Then,*

$$[\omega_{(\gamma,\delta(o))}] = (-1)^{\aleph}\, [\Omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\Omega_{(\gamma_{v_e}, o_{v_e})}]$$

where the values of $\aleph$ for separate cases are given in the following table.

$\aleph$	$l-r\geq 3$	$l-r=2$		$l-r=1$
		$\{i_1\}<\{f_e\}<\{n\}$	$\{f_e\}<\{i_{l-1}\}<\{n\}$	
$r\geq 2$	$(q+1)(r+1)$	0	$l+1$	$l+1$
$r=1$	1	1	1	1
$r=0$	$q+1$	0	0	0

Here, the third and fourth columns correspond to two possible cyclic orderings of $F^{\mathbb{R}}_\gamma(v_e)$ for $|F^{\mathbb{R}}_\gamma(v_e)|=3$ in Case **I** *of Section 5.2.*

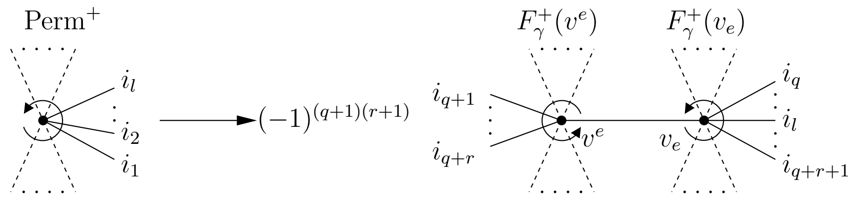

FIGURE 4. Codimension 1 boundaries of $\overline{C}_{(\tau,\delta(o))}$ where $l-r\geq 3$ & $r\geq 2$.

Proof. We will prove the statement only for the special case of $l-r\geq 3$, $r\geq 2$. The calculations for other cases are almost identical.

Let $(\Sigma^o,\mathbf{p}^o)\in C_{(\gamma,\delta(o))}$. We set $\Sigma^o_{v_e}$ to be $\{w=0\}$ and $\Sigma^o_{v^e}$ to be $\{z-x_{f_e}=0\}$. According to the convention in 5.2, the consecutive special points $(p_{i_{l-1}},p_n,p_{i_1})$ (resp. $(p_{i_{q+1}},p_{i_{q+r}},p_{f^e})$) on the component Σ_{v_e} (resp. Σ_{v^e}) are fixed at $(1,\infty,0)$. As shown in the proof of Proposition 4.7, a tubular neighborhood $\mathbb{R}V\times[0,\epsilon[$ of $(\Sigma^o,\mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{(z-x_{f_e})\cdot w+t=0 \mid t\in[0,\epsilon[\}$ with labeled points $p_{i_1}=(0,t/x_{f_e})$, $p_{i_{q+1}}=(x_{f_e}-t,1)$, $p_{i_{q+r}}=(x_{f_e},\infty)$, $p_{i_{l-1}}=(1,-t/(1-x_{f_e}))$, $p_n=(\infty,0)$, $p_{i_*}=(x_{i_*},-t/(x_{i_*}-x_{f_e}))$ for $i_*\in F^{\mathbb{R}}_\gamma\backslash\{i_1,i_{q+1},i_{q+r},i_{l-1},n\}$ and $p_\alpha=(z_\alpha,-t/(z_\alpha-x_{f_e}))$ for $\alpha\in F^+_\gamma$.

We first consider the following subcase: the special points p_{f_e} and p_n are not consecutive. According to 5.2, the differential forms $\Omega_{(\gamma_{v_e},o_{v_e})}$ and $\Omega_{(\gamma_{v^e},o_{v^e})}$ of this case are as follows:

$$\begin{aligned}\Omega_{(\gamma_{v_e},o_{v_e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v_e)|}\bigwedge_{\alpha\in F^+_\gamma(v_e)} dz_\alpha\wedge d\overline{z}_\alpha\wedge\\ &\quad dx_{i_2}\wedge\cdots\wedge dx_{i_q}\wedge dx_{f_e}\wedge\widehat{dx_{i_{q+1}}}\wedge\widehat{\cdots}\wedge\widehat{dx_{i_{q+r}}}\wedge dx_{i_{q+r+1}}\wedge\cdots\wedge dx_{i_{l-2}},\\ \Omega_{(\gamma_{v^e},o_{v^e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v^e)|}\bigwedge_{\beta\in F^+_\gamma(v^e)} dw_\beta\wedge d\overline{w}_\beta\bigwedge dy_{i_{q+2}}\wedge\cdots\wedge dy_{i_{q+r-1}}\end{aligned}$$

By using the identities $w_\beta = -t/(z_\beta - x_{f_e})$ for $\beta \in F_\gamma^+(v^e)$ and $y_i = -t/(x_i - x_{f_e})$ for $i \in F_\gamma^{\mathbb{R}}(v^e)$, we obtain the following equalities:

$$\begin{array}{ll} dt = -dx_{i_{q+1}} + dx_{f_e}, \quad dx_{f_e} = dx_{i_{q+r}}, & \\ dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \frac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in F_\gamma^\pm(v^e), \\ dy_i = -\frac{dt}{x_i - x_{f_e}} + \frac{tdx_i}{(x_i - x_{f_e})^2} - \frac{tdx_{f_e}}{(x_i - x_{f_e})^2} & \text{for } i = q+2, \cdots, q+r-1. \end{array}$$

These identities imply that $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}$ is equal to

$$(-1)^{(r-1)(q-1)} \left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|} \Theta \bigwedge_{\alpha \in F_\gamma^+} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{l-2}$$

where $\Theta = \prod_{\beta \in F_\gamma^+(v^e)} t(z_\beta - x_{f_e})^{-2} \prod_{i_{q+2}, \cdots, i_{q+r-1}} t(x_i - x_{f_e})^{-2}$. Since $\Theta > 0$, the orientation defined by $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to $(-1)^{\aleph}[\omega_{(\tau, o)}]$.

We now consider the cases $\{f_e\} < \{n\} < \{i_1\}$ (i.e., $q + r = l - 1$) and $\{i_{l-1}\} < \{n\} < \{f_e\}$ (i.e., $q = 0$). According to 5.2, the differential forms $\Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v_e}, o_{v_e})}$ are equal to

$$\left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|} \quad \left(\bigwedge\nolimits_{\beta \in F_\gamma^+(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{i_{q+2}} \wedge \cdots \wedge dy_{i_{l-2}}\right) \wedge \left(\bigwedge\nolimits_{\alpha \in F_\gamma^+(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right)$$

when $q + r = l - 1$, and

$$\left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|} \quad \left(\bigwedge\nolimits_{\beta \in F_\gamma^+(v^e)} dz_\beta \wedge d\overline{z}_\beta \bigwedge dy_{i_2} \wedge \cdots \wedge dy_{i_{r-1}}\right) \wedge \left(\bigwedge\nolimits_{\alpha \in F_\gamma^+(v_e)} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_{r+1}} \wedge \cdots \wedge dx_{i_{l-2}}\right)$$

when $q = 0$. The equation $(z - x_{f_e}) \cdot w + t = 0$ implies the following equalities:

$$\begin{array}{ll} dt = -dx_{i_{q+1}}, \quad dx_{f_e} = dx_{i_{l-1}} & \text{when } q + r = l - 1 \\ dt = dx_{i_r}, \quad dx_{f_e} = dx_{i_{q+r}} & \text{when } q = 0, \end{array}$$

and

$$\begin{array}{ll} dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \frac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in F_\gamma^\pm(v^e), \\ dy_i = -\frac{dt}{x_i - x_{f_e}} + \frac{tdx_i}{(x_i - x_{f_e})^2} - \frac{tdx_{f_e}}{(x_i - x_{f_e})^2} & \text{for } i = i_{q+2}, \cdots, i_{q+r-1}. \end{array}$$

By using these identities we obtain that $-dt \wedge \Omega_{(\gamma_{v^e}, o_{v^e})} \wedge \Omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to

$$(-1)^{(q-1)(l-q-2)} \left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|} \Theta \bigwedge_{\alpha \in F_\gamma^+} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{i_{l-2}}$$

when $q+r=l-1$, and

$$(-1)^{(r-1)}\left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|}\Theta\bigwedge_{\alpha\in F_\gamma^+}dz_\alpha\wedge d\overline{z}_\alpha\bigwedge dx_{i_2}\wedge\cdots\wedge dx_{i_{l-2}}$$

when $q=0$. Since $\Theta=\prod_{\beta\in F_\gamma^+(v^e)}t(z_\beta-x_{f_e})^{-2}\prod_{i_{q+2},\cdots,i_{q+r-1}}t(x_i-x_{f_e})^{-2}>0$, the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by $[\omega_{(\tau,o)}]$ is equal to

$$\begin{aligned}&(-1)^{\aleph}\ \left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right]=\\&\begin{cases}(-1)^{(q+1)(r-1)}\left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right] & \text{when } q+r=l-1,\\ (-1)^{(r-1)}\left[\Omega_{(\gamma_{v^e},o_{v^e})}\wedge\Omega_{(\gamma_{v_e},o_{v_e})}\right] & \text{when } q=0.\end{cases}\end{aligned}$$

□

5.3.2. Case II. Fix$(\sigma)=\emptyset$**.**

The different cases for boundaries of $C_{(\tau,o)}$ are treated separately. The proofs are essentially the same as the proof of Lemma 5.1.

Subcase $\mathbb{R}\Sigma\neq\emptyset$. Let (τ,o) be a one-vertex o-planar tree with $\mathbb{R}\Sigma\neq\emptyset$, and let $[\omega_{(\tau,o)}]$ be, in accordance with 5.1, the orientation of $C_{(\tau,o)}$ defined by the differential form

$$\omega_{(\tau,o)}:=(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_*\in\mathrm{Perm}^+\setminus(\mathrm{Perm}^+\ \ \{k-1,k,2k-1,2k\})}dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda \tag{10}$$

(which is given by the coordinates in **(C)** of 4.1.2). Here $\lambda=\Im(z_{\alpha'})$ and $\alpha'\in\{k-1,2k-1\}\bigcap\mathrm{Perm}^+$.

Lemma 5.2. *Let $(\gamma,\delta(o))$ be a two-vertex o-planar tree, and let the corresponding strata $C_{(\gamma,\delta(o))}$ be contained in the boundary of $\overline{C}_{(\tau,o)}$.*

(i) If $V_\gamma^{\mathbb{R}}=V_\gamma$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $-[\Omega_{(\gamma_{v^e},o_{v^e})}]\wedge[\Omega_{(\gamma_{v_e},o_{v_e})}]$.

(ii) If $V_\gamma^{\mathbb{R}}=\emptyset$, then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $[\Omega_{(\gamma,\delta(o))}]$.

Subcase $\mathbb{R}\Sigma=\emptyset$. Let (τ,o) be a one-vertex o-planar tree where $o=\{\mathbb{R}\Sigma=\emptyset\}$ and let $[\omega_{(\tau,o)}]$ be, in accordance with 5.1, the orientation of $C_{(\tau,o)}$ defined by

$$\omega_{(\tau,o)}:=-(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_*\in\{1,\cdots,k-2\}}dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda \tag{11}$$

(which is given by the coordinates in **(D)** of 4.1.2). Here $\lambda=\Im(z_{k-1})$.

Lemma 5.3. *Let $(\gamma,\delta(o))$ be a two-vertex o-planar tree where $V_\gamma^{\mathbb{R}}=\emptyset$, and let $C_{(\gamma,\delta(o))}$ be contained in the boundary of strata $\overline{C}_{(\tau,o)}$ given above. Then the orientation $[\omega_{(\gamma,\delta(o))}]$ induced by the orientation $[\omega_{(\tau,o)}]$ is equal to $(-1)^{\aleph}[\Omega_{(\gamma,\delta(o))}]$ where $\aleph$ is given by $|\{1,\cdots,k-1\}\bigcap F_\gamma^-|+1$.*

5.4. Conventions

Let $(\tau, o_\star)$ be the one-vertex o-planar tree where the o-planar structure $o_\star$ is given by $\mathbb{R}\Sigma \neq \emptyset$, $F_\tau^+ = \{1, 2, \cdots, k\}$, $F_\tau^- = \{k+1, \cdots, 2k\}$, and $F_\tau^{\mathbb{R}} = \{2k+1\} < \{2k+2\} < \cdots < \{2k+l := n\}$. All the other o-planar structures with $\mathbb{R}\Sigma \neq \emptyset$ on τ are obtained as follows.

Let $\varrho \in S_n$ be a permutation which commutes with σ and, if $l > 0$, preserves n. It determines an o-planar structure given by $\varrho(o_\star) = \{\mathbb{R}\Sigma \neq \emptyset; \varrho(\mathrm{Perm}^\pm);\ \mathrm{Fix}(\sigma) = \{\{\varrho(2k+1)\} < \cdots < \{\varrho(2k+l-1)\} < \{\varrho(n) = n\}\}\}$. The parity of ϱ depends only on $o = \varrho(o_\star)$ and we call it *parity* $|o|$ of $o = \varrho(o_\star)$.

5.4.1. Convention of orientations

We fix an orientation for each top-dimensional stratum as follows.

a. Case $\mathbb{R}\Sigma \neq \emptyset$. First, we select o-planar representatives for each one-vertex u-planar tree with $\mathbb{R}\Sigma \neq \emptyset$ as follows:

(1) if $l \geq 3$, we choose the representative (τ, o) of (τ, u) for which $\{2k+1\} < \{n-1\} < \{n\}$;

(2) if $l < 3$, we choose the representative (τ, o) of (τ, u) such that $k \in \mathrm{Perm}^+$.

We denote the set of these o-planar representatives of u-planar trees by $\mathcal{U}Tree(\sigma)$, and select the orientation for $C_{(\tau,u)} = C_{(\tau,o)}$ with $C_{(\tau,o)} \in \mathcal{U}Tree(\sigma)$ to be

$$(-1)^{|o|}[\omega_{(\tau,o)}], \tag{12}$$

where $\omega_{(\tau,o)}$ is the form defined according to 5.1 and $|o|$ is the parity introduced in 5.4.

b. Case $\mathbb{R}\Sigma = \emptyset$**.** Here, we choose the orientation defined by the form (11).

In what follows, if $\mathbb{R}\Sigma \neq \emptyset$ we denote the set of flags $\{2k+1, n-1, n\}$ (for $l \geq 3$ case) and $\{k, 2k, n\}$ (for $l < 3$ case) by $\mathfrak{F}$.

5.5. Adjacent top-dimensional strata with $\mathbb{R}\Sigma \neq \emptyset$

Let $C_{(\tau,u_i)}, i = 1, 2$, be a pair of adjacent top-dimensional strata with $\mathbb{R}\Sigma \neq \emptyset$, and $C_{(\gamma,u)}$ be their common codimension one boundary stratum. Let (τ, o_i) be the o-planar representatives of (τ, u_i) given in 5.4.1. Consider the pair of o-planar representatives $(\gamma, \delta(o_i))$ of (γ, u) which respectively give (τ, o_i) after contracting their edges.

Lemma 5.4. *The o-planar tree $(\gamma, \delta(o_1))$ is obtained by reversing the o-planar structure $\delta(o_2)_v$ of $(\gamma, \delta(o_1))$ at vertex v where $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$.*

Proof. Obviously, $(\gamma, \delta(o_1))$ can be obtained from $(\gamma, \delta(o_2))$ by reversing the o-planar structures at one or both its vertices v_e, v^e. If we reverse the o-planar structure of $(\gamma, \delta(o_2))$ at the vertex v such that $|F_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of its vertices v_e and v^e, then the resulting o-planar tree will not be an element of $\mathcal{U}Tree(\sigma)$ after contracting its edge: reversing the o-planar structure at the vertex v with $|F_\gamma(v) \bigcap \mathfrak{F}| > 1$, or at both of the vertices reverses cyclic order of the elements $\{2k+1, n-1, n\}$ when $l \geq 3$, and moves k from Perm$^+$ to Perm$^-$ when $l < 3$ (see Figure 5). □

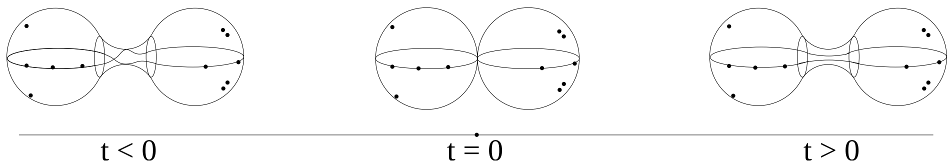

FIGURE 5. Two possible deformation of a real double point.

For a pair of two-vertex o-planar trees $(\gamma, \delta(o_i))$ as above, we calculate the differences of parities as follows.

Lemma 5.5. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be a pair of o-planar trees as above. Let $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, and let o-planar structures at the vertices v_e and v^e be*

$$\begin{aligned} \delta(o_1)_{v_e} &= \left\{ \begin{array}{l} \mathbb{R}\Sigma_{v_e} \neq \emptyset; F_\gamma^{\pm}(v_e); F_\gamma^{\mathbb{R}}(v_e) = \{\{i_1\} < \cdots < \{i_q\} < \{f_e\} < \\ < \{i_{q+r+1}\} < \cdots < \{i_{l-1}\} < \{n\}\} \end{array} \right\} \\ \delta(o_2)_{v^e} &= \{\mathbb{R}\Sigma_{v^e} \neq \emptyset; F_\gamma^{\pm}(v^e); F_\gamma^{\mathbb{R}}(v^e) = \{\{i_{q+1}\} < \cdots < \{i_{q+r}\} < \{f^e\}\}\}. \end{aligned}$$

Let v be the vertex such that $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$. Then,

$$|o_1| - |o_2| = \begin{cases} |F_\gamma^+(v^e)| + \frac{r(r-1)}{2} & \text{if } v = v^e \\ |F_\gamma^+(v_e)| + qr + rs + qs + \frac{s(s-1)}{2} + \frac{q(q-1)}{2} & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| > 3, \\ |F_\gamma^+(v_e)| + |F_\gamma^{\mathbb{R}}(v^e)| - 1 & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| = 3, \\ |F_\gamma^+(v_e)| & \text{if } v = v_e \text{ and } |F_\gamma^{\mathbb{R}}(v_e)| = 2. \end{cases}$$

Here, $r = |F_\gamma^{\mathbb{R}}(v^e)| - 1$ and $s = |F_\gamma^{\mathbb{R}}(v_e)| - q - 2$.

In Section 5.2, we have introduced differential forms $\Omega_{(\gamma_v, \delta(o_i)_v)}$ for each $v \in V_\gamma$. When we reverse the o-planar structure at the vertex v, the differential forms $\Omega_{(\gamma_v, \delta(o_2)_v)}$, $\Omega_{(\gamma_v, \delta(o_1)_v)}$ become related as follows.

Lemma 5.6. *Let $(\gamma, \delta(o_i)), i = 1, 2$ be two-vertex o-planar trees as above. Then,*

$$\Omega_{(\gamma_v, \delta(o_1)_v)} = (-1)^{\mu(v)}\ \Omega_{(\gamma_v, \delta(o_2)_v)},$$

where

$$\mu(v) = |F_\gamma^+(v)| + \frac{(|F_\gamma^{\mathbb{R}}(v)| - 2)(|F_\gamma^{\mathbb{R}}(v)| - 3)}{2}.$$

Lemmata 5.5 and 5.6 follow from straightforward calculations.

5.6. The first Stiefel-Whitney class

This section is devoted to the proof of the following theorem.

Theorem 5.7. *(i) For* $\mathrm{Fix}(\sigma) \neq \emptyset$*, the Poincaré dual of the first Stiefel-Whitney class of* $\mathbb{R}\overline{M}_{(2k,l)}$ *is*

$$[w_1] \;=\; \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}] \;=\; \sum_{\gamma} [\mathbb{R}\overline{D}_\gamma] \quad \mod 2,$$

where the both sums are taken over all two-vertex trees such that

- $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$ *and* $|v^e| = 0 \mod 2$*, or*
- $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$*,* $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ *and* $|v_e|(|v^e|-1) = 0 \mod 2$*, or*
- $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$*,* $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ *and* $|F^{\mathbb{R}}_\gamma(v^e)| = 1$*,*

and, in the first sum, in addition over all u-planar structures on γ*.*

(ii) For $\mathrm{Fix}(\sigma) = \emptyset$*, the Poincaré dual of the first Stiefel-Whitney class of* $\mathbb{R}\overline{M}_{(2k,0)}$ *vanishes.*

Proof. Fix an orientation for each top-dimensional stratum as in 5.4.1. The orientation $(-1)^{|o|}[\omega_{(\tau,o)}]$ of a top-dimensional stratum $C_{(\tau,o)}$ induces some orientation of each codimension one stratum $C_{(\gamma,\delta(o))}$ (and $C_{(\gamma,\hat{o})}$) contained in the boundary of $\overline{C}_{(\tau,o)}$. The induced orientations $(-1)^{|o|}[\omega_{(\gamma,\delta(o))}]$ and $(-1)^{|o|}[\omega_{(\gamma,\hat{o})}]$ are determined in Lemmata 5.1, 5.2 and 5.3, and they give (relative) fundamental cycles $[\overline{C}_{(\gamma,\delta(o))}]$ and $[\overline{C}_{(\gamma,\hat{o})}]$ of the codimension one strata $\overline{C}_{(\gamma,\delta(o))}$ and $\overline{C}_{(\gamma,\hat{o})}$ respectively.

The Poincaré dual of the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is given by

$$[w_1] = \begin{cases} \frac{1}{2}\sum_{(\tau,u)} \left(\sum_{(\gamma,\delta(o))}[\overline{C}_{(\gamma,\delta(o))}]\right) \mod 2, & \text{when } l>0, \\ \frac{1}{2}\sum_{(\tau,u)} \left(\sum_{(\gamma,\hat{o})}[\overline{C}_{(\gamma,\hat{o})}]\right) \mod 2, & \text{when } l=0, \end{cases} \tag{13}$$

where the external summation runs over all one-vertex u-planar trees (τ,u) and the internal one over all codimension one strata of $\overline{C}_{(\tau,o)}$ for the one-vertex o-planar tree (τ,o) which represents (τ,u) in accordance with 5.4.1. Indeed, the sum (13) detects where the orientation on $\mathbb{R}M_{(2k,l)}$ can not be extended to $\mathbb{R}\overline{M}_{(2k,l)}$.

We prove the theorem by evaluating (13).

Case $\mathrm{Fix}(\sigma) \neq \emptyset$**.** Let $C_{(\tau,o_i)}, i=1,2$ be a pair of adjacent top-dimensional strata, and $C_{(\gamma,\delta(o_i))} \subset \overline{C}_{(\tau,o_i)}$ be their common codimension one boundary stratum. We calculate $\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right]$ as follows. According to 5.4.1, the strata $C_{(\tau,o_i)}$ are oriented by $(-1)^{|o_i|}[\omega_{(\tau,o_i)}]$, and these orientations induce the orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ on $C_{(\gamma,\delta(o_i))}$. The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are given by

$$(-1)^{|o_i|+\aleph_i}[\Omega_{(\gamma_{v^e},\delta(o_i)_{v^e})} \wedge \Omega_{(\gamma_{v_e},\delta(o_i)_{v_e})}]$$

in Lemmata 5.1 and 5.2 according to the convention introduced in Section 5.2. We denote by v be the vertex such that $|F_\gamma(v) \bigcap \mathfrak{F}| \leq 1$ as in Section 5.5, and compare the induced orientations by calculating

$$\Pi(o_1, o_2) = (|o_1| + \aleph_1) - (|o_2| + \aleph_2) - \mu(v)$$

for each of the following three subcases.

First, assume that $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$. In this subcase, the o-planar structure is reversed at the vertex $v = v^e$. Therefore, $\aleph_1 = \aleph_2$ according to Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6 and using relation $r = |F^{\mathbb{R}}_\gamma(v^e)| - 1$ we obtain

$$\begin{aligned}
\Pi(o_1, o_2) = |o_1| - |o_2| - \mu(v^e) &= \frac{r(r-1)}{2} - \frac{(|F^{\mathbb{R}}_\gamma(v^e)| - 2)(|F^{\mathbb{R}}_\gamma(v^e)| - 3)}{2} \\
&= |F^{\mathbb{R}}_\gamma(v^e)| - 2 \\
&= |v^e| \mod 2.
\end{aligned}$$

The latter equality follows from the fact that $|F^{\mathbb{R}}_\gamma(v)| = |v| \mod 2$.

Second, assume that $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$ and $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$. In this subcase, the o-planar structure is reversed at the vertex $v = v_e$. Since $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$, once more $\aleph_1 = \aleph_2$ according to the Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6 and using relation $|F^{\mathbb{R}}_\gamma(v_e)| = q + s + 2$, we obtain

$$\begin{aligned}
\Pi(o_1, o_2) &= qr + rs + qs + \frac{s(s-1)}{2} + \frac{q(q-1)}{2} - \frac{(q+s)(q+s-1)}{2} \\
&= r(q+s), \\
&= (|F^{\mathbb{R}}_\gamma(v^e)| - 1)(|F^{\mathbb{R}}_\gamma(v_e)| - 2), \\
&= |v_e|(|v^e| - 1) \mod 2
\end{aligned}$$

when $|F^{\mathbb{R}}_\gamma(v_e)| > 3$, and

$$\begin{aligned}
\Pi(o_1, o_2) &= 2|F^{+}_\gamma(v_e)| = 0 \mod 2 \\
&= |v_e|(|v^e| - 1) \mod 2
\end{aligned}$$

when $|F^{\mathbb{R}}_\gamma(v_e)| = 2$.

Third, we consider $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$ and $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ case. In this subcase, the o-planar structure is reversed at the vertex v_e. Hence, $\aleph_1 = \aleph_2$ whenever $|F^{\mathbb{R}}_\gamma(v^e)| = 1, 2$, and $\aleph_1 - \aleph_2$ is $\pm(l+1) = \pm(|F^{\mathbb{R}}_\gamma(v^e)| + 2)$ whenever $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$. Finally, by applying Lemmata 5.5 and 5.6, we obtain

$$\Pi(o_1, o_2) = \begin{cases}
|F^{\mathbb{R}}_\gamma(v^e)| - 1 \pm (|F^{\mathbb{R}}_\gamma(v^e)| + 2) & = 1 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| \geq 3, \\
|F^{\mathbb{R}}_\gamma(v^e)| - 1 & = 1 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| = 2, \\
|F^{\mathbb{R}}_\gamma(v^e)| - 1 & = 0 \mod 2, \quad \text{when } |F^{\mathbb{R}}_\gamma(v^e)| = 1,
\end{cases}$$

The induced orientations $(-1)^{|o_i|}[\omega_{(\gamma,\delta(o_i))}]$ are the same if and only if $\Pi(o_1, o_2) = 0 \mod 2$. Hence, we have

$$\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right] = \frac{1 + (-1)^{\Pi(o_1,o_2)}}{2} \left[\overline{C}_{(\gamma,\delta(o_1))}\right].$$

The sum $([\overline{C}_{(\gamma,\delta(o_1))}] + [\overline{C}_{(\gamma,\delta(o_2))}])/2$ gives us the fundamental cycle $[\overline{C}_{(\gamma,\delta(o_1))}]$ when $(-1)^{|o_1|}[\omega_{(\gamma,\delta(o_1))}] = (-1)^{|o_2|}[\omega_{(\gamma,\delta(o_2))}]$, and it turns to zero otherwise. Finally, as it follows from the above case-by-case calculations of $\Pi(o_1, o_2)$, the fundamental class of a codimension one strata $\overline{C}_{(\gamma,\delta(o_1))}$ is involved in $[w_1]$ if and only if one of the three conditions given in Theorem are verified. It gives the first expression for $[w_1]$ given in Theorem. Since in this first expression the sum is taken over all u-planar structures on γ, it can be shorten to the sum of the fundamental classes of $\mathbb{R}D_\gamma$.

Case $\mathrm{Fix}(\sigma) = \emptyset$**.** Let $C_{(\tau,u_i)}, i = 1, 2$, be a pair of adjacent top-dimensional strata and $C_{(\gamma,u)}$ be their common codimension one boundary stratum. Let (τ, o_i) be the o-planar representatives of (τ, u_i) given in 5.4.1. Here, we have to consider two subcases: (i) $C_{(\gamma,u)}$ is a stratum of real curves with two real components (i.e., $|V_\gamma| = |V_\gamma^{\mathbb{R}}| = 2$), and (ii) $C_{(\gamma,u)}$ is a stratum of real curves with two complex conjugated components (i.e., $|V_\gamma| = 2$ and $|V_\gamma^{\mathbb{R}}| = 0$).

(i) Consider the pair of o-planar representatives $(\gamma, \delta(o_i))$ of (γ, u) which respectively give (τ, o_i) after contracting the edges and compare their o-planar structure. Since the both tails n and $\sigma(n)$ are in $F_\gamma(v_e)$, the o-planar structure is reversed at the vertex v^e. Therefore, $\aleph_1 = \aleph_2$ according to the Lemma 5.1. Finally, by applying Lemmata 5.5 and 5.6, we obtain

$$\Pi(o_1, o_2) = 2|F_\gamma^+(v^e)| + 1 = 1 \mod 2.$$

In other words, $\left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \left[\overline{C}_{(\gamma,\delta(o_2))}\right] = 0$ for this case.

(ii) Let $C_{(\tau,o_2)}$ be a stratum of real curves with empty real part, and let $(\gamma, \hat{o})$ be an o-planar representative of (γ, u).

The orientations of $C_{(\gamma,u)}$ induced by the orientations $(-1)^{|o_1|}[\omega_{(\tau,o_1)}]$ and $[\omega_{(\tau,o_2)}]$ of $C_{(\tau,o_1)}$ and $C_{(\tau,o_2)}$ are given in Lemmata 5.2 and 5.3. Namely, they are respectively given by the following differential forms

$$(-1)^{|o_1|} \bigwedge_{\alpha_* \in F_\gamma^+ \setminus (F_\gamma^+ \quad \{k-1,k,2k-1,2k\})} dz_{\alpha_*} \wedge d\bar{z}_{\alpha_*},$$

$$(-1)^{\aleph} \bigwedge_{\alpha_* \in F_\gamma^+ \setminus (F_\gamma^+ \quad \{k-1,k,2k-1,2k\})} dz_{\alpha_*} \wedge d\bar{z}_{\alpha_*},$$

where $|o_1| = |\{1, \cdots, k-1\} \bigcap F_\gamma^-|$ and $\aleph = |\{1, \cdots, k-1\} \bigcap F_\gamma^-| + 1$. Therefore, the orientations induced from different sides are opposite and the sum $(-1)^{\aleph-1}\left[\overline{C}_{(\gamma,u)}\right] + (-1)^{\aleph}\left[\overline{C}_{(\gamma,u)}\right]$ vanishes for all such $(\gamma, \hat{o})$. □

5.6.1. Example

Due to Theorem 5.7, the Poincaré dual of the first Stiefel-Whitney class $[w_1]$ of $\mathbb{R}\overline{M}_{(0,5)}$ can be represented by $\sum_\gamma [\mathbb{R}\overline{D}_\gamma] = \sum_{(\gamma,u)} [\overline{C}_{(\gamma,u)}]$ where γ are 5-trees with a vertex v satisfying $|v| = 4$ and $|F_\gamma(v) \bigcap \{1, 4, 5\}| = 1$. These 5-trees are given in Figure 6a, and the union corresponding strata $\bigcup_\tau \mathbb{R}\overline{D}_\gamma$ is given the three exceptional divisors obtained by blowing up the three highlighted points in Figure 6b.

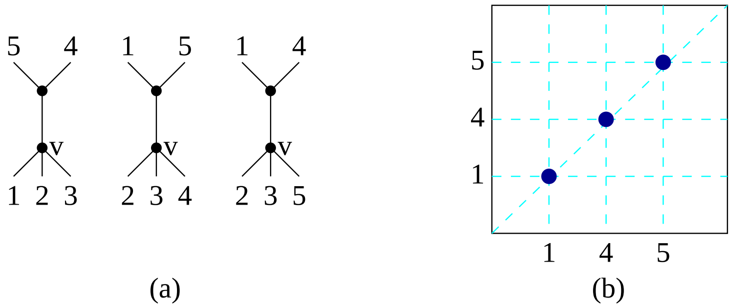

FIGURE 6. (a) The 5-trees of Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(0,5)}$ due Theorem 5.7, (b) The blown-up locus in $\mathbb{R}\overline{M}_{(0,5)}$

6. The orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$

In Section 5.6, the first Stiefel-Whitney class of $\mathbb{R}\overline{M}_{(2k,l)}$ is determined in terms of its strata. We have also proved that the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is orientable when $n = 4$ or $l = 0$. In this section, we give a combinatorial construction of orientation double covering for the rest of the cases i.e., $n > 4$ and $l > 0$. By observing the non-triviality of the orientation double cover in these cases, we prove that $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable.

6.1. Construction of orientation double covering

In Section 4.1.1, we have shown that the map $\widetilde{C}_{(2k,l)} \to \mathbb{R}M_{(2k,l)}$, which is identifying the reverse o-planar structures, is a trivial double covering. The disjoint union of closed strata $\overline{C}_{(2k,l)} = \bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$, where $|V_\tau| = 1$ and (τ, o) runs over all possible o-planar structures on τ, is a natural compactification of $\widetilde{C}_{(2k,l)}$.

To obtain the orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$ we need to get rid of the codimension one strata by pairwise gluing them. We use the following simple recipe: for each pair $(\tau, o_i), = 1, 2$, of one-vertex o-planar trees obtained by contracting the edge in a pair $(\tau, \delta(o_i)), i = 1, 2$, of two-vertex o-planar trees with the same underlying tree such that $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, $v_e = \partial_\gamma(n)$, we glue $\overline{C}_{(\tau,o_i)}$ along $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2$, if

- $\mathcal{A}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,
- $\mathcal{B}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,

$\mathcal{C}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,
$\mathcal{D}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,
$\mathcal{E}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| \neq 1$,
$\mathcal{F}$. $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F^{\mathbb{R}}_\gamma(v_e)| = 3$ and $|F^{\mathbb{R}}_\gamma(v^e)| = 1$.

We denote by $\mathbb{R}\widetilde{M}_{(2k,l)}$ the resulting factor space.

Theorem 6.1. *$\mathbb{R}\widetilde{M}_{(2k,l)}$ is the orientation double cover of $\mathbb{R}\overline{M}_{(2k,l)}$.*

Proof. Let $\widetilde{M}$ be the orientation double covering of $\mathbb{R}\overline{M}_{(2k,l)}$. The points of $\widetilde{M}$ can be considered as points in $\mathbb{R}\overline{M}_{(2k,l)}$ with local orientation. On the other hand, by using opposite o-planar structures of a one-vertex τ we can determine orientations $(-1)^{|o|}[\omega_{(\tau,o)}]$ and $(-1)^{|\bar{o}|+l-1}[\omega_{(\tau,\bar{o})}]$ on $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ where $(\tau, o) \in \mathcal{U}Tree(\sigma)$. These orientations are opposite with respect to the identification of $C_{(\tau,o)}$ and $C_{(\tau,\bar{o})}$ by the canonical diffeomorphism $-\mathbb{I}$ introduced in Subsection 4.1.1. Hence, there is a natural continuous embedding $\widetilde{C}_{(2k,l)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)}(C_{(\tau,o)} \sqcup C_{(\tau,\bar{o})}) \to \widetilde{M}$. It extends to a surjective continuous map $\overline{C}_{(2k,l)} = \bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)}(\overline{C}_{(\tau,o)} \sqcup \overline{C}_{(\tau,\bar{o})}) \to \widetilde{M}$. Since $\overline{C}_{(2k,l)}$ is compact and $\widetilde{M}$ is Hausdorff, the orientation double covering $\widetilde{M}$ is a quotient space $\overline{C}_{(2k,l)}/R$ of $\overline{C}_{(2k,l)}$ under the equivalence relation R defined by the map $\overline{C}_{(2k,l)} \to \widetilde{M}$.

This equivalence relation is uniquely determined by its restriction to the codimension one faces of $\overline{C}_{(2k,l)}$, which cover the codimension one strata of $\mathbb{R}\overline{M}_{(2k,l)}$ under the composed map $\overline{C}_{(2k,l)} \to \widetilde{M} \to \mathbb{R}\overline{M}_{(2k,l)}$. On the other hand, the equivalence relation on the codimension one faces is determined by the first Stiefel-Whitney class: A partial section of the induced map $\overline{C}_{(2k,l)}/R \to \mathbb{R}\overline{M}_{(2k,l)}$ given by distinguished strata $\bigsqcup_{(\tau,o)\in\mathcal{U}Tree(\sigma)} C_{(\tau,o)}$. Over a neighborhood of a codimension one stratum of $\mathbb{R}\overline{M}_{(2k,l)}$, a partial section extends to a section if this strata is not involved in the expression for the first Stiefel-Whitney class given in Theorem 5.7, and it should not extend, otherwise. Notice that the faces $\overline{C}_{(\tau,\delta(o_i))}$ considered in relations $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{E}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which do not contribute to the expression $[w_1]$ given in Theorem 5.7, and the faces $\overline{C}_{(\tau,\delta(o_i))}$ in relations $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{F}$ are mapped onto the strata $\overline{C}_{(\tau,\delta(u))}$ which contribute to the expression $[w_1]$. There are four different faces $\overline{C}_{(\tau,\delta(o)_i)}, i = 1, \cdots, 4$ over each codimension one stratum $\overline{C}_{(\tau,\delta(u))}$. Lemma 5.4 determines the pairs $\overline{C}_{(\tau,\delta(o)_i)}$, $\overline{C}_{(\tau,\delta(o)_j)}$ to be glued to each other. ☐

Corollary 6.2. *The moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable when $2k + l > 4$ and $l > 0$.*

Proof. Let $l \geq 3$, and (τ, o) be an o-planar structure with $\{2k+1\} < \{n-1\} < \{n\}$. It is clear that, we can produce any o-planar structure on τ with $\{2k+1\} < \{n-1\} < \{n\}$ by applying following operations consecutively:

- interchanging the order of two consecutive tails $\{i, i+1\}$ for $|\{i, i+1\} \bigcap \mathfrak{F}| \leq 1$ and $n \notin \{i, i+1\}$,
- swapping $j \in \mathrm{Perm}^+$ with $\bar{j} \in \mathrm{Perm}^-$ for $j \neq k, 2k$.

The one-vertex o-planar trees with $\{n-1\} < \{2k+1\} < \{n\}$ can be produced from the o-planar tree $(\tau, \bar{o})$ via same procedure.

Let $l = 1, 2$. Similarly, if we start with o-planar tree (τ, o) with $k \in \mathrm{Perm}^+$ ($k \in \mathrm{Perm}^-$), we can produce any o-planar structure on τ with $k \in \mathrm{Perm}^+$ ($k \in \mathrm{Perm}^-$) by swapping $j \in \mathrm{Perm}^+$ with $\bar{j} \in \mathrm{Perm}^-$ for $j \neq k, 2k$.

Note that, these operations correspond to passing from one top-dimensional stratum to another in $\mathbb{R}\widetilde{M}_{(2k,l)}$ through certain faces. These faces correspond to the one-edge o-planar trees $(\gamma, \delta(o)_i)$ with $F_\gamma(v^e) = \{i, i+1, f^e\}$ (resp. $F_\gamma(v^e) = \{j, \bar{j}, f^e\}$) which are faces glued according to the relations of type $\mathcal{A}$. Hence, any two top-dimensional strata in $\mathbb{R}\widetilde{M}_{(2k,l)}$ with same cyclic ordering of $\mathfrak{F}$ (resp. with k is in same set $\mathrm{Perm}^\pm$) can be connected through a path passing through these codimension faces $\overline{C}_{(\gamma,\delta(o)_i)}$. The quotient space $\overline{C}_{(2k,l)}/\mathcal{A}$ has two connected components since there are two possible cyclic orderings of $\mathfrak{F}$ when $l \geq 3$ (resp. two possibilities for $l = 1, 2$ case: $k \in \mathrm{Perm}^+$ and $k \in \mathrm{Perm}^-$).

The set of relations of type $\mathcal{B}$ is not empty when $2k + l > 4$ and $l > 0$. Moreover, the relation of type $\mathcal{B}$ reverses the cyclic ordering on $\mathfrak{F}$ (resp. moves k from $\mathrm{Perm}^\pm$ to $\mathrm{Perm}^\mp$). Hence, the faces glued according to the relations of type $\mathcal{B}$ connect two different components of $\overline{C}_{(2k,l)}/\mathcal{A}$. Therefore, the orientation double cover $\mathbb{R}\overline{M}_{(2k,l)}$ is nontrivial when $2k + l > 4$ and $l > 0$ which simply means that the moduli space $\mathbb{R}\overline{M}_{(2k,l)}$ is not orientable in this case. □

6.1.1. Examples

In Example 4.5.3, we obtained that $\mathbb{R}\overline{M}_{(0,5)}, \mathbb{R}\overline{M}_{(2,3)}$, and $\mathbb{R}\overline{M}_{(4,1)}$ are respectively, a torus with three points blown up, a sphere with three points blown up, and a sphere with one point blown up. The coverings $\mathbb{R}\widetilde{M}_{(0,5)}, \mathbb{R}\widetilde{M}_{(2,3)}$ and $\mathbb{R}\widetilde{M}_{(4,1)}$ are obtained by taking the two copies of the corresponding moduli space of real curves and replacing the blown up loci by annuli. Therefore, $\mathbb{R}\widetilde{M}_{(0,5)}, \mathbb{R}\widetilde{M}_{(2,3)}$ and $\mathbb{R}\widetilde{M}_{(4,1)}$ are surfaces of genus 4, genus 2 and genus 0, respectively (see Figure 7 which illustrates the case $(k, l) = (0, 5)$).

6.2. Combinatorial types of strata of $\mathbb{R}\widetilde{M}_{(2k,l)}$

While constructing $\mathbb{R}\widetilde{M}_{(2k,l)}$, the closure of the each codimension one strata are glued in a consistent way. This identification of codimension strata gives an equivalence relation among the o-planar trees when $l \neq 0$.

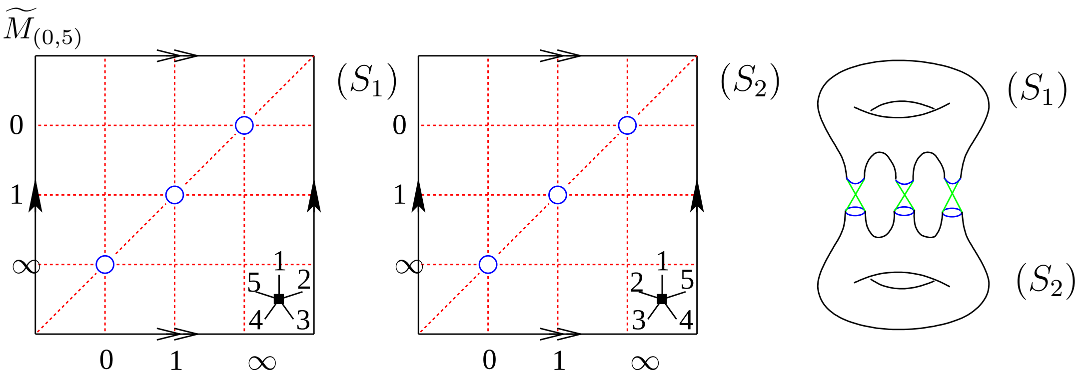

FIGURE 7. Stratification of $\mathbb{R}\widetilde{M}_{(0,5)}$

We define the notion of *R-equivalence* on the set of such o-planar trees by treating different cases separately. Let $(\gamma_1, o_1), (\gamma_2, o_2)$ be o-planar trees.

(1) If $|V_{\gamma_i}^{\mathbb{R}}| = 1$, then we say that they are R-equivalent whenever γ_1, γ_2 are isomorphic (i.e., $\gamma_1 \approx \gamma_2$) and the o-planar structures are the same.

(2) If γ_i have an edge corresponding to real node (i.e. $E_{\gamma_i}^{\mathbb{R}} = \{e\}$ and $V_{\gamma_i}^{\mathbb{R}} = \partial_\gamma(e) = \{v^e, v_e\}$), we first obtain $(\gamma_i(e), o_i(e))$ by contracting conjugate pairs of edges until there will be none. We say that (γ_1, o_1) and (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 1 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v^e) \bigcap \mathfrak{F}| \leq 1$, and $|v^e| = 0 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 1 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| \neq 3$ and $|v_e|(|v^e - 1|) = 0 \mod 2$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v_e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| = 3$ and $|F_\gamma^{\mathbb{R}}(v^e)| \neq 1$,
   - $(\gamma_1(e), o_1(e))$ produces $(\gamma_1(e), o_2(e))$ by reversing the o-planar structure at the vertex v^e, $|F_\gamma(v_e) \bigcap \mathfrak{F}| \leq 1$, $|F_\gamma^{\mathbb{R}}(v_e)| = 3$ and $|F_\gamma^{\mathbb{R}}(v^e)| = 1$,

(3) Otherwise, if γ_i have more than one invariant edge (i.e. $|E_{\gamma_i}|^{\mathbb{R}} > 1$), we say that (γ_1, o_1), (γ_2, o_2) are R-equivalent whenever $\gamma_1 \approx \gamma_2$ and there exists an edge $e \in E_{\gamma_i}^{\mathbb{R}}$ such that the o-planar trees $(\gamma_i(e), o_i(e))$, which are obtained by contracting all edges but e, are R-equivalent in the sense of the Case (2).

We call the maximal set of pairwise R-equivalent o-planar trees by *R-equivalence classes* of o-planar trees.

Theorem 6.3. *A stratification of the orientation double cover* $\mathbb{R}\widetilde{M}_{(2k,l)}$ *is given by*

$$\mathbb{R}\widetilde{M}_{(2k,l)} = \bigsqcup_{\substack{\text{R-equivalance classes} \\ \text{of o-planar } (\gamma,o)}} C_{(\gamma,o)}.$$

6.3. Some other double coverings of $\mathbb{R}\overline{M}_{(2k,l)}$

In [11], Kapranov constructed a different double covering $\widehat{\mathbb{R}M}_{(0,l)}$ of $\mathbb{R}\overline{M}_{(0,l)}$ having no boundaries. He has applied the following recipe to obtain the double covering: Let $\overline{C}_{(0,l)}$ be the disjoint union of closed strata $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ as above. Let $(\gamma,\delta(o_i)), i = 1,2$ be two-vertex o-planar trees representing the same u-planar tree (γ,u), and let (τ,o_i) be the one-vertex trees obtained by contracting the edges of $(\gamma,\delta(o_i))$. The strata $\overline{C}_{(\gamma,\delta(o_i))}, i = 1,2$ are glued if $(\gamma,\delta(o_1))$ produces $(\gamma,\delta(o_2))$ by reversing the o-planar structure at vertex $v^e \neq \partial_\gamma(n)$. We obtain first Stiefel-Whitney class of $\widehat{\mathbb{R}M}_{(0,l)}$ by using the same arguments in Theorem 5.7.

Proposition 6.4. *The Poincaré dual of the first Stiefel-Whitney class of* $\widehat{\mathbb{R}M}_{(0,l)}$ *is*

$$[\widehat{w_1}] = \frac{1}{2} \sum_{(\tau,o)} \sum_{(\gamma,\delta(o)):|v^e|=0 \mod 2} \left[\overline{C}_{(\gamma,\delta(o))}\right] \mod 2.$$

It is well-known that these spaces are not orientable when $l \geq 5$.

6.3.1. A double covering from open-closed string theory

In [5, 17], a different 'orientation double covering' is considered. It can be given as the disjoint union $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ where $F_\tau^+ = \{1, \cdots, k\}$, and $F_\tau^{\mathbb{R}}$ carries all possible oriented cyclic ordering. It is a disjoint union of manifolds with corners. The covering map $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)} \to \mathbb{R}\overline{M}_{(2k,l)}$ is two-to-one only over a subset of the open space $\mathbb{R}M_{(2k,l)}$. It only covers the subset $\bigsqcup_{(\tau,u)} \overline{C}_{(\tau,u)}$ of $\mathbb{R}\overline{M}_{(2k,l)}$ where u-planar trees (τ,u) have the partition $\{\{1,\cdots,k\},\{k+1,\cdots,2k\}\}$ of $F_\tau \setminus F_\tau^{\mathbb{R}}$. Moreover, the covering map is not two-to-one over the strata with codimension higher than zero.

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