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The Kähler-Ricci flow on Kähler surfaces

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Abstract. The existence of Kähler-Einstein metrics on a compact Kähler manifold of definite or vanishing first Chern class has been the subject of intense study over the last few decades, following Yau's solution to Calabi's conjecture. The Kähler-Ricci flow is a canonical deformation for Kähler metrics. In this expository note, we apply some known results of the Kähler-Ricci flow and give a metric classification for Kähler surfaces with semi-negative or positive first Chern class.

1. Introduction

The problem of finding Kähler-Einstein metrics on a compact Kähler manifold has been the subject of intense study over the last few decades. In his solution to Calabi's conjecture, Yau [Ya1] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with vanishing or negative first Chern class. A proof of Yau's theorem is given by Cao [Ca] using the Kähler-Ricci flow.

As is well-known, Hamilton's Ricci flow has become one of the most powerful tools in geometric analysis [Ha1]. The Ricci flow can be applied to give an independent proof of the classical uniformization for Rieman surfaces (c.f. [Ha2, Ch, ChLuTi]). Recently Perelman [Pe] has made a major breakthrough in studying the Ricci flow with remarkable applications to the study of 3-manifolds. The convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds with positive first Chern class was claimed by Perelman and it has been generalized to any Kähler manifolds admitting a Kähler-Ricci soliton by Tian and Zhu with certain assumptions on the initial metrics [TiZhu]. Previously, it was proved in [ChTi] that the Kähler-Ricci flow converges to a Kähler-Einstein metric if the bisectional curvature of the initial Kähler metric is non-negative and positive at least at one point.

Most algebraic varieties do not admit Kähler-Einstein metrics, for example, those with indefinite first Chern class, so it is a natural question to ask if there exist any well-defined canonical metrics on these varieties or on their canonical models. Tsuji [Ts] applied the Kähler-Ricci flow to produce a canonical singular Kähler-Einstein metric on non-singular minimal algebraic varieties of general type. In [SoTi], new canonical metrics on the canonical models of projective varieties of positive Kodaira dimension were defined and such metrics were constructed by the Kähler-Ricci flow on Kähler surfaces.

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In this expository note, we present a metric classification for Kähler surfaces with non-negative Kodaira dimension or positive first Chern class by the Kähler-Ricci flow.

2. Preliminaries

Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $(z_1, ..., z_n)$, ω can be written in the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. The Kähler form ω is a closed $(1,1)$-form by the Kähler condition. In other words, for i, j, $k = 1, ..., n$,

$$\frac{\partial g_{i\bar{k}}}{\partial z_j} = \frac{\partial g_{j\bar{k}}}{\partial z_i} \text{ and } \frac{\partial g_{k\bar{i}}}{\partial \bar{z}_j} = \frac{\partial g_{k\bar{j}}}{\partial \bar{z}_i}.$$

We also define the space of plurisubharmonic functions with respect to a Kähler form ω by

$$\mathcal{P}(X, \omega) = \{\varphi \in C^\infty(X) \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

The curvature tensor for g is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}, \ i, j, k = 1, 2, ..., n.$$

The bisectional curvature of ω is positive if

$$R_{i\bar{j}k\bar{l}} v^i \bar{v}^j w^k \bar{w}^l \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of X. The Ricci tensor is given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j}, \ i, j = 1, 2, ..., n.$$

So its Ricci curvature form is

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}} dz_i \wedge d\bar{z}_j = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{k\bar{l}}).$$

Definition 2.1. *The Kähler metric ω is called a Kähler-Einstein metric on X if*

$$Ric(\omega) = \lambda\omega,$$

for some constant $\lambda \in \mathbf{R}$.

We can always scale the Kähler-Einstein metric ω so that $\lambda = -1$, 0 or 1. By the Einstein equation, the first Chern class of X has to be definite or vanishing if there exists a Kähler-Einstein metric on X.

The Ricci flow introduced by Hamilton ([Ha1]) on a Riemannian manifold is defined by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}. \tag{2.1}$$

On a Kähler manifold X, the Kähler condition is preserved by the Ricci flow if the initial metric is Kähler, so that the Ricci flow is called the Kähler-Ricci flow. We define the following normalized Kähler-Ricci flow

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} + \lambda g_{i\bar{j}}, \tag{2.2}$$

where $\lambda = -1,\ 0,\ 1$.

Let X be a compact complex manifold of complex dimension n and K_X be the canonical line bundle on X. The canonical ring of X is defined by

$$R(X) = \oplus_{m=0}^{\infty} H^0(X, K_X^m)$$

with the pairing

$$H^0(X, K_X^{m_1}) \otimes H^0(X, K_X^{m_2}) \to H^0(X, K_X^{m_1+m_2}).$$

Then we can define the Kodaira dimension $\mathrm{kod}(X)$ of X by

$$\mathrm{kod}(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbf{C} \\ tr(R(X)) - 1 & \text{otherwise,} \end{cases}$$

where $tr(R(X))$ is the transcendental degree of the canonical ring $R(X)$.

We always have $\mathrm{kod}(X) \leq \dim(X)$. In fact, if $\mathrm{kod}(X) \geq 0$, we can define the following meromorphic pluricanonical map for sufficiently large m

$$\Phi_m : X \to \mathbf{CP}^{N_m}$$

by

$$\Phi_m(z) = [S_0^{(m)}(z), S_1^{(m)}(z), ..., S_{N_m}^{(m)}(z)],$$

where $\{S_0^{(m)}, S_1^{(m)}, ..., S_{N_m}^{(m)}\}$ is a basis of $H^0(X, K_X^m)$. The Kodaira dimension of X is exactly the complex dimension of the image of X by Φ_m for sufficiently large m.

The compact complex manifolds can be classified according to their Kodaira dimension by $\mathrm{kod}(X) = -\infty, 0, 1, ..., \dim(X)$.

In the case of smooth compact complex surfaces, the Kodaira dimension must be $-\infty$, 0, 1 or 2. We have the Enriques-Kodaira classification by Kodaira dimension dividing the minimal surfaces into ten classes. Nonsingular rational curves with self-intersection -1 are called exceptional curves of first kind or simply (-1)-curves. A smooth compact complex surface is called a minimal surface, if it does not contain any (-1)-curve. Any smooth compact complex surface is birationally equivalent to a minimal surface by contracting the (-1)-curves.

3. $\mathrm{kod}(X) = 2$

A compact complex surface X of $\mathrm{kod}(X) = 2$ is called a surface of general type. A surface of general type X is minimal if and only if its canonical line bundle K_X is semi-positive, so that $c_1(X) \leq 0$.

If $c_1(X) < 0$, K_X is a positive line bundle so that X must be a minimal algebraic surface. It is proved by Yau [Ya1] and Aubin [Au] independently that there always exists a unique Kähler-Einstein metric on X. Cao [Ca] gave an alternative proof by applying the following Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) - \omega \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{3.1}$$

where ω_0 is a Kähler metric on X.

Theorem 3.1 (Cao)**.** *Let X be a minimal complex surface with $c_1(X) < 0$. Then the Kähler-Ricci flow converges for any initial Kähler metric to the unique Kähler-Einstein metric ω_{KE} with*

$$Ric(\omega_{KE}) = -\omega_{KE}.$$

In general, a minimal surface of general type might contain a finite number of (-2)-curves and $c_1(X)$ is not negative but semi-negative. Let C be the union of these curves whose connected components are rational curves of A-D-E-type (see [BHPV]). The canonical model X_{can} of X is then obtained by blowing down C

$$f : X \to X_{can}.$$

X_{can} is an orbifold surface with singularities of A-D-E-type. We can still apply the normalized Kähler-Ricci flow (3.1) on a minimal surface of general type. Let $Ka(X)$ denote the Kähler cone of X, that is,

$$K_a(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

The Kähler class will change along the Kähler-Ricci flow by the following ordinary differential equation

$$\begin{cases} \dfrac{\partial [\omega]}{\partial t} = -c_1(X) - [\omega] \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \tag{3.2}$$

It follows that

$$[\omega(t, \cdot)] = -c_1(X) + e^{-t}([\omega_0] + c_1(X)).$$

Let $\chi \in -c_1(X)$ be a closed semi-positive $(1,1)-$form and Ω be the smooth volume form on X such that

$$\sqrt{-1}\partial\overline{\partial} \log \Omega = \chi.$$

We choose the reference Kähler metric ω_t by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi)$$

so that

$$\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Then the Kähler-Ricci flow can be reduced to a Monge-Ampère flow for the potential φ given by

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \tag{3.3}$$

Tsuji [Ts] proved the following convergence result for the Kähler-Ricci flow.

Theorem 3.2 (Tsuji)**.** *Let X be a minimal complex surface of general type. Then the Kähler-Ricci flow converges to a unique singular Kähler-Einstein metric ω_∞ smooth outside the (-2)-curves with*

$$Ric(\omega_\infty) = -\omega_\infty$$

for any initial Kähler metric ω_0 satisfying $[\omega_0] > -c_1(X)$.

The key observation in Tsuji's proof is that the canonical bundle K_X on a minimal surface of general type is big and nef, so that for any sufficiently small $\epsilon > 0$

$$[K_X] - \epsilon[C] > 0.$$

The initial class condition is removed by Tian and Zhang [TiZha] and a stronger uniform C^0-estimate is obtained.

Theorem 3.3 (Tian-Zhang)**.** *Let X be a minimal complex surface of general type and $f : X \to X_{can}$ be the holomorphic map from X to its canonical model contracting the (-2)-curves. Then the Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric ω_∞ as in Theorem 3.2 for any initial Kähler-metric. Furthermore*

$$\omega_\infty = f^*\omega_{KE},$$

where ω_{KE} is the unique smooth Kähler-Einstein orbiford metric on X_{can}. In particular, ω_∞ has local continuous potential, i.e.,

$$\omega_\infty = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$$

for some $\varphi_\infty \in C^0(X)$.

The critical equation for the Monge-Ampère flow on X is given by

$$\frac{(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty)^2}{\Omega} = e^{\varphi_\infty}.$$

The following is an immediate corollary from Theorem 3.3 by the fact that $\omega_\infty^2 = \Omega e^{\varphi_\infty}$ and φ_∞ is continuous.

Corollary 3.1. *The Kähler-Einstein volume form on X defined by $\Omega_{KE} = \omega_\infty^2$ is a continuous and nonvanishing volume form on X such that*

$$\sqrt{-1}\partial\overline{\partial}\log\Omega_{KE} = \omega_\infty.$$

4. $\mathrm{kod}(X) = 1$

An elliptic fibration of a surface X is a proper, connected holomorphic map $f : X \to \Sigma$ from X to a curve Σ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface X of $\mathrm{kod}(X) = 1$ must be an elliptic surface. Such an elliptic surface is sometimes called a properly elliptic surface. Since we assume that X is minimal, all fibres are free of (-1)-curves. A very simple example is the product of two curves, one elliptic and the other of genus ≥ 2.

Let $f : X \to \Sigma$ be an elliptic surface. The differential *df* can be viewed as an injection of sheaves $f^*(K_\Sigma) \to \Omega^1_X$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then *df* vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of f we have the following exact sequence

$$0 \to f^*(K_\Sigma) \to \Omega^1_X \to \Omega_{X/\Sigma} \to 0$$

including an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of f on X. The following theorem is well-known (c.f. [BHPV]).

Theorem 4.1 (Kodaira)**.** *Let $f : X \to \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1F_1$, .., $X_{s_k} = m_kF_k$. Then*

$$K_X = f^*(K_\Sigma \otimes (f_{*1}\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(\sum(m_i - 1)F_i), \tag{4.1}$$

or

$$K_X = f^*(L \otimes \mathcal{O}_X(\sum(m_i - 1)F_i),$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on Σ.

Note that $\deg(f_{*1}\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0$ and the equality holds if and only if f is locally trivial. The following invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(\Sigma) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i})\right)$$

determines the Kodaira dimension of X.

Proposition 4.1. (cf. [BHPV]) *Let $f : X \to \Sigma$ be a relatively minimal elliptic fibration and X be compact. Then* $\mathrm{kod}(X) = 1$ *if and only if* $\delta(f) > 0$.

Kodaira classified all possible singular fibres for f. A fibre X_s is stable if

(1) X_s is reduced,
(2) X_s contains no (-1)-curves,
(3) X_s has only node singularities.

The only stable singular fibres are of type I_b for $b > 0$, therefore such singular fibres are particularly interesting. Let $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \mathrm{Im} z > 0\}$ be the upper half plane and $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})/\{\pm 1\}$ be the modular group acting by $z \to \frac{az+b}{cz+d}$. Then $\mathcal{S}_1/\Gamma_1 \cong \mathbf{C}$ is the period domain or the moduli space of elliptic curves. The j-function gives an isomorphism $\mathcal{S}_1/\Gamma_1 \to \mathbf{C}$ with

(1) $j(z) = 0$ if $z = e^{\frac{\pi}{3}\sqrt{-1}}$ modular Γ_1,
(2) $j(z) = 1$ if $z = \sqrt{-1}$ modular Γ_1.

Let $\Sigma_{reg} = \{s \in \Sigma \mid X_s \text{ is a nonsingular fibre}\}$ Then any elliptic surface $f : X \to \Sigma$ gives a period map $p : \Sigma_{reg} \to \mathcal{S}_1/\Gamma_1$. Set $J : \Sigma_{reg} \mapsto \mathbf{C}$ by $J(s) = j(p(s))$. For a stable fibre X_s of type I_b, the functional invariant J has a pole of order b at s and the monodromy is given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Again we apply the normalized Kähler-Ricci flow (3.1) on such elliptic surfaces of Kodaira dimension one. We choose a semi-positive $(1,1)-$form $\chi \in -c_1(X)$ to be the pullback of a Kähler form on the base Σ and χ might vanish somewhere due to the presence of singular fibres. Let Ω be the smooth volume form on X such that $\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi$. The Kähler class of ω deformed by the Kähler-Ricci flow is given by

$$[\omega] = (1 - e^{-t})[\chi] + e^{-t}[\omega_0],$$

with the initial Kähler metric ω_0. So we let $\omega_t = (1 - e^{-t})\chi + e^{-t}\omega_0$ be the reference metric and $\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi$.

As we discussed in the previous section, one can reduce the Kähler-Ricci flow (2.2) to the following Monge-Ampère flow on Kähler potentials.

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \log \frac{e^{-t}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \tag{4.2}$$

The simple example is the Kähler-Ricci flow on $X = E \times C$, where E is an elliptic curve and C is a curve of genus greater than one. Let $\pi_1 : X \to E$ and $\pi_2 : X \to C$ be the projection maps. Let ω_E be a flat metric on E, ω_C be the hyperbolic metric on C with $\omega_C \in -c_1(C)$ and

$$\omega_0 = \pi_1^*\omega_E + \pi_2^*\omega_C$$

be the initial Kähler metric on X for the Kähler-Ricci flow. Then

$$\omega_t = \pi_2^*\omega_C + e^{-t}\pi_1^*\omega_E$$

solves the Kähler-Ricci flow by

$$\frac{\partial\omega_t}{\partial t} = -e^{-t}\pi_1^*\omega_E = -Ric(\omega_t) - \omega_t.$$

And

$$\lim_{t\to\infty} \omega_t = \pi_2^*\omega_C.$$

The following general result is proved in [SoTi]

Theorem 4.2 (Song-Tian). *Let $f : X \to \Sigma$ be a minimal elliptic surface of* $\mathrm{kod}(X) = 1$. *Let $\Sigma_{reg} = \Sigma \setminus \{s_1, \ldots, s_k\}$. Then the Kähler-Ricci flow has a global solution $\omega(t, \cdot)$ for any initial Kähler metric satisfying:*

(1) *$\omega(t,\cdot)$ converges to $f^*\omega_\infty \in -c_1(X)$ as currents for a positive current ω_∞ on Σ,*
(2) *ω_∞ is smooth on Σ_{reg} and satisfies as currents on Σ*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{WP} + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}[s_i],$$

(3) *for any compact subset $K \in f^{-1}(\Sigma_{reg})$, there is a constant C_K such that*

$$||\omega||_{L^\infty(K)} + \sup_{s \in f(K)} ||e^t \omega|_{f^{-1}(s)}||_{L^\infty(f^{-1}(s))} + ||S||_{L^\infty(K)} \leq C_K,$$

where S is the scalar curvature of $\omega(t,\cdot)$.

The Kähler-Ricci flow collapses exponentially fast along the vertical direction and the limit metric satisfies a generalized Kähler-Einstein equation (2) with the correction term as the curvature of the fibration. The generalized Kähler-Einstein equation can also be considered as a local version of Kodaira's adjunction formula (4.1).

Corollary 4.1. *Let $f : X \to \Sigma$ be an elliptic fibre bundle over a curve Σ of genus greater than one. Then the Kähler-Ricci flow (2.2) has a global solution with any initial Kähler metric. Furthermore, $\omega(t,\cdot)$ converges with uniformly bounded scalar curvature to the pullback of the Kähler-Einstein metric on Σ.*

If $f : X \to \Sigma$ has only singular fibres of type mI_0, all the smooth fibres of f are isomorphic so that the period map p is trivial with its image as a point in $\mathcal{M}$. The Kähler-Ricci flow will then converge to a cone hyperbolic metric on Σ and the cone singularities appear from where the multiple fibres sit.

5. $\mathrm{kod}(X) = 0$

If X is a minimal Kähler surface of $\mathrm{kod}(X) = 0$, the canonical line bundle K_X is numerically trivial so that $c_1(X) = 0$. Yau's solution to Calabi's conjecture shows that there always exists a Ricci-flat Kähler metric in any given Kähler class on X. An alternative proof is given by Cao [Ca] by applying the following Kähler-Ricci flow,

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{5.1}$$

where ω_0 is a Kähler metric on X.

Theorem 5.1 (Cao). *Let X be a minimal complex surface with $c_1(X) = 0$. Then the Kähler-Ricci flow (5.1) converges for any initial Kähler metric ω_0 to the unique Ricci-flat metric $\omega_{KE} \in [\omega_0]$ with*

$$Ric(\omega_{KE}) = 0.$$

6. Fano surfaces

A compact complex surface X with $c_1(X) > 0$ is called a Fano surface. The anti-canonical bundle K_X^{-1} is thus positive. By the Enriques-Kodaira classification, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$ for $0 \leq n \leq 8$ are the only compact differential four-manifolds on which there is a complex structure with positive first Chern class. Tian [Ti2] proves the following theorem for the existence of a Kähler-Einstein metric on Fano surfaces.

Theorem 6.1 (Tian). *Any compact complex surface X with $c_1(X) > 0$ admits a Kähler-Einstein metric if the Lie algebra of the automorphism group on X is reductive.*

So there always exists a Kähler-Einstein metric on $\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$ for $3 \leq n \leq 8$.

However, there does not exist any Kähler-Einstein metric neither on $\mathbf{CP}^2 \# 1\overline{\mathbf{CP}^2}$ nor on $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}^2}$ since the Lie algebra of the automorphism group on these toric Fano surfaces is not reductive. One can also disprove the existence of any Kähler-Einstein metric by showing the Futaki invariant on these two surfaces is non-zero.

Definition 6.1. *Let X be a complex surface with $c_1(X) > 0$. A Kähler metric ω is called a Kähler-Ricci soliton if it satisfies*

$$Ric(\omega) = \omega + L_V \omega, \tag{6.1}$$

where V is a holomorphic vector field on X and L_V denotes the Lie derivative along V.

A Kähler-Einstein metric is also a special Kähler-Ricci soliton by taking $V = 0$. In [Koi], Koiso constructed a Kähler-Ricci soliton on $\mathbf{CP}^2 \# 1\overline{\mathbf{CP}^2}$. A general result on the existence of Kähler-Einstein metrics and Kähler-Ricci soliton on toric manifolds is proved in [WaZh].

Theorem 6.2 (Wang-Zhu). *There exists a Kähler-Ricci soliton on a toric Kähler manifold with positive first Chern class.*

Therefore there always exists a Kähler-Einstein metric or a Kähler-Ricci soliton on a complex surface of positive first Chern class.

We apply the the following normalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -Ric(\omega) + \omega \\ \omega\,|_{t=0} = \omega_0 \end{cases} \tag{6.2}$$

with the initial Kähler metric $\omega_0 \in c_1(X)$ so that the Kähler class $[\omega]$ will not change.

Perelman proves a gradient estimate on the Kähler-Ricci flow (6.2), which implies the scalar curvature will stay uniformly bounded (see [SeTi]). He has also claimed that the Kähler-Ricci flow converges (6.2) on a Kähler-Einstein manifold X with positive first Chern class to a Kähler-Einstein metric. This is proved by Tian and Zhu [TiZhu] and generalized to the Kähler-Ricci solitons. Suppose X admits a Kähler-Ricci soliton with respect to the holomorphic vector field V. Then the imaginary part of V induces a one-parameter subgroup G_V in the automorphism group of X.

Theorem 6.3 (Tian-Zhu)**.** *Let X be a compact complex surface of $c_1(X) > 0$. Then the Kähler-Ricci flow will converge to a Kähler-Einstein metric or a Kähler-Ricci soliton if the initial Kähler metric is G_V-invariant.*

7. Generalizations

The generalized Kähler-Einstein metric on the canonical model of an elliptic surface with Kodaira dimension one can be generalized and defined on a family of surfaces with a fibration structure.

Mirror symmetry and the SYZ conjecture make predictions for Calabi-Yau manifolds with "large complex structure limit point" (cf. [StYaZa]). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov-Hausdorff sense to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. [GrWi]) in the case of $K3$ surfaces. The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Oogrui-Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of $K3$ surfaces is McLean's metric.

We will apply a deformation for a family of Calabi-Yau metrics and derive McLean's metric [Mc] without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a $K3$ surface $\hat{X}$ can be identified as the mirror to the large Kähler limit of X as shown in [GrWi], so we can fix the complex structure on X and deform the Kähler class to infinity. Let $f : X \to \mathbf{CP}^1$ be an elliptic $K3$ surface. Let $\chi \geq 0$ be the pullback of a Kähler form on $\mathbf{CP}^1$ and ω_0 be a Kähler form on X. We construct a reference Kähler metric $\omega_t = \chi + e^{-t}\omega_0$ and $[\omega_t]$ tends to $[\chi]$ as $t \to \infty$. We can always scale ω_0 so that the volume of each fibre of f with respect to ω_t is e^{-t}. Suppose Ω is a Ricci-flat volume form on X with $\partial\overline{\partial}\log\Omega = 0$. Then Yau's proof [Ya1] of Calabi's conjecture yields a unique solution φ_t to the following Monge-Ampère equation for $t \in [0, \infty)$

$$\begin{cases} \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^2}{\Omega} = C_t \\ \int_X \varphi_t \Omega = 0, \end{cases} \tag{7.1}$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t, \cdot) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$. The following theorem is proved in [SoTi].

Theorem 7.1. *Let $f : X \to \mathbf{CP}^1$ be an elliptically fibred $K3$ surface with 24 singular fibres of type I_1. Then the Ricci-flat metrics $\omega(t, \cdot)$ converges to the pullback of a Kähler metric ω_∞ on $\mathbf{CP}^1$ in any compact set of X_{reg} in $C^{1,1}$ as $t \to \infty$. The Kähler metric ω_∞ on $\mathbf{CP}^1$ satisfies the equation*

$$Ric(\omega_\infty) = \omega_{WP}. \tag{7.2}$$

This limit metric ω_∞ coincides with McLean's metric as obtained by Gross and Wilson [GrWi]. Their construction is certainly much more delicate and gives an accurate approximation near the singular fibres by the Ooguri-Vafa metrics. Also McLean's metric is an example of the generalized Kähler-Einstein metric defined as

$$Ric(\omega) = -\lambda\omega + \omega_{WP}$$

when $\lambda = 0$.

In fact, these canonical metrics belong to a class of Kähler metrics defined in [SoTi], which generalize Calabi's extremal metrics. Let Y be a Kähler manifold of complex dimension n together with a fixed closed (1,1)-form θ. Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. One may consider the following equation

$$\bar{\partial}V_\varphi = 0, \tag{7.3}$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar{\partial}(S(\omega_\varphi) - tr_{\omega_\varphi}(\theta)). \tag{7.4}$$

Clearly, when $\theta = 0$, (7.3) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (7.3) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - tr_{\omega_\varphi}(\theta) = \mu, \tag{7.5}$$

where μ is the constant given by

$$\mu = \frac{n(c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$Ric(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler-Einstein metric.

Another example of such extremal metrics is constructed by Fine [Fi]. Let $f : X \to \Sigma$ be a Kähler surface admitting a non-singular holomorphic fibration over Σ, with fibres of genus at least 2. We also assume $c_1(\Sigma) \leq 0$. Let V be the vertical tangent bundle of X and $[\omega_t] = -f^*c_1(\Sigma) - e^{-t}c_1(V)$.

Let χ be a Kähler form in $-c_1(\Sigma)$ and $\bar{\omega} \in -c_1(V)$. Then $\bar{\omega} = \omega_H \oplus \theta\chi$, where ω_H is the hyperbolic Kähler form on each fiber and θ is a smooth function on X. We then set

$$\omega_t = \chi + e^{-t}\bar{\omega}.$$

The following theorem is proved by Fine in [Fi].

Theorem 7.2. *For sufficiently large $t \geq 0$, there exists a constant scalar curvature Kähler metric in $[\omega_t]$. Furthermore, such a family of metrics converge to a Kähler metric ω_∞ on Σ defined by*

$$S(\omega_\infty) = tr_{\omega_\infty}(\omega_{WP}) + const, \tag{7.6}$$

where ω_{WP} is the pullback of the Weil-Petersson metric from the moduli spaces of the fibre curves with a certain polarization.

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