# Moduli spaces of rational tropical curves 

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#### Abstract

This note is devoted to the definition of moduli spaces of rational tropical curves with $n$ marked points. We show that this space has a structure of a smooth tropical variety of dimension $n-3$. We define the Deligne-Mumford compactification of this space and tropical $\psi$-class divisors.


This paper gives a detailed description of the moduli space of tropical rational curves mentioned in [4]. The survey [4] was prepared under rather sharp time and volume constraints. As a result the coordinate presentation of this moduli space from [4] contains a mistake (it was over-simplified). In this paper we'll correct the mistake and give a detailed description on $\overline{\mathcal{M}}_{0,5}$ as our main example.

## 1. Introduction: smooth tropical varieties

In this section we follow the definitions of [5] and [4].
The underlying algebra of tropical geometry is given by the semifield $\mathbb{T}=\mathbb{R} \cup\{-\infty\}$ of tropical numbers. The tropical arithmetic operations are " $a+b$ " $=\max \{a, b\}$ and $" a b "=a+b$. The quotation marks are used to distinguish between the tropical and classical operations. With respect to addition $\mathbb{T}$ is a commutative semigroup with zero " $0_{\mathbb{T}}$ " $=-\infty$. With respect to multiplication $\mathbb{T}^{\times}=\mathbb{T} \backslash\left\{0_{\mathbb{T}}\right\} \approx \mathbb{R}$ is an honest commutative group with the unit " $1_{\mathbb{T}}$ " $=0$. Furthermore, the addition and multiplication satisfy the distribution law " $a(b+c)$ " " " $a b+a c$ ", $a, b, c \in \mathbb{T}$. These operations may be viewed as a result of the so-called dequantization of the classical arithmetic operations that underlies the patchworking construction, see [3] and [8].

These tropical operations allow one to define tropical Laurent polynomials. Namely, a tropical Laurent polynomial is a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$,

$$
f(x)=" \sum_{j} a_{j} x^{j} "=\max _{j}\left(a_{j}+j x\right),
$$

where $j x$ denotes the scalar product, $x \in\left(\mathbb{T}^{\times}\right)^{n} \approx \mathbb{R}^{n}, j \in \mathbb{Z}^{n}$ and only finitely many coefficients $a_{j} \in \mathbb{T}$ are non-zero (i.e. not $-\infty$ ).

Affine-linear functions with integer slopes (for brevity we call them simply affine functions) form an important subcollection of all Laurent polynomials. Namely, these are such functions $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ that both $f$ and " $\frac{1_{T}}{f} "=-f$ are tropical Laurent polynomials.

We equip $\mathbb{T}^{n} \approx[-\infty, \infty)^{n}$ with the Euclidean topology. Let $U \subset \mathbb{T}^{n}$ be an open set.

Definition 1.1. A continuous function $f: U \rightarrow \mathbb{T}$ is called regular if its restriction to $U \cap \mathbb{R}^{n}$ coincides with a restriction of some tropical Laurent polynomial to $U \cap \mathbb{R}^{n}$.

We denote the sheaf of regular functions on $\mathbb{T}^{n}$ with $\mathcal{O}$ (or sometimes $\mathcal{O}_{\mathbb{T}^{n}}$ to avoid confusion). Any subset $X \subset \mathbb{T}^{n}$ gets an induced regular sheaf $\mathcal{O}_{X}$ by restriction. For our purposes we restrict our attention only to the case when $X$ is a polyhedral complex, i.e. when $X$ is the closure of a union of convex polyhedra (possibly unbounded) in $\mathbb{R}^{n}$ such that the intersection of any number of such polyhedra is their common face. We say that $X$ is an $k$-dimensional polyhedral complex if it is obtained from a union of $k$-dimensional polyhedra. These polyhedra are called the facets of $X$.

Let $V \subset X$ be an open set and $f \in \mathcal{O}_{X}(V)$ be a regular function in $V$. A point $x \in V$ is called a "zero point" of $f$ if the restriction of " $\frac{1_{\mathrm{T}}}{f}$ " $=-f$ to $W \subset V$ is not regular for any open neighborhood $W \ni x$. Note that it may happen that $x$ is a "zero point" for $\phi: U \rightarrow \mathbb{T}$, but not for $\left.\phi\right|_{X \cap U}$. It is easy to see that if $X$ is a $k$-dimensional polyhedral complex then the "zero locus" $Z_{f}$ of $f$ is a $(k-1)$-dimensional polyhedral subcomplex.

To each facet of $Z_{f}$ we may associate a natural number, called its weight (or degree). To do this we choose a "zero point" $x$ inside such a facet. We say that $x$ is a "simple zero" for $f$ if for any local decomposition into a sum (i.e. the tropical product) of regular function $f=$ " $g h "=g+h$ on $V$ near $x$ we have either $g$ or $h$ affine (i.e. without a "zero"). We say that the weight is $l$ if $f$ can be locally decomposed into a tropical product of $l$ functions with a simple zero at $x$.

A regular function $f$ allows us to make the following modification on its domain $V \subset X \subset \mathbb{T}^{n}$. Consider the graph

$$
\Gamma_{f} \subset V \times \mathbb{T} \subset \mathbb{T}^{n+1}
$$

It is easy to see that the "zero locus"

$$
\bar{\Gamma}_{f} \subset V \times \mathbb{T}
$$

of the (regular) function " $y+f(x)$ " (defined on $V \times \mathbb{T}$ ), where $x$ is the coordinate on $V$ and $y$ is the coordinate on $\mathbb{T}$, coincides with the union of $\Gamma_{f}$ and the undergraph

$$
U \Gamma_{f, Z}=\left\{(x, y) \in V \times \mathbb{T} \mid x \in Z_{f}, y \leq f(x)\right\}
$$

Furthermore, the weight of a facet $F \subset \bar{\Gamma}_{f}$ is 1 if $F \in \Gamma_{f}$ (recall that as $V$ is an unweighted polyhedral complex all the weights of its facets are equal to one) and is the weight of the corresponding facet of $Z_{f}$ if $F \in U \Gamma_{f, Z}$. We view $\bar{\Gamma}_{f}$ as a "tropical closure" of the settheoretical graph $\Gamma_{f}$. Note that we have a map $\bar{\Gamma}_{f} \rightarrow V$. We set $\tilde{V}=\bar{\Gamma}_{f}$ to be the result of the tropical modification $\mu_{f}: \tilde{V} \rightarrow V$ along the regular function $f$. The locus $Z_{f}$ is called the center of tropical modification.

The weights of the facets of $\tilde{V}$ supplies us with some inconvenience as they should be incorporated in the definition of the regular sheaf $\mathcal{O}_{\tilde{V}}$ on $\tilde{V}$. Namely, the affine functions defined by $\mathcal{O}_{\tilde{V}}$ on a facet of weight $w$ should contain the group of functions that come as restrictions to this facet of the affine functions on $\mathbb{T}^{n+1}$ as a subgroup of index $w$.

Sometimes one can get rid of the weights of $\tilde{V}$ by a reparameterization with the help of a map $\bar{V} \rightarrow \tilde{V}$ that is given by locally linear maps in the corresponding charts. Indeed, the restriction of $\mu_{g}: \bar{V} \rightarrow \tilde{V}$ to a facet is locally given by a linear function between two $k$-dimensional affine-linear spaces defined over $\mathbb{Z}$. If its determinant equals to $w$ then the push-forward of $\mathcal{O}_{\bar{V}}$ supplies an extension of $\mathcal{O}_{\tilde{V}}$ required by the weights. Note however that if $w>1$ then the converse map is not defined over $\mathbb{Z}$ and thus is not given by elements of $\mathcal{O}_{\tilde{V}}$.

Tropical modifications give the basic equivalence relation in Tropical Geometry. It can be shown that if we start from $\mathbb{T}^{k}$ and do a number of tropical modifications on it then the result is a $k$-dimensional polyhedral complex $Y \subset \mathbb{T}^{n}$ that satisfies to the following balancing property (cf. Property 3.3 in [4] where balancing is restated in an equivalent way).

Property 1.2. Let $E \subset Y \cap \mathbb{R}^{N}$ be a $(k-1)$-dimensional face and $F_{1}, \ldots, F_{l}$ be the facets of $Y$ adjacent to $F$ whose weights are $w_{1}, \ldots, w_{l}$. Let $L \subset \mathbb{R}^{N}$ be a $(N-k)$-dimensional affine-linear space with an integer slope and such that it intersects $E$. For a generic (real) vector $v \in \mathbb{R}^{N}$ the intersection $F_{j} \cap(L+v)$ is either empty or a single point. Let $\Lambda_{F_{j}} \subset \mathbb{Z}^{N}$ be the integer vectors parallel to $F_{j}$ and $\Lambda_{L} \subset \mathbb{Z}^{N}$ be the integer vectors parallel to $L$. Set $\lambda_{j}$ to be the product of $w_{j}$ and the index of the subgroup $\Lambda_{F_{j}}+\Lambda_{L} \subset \mathbb{Z}^{N}$. We say that $Y \subset \mathbb{T}^{n}$ is balanced if for any choice of $E, L$ and a small generic $v$ the sum

$$
\iota_{L}=\sum_{j \mid F_{j} \cap(L+v) \neq \emptyset} \lambda_{j}
$$

is independent of $v$. We say that $Y$ is simply balanced if in addition for every $j$ we can find $L$ and $v$ so that $F_{j} \cap(L+v) \neq \emptyset, \iota_{L}=1$ and for every small $v$ there exists an affine hyperplane $H_{v} \subset L$ such that the intersection $Y \cap(L+v)$ sits entirely on one side of $H_{v}+v$ in $L+v$ while the intersection $Y \cap\left(H_{v}+v\right)$ is a point.

Definition 1.3 (cf. [5],[4]). A topological space $X$ enhanced with a sheaf of tropical functions $\mathcal{O}_{X}$ is called a (smooth) tropical variety of dimension $k$ if for every $x \in X$ there exist an open set $U \ni x$ and an open set $V$ in a simply balanced polyhedral complex $Y \subset \mathbb{T}^{N}$ such that the restrictions $\left.\mathcal{O}_{X}\right|_{U}$ and $\left.\mathcal{O}_{Y}\right|_{V}$ are isomorphic.

Tropical varieties are considered up to the equivalence generated by tropical modifications. It can be shown that a smooth tropical variety of dimension $k$ can be locally obtained from $\mathbb{T}^{k}$ by a sequence of tropical modifications centered at smooth tropical varieties of dimension $(k-1)$. This follows from the following proposition.

Proposition 1.4. Any $k$-dimensional simply balanced polyhedral complex $X \subset \mathbb{R}^{n}$ can be obtained from $\mathbb{T}^{k}$ by a sequence of consecutive tropical modifications whose centers are simply balanced $(k-1)$-dimensional polyhedral complexes.

Proof. We prove this proposition inductively by $n$. Without the loss of genericity we may assume that $X$ is a fan, i.e. each convex polyhedron of $X$ is a cone centered at the origin.

The base of the induction, when $n=k$, is trivial. When $n>k$ let us take an $(n-k)$-dimensional affine-linear subspace $L \subset \mathbb{R}^{n}$ given by Property 1.2. Choose a linear projection

$$
\lambda: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n-1}
$$

defined over $\mathbb{Z}$ and such that $\operatorname{ker}(\lambda)$ is a line contained in $L$.
The image $\lambda(X) \subset \mathbb{R}^{n-1}$ is a $k$-dimensional polyhedral complex since $L$ is transversal to some facets of $X$. We claim that

$$
\left.\lambda\right|_{X}: X \rightarrow \lambda(X)
$$

is a tropical modification once we identify $\mathbb{R}^{n}$ and $\mathbb{R}^{n-1} \times \mathbb{R}$. The center of this modification is the locus

$$
Z_{f}=\left\{x \in \mathbb{R}^{n-1} \mid \operatorname{dim}\left(\lambda^{-1}(x) \cap X\right)>0\right\}
$$

Here we use the dimension in the usual topological sense. Note that the ( $k-1$ )-dimensional complex $Z_{f} \subset \mathbb{R}^{n-1}$ is simply balanced, existence of the needed ( $n-k$ )-dimensional affinelinear spaces follows from the fact that $X \subset \mathbb{R}^{n}$ is simply balanced.

To justify our claim we note that near any point $x \in Z_{f}$ the subcomplex $Y \subset X$ obtained as the (Euclidean topology) closure of $X \backslash \lambda^{-1}\left(Z_{f}\right)$ is a (set-theoretical) graph of a convex function. This, once again, follows from the fact that $X \subset \mathbb{R}^{n}$ is simply balanced, this time applied to the points in the facets on $X \backslash Y$. Thus it gives a regular tropical function $f$ and it remains only to show that the the weight of any facet of $E \subset Z_{f}$ is 1 . But this follows, in turn, from the balancing condition at $\lambda^{-1}(E) \cap Y$.

## 2. Tropical curves and their moduli spaces

The definition of tropical variety is especially easy in dimension 1. Tropical modifications take a graph into a graph (with arbitrary valence of its vertices) and the tropical structure carried by the sheaf $\mathcal{O}_{X}$ amounts to a complete metric on the complement of the set of 1 -valent vertices of the graph $X$ (cf. [5], [6], [1]). Thus, each 1-valent vertex of a tropical curve $X$ is adjacent to an edge of infinite length.

A tropical modification allows one to contract such an edge or to attach it at any point of $X$ other than a 1 -valent vertex. If we have a finite collection of marked points on $X$ then by passing to an equivalent model if needed we may assume that the set of marked points coincides with the set of 1 -valent vertices. (Of course, if $X$ is a tree then we have to have at least two marked points to make such assumption.)

The genus of a tropical curve $X$ is $\operatorname{dim} H_{1}(X)$. Let $\mathcal{M}_{g, n}$ be the set of all tropical curves $X$ of genus $g$ with $n$ distinct marked points. Fixing a combinatorial type of a graph $\Gamma$ with $n$ marked leaves defines a subset $U_{\Gamma} \subset \mathcal{M}_{g, n}$ consisting of marked tropical curves with this combinatorics. A length of any non-leaf edge of $\Gamma$ defines a real-valued function on $U_{\Gamma}$. Such functions are called edge-length functions. To avoid difficulties caused by self-automorphisms of $X$ from now on we restrict our attention to the case $g=0$.

Definition 2.1. The combinatorial type of a tropical curve $X$ is its equivalence class up to homeomorphisms respecting the markings.

Combinatorial types partition the set $\mathcal{M}_{0, n}$ into disjoint subsets. The edge-length functions define the structure of the polyhedral cone $\mathbb{R}_{\geq 0}^{M}$ in each of those subsets (as the lengths have to be positive). The number $M$ here is the number of the bounded (non-leaf) edges in $X$. By the Euler characteristic reasoning it is equal to $n-3$ if $X$ is (1- and) 3 -valent, it is smaller if $X$ has vertices of higher valence.

Furthermore, any face of the polyhedral cone $\mathbb{R}_{\geq 0}^{M}$ coincides with the cone corresponding to another combinatorial type, the one where we contract some of the edges of $X$ to points. This gives the adjacency (fan-like) structure on $\mathcal{M}_{0, n}$, so $\mathcal{M}_{0, n}$ is a (non-compact) polyhedral complex. In particular, it is a topological space.
Theorem 1. The set $\mathcal{M}_{0, n}$ for $n \geq 3$ admits the structure of an $(n-3)$-dimensional tropical variety such that the edge-length functions are regular within each combinatorial type. Furthermore, the space $\mathcal{M}_{0, n}$ can be tropically embedded in $\mathbb{R}^{N}$ for some $N$ (i.e. $\mathcal{M}_{0, n}$ can be presented as a simply balanced complex).
Proof. This theorem is trivial for $n=3$ as $\mathcal{M}_{0,3}$ is a point. Otherwise, any two disjoint ordered pairs of marked points can be used to define a global regular function on $\mathcal{M}_{0, n}$ with values in $\mathbb{R}=\mathbb{T}^{\times}$. Namely, each such ordered pair defines the oriented path on the tropical curve $X$ connecting the corresponding marked points. These paths can be embedded.

Since the two pairs of marked points are disjoint, the intersection of the two corresponding paths has to have finite length. We take this length with the positive sign if the orientations agree and with the negative sign otherwise. This defines a function on $\mathcal{M}_{0, n}$. We call such functions the double ratio functions.

Take all possible disjoint pairs of marked points and use them as coordinates for our embedding

$$
\iota: \mathcal{M}_{0, n} \rightarrow \mathbb{R}^{N}
$$

where $N$ is the number of all possible decompositions of $n$ into two disjoint pairs. The theorem now follows from the following two lemmas.

Note that, strictly speaking, each coordinate in $\mathbb{R}^{N}$ depends not only on the choice of two disjoint pairs of marked points but also on the order of points in each pair. However, changing the order in one of the pairs only reverses the sign of the double ratio. Taking an extra coordinate for such a change of order would be redundant. Indeed, for any balanced complex $Y \subset \mathbb{R}^{N}$ and any affine-linear function $\lambda: \mathbb{R}^{N} \rightarrow \mathbb{R}$ with an integer slope the graph of $\lambda$ is a balanced complex in $\mathbb{R}^{N+1}$ isomorphic to the initial complex $Y$.

Lemma 2.2. The map ८ is a topological embedding.
Proof. First, let us prove that $\iota$ is an embedding. The combinatorial type of $X$ is determined by the set of the coordinates that do not vanish on $X$. Indeed, any non-leaf edge $E$ of the tree $X$ separates the leaves (i.e. the set of markings) into two classes corresponding
to the components of $X \backslash E$. Let us take a coordinate in $\mathbb{R}^{n}$ that corresponds to four marking points (union of the two disjoint pairs) such that two of these points belong to one class and two to the other class. We call such a coordinate an E-compatible coordinate. Note that an $E$-compatible coordinate vanishes on $X$ if and only if the pairs of markings defined by the coordinate agree with the pairs defined by the classes.

This observation suffices to reconstruct the combinatorial type of $X$. Furthermore, the length of $E$ equals to the minimal non-zero absolute value of the $E$-compatible coordinates. This implies that $\iota$ is an embedding.

Lemma 2.3. The image $\iota\left(\mathcal{M}_{0, n}\right)$ is a simply balanced complex in $\mathbb{R}^{N}$.
Proof. This is a condition on codimension 1 faces of $\mathcal{M}_{0, n}$. First we shall check it for the case $n=4$. There are three ways to split the four marking points into two disjoint pairs. Accordingly, there are three combinatorial types of 3-valent trees with three marked leaves. Thus our space $\mathcal{M}_{0,4}$ is homeomorphic to the tripod, or the "interior" of the letter $Y$, see Figure 1. Each ray of this tripod correspond to a combinatorial type of a 3 -valent tree with 4 leaves while the vertex correspond to the 4 -valent tree.


Figure 1. The tropical moduli space $\mathcal{M}_{0,4}$ and its points on the corresponding edges.

Up to the sign we have the total of three double ratios for $n=4$. Let us e.g. take those defined by the following ordered pairs: $\{(12),(34)\},\{(13),(24)\}$ and $\{(14),(23)\}$ Each is vanishing on the corresponding ray of the tripod. Let us parameterize each ray of the tripod by its only edge-length $t \geq 0$ and compute the corresponding map to $\mathbb{R}^{3}$.

We have the following embeddings on the three rays

$$
t \mapsto(0, t, t), t \mapsto(t, 0,-t), t \mapsto(-t,-t, 0)
$$

The sum of the primitive integer vectors parallel to the resulting directions is 0 and thus $\iota\left(\mathcal{M}_{0,4}\right)$ is balanced.

In the case $n>4$ the codimension 1 faces of $\mathcal{M}_{0, n}$ correspond to the combinatorial types of $X$ with a single 4 -valent vertex. Near a point inside of such face $F$ the space $\mathcal{M}_{0, n}$ looks like the product of $\mathcal{M}_{0,4}$ and $\mathbb{R}^{n-4}$. The factor $\mathbb{R}^{n-4}$ comes from the edge-lengths on $F$ (its combinatorial type has $n-4$ bounded edges) while the factor $\mathcal{M}_{0,4}$ comes from perturbations of the 4 -valent vertex (which result in a new bounded edge in one of the three possible combinatorial types of the result).

We have a well-defined map from the union $U$ of the $F$-adjacent facets to $F$ by contracting the new edge to a point. Note that the edge-length functions exhibit $F$ as the positive quadrant in $\mathbb{R}^{n-4}$. Furthermore, in the combinatorial type of $F$ we may choose 4 leaves such that contracting all other leaves will take place outside of the 4 -valent vertex (see Figure 2). This contraction defines a map $U \rightarrow \mathcal{M}_{0,4}$.


Figure 2. One of the possible contractions of a tree with a 4 -valent vertex to the tree corresponding to the origin $O \in \mathcal{M}_{0,4}$.

The lemma now follows from the observation that the resulting decomposition into $\mathcal{M}_{0,4} \times \mathbb{R}^{n-4}$ agrees with the double ratio functions. Indeed, note that the complement of the 4 -valent vertex for a curve in the combinatorial type $F$ is composed of four components. If the double ratio is such that its four markings are in one-to-one correspondence with these components then at $U$ it coincides with sum of the pull-back of the corresponding double ratio in $F$ with the pull-back of the corresponding double ratio in $\mathcal{M}_{0,4}$. If one of the four components is lacking a marking from the double ratio $\rho$ then $\left.\rho\right|_{U}$ coincides with the corresponding pull-back from $F$.

Remark 2.4. The functions $Z_{x_{i}, x_{j}}$ from [4] do not define regular functions on $\mathcal{M}_{0, n}$, contrary to what is written in [4]. These functions were a result of an erroneous simplification of the double ratio functions. But these functions cannot be regular as they are always
positive and Proposition 5.12 of [4] is not correct. Even the projectivization of the embedding is not a balanced complex already for $\mathcal{M}_{0,5}$. One should use the (non-simplified) double ratios instead.

Clearly, the space $\mathcal{M}_{0, n}$ is non compact. However it is easy to compactify it by allowing the lengths of bounded edges to assume infinite values. Let $\overline{\mathcal{M}}_{0, n}$ be the space of connected trees with $n$ (marked) leaves such that each edge of this tree is assigned a length $0<l \leq+\infty$ so that each leaf has length necessarily equal to $+\infty$.
Corollary 2.5. The space $\overline{\mathcal{M}}_{0, n}$ is a smooth compact tropical variety.
To verify that $\overline{\mathcal{M}}_{0, n}$ is smooth near a point $x$ at the boundary

$$
\partial \overline{\mathcal{M}}_{0, n}=\overline{\mathcal{M}}_{0, n} \backslash \mathcal{M}_{0, n}
$$

we need to examine those double ratios that are equal to $\pm \infty$ at $x$. There we use only those signs that result in $-\infty$ do that the map takes values in $\mathbb{T}^{N}$.
Remark 2.6. Note that the compactification $\overline{\mathcal{M}}_{0, n} \supset \mathcal{M}_{0, n}$ corresponds to the DeligneMumford compactification in the complex case as under the 1-parametric family collapse of a Riemannian surface to a tropical curve the tropical length of an edge corresponds to the rate of growth of the complex modulus of the holomorphic annulus collapsing to that edge.

Furthermore, similarly to the complex story the infinite edges decompose a tropical curve into components (where the non-leaf edges are finite). Any tropical map from an infinite edge which is bounded would have to be constant and thus the image would have to split as a union of several tropical curves in the target. Such decompositions were used by Gathmann and Markwig in their deduction of the tropical WDVV equation in $\mathbb{R}^{2}$, see [1].

## 3. Tropical $\psi$-classes

Note that we do have the forgetting maps

$$
\mathrm{ft}_{j}: \overline{\mathcal{M}}_{0, n+1} \rightarrow \overline{\mathcal{M}}_{0, n}
$$

for $j=1, \ldots, n+1$ by contracting the leaf with the $j$-marking. This map is sometimes called the universal curve. Each marking $k \neq j$ defines a section $\sigma_{k}$ of $\mathrm{ft}_{j}$. The conormal bundle to $\sigma_{k}$ defines the $\psi_{k}$-class in complex geometry (to avoid ambiguity we take $j=$ $n+1)$. This notion can be adapted to our tropical setup.

Recall that so far our choice of tropical models in their equivalence class was such that the leaves of the tropical curves were in 1-1 correspondence with the markings. For this choice we have the images $\sigma_{k}\left(\overline{\mathcal{M}}_{0, n}\right)$ contained in the boundary part of $\overline{\mathcal{M}}_{0, n+1}$. This presentation is compatible with the point of view when we think about line bundles in tropical geometry to be given by $H^{1}\left(X, \mathcal{O}^{\times}\right)$. Here $X$ is the base of the bundle and $\mathcal{O}^{\times}$ is the sheaf of "non-vanishing" tropical regular functions. Such functions are given in the charts to $\mathbb{R}^{N}$ by affine-linear functions with integer slopes, see [6]. (Recall that $\mathbb{T}^{\times}=\mathbb{R}$ is an honest group with respect to tropical multiplication, i.e. the classical addition.)

However, the following alternative construction allows one to obtain the $\psi$-classes more geometrically (as we'll illustrate in an example in the next section). This approach is based on contracting the leaves marked by number $k$.

The canonical class of a tropical curve is supported at its vertices, namely we take each vertex with the multiplicity equal to its valence minus 2, cf. [6]. Furthermore, the cotangent bundle near a 3 -valent vertex point can be viewed as a neighborhood of the origin for the line given by the tropical polynomial " $x+y+1_{\mathbb{T}}$ " in $\mathbb{R}^{2}$, so the +1 self-intersection of the line gives the required multiplicity for the canonical class at any 3 -valent vertex. Thus we can use the intersections with the corresponding codimension 1 faces in $\mathcal{M}_{0, n}$ to define the $\psi$-classes there. In other words, tropical $\psi$-classes will be supported on the $(n-4)$-dimensional faces in $\mathcal{M}_{0, n}$.

Namely, for a $\psi_{k}$-class we have to collect those codimension 1 faces in $\mathcal{M}_{0, n}$ whose only 4 -valent vertex is adjacent to the leaf marked by $k$. After a contraction of this leaf we get a 3 -valent vertex, thus the multiplicity of every face in a $\psi$-divisor is 1 . We arrive at the following definition.

Definition 3.1. The tropical $\psi_{k}$-divisor $\Psi_{k} \subset \mathcal{M}_{0, n}$ is the union of those $(n-4)$ dimensional faces that correspond to tropical curves with a 4 -valent vertex adjacent to the leaf marked by $k, k=1, \ldots, n$. Each such face is taken with the multiplicity 1.

Proposition 3.2. The subcomplex $\Psi_{k}$ is a divisor, i.e. satisfies the balancing condition.
Proof. Recall that the balancing condition is a condition at $(n-5)$-dimensional faces. In $\mathcal{M}_{0, n}$ there are two types of such faces, one corresponding to tropical curves with two 4 -valent vertices and one corresponding to a tropical curve with a 5 -valent vertex.

Near the faces of the first type the moduli space $\mathcal{M}_{0, n}$ is locally a product of two copies of $\mathcal{M}_{0,4}$ and $\mathbb{R}^{n-5}$. The $\Psi$-divisor is a product of $\mathbb{R}^{n-5}$, one copy of $\mathcal{M}_{0,4}$ and the central (3-valent) point in the other copy of $\mathcal{M}_{0,4}$ (this is the point corresponding to the 4 -valent vertex adjacent to the leaf marked by $k$ ). Thus the balancing condition holds trivially in this case.

Near the faces of the second type the moduli space $\mathcal{M}_{0, n}$ is locally a product of $\mathcal{M}_{0,5}$ and $\mathbb{R}^{n-5}$. As in the proof of Theorem 1 each double ratio decomposes to the sum of the corresponding double ration in $\mathcal{M}_{0,5}$ (perhaps trivial if two of the markings for the double ratio correspond to the same edge adjacent to the 5 -valent vertex) and an affine-linear function in $\mathbb{R}^{n-5}$. Thus it suffices to check only the balancing condition for the $\Psi$-divisors in $\mathcal{M}_{0,5}$. This example is considered in details in the next section. The balancing condition there follows from Proposition 4.1.

Conjecturally, the tropical $\Psi$-divisors are limits of some natural representatives of the divisors for the complex $\psi$-classes under the collapse of the complex moduli space onto the corresponding tropical moduli space $\mathcal{M}_{0, n}$. Note that our choice for the tropical $\Psi$-divisor is not contained in the boundary $\partial \overline{\mathcal{M}}_{0, n} \subset \overline{\mathcal{M}}_{0, n}$ (cf. the calculus of the complex boundary classes in [2]), but comes as a closure of a divisor in $\mathcal{M}_{0, n}$.

## 4. The space $\overline{\mathcal{M}}_{0,5}$

We have already described the moduli space $\mathcal{M}_{0,4}$ as the tripod of Figure 1 . It has only one 0 -dimensional face $O \in \mathcal{M}_{0,4}$. This point (considered as a divisor) coincides with the divisors $\Psi_{1}=\Psi_{2}=\Psi_{3}=\Psi_{4}$. The description of $\mathcal{M}_{0,5}$ is somewhat more interesting.

There are 15 combinatorial types of 3 -valent trees with 5 marked leaves. If we forget about the markings there is only one homeomorphism class for such a curve (see Figure 3). To get the number of non-isomorphic markings we take the number all possible reordering of vertices (equal to $5!=120$ ) and divide by $2^{3}=8$ as there is an 8 -fold symmetry of reordering. Indeed there is one symmetry interchanging the left two leaves, one interchanging the right two leaves and the central symmetry around the central leave of the 3 -valent tree on top of Figure 3.


Figure 3. Adjunction of combinatorial types corresponding to the quadrant connecting the rays (45) and (12).

Thus the space $\mathcal{M}_{0,5}$ is a union of 15 quadrants $\mathbb{R}_{>0}^{2}$. These quadrants are attached along the rays which correspond to the combinatorial types of curves with one 4 -valent vertex. Such curves also have one 3 -valent vertex which is adjacent to two leaves and the only bounded edge of the curve, see the bottom of Figure 3. Such combinatorial types are determined by the markings of the two leaves emanating from the 3 -valent vertex. Thus we have a total of $\binom{5}{2}=10$ of such rays.


Figure 4. The link of the origin in $\mathcal{M}_{0,5}$.

The two boundary edges of the quadrant correspond to contractions of the bounded edges of the combinatorial type as shown on Figure 3. The global picture of adjacency of quadrants and rays is shown on Figure 4 where the reader may recognize the well-known Petersen graph, cf. the related tropical Grassmannian picture in [7]. Vertices of this graph correspond to the rays of $\mathcal{M}_{0,5}$ while the edges correspond to the quadrants. Thus the whole picture may be interpreted as the link of the only vertex $O \in \mathcal{M}_{0,5}$ (the point $O$ corresponds to the tree with a 5 -valent vertex adjacent to all the leaves).

To locate the $\Psi_{k}$-divisor we recall that the $k$ th leaf has to be adjacent to a 4 -valent vertex if it appears in $\Psi_{k}$. This means that $\Psi_{k}$ consists of 6 rays that are marked by pairs not containing $k$.
Proposition 4.1. The subcomplex $\Psi_{k} \subset \mathcal{M}_{0,5}$ is a divisor.
Proof. Since the whole $\mathcal{M}_{0,5}$ is $S_{5}$-symmetric it suffices to check the balancing condition only for $\Psi_{1}$. The embedding $\mathcal{M}_{0,5} \subset \mathbb{R}^{N}$ is given by the double ratios, so it suffices to check that for each double ratio function the sum of its gradients on the six rays of $\Psi_{1}$ vanishes.

If the double ratio is determined by two pairs disjoint from the marking 1, e.g. by $\{(23),(45)\}$ then its restriction onto the six rays of $\Psi_{1}$ is the same as its restriction to
the three rays $\mathcal{M}_{0,4}$ taken twice and thus balanced. Namely its gradient is 1 on the rays (24) and (35); -1 on the rays (25) and (34); and 0 on the rays (23) and (45).

If the four markings of the double ratio contain the marking 1 then thanks to the symmetry we may assume that the double ratio is given by $\{(12),(34)\}$. It vanishes on the rays (34), (35), (45) and (25); it has gradient +1 on the ray (24) and the gradient -1 on the ray (23). Once again, the balancing condition holds.

As our final example of the paper we would like to describe explicitly the universal curve

$$
\mathrm{ft}_{5}: \mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4}
$$

This is presented on Figure 5. Once again, we interpret the Peterson graph as the link $L$


Figure 5. The three fibers and four sections of the universal curve $\mathrm{ft}_{5}: \mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4}$.
of the vertex $O \in \mathcal{M}_{0,5}$. Similarly, the link of the origin in $\mathcal{M}_{0,4}$ consists of three points. Thus $L$ is the union of the fibers of $\mathrm{ft}_{5}$ (away from a neighborhood of infinity) over these three points and four copies of a neighborhood of the origin in $\mathcal{M}_{0,4}$ corresponding to the four sections $\sigma_{1}, \sigma_{2}, \sigma_{3}$ and $\sigma_{4}$ of the universal curve. Figure 5 depicts the fibers in $L$ with solid lines and the sections with dashed lines.

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