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Asymptotically maximal real algebraic hypersurfaces of projective space

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Abstract. Using the combinatorial patchworking, we construct an asymptotically maximal (in the sense of the generalized Harnack inequality) family of real algebraic hypersurfaces in an n-dimensional real projective space. This construction leads to a combinatorial asymptotic description of the Hodge numbers of algebraic hypersurfaces in the complex projective spaces and to asymptotically sharp upper bounds for the individual Betti numbers of primitive T-hypersurfaces in terms of Hodge numbers of the complexifications of these hypersurfaces.

1. Introduction

In 1876 A. Harnack published a paper [Har76] where he found an exact upper bound for the number of connected components for a curve of a given degree. Harnack proved that the number of components of a real plane projective curve of degree m is at most $\frac{(m-1)(m-2)}{2}+1$. On the other hand, for each natural number m he constructed a non-singular real projective curve of degree m with $\frac{(m-1)(m-2)}{2}+1$ components, which shows that his estimate cannot be improved without introducing new ingredients.

It is natural to ask whether there exists a similar inequality for surfaces in the three-dimensional projective space. This question is known as the Harnack problem. Understood literally, *i.e.* as a question about the number of components, it has appeared to be a difficult problem. The maximal number of components is found only for degree ≤ 4. However Harnack Inequality has been generalized in another way.

Theorem 1.1 (Generalized Harnack Inequality)**.** *If X is a real algebraic variety, then*

$$\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X;\mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_*(\mathbb{C}X;\mathbb{Z}_2), \tag{1}$$

where $\mathbb{R}X$ and $\mathbb{C}X$ are the sets of real and complex points of X, respectively.

Since $\mathbb{R}X$ is the fixed point set of involution $conj : \mathbb{C}X \to \mathbb{C}X$, Theorem 1.1, in turn, is a special case of the following theorem.

Key words and phrases. M-hypersurfaces, Hodge numbers, combinatorial patchworking, tropical hypersurfaces.

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Theorem 1.2 (Smith - Floyd Inequality)**.** *Let X be a topological space, $\tau : X \to X$ an involution and F the fixed point set of τ. Then*

$$\dim_{\mathbb{Z}_2} H_*(F; \mathbb{Z}_2) \le \dim_{\mathbb{Z}_2} H_*(X; \mathbb{Z}_2).$$

(See, e. g., Bredon [Bre72]. To avoid a discussion of choice of homology theory, one can suppose that X and τ are simplicial.)

Although Theorem 1.2 was first stated by E. E. Floyd [Flo52], all arguments needed for the proof appeared in earlier works by P. A. Smith, see [Smi38]. Theorem 1.1 was formulated first by R. Thom [Tho65]. He got the inequality (1) as a corollary of Theorem 1.2. He did not observe however that the inequality (1) gives the best estimates of $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X; \mathbb{Z}_2)$. It was V. M. Kharlamov [Kha72] and V. A. Rokhlin [Rok72] who acknowledged the strength and importance of Generalized Harnack Inequality. They turned the Smith theory into a powerful tool for studying the topology of real algebraic varieties ([Kha72], [Kha73], [Kha75], [Rok72]).

If X is a nonsingular curve of degree m, then $\mathbb{C}X$ is homeomorphic to a sphere with $(m-1)(m-2)/2 = (m^2 - 3m + 2)/2$ handles, and

the right hand side of the inequality (1) is $m^2 - 3m + 4$. In this case the left hand side is the doubled number of components of $\mathbb{R}X$. Hence Theorem 1.1 generalizes Harnack Inequality.

A real algebraic variety for which the left and right hand sides of the inequality (1) are equal is called an *M-variety* or a *maximal variety.*

In [IV] we proved the following statement.

Theorem 1.3. *For any positive integers m and n, there exists a nonsingular hypersurface X of degree m in $\mathbb{R}P^n$ such that*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X; \mathbb{Z}_2) = \sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X),$$

where $h^{p,q}$ are Hodge numbers.

In particular, for any positive integers m and n, there exists an M-hypersurface of degree m in $\mathbb{R}P^n$. Notice that, for a nonsingular hypersurface X of degree m in $\mathbb{C}P^n$, one has $\sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X) = h^{p,n-1-p}(\mathbb{C}X)$ if $2p = n-1$, and $\sum_{q=0}^{n-1} h^{p,q}(\mathbb{C}X) = h^{p,n-1-p}(\mathbb{C}X) + 1$ otherwise.

The construction presented in [IV] can be seen as a combinatorial version of the construction mentioned in [Vir79a]. The M-hypersurfaces in [IV] are constructed using the *primitive patchworking.* It is a particular case of the *combinatorial patchworking,* which in turn is a particular case of the Viro method of construction of real algebraic varieties, see [Vir83], [Vir84], [Vir94], [Ris92], [Stu94], [IV96], and Section 2 below. The combinatorial patchworking provides piecewise-linear models of hypersurfaces. In the case of the primitive patchworking, these models are *nonsingular real tropical hypersurfaces* (*cf.* [Mi05]). A nonsingular algebraic hypersurface X in $\mathbb{R}P^n$ constructed by means of the primitive patchworking is called a *primitive T-hypersurface.*

Let n be a positive integer, P a real polynomial of degree n in one variable, b a vector $(b_0, b_1, \dots, b_{n-1})$ in $\mathbb{Z}^n$, and $\mathcal{C}$ a class of algebraic hypersurfaces in $\mathbb{R}P^n$. We say that $\mathcal{C}$ *satisfies the condition* $b \overset{n}{\leq} P$ (respectively, $b \overset{n}{\geq} P$) if there exists a real univariate polynomial Q of degree $n-1$ such that, for any hypersurface X in $\mathcal{C}$, one has the inequality $\sum_{p=0}^{n-1} b_i \dim_{\mathbb{Z}_2} H_p(\mathbb{R}X;\, \mathbb{Z}_2) \leq P(m) + Q(m)$ (respectively, $\sum_{p=0}^{n-1} b_i \dim_{\mathbb{Z}_2} H_p(\mathbb{R}X;\, \mathbb{Z}_2) \geq P(m) + Q(m)$), where m is the degree of X. We say that $\mathcal{C}$ satisfies the condition $b \overset{n}{=} P$ if $\mathcal{C}$ satisfies the both conditions $b \overset{n}{\leq} P$ and $b \overset{n}{\geq} P$.

Let $B \in \mathbb{Z}^n$ be the vector with all the coordinates equal to 1. Since for any nonsingular hypersurface X of degree m in $\mathbb{C}P^n$ one has

$$\dim_{\mathbb{Z}_2} H_*(\mathbb{C}X; \mathbb{Z}_2) = \frac{(m-1)^{n+1} - (-1)^{n+1}}{m} + n + (-1)^{n+1}$$

(see, for example, [Far57]), the Generalized Harnack Inequality implies that the class of all nonsingular algebraic hypersurfaces in $\mathbb{R}P^n$ verifies the condition $B \overset{n}{\leq} x^n$. We say that a sequence $(X_m)_{m\in\mathbb{N}}$, where X_m is a nonsingular hypersurface of degree m in $\mathbb{R}P^n$, is *asymptotically maximal*, if this sequence verifies the condition $B \overset{n}{=} x^n$, *i.e.*, if $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}X_m;\, \mathbb{Z}_2) = m^n + O(m^{n-1})$.

For any integer $p = 0, \dots, n-1$, put

$$H_p(x) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \binom{x(p+1) - (x-1)i - 1}{n}.$$

If X is a nonsingular hypersurface of degree m in $\mathbb{C}P^n$, then $H_p(m) = h^{p,n-1-p}(\mathbb{C}X) - 1$ in the case $n - 1 = 2p$, and $H_p(m) = h^{p,n-1-p}(\mathbb{C}X)$ otherwise (see [DKh86]).

For any integer $p = 0, \dots, n-1$, denote by B_p the vector in $\mathbb{Z}^n$ such that all the coordinates of B_p are equal to 0 except the p-th coordinate which is equal to 1.

The main result of the present paper is the following theorem.

Theorem 1.4. *For any positive integer n and any integer $p = 0, \dots, n-1$, the class of primitive T-hypersurfaces in $\mathbb{R}P^n$ satisfies the condition $B_p \overset{n}{\leq} H_p$.*

Theorem 1.4 immediately implies the following statement.

Corollary 1.5. *For any positive integer n, any integer $p = 0, \dots, n-1$, and any asymptotically maximal sequence $(X_m)_{m\in\mathbb{N}}$ such that X_m is a primitive T-hypersurface of degree m in $\mathbb{R}P^n$, the sequence $(X_m)_{m\in\mathbb{N}}$ satisfies the condition $B_p \overset{n}{=} H_p$, i.e.,*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m;\, \mathbb{Z}_2) = h^{p,n-1-p}(\mathbb{C}X_m) + O(m^{n-1}).$$

Remark 1.1. As it was shown by B. Bertrand [Ber06], for any primitive T-hypersurface X in $\mathbb{R}P^n$ (the projective space can be replaced by any nonsingular projective toric variety) the Euler characteristic of $\mathbb{R}X$ is equal to the signature of $\mathbb{C}X$.

Remark 1.2. The statement of Theorem 1.4 (and the statement of Corollary 1.5) becomes false if one replaces the class of primitive T-hypersurfaces by the class of all nonsingular algebraic hypersurfaces in $\mathbb{R}P^n$. For example, there exists an asymptotically maximal sequence $(Y_m)_{m\in\mathbb{N}}$ of nonsingular surfaces in $\mathbb{R}P^3$ such that Y_m is of degree m and $\dim_{\mathbb{Z}_2} H_0(\mathbb{R}Y_m;\mathbb{Z}_2) = \frac{7}{24}m^3+O(m^2)$; see [Vir79b] (note that $h^{0,2}(\mathbb{C}Y_m) = \frac{1}{6}m^3+O(m^2)$). More detailed information concerning the asymptotic behavior of Betti numbers of algebraic hypersurfaces in $\mathbb{R}P^n$ can be found in [Bih03].

The paper is organized as follows. Section 2 is devoted to the combinatorial patchworking. The key upper bounds used in the proof of Theorem 1.4 are based on the results of [Sh96, ISh03] and are presented in Sections 3 and 4. These upper bounds together with a combinatorial description of Hodge numbers of algebraic hypersurfaces in $\mathbb{C}P^2$ (Corollary 5.2) give a proof of Theorem 1.4. The combinatorial description of Hodge numbers is proved in Sections 5 - 8. Section 5 contains a construction of an asymptotically maximal sequence of hypersurfaces in $\mathbb{R}P^n$. This construction is a simplified version of the construction described in [IV] (the latter construction produces maximal hypersurfaces). To prove that the constructed sequence of hypersurfaces satisfies the condition $B_p \overset{n}{=} H_p$, we present a collection of cycles of these hypersurfaces (Section 6), and prove a recurrent relation for the Hodge numbers (Section 7).

2. Combinatorial Patchworking of Hypersurfaces in $\mathbb{R}P^n$

Let m be a positive integer number (it would be the degree of the hypersurface under construction) and $T^n(m)$ be the simplex in $\mathbb{R}^n$ with vertices $(0,0,\dots,0)$, $(0,0,\dots,0,m)$, $(0,\dots,0,m,0)$, $\dots$, $(m,0,\dots,0)$. We shorten the notation of $T^n(m)$ to T, when n and m are unambiguous and call $T^n(m)$ the *standard n-simplex of size m*. Take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates $(i_1,\dots,i_n)$ is denoted by $\alpha_{i_1,\dots,i_n}$.

Denote by T_* the union of all the symmetric copies of T under reflections and compositions of reflections with respect to coordinate hyperplanes. Extend the triangulation τ to a symmetric triangulation τ_* of T_*, and the distribution of signs $\alpha_{i_1,\dots,i_n}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If an n-simplex of the triangulation of T_* has vertices of different signs, select a piece of hyperplane being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by Γ the union of the selected pieces. It is a piecewise-linear hypersurface contained in T_*. It is not a simplicial subcomplex of T_*, but can be deformed by an isotopy preserving τ_* to a subcomplex K of the first barycenter subdivision τ'_* of τ_*. Each n-simplex of τ'_* has a unique vertex belonging to τ_*. Denote by τ_*^+ the union of all the n-simplices of τ'_* containing positive vertices of τ_* and by τ_*^- the union of all the rest n-simplices. The subcomplex K is the intersection of τ_*^+ and τ_*^-. A point of Γ

contained in a simplex σ of τ_* belongs to a unique segment connecting the face of σ with positive vertices and the face with negative ones. This segment meets K also in a unique point and the deformation of Γ to K can be done along those segments.

Identify by the symmetry with respect to the origin the faces of T_*. The quotient space $\widetilde{T}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Denote by $\widetilde{\Gamma}$ the image of Γ in $\widetilde{T}$.

A triangulation τ of T is said to be *convex* if there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the n-simplices of τ. Sometimes, such triangulations are also called coherent (see [GKZ94]) or regular (see [Zie94]).

Theorem 2.1 (see [Vir83], [Vir94])**.** *If τ is convex, there exists a nonsingular hypersurface X of degree m in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \widetilde{T}$ mapping the set of real points $\mathbb{R}X$ of X onto $\widetilde{\Gamma}$.*

A hypersurface defined by a polynomial

$$\sum_{(i_1,\ldots,i_n)\in V} \alpha_{i_1,\ldots i_n} x_0^{m-i_1-\ldots i_n} x_1^{i_1} \ldots x_n^{i_n} t^{\nu(i_1,\ldots i_n)},$$

where V is the set of vertices of τ, and t is positive and sufficiently small, satisfies the properties described in Theorem 2.1. The polynomial above and its affine version

$$P_t^{\nu,\alpha}(x_1,\ldots,x_n) = \sum_{(i_1,\ldots,i_n)\in V} \alpha_{i_1,\ldots i_n} x_1^{i_1} \ldots x_n^{i_n} t^{\nu(i_1,\ldots i_n)},$$

are called *T-polynomials* associated with the function ν and the distribution of signs $\alpha : V \to \mathbb{R}$, $\alpha(i_1,\ldots,i_n) = \alpha_{i_1,\ldots,i_n}$. The hypersurface X defined by a T-polynomial is called a *T-hypersurface.* If the triangulation τ is *primitive* (that is, each n-simplex of τ is of volume $\frac{1}{n!}$), then X is called a *primitive T-hypersurface.*

3. Critical Points of T-polynomials

To any orthant O in $\mathbb{R}^n$ we associate the map $\mathcal{S}_O = s_{(i_1)} \circ s_{(i_2)} \circ \ldots \circ s_{(i_k)}$, where i_1, …, i_k are the indices of all negative coordinates of a point in the interior of O, and $s_{(i_j)}$, $j = 1$, …, k, is the reflection with respect to the i_j-th coordinate hyperplane in $\mathbb{R}^n$.

Let q be a point in $\mathbb{R}^n$. An n-dimensional lattice simplex δ in an orthant O of $\mathbb{R}^n$ is called *q-generic* if the point $\mathcal{S}_O(q)$ belongs neither to δ, nor to any hyperplane containing an $(n-1)$-dimensional face of δ. Let $\delta \subset O$ be a q-generic simplex. An $(n-1)$-dimensional face of δ is called *q-visible* (resp., *non-q-visible*) if the cone over this face with the vertex at $\mathcal{S}_O(q)$ does not intersect (resp., does intersect) the interior of δ. The q-index $i^q(\delta)$ of δ is the number of q-visible $(n-1)$-dimensional faces of δ. The co-q-index of δ is the number $n - i^q(\delta)$. Denote by $V_+^q(\delta)$ (resp., $V_-^q(\delta)$) the set of vertices of δ which belong to all q-visible (resp., non-q-visible) $(n-1)$-dimensional faces of δ. A q-generic simplex δ whose vertices are equipped with signs is called *real q-critical* if all the vertices in $V_+^q(\delta)$ have the same sign and the vertices in $V_-^q(\delta)$ have the sign opposite to that of the vertices in $V_+^q(\delta)$.

A triangulation τ of $T^n(m)$ is called q-generic, if all its n-simplices are q-generic. A simplex of τ is called *q-terminal* if it is contained in an $(n-1)$-dimensional non-q-visible face of $T^n(m)$. Associate to any non-q-terminal simplex σ of τ an n-simplex of τ in the following way. Let v be a point in the relative interior of σ. Take a point $\hat{v}$ such that

- $\hat{v}$ belongs to the ray which starts at q and passes through v,
- the distance between q and $\hat{v}$ is greater than the distance between q and v,
- the segment joining v and $\hat{v}$ is contained in an n-simplex of τ.

The latter n-simplex does not depend on the choice of v in the relative interior of σ and is called the *q-upper* simplex of σ. A T-polynomial of degree m is called q-generic, if the corresponding triangulation of $T^n(m)$ is q-generic.

Theorem 3.1 (see [Sh96, ISh03]). *Let $q = (-q_1, \ldots, -q_n)$ be a point with negative integer coordinates in $\mathbb{R}^n$, and $P_t^{\nu,\alpha}$ a (non-homogeneous) q-generic T-polynomial of degree m in n variables. Then, there is a one-to-one correspondence between the real critical points in $(\mathbb{R}^*)^n$ of the polynomial*

$$x_1^{q_1} \ldots x_n^{q_n} P_t^{\nu,\alpha}(x_1, \ldots, x_n)$$

and the real q-critical n-simplices of τ_* (where τ is the triangulation defined by ν) such that the index of a real critical point of $P_t^{\nu,\alpha}$ with positive (resp., negative) critical value is equal to the q-index (resp., co-q-index) of the corresponding simplex. If τ is primitive, each n-simplex of τ has exactly one real critical symmetric copy in τ_*.

Proposition 3.2. *Let q be a point in $\mathbb{R}^n$, and τ_1, τ_2 convex primitive q-generic triangulations of a standard simplex $T^n(m)$. Then, for any integer $i = 1, \ldots, n$, the numbers of simplices of q-index i in τ_1 and in τ_2 coincide.*

Proof. As is known (see, for example, [Dai00, Ber06]), for any integer $j = 0, \ldots, n$, the numbers of j-dimensional simplices in τ_1 and τ_2 coincide. Thus, the numbers S_1^j and S_2^j of j-dimensional non-q-terminal simplices in τ_1 and τ_2, respectively, also coincide. For any j-dimensional non-q-terminal simplex σ in τ_k, $k = 1, 2$, the q-index of the q-upper simplex of σ is at least $n - j$. Denote by $C_{i,1}$ and $C_{i,2}$ the numbers of n-simplices of q-index i in τ_1 and τ_2, respectively. Since $C_{n,k} = S_k^0$, $k = 1, 2$, we obtain $C_{n,1} = C_{n,2}$. Furthermore, for any integer $j = 1, \ldots, n-1$, we have $C_{n-j,k} = S_k^j - \sum_{s=0}^{j-1} \binom{n-s}{j-s} C_{n-s,k}$, $k = 1, 2$. Thus, $C_{i,1} = C_{i,2}$ for any integer number $i = 1, \ldots, n$. □

4. Upper Bounds for Betti Numbers of Primitive T-hypersurfaces

Let $\mathcal{P}$ be the product of a real polynomial of degree m in n variables and any monomial in n variables. Let $X_{\mathcal{P}} \subset \mathbb{C}P^n$ be the projective closure of $\{\mathcal{P} = 0\} \cap (\mathbb{C}^*)^n$. Assume that $\mathcal{P}$ has only nondegenerate critical points in $(\mathbb{R}^*)^n$ and that the hypersurface $X_{\mathcal{P}}$ is nonsingular. Denote by c_p^+ (respectively, c_p^-) the number of real critical points of $\mathcal{P}$ in $(\mathbb{R}^*)^n$ of index p and with positive (respectively, negative) critical value.

The following statement is well known and can be found, for example, in [ISh03].

Proposition 4.1. *There exists a real univariate polynomial R of degree $n-1$ such that, for any polynomial $\mathcal{P}$ as above and any integer number $p = 0, 1, \ldots, n-1$, the following inequality holds:*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_{\mathcal{P}}; \mathbb{Z}_2) \leq c_p^- + c_{n-p}^+ + R(m).$$

Let

- $q \in \mathbb{R}^n$ be a point with negative integer coordinates,
- τ a convex primitive q-generic triangulation of $T^n(m)$,
- $\nu : T^n(m) \to \mathbb{R}$ a convex piecewise-linear function certifying the convexity of τ,
- α a distribution of signs at the integer points of $T^n(m)$,
- $P_t^{\nu,\alpha}$ a (non-homogeneous) T-polynomial associated with ν and α,
- X a hypersurface of degree m in $\mathbb{R}P^n$ defined by (the homogenization of) $P_t^{\nu,\alpha}$.

Denote by $C_i(m)$ the number of n-simplices of τ of q-index i. Theorem 3.1 and Proposition 4.1 imply the following statement.

Theorem 4.2 (*cf.* [Sh96, ISh03])**.** *For any integer $p = 0, \ldots, n-1$, the following inequality holds:*

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X; \mathbb{Z}_2) \leq C_{n-p}(m) + R(m),$$

where R is a polynomial described in Proposition 4.1.

According to Proposition 3.2, the numbers $C_n(m), \ldots, C_1(m)$ do not depend on the choice of a convex primitive q-generic triangulation τ of $T^n(m)$. To prove Theorem 1.4, it remains to compare the numbers $C_n(m), \ldots, C_1(m)$ with the numbers $H_0(m), \ldots, H_{n-1}(m)$.

If $\{X_m\}_{m \in \mathbb{N}}$ is an asymptotically maximal sequence of primitive T-hypersurfaces, then due to Theorem 4.2 and the equality $\sum_{p=0}^{n-1} C_{n-p}(m) = m^n$, we obtain

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = C_{n-p}(m) + O(m^{n-1}),$$

for any integer $p = 0, \ldots, n-1$. In the remaining part of the paper, we construct an asymptotically maximal sequence of primitive T-hypersurfaces and show the equality

$$\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = H_p(m) + O(m^{n-1})$$

for the hypersurfaces X_m of the sequence.

5. Triangulation and Signs Generating Asymptotically Maximal Sequence of Hypersurfaces

In this section we describe for each positive integer n and for each positive integer m a triangulation $\tau^n(m)$ of the standard simplex $T^n(m)$ and a distribution of signs at the vertices of $\tau^n(m)$ which provide via Theorem 2.1 an asymptotically maximal sequence of hypersurfaces in $\mathbb{R}P^n$.

To construct the triangulation $\tau^n(m)$, we use induction on n. If $n = 1$, the triangulation $\tau^1(m)$ of $[0, m]$ is formed by m intervals $[0, 1], \ldots, [m-1, m]$ for any m.

Assume that the triangulations of the standard simplices of dimensions less than n and all the sizes are constructed and consider the n-dimensional standard simplex $T^n(m)$ of size m.

Denote by $x_1, \dots, x_n$ the coordinates in $\mathbb{R}^n$. Let $T_j^{n-1} = T^n(m) \cap \{x_n = m - j\}$, and T_j be the image of T_j^{n-1} under the orthogonal projection to the coordinate hyperplane $\{x_n = 0\}$. Numerate the vertices of each simplex $T_1, \dots, T_m$ as follows: assign 1 to the vertex at the origin and $i + 1$ to the vertex with nonzero coordinate at the i-th place. Assign to the vertices of $T_1^{n-1}, \dots, T_{m-1}^{n-1}$ the numbers of their projections. A triangulation of each simplex $T_1, \dots, T_m$ is already constructed. Take the corresponding triangulations in the simplices T_j^{n-1}, if $m - j$ is even. If $m - j$ is odd, take the linear map $T_j^{n-1} \to T_j$ sending the i-th vertex of T_j^{n-1} to the vertex number $n + 1 - i$ of T_j $(i = 1, \dots, n)$. The preimages of simplices of the triangulation of T_j form a triangulation of T_j^{n-1}.

Let l be a nonnegative integer not greater than $n - 1$. If $m - j$ is even, denote by T_j^l the l-face of T_j^{n-1} which is the convex hull of the vertices with numbers $1, \dots, l + 1$. If $m - j$ is odd, denote by T_j^l the l-face of T_j^{n-1} which is the convex hull of the vertices with numbers $n - l, \dots, n$.

Now for any integer $0 \le j \le m - 1$ and any integer $0 \le l \le n - 1$, take the join $T_{j+1}^l * T_j^{n-1-l}$. The triangulations of T_{j+1}^l and T_j^{n-1-l} constructed by the inductive assumption define a triangulation of $T_{j+1}^l * T_j^{n-1-l}$. This gives rise to the desired triangulation $\tau^n(m)$ of $T^n(m)$. It easy to see that $\tau^n(m)$ is convex: a convex piecewise-linear function certifying the convexity of $\tau^n(m)$ can be obtained combining the following functions:

- a convex piecewise-linear function whose domains of linearity are the convex hulls of T_j^{n-1} and T_{j+1}^{n-1}, $j = 0, \dots, m - 1$;
- affine-linear functions $L_j(\varepsilon) : T_j^{n-1} \to \mathbb{R}$ (here j runs over all the integers $1 \le j \le m$ such that $m - j$ is even, and ε is a sufficiently small positive number); any function $L_j(\varepsilon)$ sends a vertex with number i of T_j^{n-1} to εi;
- convex piecewise-linear functions (multiplied by appropriate constants) certifying the convexity of the triangulations of $T_1^{n-1}, \dots, T_m^{n-1}$.

The distribution of signs at the vertices of $\tau^n(m)$ is as follows: all the vertices get the sign "+".

Let $(X_m)_{m \in \mathbb{N}}$ be the sequence of hypersurfaces in $\mathbb{R}P^n$ provided according to Theorem 2.1 by the triangulations $\tau^n(m)$ and the distribution of signs described above.

Theorem 5.1. *For any positive integer n and any integer $p = 0, \dots, n - 1$, the sequence $(X_m)_{m \in \mathbb{N}}$ satisfies* $\dim_{\mathbb{Z}_2} H_p(\mathbb{R}X_m; \mathbb{Z}_2) = H_p(m) + O(m^{n-1})$.

Corollary 5.2. *For any positive integer n and any integer $p = 0, \dots, n - 1$, one has $H_p(m) = C_{n-p}(m) + O(m^{n-1})$.*

Proof. The statement immediately follows from Theorems 5.1 and 4.2 and the equality $\sum_{p=0}^{n-1} C_{n-p}(m) = m^n$. □

Theorem 5.1 is proved in Section 8. We precede the proof by a description of a certain collection of cycles of $\mathbb{R}X_m$ (Section 6) and a recurrent relation for the Hodge numbers of algebraic hypersurfaces in $\mathbb{C}P^n$ (Section 7).

6. Narrow Cycles

For any positive integers n and m, and any integer $p = 0, \ldots, n-1$, we define a collection c_i, $i \in I^{n,p}(m)$ of p-cycles of $\widetilde{\Gamma}^n(m) \subset \widetilde{T}$, where $T = T^n(m)$ and $\widetilde{\Gamma}^n(m)$ is the piecewise-linear hypersurface provided by the triangulation $\tau^n(m)$ and the distribution of signs described in Section 5 (in fact, any c_i is also a p-cycle of the hypersurface $\Gamma^n(m) \subset T_*$). The cycles c_i are called *narrow*.

The collection of narrow cycles c_i is constructed together with a collection of *axes* b_i. Any axis b_i is a $(n-1-p)$-cycle in $\widetilde{T} \setminus \widetilde{\Gamma}^n(m)$ (where p is the dimension of c_i) composed by simplices of the triangulation $\tau^n_*(m)$ of T_* and representing a homological class such that its linking number with any p-dimensional narrow cycle c_k is δ_{ik}.

Let us fix some notations. For any simplex T^l_j (where $1 \le j \le m$ and $0 \le l \le n-1$), denote by $(T^l_j)_*$ the union of the symmetric copies of T^l_j under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$, where $i = 1, \ldots, l$, if $m - j$ is even, and $i = n-l, \ldots, n-1$, if $m - j$ is odd, and compositions of these reflections.

Any simplex T^l_j is naturally identified with the standard simplex $T^l(j)$ in $\mathbb{R}^l$ with vertices $(0, \ldots, 0)$, $(j, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, j)$ via the linear map $\mathcal{L}^l_j : T^l_j \to T^l(j)$ sending

(1) the vertex with number i of T^l_j to the vertex of $T^l(j)$ with the same number, if $m - j$ is even,

(2) the vertex with number i of T^l_j to the vertex of $T^l(j)$ with the number $i-n+l+1$, if $m - j$ is odd.

It is easy to see that $\mathcal{L}^l_j$ is simplicial with respect to the chosen triangulations of T^l_j and $T^l(j)$. The natural extension of $\mathcal{L}^l_j$ to $(T^l_j)_*$ identifies $(T^l_j)_*$ with $(T^l(m))_*$ and respects the chosen triangulations.

By *a symmetry* we mean a composition of reflections with respect to coordinate hyperplanes. Let $s_{(i)}$ be the reflection of $\mathbb{R}^n$ with respect to the hyperplane $\{x_i = 0\}$, $i = 1, \ldots, n$. Denote by s^l_j the symmetry of $(T^{l+1}_j)_*$ which is identical if $m - j$ is even, and coincides with the restriction of $s_{(n-l-1)} \circ \ldots \circ s_{(n-1)}$ on $(T^{l+1}_j)_*$ if $m - j$ is odd.

The narrow cycles and their dual cycles are defined below using induction on n. For $n = 1$ and $m \ge 3$, the narrow cycles are the pairs of points

$$(-1/2, -3/2), \ldots, (-(2m-5)/2, -(2m-3)/2),$$

(The set of narrow cycles is empty if $n = 1$ and $m = 1, 2$.) The axes are pairs of vertices

$$(-1, -m+1), (-2, -m), (-3, -m+1), \ldots, (-m+2, -m),$$

if m is even, and pairs of vertices

$$(-1, -m), (-2, -m+1), (-3, -m), \ldots, (-m+2, -m),$$

if m is odd.

Assume that for all natural m and all natural $k < n$ the narrow cycles c_i in the hypersurface $\widetilde{\Gamma}^k(m) \subset \widetilde{T}^k(m)$ and the axes b_i in $\widetilde{T}^k(m) \setminus \widetilde{\Gamma}^k(m)$ are constructed. The narrow cycles of the hypersurface in $\widetilde{T}^n_m$ are divided into 3 families.

Horizontal Cycles. The initial data for constructing a cycle of the first family consist of an integer j satisfying inequality $1 \le j \le m-1$ and a narrow cycle of the hypersurface in $(T^{n-1}(j))_*$ constructed at the previous steps. In the copy $(T^{n-1}_j)_*$ of $(T^{n-1}(j))_*$, take the copy c of this cycle and the copy b of its axis.

There exists exactly one symmetric copy of T^0_{j+1} incident to b. It is T^0_{j+1} itself, if $m-j$ is odd, and either T^0_{j+1}, or $s_{(n-1)}(T^0_{j+1})$, if $m-j$ is even. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b is opposite to the sign of c, we include c in the collection of narrow cycles of $\widetilde{\Gamma}$. Otherwise take $s_{(n)}(c)$ as a narrow cycle of $\widetilde{\Gamma}$. The axis of c (resp., $s_{(n)}(c)$) is the suspension of b (resp., $s_{(n)}(b)$) with the vertex $s(T^0_{j+1})$ (resp., $s_{(n)}(s(T^0_{j+1}))$) and with the vertex $s(T^0_{j-1})$ (resp., $s_{(n)}(s(T^0_{j-1}))$).

Co-Horizontal Cycles. The initial data for constructing a cycle of the second family are the same as in the case of the horizontal cycles: the data consist of an integer j satisfying inequality $1 \le j \le m-1$ and a narrow cycle of the hypersurface in $(T^{n-1}(j))_*$.

In the copy $(T^{n-1}_j)_*$ of $(T^{n-1}(j))_*$, take the copy c of this cycle and the copy b of its axis. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b coincides with the sign of c, take b as axis of a narrow cycle of $\widetilde{\Gamma}$. Otherwise take $s_{(n)}(b)$. The corresponding narrow cycle is a suspension of c (resp., $s_{(n)}(c)$).

Join Cycles. The initial data consist of integers j and l satisfying inequalities $1 \le j \le m-1$, $1 \le l \le n-2$, the copy $c_1 \subset (T^l_{j+1})_*$ of a narrow cycle of the hypersurface in $(T^l(j+1))_*$, the copy $c_2 \subset (T^{n-1-l}_j)_*$ of a narrow cycle of the hypersurface in $(T^{n-1-l}(j))_*$ and the copies $b_1 \subset (T^l_{j+1})_*$ and $b_2 \subset (T^{n-1-l}_j)_*$ of the axes of these narrow cycles.

One of the joins $b_1 * b_2$ and $s^l_{j+1}(b_1) * s^{n-1-l}_j(b_2)$, belongs to $\tau^n_*(m)$; denote this join by J. If the signs of c_1 and c_2 coincide, take J as the axis of a cycle of $\widetilde{\Gamma}^n(m)$. Otherwise take $s_{(n)}(J)$. The corresponding narrow cycle is either $c_1 * c_2$, or $s^l_{j+1}(c_1) * s^{n-1-l}_j(c_2)$, or $s_{(n)}(c_1 * c_2)$, or $s_{(n)}(s^l_{j+1}(c_1) * s^{n-1-l}_j(c_2))$.

Proposition 6.1. *For any integer $p = 0, \ldots, n-1$, the $\mathbb{Z}_2$-homology classes of the narrow cycles c_i, $i \in I^{n,p}(m)$, are linearly independent in $H_p(\widetilde{\Gamma}^n(m); \mathbb{Z}_2)$.*

Proof. Both c_i and b_i with $i \in I^{n,p}(m)$ are $\mathbb{Z}_2$-cycles homologous to zero in $\widetilde{T}$, which is homeomorphic to the projective space of dimension n. The sum of dimensions of c_i and b_i is $n-1$. Thus we can consider the linking number of c_i, $i \in I^{n,p}(m)$, and b_k, $k \in I^{n,p}(m)$, taking values in $\mathbb{Z}_2$. Each c_i bounds an obvious ball in $\widetilde{T}$. This ball meets b_i in a single point transversally and is disjoint with b_k for $k \neq i$ and $i, k \in I^{n,p}(m)$. Hence the linking number of c_i and b_k is δ_{ik}. This proves that the cycles c_i, $i \in I^{n,p}(m)$, realize linearly independent $\mathbb{Z}_2$-homology classes of $\widetilde{\Gamma}^n(m)$. □

7. Recurrent Relation for Hodge Numbers

For positive integers n and m, and an integer $p = 0, \ldots, n-1$, denote by $A^{n,p}_m$ the number of ordered $(n+1)$-partitions of $m(p+1)$ such that each of the summands does not exceed $m-1$. In other words, this is the number of interior integer points in the section of the cube $[0,m]^{n+1}$ by the hyperplane $\sum_{i=1}^{n+1} x_i = m(p+1)$.

We have $A^{n,p}_m = h^{p,n-1-p}(\mathbb{C}X) - 1$, if $n-1 = 2p$, and $A^{n,p}_m = h^{p,n-1-p}(\mathbb{C}X)$ otherwise, where X is a nonsingular surface of degree m in $\mathbb{C}P^n$ (see [DKh86]). Furthermore,

$$A^{n,p}_m = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \binom{m(p+1) - (m-1)i - 1}{n}.$$

If either $n < 0$ or $p < 0$, put $A^{n,p}_m = 0$. If $n = 0$ and $p \neq 0$, put $A^{n,p}_m = 0$. Finally, if $n = 0$ and $p = 0$, put $A^{n,p}_m = 1$.

Proposition 7.1. *Let n and m be positive integers, and p a nonnegative integer not greater than $n-1$. The following recurrent relation holds true:*

$$\begin{aligned}
A^{n,p}_m = &\sum_{j=1}^{m-1} A^{n-1,p}_j + \sum_{j=1}^{m-1} A^{n-1,p-1}_j + \sum_{j=1}^{m-1} A^{n-2,p-1}_{j+1} + \sum_{j=1}^{m-1} A^{n-2,p-1}_j + \\
&\sum_{j=1}^{m-1}\sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A^{l,k}_{j+1} A^{n-1-l,p-1-k}_j + \sum_{j=1}^{m-1}\sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A^{l,k}_{j+1} A^{n-2-l,p-1-k}_j + \\
&\sum_{j=1}^{m-1}\sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A^{l,k}_{j+1} A^{n-2-l,p-2-k}_j + \sum_{j=1}^{m-1}\sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A^{l,k}_{j+1} A^{n-3-l,p-2-k}_j.
\end{aligned}$$

Proof. We prove the statement using induction on m. If $m = 1$ the statement is evident. For the inductive step, we need to show that

$$\begin{aligned}
A^{n,p}_m = &A^{n,p}_{m-1} + A^{n-1,p}_{m-1} + A^{n-1,p-1}_{m-1} + A^{n-2,p-1}_m + A^{n-2,p-1}_{m-1} + \\
&\sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A^{l,k}_m A^{n-1-l,p-1-k}_{m-1} + \sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A^{l,k}_m A^{n-2-l,p-1-k}_{m-1} + \\
&\sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A^{l,k}_m A^{n-2-l,p-2-k}_{m-1} + \sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A^{l,k}_m A^{n-3-l,p-2-k}_{m-1}.
\end{aligned}$$

We call an ordered $(n+1)$-partition of $m(p+1)$ *appropriate*, if each of its summands does not exceed $m-1$. A partition $a_1 + \ldots + a_s$ of mr such that all the summands do not exceed $m-1$ is called *reducible*, if there exist integers k and l such that $l < s-1$ and

$$\sum_{i=1}^{l+1} a_i = m(k+1).$$

For any reducible partition, denote by L the largest $l < s-1$ such that $\sum_{i=1}^{l+1} a_i$ is divisible by m. A partition of mr such that all the summands do not exceed $m-1$ and which is not reducible is called *irreducible*.

Denote the summands of an appropriate partition by $a_1, \ldots, a_{n+1}$. Let us prove that

(1) $A_{m-1}^{n,p}$ is the number of appropriate irreducible partitions with $a_1 < m-1$ and $a_{n+1} > 1$,
(2) $A_{m-1}^{n-1,p}$ is the number of appropriate irreducible partitions with $a_1 < m-1$ and $a_{n+1} = 1$,
(3) $A_{m-1}^{n-1,p-1}$ is the number of appropriate irreducible partitions with $a_1 = m-1$ and $a_{n+1} > 1$,
(4) $A_{m-1}^{n-2,p-1}$ is the number of appropriate irreducible partitions with $a_1 = m-1$ and $a_{n+1} = 1$,
(5) $\sum_{l=1}^{n-2}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-1-l,p-1-k}$ is the number of appropriate reducible partitions with $a_{L+2} < m-1$ and $a_{n+1} > 1$,
(6) $\sum_{l=1}^{n-3}\sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-2-l,p-1-k}$ is the number of appropriate reducible partitions with $a_{L+2} < m-1$ and $a_{n+1} = 1$,
(7) $\sum_{l=1}^{n-3}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-2-l,p-2-k}$ is the number of appropriate reducible partitions with $a_{L+2} = m-1$ and $a_{n+1} > 1$,
(8) $\sum_{l=1}^{n-4}\sum_{k=0}^{p-2} A_m^{l,k} A_{m-1}^{n-3-l,p-2-k} + A_m^{n-2,p-1}$ is the number of appropriate reducible partitions with $a_{L+2} = m-1$ and $a_{n+1} = 1$.

Let Π be an ordered s-partition $a_1+\ldots+a_s$ of $(m-1)r$, where $a_i \le m-2$ for $i = 1, \ldots, s$. This partition defines in the following way an ordered s-partition $f(\Pi) : a'_1 + \ldots + a'_s$ of mr with $a'_i \le m-1$ and an ordered $(s+1)$-partition $g(\Pi) : a''_1 + \ldots + a''_{s+1}$ of mr with $a''_i \le m-1$. Let $i_1, \ldots, i_{r-1}$ be the integers such that

$$\sum_{j=1}^{i_q} a_j \le (m-1)q, \quad \sum_{j=1}^{i_q+1} a_j > (m-1)q,$$

for any $q = 1, \ldots, r-1$. Take $a'_i = a_i + 1$ if $i = i_q + 1$ (for some $q = 1, \ldots, r-1$) or $i = s$, and $a'_i = a_i$ otherwise. Take $a''_i = a_i + 1$, if $i = i_q + 1$ (for some $q = 1, \ldots, r-1$), and $a''_i = a_i$ otherwise. Take, in addition, $a''_{s+1} = 1$. Note that the partitions $f(\Pi)$ and $g(\Pi)$ are both irreducible, $a'_1 < m-1$, $a'_s > 1$, and $a''_1 < m-1$. For any irreducible ordered s-partition $\Phi : a'_1 + \ldots + a'_s$ of mr such that $a'_1 < m-1$, $a'_s > 1$, and $a'_i \le m-1$, $i = 2$, $\ldots$, s, there exists a unique partition Π such that $f(\Pi) = \Phi$. Indeed, let $i_1, \ldots, i_{r-1}$ be the integers such that

$$\sum_{j=1}^{i_q} a'_j \leq mq - 1, \quad \sum_{j=1}^{i_q+1} a'_j > mq - 1;$$

for any $q = 1, \ldots, r-1$; take $a_i = a'_i - 1$, if $i = i_q + 1$ (for some $q = 1, \ldots, r-1$) or $i = s$, and $a_i = a'_i$ otherwise. (Note that $a'_{i_q+1} > 1$, because Φ is irreducible.) Similarly, for any irreducible ordered $(s+1)$-partition $\Psi : a''_1 + \ldots + a''_{s+1}$ of mr such that $a''_1 < m-1$, $a''_{s+1} = 1$, and $a''_i \leq m-1$, $i = 2, \ldots, s$, there exists a unique partition Π such that $g(\Pi) = \Psi$.

The constructions of $f(\Pi)$ and $g(\Pi)$ described above give immediately (1) and (2). To prove (3) (respectively, (4)), one can apply the construction of $f(\Pi)$ (respectively, $g(\Pi)$) to ordered $(n+1)$-partitions $a_1 + \ldots + a_{n+1}$ (respectively, to ordered n-partitions $a_1 + \ldots + a_n$) of $(m-1)(p+1)$ such that $a_1 = m-1$ and $a_i \leq m-2$ for $i = 2, \ldots, n+1$ (resp., $i = 2, \ldots, n$).

The statements (5) - (8) follow from (1) - (4): to any appropriate reducible partition $a_1 + \ldots + a_{n+1}$, one can associate the irreducible partition $a_{L+2} + \ldots + a_{n+1}$. □

8. Proofs of Theorems 5.1 and 1.4

Proof of Theorem 5.1. For positive integers n and m, and an integer $p = 0, \ldots, n-1$, denote by $N_m^{n,p}$ the number of narrow p-cycles c_i, $i \in I^{n,p}(m)$ constructed in Section 6. If either $n \leq 0$ or $p < 0$, put $N_m^{n,p} = 0$. If $n = 0$ and $p \neq 0$, put $N_m^{n,p} = 0$. Finally, if $n = 0$ and $p = 0$, put $N_m^{n,p} = 1$.

According to the construction of narrow cycles, the numbers $N_m^{n,p}$ satisfy the following recurrent relation:

$$N_m^{n,p} = N_{m-1}^{n,p} + N_{m-1}^{n-1,p} + N_{m-1}^{n-1,p-1} + \sum_{l=1}^{n-2} \sum_{k=0}^{p-1} N_m^{l,k} N_{m-1}^{n-1-l,p-1-k}.$$

In addition, $N_1^{1,0} = N_2^{1,0} = 0$ and $N_m^{1,0} = m-2$ for any integer $m \geq 3$.

Fix a positive integer n and an integer $p = 0, \ldots, n-1$. Notice that $A_m^{n,p} \leq (m-1)^n$ for any positive integer m. Thus, Proposition 7.1 implies that, for $n \geq 2$, one has

$$A_m^{n,p} = A_{m-1}^{n,p} + A_{m-1}^{n-1,p} + A_{m-1}^{n-1,p-1} + \sum_{l=1}^{n-2} \sum_{k=0}^{p-1} A_m^{l,k} A_{m-1}^{n-1-l,p-1-k} + O(m^{n-2}).$$

In addition, $A_m^{1,0} = m-1$ for any positive integer m. Comparing the two recurrent relations, we obtain $N_m^{n,p} = A_m^{n,p} + O(m^{n-1})$. This proves Theorem 5.1, since, according to Proposition 6.1, the cycles c_i, $i \in I^{n,p}(m)$, realize linearly independent $\mathbb{Z}_2$-homology classes in $H_p(\widetilde{\Gamma}^n(m); \mathbb{Z}_2)$. □

Proof of Theorem 1.4. The statement immediately follows from Theorem 4.2 and Corollary 5.2. □

References

[Ber06] B. Bertrand, *Euler characteristic of primitive T-hypersurfaces and maximal surfaces*, Preprint arxiv: math.AG/0602534, 2006.

[Bih03] F. Bihan, *Asymptotiques de nombres de Betti d'hypersurfaces projectives réelles*, Preprint arxiv: math.AG/0312259, 2003.

[Bre72] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, N. Y. - London, 1972.

[Dai00] D. Dais, Über unimodulare, kohärente Triangulierungen von Gitterpolytopen. Beispiele und Anwendungen. Lecture notes of the summer school "Géométrie des variétés toriques", Grenoble, 2000.

[DKh86] V. I. Danilov, A. G. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Izv. Acad. Nauk SSSR **50** (1986), (Russian), English transl., Math. USSR Izvestiya **29:2** (1987), 279–298.

[Far57] I. Fary, *Cohomologie des variétés algébriques*, Ann. Math. **65** (1957), 21–73.

[Flo52] E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. AMS **72** (1952), 138–147.

[GKZ94] I. Gelfand, M. Kapranov and A. Zelevinski, *Discriminants, resultants and multidimensional determinants*. Birkhäuser. Boston, 1994.

[Har76] A. Harnack, *Über Vieltheiligkeit der ebenen algebraischen Curven*, Math. Ann. **10** (1876), 189–199.

[ISh03] I. Itenberg and E. Shustin, *Critical points of real polynomials and topology of real algebraic T-surfaces*, Geom. Dedicata. **101** (2003), no. 1, 61 - 91.

[IV96] I. Itenberg and O. Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Math. Intelligencer **18:4** (1996), 19–28.

[IV] I. Itenberg and O. Viro, *Maximal real algebraic hypersurfaces of projective space*, (in preparation).

[Kha72] V. M. Kharlamov, *The maximal number of components of a 4th degree surface in* $\mathbb{R}P^3$, Funksional. Anal. i Prilozhen. **6** (1972), 101 (Russian).

[Kha73] V. M. Kharlamov, *New congruences for the Euler characteristic of real algebraic varieties*, Funksional. Anal. i Prilozhen. **7** (1973), 74–78 (Russian).

[Kha75] V. M. Kharlamov, *Additional congruences for the Euler characteristic of real algebraic variety of even degree*, Funksional. Anal. i Prilozhen. **9** (1975), 51–60 (Russian).

[Mi05] G. Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$. J. Amer. Math. Soc. **18** (2005), 313–377.

[Ris92] J.-J. Risler, *Construction d'hypersurfaces réelles [d'après Viro]*, Séminaire N. Bourbaki, no. 763, vol. 1992–93, November 1992.

[Rok72] V. A. Rokhlin, *Congruences modulo* 16 *in Hilbert's sixteenth problem*, Funktsional. Anal. i Prilozhen. **6** (1972), 58–64 (Russian), English transl., Functional Anal. Appl.

[Smi38] P. A. Smith, *Transformations of finite period*, Ann. of Math. **39** (1938), 127–164.

[Sh96] E. Shustin, *Critical points of real polynomials, subdivisions of Newton polyhedra and topology of real algebraic hypersurfaces*, Amer. Math. Soc. Transl. (2) **173** (1996), 203–223.

[Stu94] B. Sturmfels, *Viro's theorem for complete intersections.* Annali della Scoula Normale Superiore di Pisa (4), **21:3** (1994), 377–386.

[Tho65] R. Thom, *Sur l'homologie des variétés algébriques réelles*, Diff. and comb. top., A Symp. in honor of M. Morse, Prinston Univ. Press, 1965, pp. 255–265.

[Vir79a] O. Viro, *Constructing M-surfaces*, Funktsional. Anal. i Prilozhen. **13** (1979), 71–72 (Russian), English transl., Functional Anal. Appl.

[Vir79b] O. Viro, *Construction of multicomponent real algebraic surfaces*, Dokl. Akad. Nauk SSSR **248** (1979), no. 2 (Russian), English transl., Soviet Math. Dokl. **20** (1979), no. 5, 991–995.

[Vir83] O. Viro, *Gluing of algebraic hypersurfaces, smoothing of singularities and construction of curves*, Proc. Leningrad Int. Topological Conf. (Leningrad, Aug. 1983), Nauka, Leningrad, 1983, pp. 149–197 (in Russian).

[Vir84] O. Viro, *Gluing of plane real algebraic curves and construction of curves of degrees 6 and 7*, Lect. Notes Math. **1060** Springer-Verlag, Berlin Heidelberg, 1984, pp. 187–200.

[Vir94] O. Viro, *Patchworking real algebraic varieties*, Preprint. Uppsala University. 1994.

[Zie94] G. Ziegler, *Lectures on Polytopes.* Springer-Verlag. New York, 1994.

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