Planar Contact Structures
with Binding Number Three

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ABSTRACT. In this article, we find the complete list of all contact structures (up to iso-
topy) on closed three-manifolds which are supported by an open book decomposition
having planar pages with three (but not less) boundary components. We distinguish
them by computing their first Chern classes and three dimensional invariants (when-
ever possible). Among these contact structures we also distinguish tight ones from
those which are overtwisted.

1. Introduction

Let (M, ξ) be a closed oriented 3-manifold with the contact structure ξ, and let (S, h) be
an open book (decomposition) of M which is compatible with ξ. In this case, we also say
that (S, h) supports ξ (for the definitions of these terms see the next section). Based on
Giroux's correspondence theorem (Theorem 2.3), two natural questions have been asked
in [EO]:

(1) What is the possible minimal page genus $g(S) =$genus(S)?

(2) What is the possible minimal number of boundary components of a page S with $g(S)$
minimal?

In [EO], two topological invariants $sg(\xi)$ and $bn(\xi)$ were defined to be the answers. More
precisely, we have:

$$sg(\xi) = \min\{ \ g(S) \mid (S, h) \text{ an open book decomposition supporting } \xi\},$$

called *the support genus* of ξ, and

$$bn(\xi) = \min\{ \ |\partial S| \mid (S, h) \text{ an open book decomposition supporting } \xi \text{ and } g(S) = sg(\xi)\},$$

called *the binding number* of ξ. There are some partial results about these invariants. For
instance, it is proved in [Et1] that if (M, ξ) is overtwisted, then $sg(\xi) = 0$.

Unlike the overtwisted case, there is not much known yet for $sg(\xi)$ if ξ is tight. The
algorithm given in [Ar] finds a reasonable upper bound for $sg(\xi)$ using the given contact
surgery diagram of ξ. However, there is no systematic way to obtain actual values of
$sg(\xi)$ and $bn(\xi)$ yet.

One of the ways to work on the above questions is to get a complete list of contact manifolds corresponding to a fixed support genus and a fixed binding number. To get such complete list, we consider all possible monodromy maps h. The first step in this direction is the following result given in [EO]. Throughout the paper $L(m,n)$ stands for the lens space obtained by $-m/n$ rational surgery on an unknot.

Theorem 1.1 ([EO]). *Suppose ξ is a contact structure on a 3-manifold M that is supported by a planar open book (i.e., $sg(\xi) = 0$). Then*

(1) *If $bn(\xi) = 1$, then ξ is the standard tight contact structure on S^3.*
(2) *If $bn(\xi) = 2$ and ξ is tight, then ξ is the unique tight contact structure on the lens space $L(m, m-1) = L(m, -1)$ for some $m \in \mathbb{Z}_+ \cup \{0\}$.*
(3) *If $bn(\xi) = 2$ and ξ is overtwisted then ξ is the overtwisted contact structure on $L(m, 1)$, for some $m \in \mathbb{Z}_+$, with $e(\xi) = 0$ and $d_3(\xi) = -\frac{1}{4}m + \frac{3}{4}$ where $e(\xi)$ and $d_3(\xi)$ denotes the Euler class and d_3-invariant of ξ, respectively. When m is even then the refinement of $e(\xi)$ is given by $\Gamma(\xi)(\mathfrak{s}) = \frac{m}{2}$ where $\mathfrak{s}$ is the unique spin structure on $L(m, 1)$ that extends over a two handle attached to a μ with framing zero. Here we are thinking of $L(m, 1)$ as $-m$ surgery on an unknot and μ is the meridian to the unknot.*

We remark that Theorem 1.1 gives the complete list of all contact 3-manifolds which can be supported by planar open books whose pages have at most 2 boundary components. Next step in this direction should be to find all contact 3-manifolds (M, ξ) such that $sg(\xi) = 0$ and $bn(\xi) = 3$. In the present paper, we will get all such contact structures, and also distinguish tight ones by looking at the monodromy maps of their corresponding open books (See Theorem 1.2 and Theorem 1.3). After the preliminary section (Section 2), we prove the main results in Section 3. Although some ideas in the present paper have been already given or mentioned in [EO], we will give their explicit statements and proofs in our settings. We finish this section by stating the main results.

Let Σ be the compact oriented surface with $|\partial\Sigma| = 3$, and consider the boundary parallel curves a, b, c in Σ as in the Figure 1. Through out the paper, Σ will always stand for this surface whose abstract picture is given below. Let $Aut(\Sigma, \partial\Sigma)$ be the group of (isotopy classes of) diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. (Such diffeomorphisms are automatically orientation-preserving).

It is known (see [Bi]) that
$$Aut(\Sigma, \partial\Sigma) = \mathbb{Z}\langle D_a \rangle \oplus \mathbb{Z}\langle D_b \rangle \oplus \mathbb{Z}\langle D_c \rangle \cong \mathbb{Z}^3$$
where D_a, D_b, D_c denote positive Dehn twists along the curves a, b, c given as in Figure 1. In the rest of the paper, we will not make any distinction between isotopy classes of arcs/curves/maps and the individual arcs/curves/maps.

We start with studying the group $Aut(\Sigma, \partial\Sigma)$ in details. Since generators commute with each other, we have that
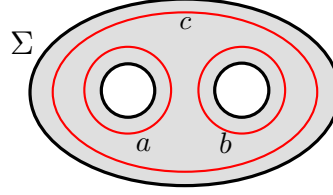
FIGURE 1. The surface Σ and the curves giving the generators of $Aut(\Sigma, \partial\Sigma)$.

$$Aut(\Sigma, \partial\Sigma) = \{D_a{}^p D_b{}^q D_c{}^r \mid p, q, r \in \mathbb{Z}\}.$$

For any given $p, q, r \in \mathbb{Z}$, let $Y(p, q, r)$ denote the smooth 3-manifold given by the smooth surgery diagram in Figure 2 (diagram on the left). It is an easy exercise to check that $Y(p, q, r)$ is indeed diffeomorphic to Seifert fibered manifold given in Figure 2 (diagram on the right).

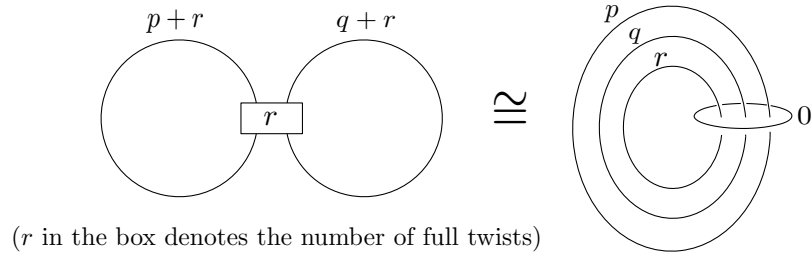(r in the box denotes the number of full twists)

FIGURE 2. Seifert fibered manifold $Y(p, q, r)$.

Now we state the following theorem characterizing all closed contact 3-manifolds whose contact structures supported by open books $(\Sigma, \phi = D_a{}^p D_b{}^q D_c{}^r)$.

Theorem 1.2. *Let (M, ξ) be a contact manifold supported by the open book (Σ, ϕ) where $\phi = D_a{}^p D_b{}^q D_c{}^r \in Aut(\Sigma, \partial\Sigma)$ for fixed integers p, q, r. Then (M, ξ) is contactomorphic to $(Y(p, q, r), \xi_{p,q,r})$ where $\xi_{p,q,r}$ is the contact structure on $Y(p, q, r)$ given by the contact surgery diagram in Figure 3. Moreover,*

(1) ξ is tight (in fact holomorphically fillable) if $p \geq 0, q \geq 0, r \geq 0$, and

(2) ξ is overtwisted otherwise.

Remark 1.1. In Figure 3, if $r = 0$, then we completely delete the family corresponding to r from the diagram, so we are left with two families of Legendrian curves which do not link to each other, and so the contact surgery diagram gives a contact structure on the connected sum of two lens spaces. However, if $p = 0$ (or $q = 0$), then we replace
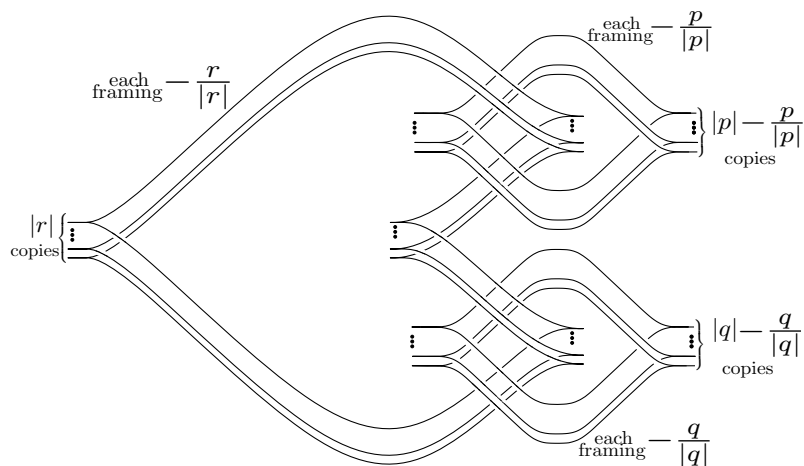
FIGURE 3. Contact manifold $(Y(p,q,r), \xi_{p,q,r})$.

the Legendrian family corresponding to p (or q) by a single Legendrian unknot with tb number equal to -1, and we do $(+1)$-contact surgery on the new unknot. Note also that Figure 3 is symmetric with respect to p and q. This reduces the number of cases in the proof of Theorem 1.3.

Of course not all $\xi_{p,q,r}$ have binding number three:

Theorem 1.3. *Let (M, ξ) be a closed contact 3-manifold for which $sg(\xi) = 0$ and $bn(\xi) = 3$. Then (M, ξ) is contactomorphic to some $(Y(p,q,r), \xi_{p,q,r})$ satisfying the following conditions:*

(1) *If $r = 0$, then $p \neq 1$ and $q \neq 1$.*
(2) *If $r = 1$, then $p \notin \{-1, 0\}$ and $q \notin \{-1, 0\}$.*
(3) *If $r = -1$, then $p \neq 1$ and $q \neq 1$.*
(4) *If $|r| \geq 2$, then $pq \neq -1$ and $(p, q) \notin \{(1, 0), (0, 1)\}$.*

Suppose that (M, ξ) is a closed contact 3-manifold with $sg(\xi) = 0$ and $bn(\xi) = 3$, and let $c_1 = c_1(\xi) \in H^2(M; \mathbb{Z})$ denote the first Chern class, and $d_3 = d_3(\xi)$ denote the 3-dimensional invariant (which lies in $\mathbb{Q}$ whenever c_1 is a torsion class in $H^2(M; \mathbb{Z})$). Using c_1 and d_3, we can distinguish these structures in most of the cases. In fact, we have either M is a lens space, or a connected sum of lens spaces, or a Seifert fibered manifold with three singular fibers. If one of the first two holds, then using the tables given in Section 3 and 4, one can get the complete list of all possible (M, ξ) without any repetition. That is, the contact structures in the list are all distinct pairwise and unique up to isotopy. On the other hand, if the third holds, we can also study them whenever c_1 is a torsion class. More discussion will be given in Section 4.

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2. Preliminaries

2.1. Contact structures and open book decompositions

A 1-form $\alpha \in \Omega^1(M)$ on a 3-dimensional oriented manifold M is called a *contact form* if it satisfies $\alpha \wedge d\alpha \neq 0$. An *oriented contact structure* on M is then a hyperplane field ξ which can be globally written as kernel of a contact 1-form α. We will always assume that ξ is a *positive* contact structure, that is, $\alpha \wedge d\alpha > 0$. Two contact structures ξ_0, ξ_1 on a 3-manifold are said to be *isotopic* if there exists a 1-parameter family ξ_t ($0 \leq t \leq 1$) of contact structures joining them. We say that two contact 3-manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $f : M_1 \longrightarrow M_2$ such that $f_*(\xi_1) = \xi_2$. Note that isotopic contact structures give contactomorphic contact manifolds by Gray's Theorem. Any contact 3-manifold is locally contactomorphic to $(\mathbb{R}^3, \xi_0)$ where *standard contact structure* ξ_0 on $\mathbb{R}^3$ with coordinates (x, y, z) is given as the kernel of $\alpha_0 = dz + x dy$. The standard contact structure ξ_{st} on the 3-sphere $S^3 = \{(r_1, r_2, \theta_1, \theta_2) : r_1^2 + r_2^2 = 1\} \subset \mathbb{C}^2$ is given as the kernel of $\alpha_{st} = r_1^2 d\theta_1 + r_2^2 d\theta_2$. One basic fact is that $(\mathbb{R}^3, \xi_0)$ is contactomorphic to $(S^3 \setminus \{pt\}, \xi_{st})$. For more details on contact geometry, we refer the reader to [Ge], [Et3].

An *open book decomposition* of a closed 3-manifold M is a pair (L, f) where L is an oriented link in M, called the *binding*, and $f : M \setminus L \to S^1$ is a fibration such that $f^{-1}(t)$ is the interior of a compact oriented surface $S_t \subset M$ and $\partial \Sigma_t = L$ for all $t \in S^1$. The surface $S = S_t$, for any t, is called the *page* of the open book. The *monodromy* of an open book (L, f) is given by the return map of a flow transverse to the pages (all diffeomorphic to S) and meridional near the binding, which is an element $h \in Aut(S, \partial S)$, the group of (isotopy classes of) diffeomorphisms of S which restrict to the identity on ∂S. The group $Aut(S, \partial S)$ is also said to be the mapping class group of S, and denoted by $\Gamma(S)$.

An open book can also be described as follows. First consider the mapping torus

$$S(h) = [0, 1] \times S / (1, x) \sim (0, h(x))$$

where S is a compact oriented surface with $n = |\partial S|$ boundary components and h is an element of $Aut(S, \partial S)$ as above. Since h is the identity map on ∂S, the boundary $\partial S(h)$ of the mapping torus $S(h)$ can be canonically identified with n copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0, 1]/(0 \sim 1)$ and the second one comes from a component of ∂S. Now we glue in n copies of $D^2 \times S^1$ to cap off $S(h)$ so that ∂D^2 is identified with $S^1 = [0, 1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of ∂S. Thus we get a closed 3-manifold

$$M = M_{(S,h)} := S(h) \cup_n D^2 \times S^1$$

equipped with an open book decomposition (S, h) whose binding is the union of the core circles in the $D^2 \times S^1$'s that we glue to $S(h)$ to obtain M. See [Gd], [Et2] for details.

2.2. Legendrian knots and contact surgery

A *Legendrian knot K* in a contact 3-manifold (M, ξ) is a knot that is everywhere tangent to ξ. Any Legendrian knot comes with a canonical *contact framing* (or *Thurston-Bennequin framing*), which is defined by a vector field along K that is transverse to ξ. We call (M, ξ) (or just ξ) *overtwisted* if it contains an embedded disc $D \approx D^2 \subset M$ with boundary $\partial D \approx S^1$ a Legendrian knot whose contact framing equals the framing it receives from the disc D. If no such disc exists, the contact structure ξ is called *tight*. Also if a contact 3-manifold (M, ξ) is the boundary of a Stein manifold (resp. a symplectic manifold) with certain compatibility conditions satisfied, then ξ is called *Stein (holomorphically) fillable* (resp. *symplectically fillable*). See [Et2] or [OS] for the complete definitions of fillability, and related facts. We will use the following fact later.

Theorem 2.1 ([EG])**.** *Any symplectically fillable contact structure is tight.*
($\Rightarrow$ *Any holomorphically fillable contact structure is tight.*)

For any $p, q \in \mathbb{Z}$, a contact (r)-surgery $(r = p/q)$ along a Legendrian knot K in a contact manifold (M, ξ) was first described in [DG]. It was proved in [Ho] that if $r = 1/k$ with $k \in \mathbb{Z}$, then the resulting contact structure is unique up to isotopy. In particular, a contact ± 1-surgery along a Legendrian knot K on a contact manifold (M, ξ) determines a unique surgered contact manifold which will be denoted by $(M, \xi)_{(K, \pm 1)}$.

The most general result along these lines is:

Theorem 2.2 ([DG])**.** *Every closed contact 3-manifold (M, ξ) can be obtained via contact (± 1)-surgery on a Legendrian link in (S^3, ξ_{st}).*

Any closed contact 3-manifold (M, ξ) can be described by a *contact surgery diagram* drawn in $(\mathbb{R}^3, \xi_0) \subset (S^3, \xi_{st})$. By Theorem 2.2, there is a contact surgery diagram for (M, ξ) such that the contact surgery coefficient of any Legendrian knot in the diagram is ± 1. For any oriented Legendrian knot K in $(\mathbb{R}^3, \xi_0)$, we compute the *Thurston-Bennequin number $tb(K)$*, and the *rotation number $rot(K)$* as

$$tb(K) = bb(K) - (\# \text{ of left cusps of K}),$$
$$rot(K) = \frac{1}{2}[(\# \text{ of downward cusps}) - (\# \text{ of upward cusps})]$$

where $bb(K)$ is the blackboard framing of K.

If a contact surgery diagram for (M, ξ) is given, we can also get the smooth surgery diagram for the underlying 3-manifold M. Indeed, for a Legendrian knot K in a contact surgery diagram, we have:

$$\text{Smooth surgery coefficient of } K = \text{Contact surgery coefficient of } K + tb(K)$$

For more details see [OS] and [Gm].

2.3. Compatibility and stabilization

A contact structure ξ on a 3-manifold M is said to be *supported by an open book* (L, f) if ξ is isotopic to a contact structure given by a 1-form α such that

(1) $d\alpha$ is a positive area form on each page $S \approx f^{-1}(\text{pt})$ of the open book and
(2) $\alpha > 0$ on L (Recall that L and the pages are oriented.)

When this holds, we also say that the open book (L, f) is *compatible with the contact structure ξ on M*.

Definition 2.1. A positive (resp., negative) stabilization $S_K^+(S, h)$ (resp., $S_K^-(S, h)$) of an abstract open book (S, h) is the open book

(1) with page $S' = S \cup$ 1-handle and
(2) monodromy $h' = h \circ D_K$ (resp., $h' = h \circ D_K^{-1}$) where D_K is a right-handed Dehn twist along a curve K in S' that intersects the co-core of the 1-handle exactly once.

Based on the result of Thurston and Winkelnkemper [TW] which introduced open books into the contact geometry, Giroux proved the following theorem strengthening the link between open books and contact structures.

Theorem 2.3 ([Gi])**.** *Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilizations: Two contact structures supported by the same open book are isotopic, and two open books supporting the same contact structure have a common positive stabilization.*

Following fact was first implied in [LP], and then in [AO]. The given version below is due to Giroux and Matveyev. For a proof, see [OS].

Theorem 2.4. *A contact structure ξ on M is holomorphically fillable if and only if ξ is supported by some open book whose monodromy admits a factorization into positive Dehn twists only.*

For a given fixed open book (S, h) of a 3-manifold M, there exists a unique compatible contact structure up to isotopy on $M = M_{(S,h)}$ by Theorem 2.3. We will denote this contact structure by $\xi_{(S,h)}$. Therefore, an open book (S, h) determines a unique contact manifold $(M_{(S,h)}, \xi_{(S,h)})$ up to contactomorphism.

Taking a positive stabilization of (S, h) is actually taking a special Murasugi sum of (S, h) with the positive Hopf band (H^+, D_γ) where $\gamma \subset H^+$ is the core circle. Taking a Murasugi sum of two open books corresponds to taking the connect sum of 3-manifolds associated to the open books. The proofs of the following facts can be found in [Gd], [Et2].

Theorem 2.5. $(M_{S_K^+(S,h)}, \xi_{S_K^+(S,h)}) \cong (M_{(S,h)}, \xi_{(S,h)}) \# (S^3, \xi_{st}) \cong (M_{(S,h)}, \xi_{(S,h)})$.

Theorem 2.6. *Let (S, h) be an open book supporting the contact manifold (M, ξ). If K is a Legendrian knot on the page S of the open book, then*

$$(M, \xi)_{(K, \pm 1)} = (M_{(S, h \circ D_K^\mp)}, \xi_{(S, h \circ D_K^\mp)}).$$

2.4. Homotopy invariants of contact structures

The set of oriented $2-$plane fields on a given 3-manifold M is identified with the space $Vect(M)$ of nonzero vector fields on M. $v_1, v_2 \in Vect(M)$ are called *homologous* (denoted by $v_1 \sim v_2$) if v_1 is homotopic to v_2 in $M \setminus B$ for some $3-$ball B in M. The space $Spin^c(M)$ of all spin^c structures on M is the defined to be the quotient space $Vect(M)/\sim$. Therefore, any contact structure ξ on M defines a spin^c structure $\tau_\xi \in Spin^c(M)$ which depends only on the homotopy class of ξ. As the first invariant of ξ, we will use the first Chern class $c_1(\xi) \in H^2(M; \mathbb{Z})$ (considering ξ as a complex line bundle on M). For a spin^c structure τ_ξ, whose first Chern class $c_1(\tau_\xi)(:= c_1(\xi))$ is torsion, the obstruction to homotopy of two 2-plane fields (contact structures) both inducing τ_ξ can be captured by a single number. This obstruction is the 3-*dimensional invariant* $d_3(\xi)$ of ξ. To compute $d_3(\xi)$, suppose that a compact almost complex 4-manifold (X, J) is given such that $\partial X = M$, and ξ is the complex tangencies in TM, i.e., $\xi = TM \cap J(TM)$. Let $\sigma(X), \chi(X)$ denote the signature and Euler characteristic of X, respectively. Then we have

Theorem 2.7 ([Gm]). *If $c_1(\xi)$ is a torsion class, then the rational number*

$$d_3(\xi) = \frac{1}{4}\big(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)\big)$$

is an invariant of the homotopy type of the 2-plane field ξ. Moreover, two 2-plane fields ξ_1 and ξ_2 with $\mathfrak{t}_{\xi_1} = \tau_{\xi_2}$ and $c_1(\tau_{\xi_i}) = c_1(\xi_i)$ a torsion class are homotopic if and only if $d_3(\xi_1) = d_3(\xi_2)$. $\qquad \square$

As a result of this fact, if (M, ξ) is given by a contact ± 1-surgery on a link, then we have

Corollary 2.8 ([DGS]). *Suppose that (M, ξ), with $c_1(\xi)$ torsion, is given by a contact (± 1)-surgery on a Legendrian link $\mathbb{L} \subset (S^3, \xi_{st})$ with $tb(K) \neq 0$ for each $K \subset \mathbb{L}$ on which we perform contact $(+1)$-surgery. Let X be a 4-manifold such that $\partial X = M$. Then*

$$d_3(\xi) = \frac{1}{4}\big(c^2 - 3\sigma(X) - 2\chi(X)\big) + s,$$

where s denotes the number of components in $\mathbb{L}$ on which we perform $(+1)$-surgery, and $c \in H^2(X; \mathbb{Z})$ is the cohomology class determined by $c(\Sigma_K) = rot(K)$ for each $K \subset \mathbb{L}$, and $[\Sigma_K]$ is the homology class in $H_2(X)$ obtained by gluing the Seifert surface of K with the core disc of the 2-handle corresponding K.

We use the above formula as follows: Suppose $\mathbb{L}$ has k components. Write $\mathbb{L} = \sqcup_i^k K_i$. By converting all contact surgery coefficients to the topological ones, and smoothing each cusp in the diagram, we get a framed link (call it $\mathbb{L}$ again) describing a simply connected 4-manifold X such that $\partial X = M$. Using this description, we compute

$$\chi(X) = 1 + k, \text{ and } \sigma(X) = \sigma(\mathcal{A}_{\mathbb{L}})$$

where $\mathcal{A}_{\mathbb{L}}$ is the linking matrix of $\mathbb{L}$. Using the duality, the number c^2 is computed as

$$c^2 = (PD(c))^2 = [b_1 \ b_2 \cdots b_k] \mathcal{A}_{\mathbb{L}} [b_1 \ b_2 \cdots b_k]^T$$

where $PD(c) \in H_2(X, \partial X; \mathbb{Z})$ is the Poincaré dual of c, the row matrix $[b_1 \ b_2 \cdots b_k]$ is the unique solution to the linear system

$$\mathcal{A}_{\mathbb{L}} [b_1 \ b_2 \cdots b_k]^T = [rot(K_1) \ rot(K_2) \cdots rot(K_k)]^T.$$

Here the superscript "T" denotes the transpose operation in the space of matrices. See [DGS], [Gm] for more details.

2.5. Right-veering diffeomorphisms

For a given compact oriented surface S with nonempty boundary ∂S, let $Dehn^+(S, \partial S) \subset Aut(S, \partial S)$ be the submonoid of product of all positive Dehn twists. In [HKM], another submonoid $Veer(S, \partial S)$ of all right-veering elements in $Aut(S, \partial S)$ was introduced and studied. They defined right-veering elements of $Aut(S, \partial S)$ as follows: Let α and β be isotopy classes (relative to the endpoints) of properly embedded oriented arcs $[0, 1] \to S$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial S$. Let $\pi : \tilde{S} \to S$ be the universal cover of S (the interior of $\tilde{S}$ will always be $\mathbb{R}^2$ since S has at least one boundary component), and let $\tilde{x} \in \partial \tilde{S}$ be a lift of $x \in \partial S$. Take lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}$. $\tilde{\alpha}$ divides $\tilde{S}$ into two regions – the region "to the left" (where the boundary orientation induced from the region coincides with the orientation on $\tilde{\alpha}$) and the region "to the right". We say that β is *to the right* of α if either $\alpha = \beta$ (and hence $\tilde{\alpha}(1) = \tilde{\beta}(1)$), or $\tilde{\beta}(1)$ is in the region to the right (see Figure 4).
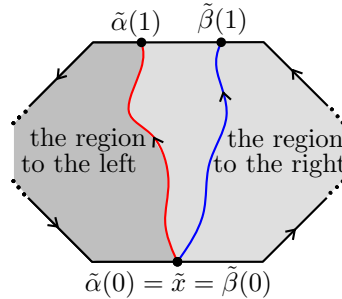

FIGURE 4. Lifts of α and β in the universal cover $\tilde{S}$.

Alternatively, isotop α and β, while fixing their endpoints, so that they intersect transversely (this include the endpoints) and with the fewest possible number of intersections. Assume that $\alpha \neq \beta$. Then in the universal cover $\tilde{S}$, $\tilde{\alpha}$ and $\tilde{\beta}$ will meet only at $\tilde{x}$. If not, subarcs of $\tilde{\alpha}$ and $\tilde{\beta}$ would cobound a disk D in $\tilde{S}$, and we could use an innermost disk

argument on $\pi(D) \subset S$ to reduce the number of intersections of α and β by isotopy. Then β is to the right of α if $int(\tilde{\beta})$ lies in the region to the right. As an alternative to passing to the universal cover, we simply check to see if the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x.

Definition 2.2. Let $h : S \to S$ be a diffeomorphism that restricts to the identity map on ∂S. Let α be a properly embedded oriented arc starting at a basepoint $x \in \partial S$. Then h is *right-veering* (that is, $h \in Veer(S, \partial S)$) if for every choice of basepoint $x \in \partial S$ and every choice of α based at x, $h(\alpha)$ is to the right of α (at x). If C is a boundary component of S, we say is h is *right-veering with respect to C* if $h(\alpha)$ is to the right of α for all α starting at a point on C.

It turns out that $Veer(S, \partial S)$ is a submonoid and we have the inclusions:

$$Dehn^+(S, \partial S) \subset Veer(S, \partial S) \subset Aut(S, \partial S).$$

In [HKM], they proved the following theorem which is hard to use but still can be used to distinguish tight structure in some cases.

Theorem 2.9 ([HKM])**.** *A contact structure (M, ξ) is tight if and only if all of its compatible open book decompositions (S, h) have right-veering $h \in Veer(S, \partial S) \subset Aut(S, \partial S)$.*

3. The proofs of results

We first prove that the submonoids $Dehn^+(\Sigma, \partial\Sigma)$ and $Veer(\Sigma, \partial\Sigma)$ are actually the same in our particular case.

Lemma 3.1. $Dehn^+(\Sigma, \partial\Sigma) = Veer(\Sigma, \partial\Sigma)$ *for the surface Σ given in Figure 1.*

Proof. The inclusion $Dehn^+(S, \partial S) \subset Veer(S, \partial S)$ is true for a general compact oriented surface S with boundary (see Lemma 2.5. in [HKM] for the proof). Now, suppose that $\phi \in Veer(\Sigma, \partial\Sigma) \subset Aut(\Sigma, \partial\Sigma)$. Then we can write ϕ in the form

$$\phi = D_a{}^p D_b{}^q D_c{}^r \text{ for some } p, q, r \in \mathbb{Z}.$$

We will show that $p \geq 0, q \geq 0, r \geq 0$. Consider the properly embedded arc $\alpha \subset \Sigma$ one of whose end points is $x \in \partial\Sigma$ as shown in the Figure 5. Note that, for any $p, q, r \in \mathbb{Z}$, $D_c{}^r$ fixes α, and also any image $D_a{}^p D_b{}^q(\alpha)$ of α because c does not intersect any of these arcs. Assume at least one of p, q, or r is strictly negative. First assume that $p < 0$. Then consider two possible different images $\phi(\alpha) = D_a{}^p D_b{}^q(\alpha)$ of α corresponding to whether $q < 0$ or $q > 0$ (See Figure 5). Since we are not allowed to rotate any boundary component, clearly $\phi(\alpha)$ is to the left of α at the boundary point x. Equivalently, $\phi(\alpha)$ is not to the right of α at x which implies that h is not right-veering with respect to the boundary component parallel to a. Therefore, $\phi \notin Veer(\Sigma, \partial\Sigma)$ which is a contradiction. Now by symmetry, we are also done for the case $q < 0$. Finally, exactly the same argument (with a different choice of arc one of whose end points is on the boundary component parallel to the curve c) will work for the case when $r < 0$. $\square$
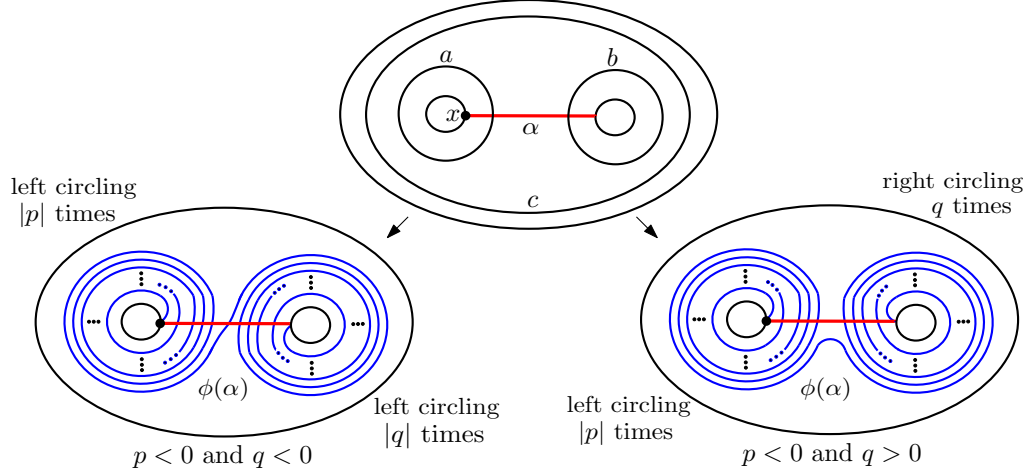
FIGURE 5. The arc α and its image $\phi(\alpha) = D_a{}^p D_b{}^q(\alpha)$.

Lemma 3.2. *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then ξ is tight if and only if ξ is holomorpfically fillable.*

Proof. Assume that ξ is tight. Since $\phi \in Aut(\Sigma, \partial\Sigma)$, there exists integers p, q, r such that $\phi = D_a{}^p D_b{}^q D_c{}^r$. As ξ is tight, the monodromy of any open book supporting ξ is right-veering by Theorem 2.9. In particular, we have $\phi \in Veer(\Sigma, \partial\Sigma)$ since (Σ, ϕ) supports ξ. Therefore, $\phi \in Dehn^+(\Sigma, \partial\Sigma)$ by Lemma 3.1, and so $p \geq 0, q \geq 0, r \geq 0$. Thus, ξ is holomorphically fillable by Theorem 2.4. Converse statement is a consequence of Theorem 2.1. $\square$

Now, the following corollary of Lemma 3.2 is immediate:

Corollary 3.3. *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then*

$$\xi \text{ is tight} \iff \phi = D_a{}^p D_b{}^q D_c{}^r \text{ with } p \geq 0, q \geq 0, r \geq 0. \qquad \square$$

Proof of Theorem 1.2. Let (M, ξ) be a contact manifold supported by the open book $(\Sigma, \phi_{p,q,r})$ where $\phi_{p,q,r} = D_a{}^p D_b{}^q D_c{}^r \in Aut(\Sigma, \partial\Sigma)$ for $p, q, r \in \mathbb{Z}$. As explained in [EO], $(M, \xi) = (M_{(\Sigma, \phi_{p,q,r})}, \xi_{(\Sigma, \phi_{p,q,r})})$ is given by the contact surgery diagram in Figure 6. Then we apply the algorithm given in [DG] and [DGS] to convert each rational coefficient into ± 1's, and obtain the diagram given in Figure 3.

To determine the topological (or smooth) type of (M, ξ), we start with the diagram in Figure 3. Then by converting the contact surgery coefficients into the smooth surgery
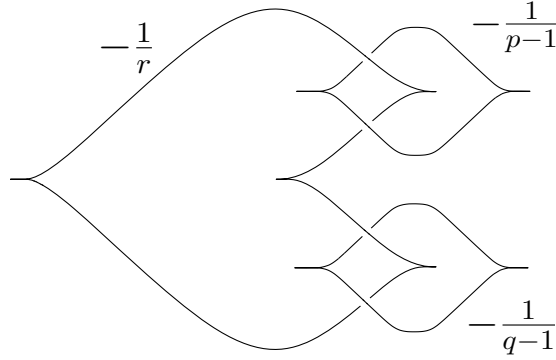
FIGURE 6. Contact surgery diagram corresponding to $(\Sigma, \phi_{p,q,r})$.
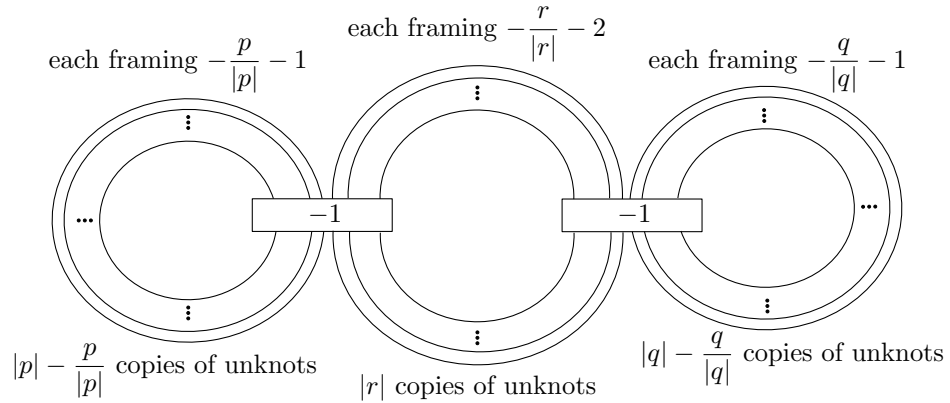

FIGURE 7. Smooth surgery diagram corresponding to Figure 3.

coefficients, we get the corresponding smooth surgery diagram in Figure 7 where each curve is an unknot.

Now we modify this diagram using a sequence of blow-ups and blow-downs. These operations do not change smooth type of M. We first blow up the diagram twice so that we unlink two -1 twists. Then we blow down each unknot in the most left and the most right families. Finally we blow down each unknot of the family in the middle. We illustrate these operations in Figure 8. To keep track the surgery framings, we note that each blow-up increases the framing of any unknot by 1 if the unknot passes through the corresponding twist box in Figure 7. So we get the first diagram in Figure 8. Blowing
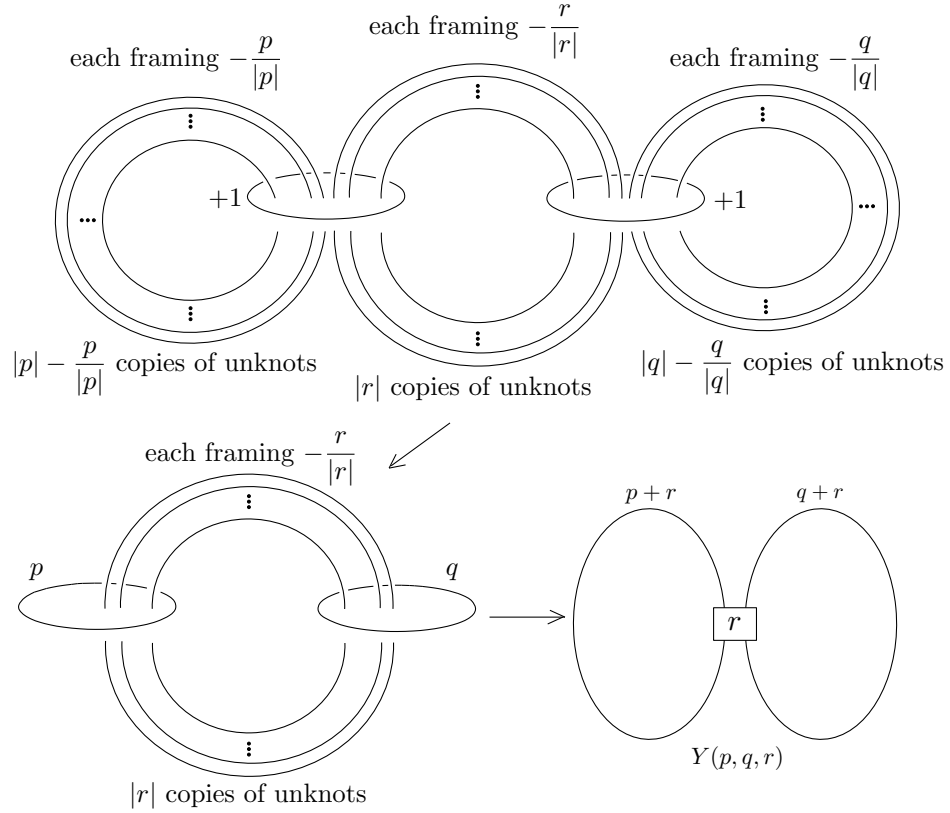
FIGURE 8. Squence of blow-ups and blow-downs.

each member down on the left (resp. right) decreases the framing of the left (resp. right) $+1$-unknot by $-\frac{p}{|p|}$ (resp. $-\frac{q}{|q|}$). Since there are $|p| - \frac{p}{|p|}$ blow-downs on the left and $|q| - \frac{q}{|q|}$ blow-downs on the right, we get the second diagram in Figure 8. Finally, if we blow down each $(-\frac{r}{|r|})$-unknot in the middle family, we get the last diagram. Note that each blow-down decreases the framing by $-\frac{r}{|r|}$, and introduces a $\frac{r}{|r|}$ full twist. Hence, we showed that (M, ξ) is contactomorphic to $(Y(p,q,r), \xi_{p,q,r})$. The statements (1) and (2) are the consequences of Corollary 3.3. □

We now examine the special case where $Y(p, q, r)$ is homeomorphic to 3-sphere S^3. The following lemma lists all planar contact structures on S^3 with binding number less than or equal to three.

Lemma 3.4. *Suppose that $(Y(p,q,r), \xi_{p,q,r})$ is contactomorphic to (S^3, ξ) for some contact structure ξ on S^3. Then Table 1 lists all possible values of (p,q,r), the corresponding ξ (in terms of the d_3-invariant), and its binding number.*

r	p	q	$d_3(\xi)$	$bn(\xi)$
-3	-2	1	$-1/2$	3
-2	-3	1	$-1/2$	3
-1	1	any q	$1/2$	2
-1	3	2	$3/2$	3
0	1	1	$-1/2$ (tight)	1
0	-1	-1	$3/2$	3
0	1	-1	$1/2$	2
1	0	1	$-1/2$ (tight)	1
1	-1	any q	$1/2$	2
1	-2	-3	$-1/2$	3
2	3	-1	$3/2$	3
3	2	-1	$3/2$	3
$\lvert r \rvert \geq 2$	1	-1	$1/2$	2

TABLE 1. All planar contact structures on S^3 with binding number ≤ 3.

Proof. The proof is the direct consequence of the discussion given in the proof of Lemma 5.5 in [EO]. We remark that the interchanging p and q does not affect the contact structure in Figure 3, so we do not list the possibilities for (p,q,r) that differ by switching p and q. Note that in Table 1 there are only two contact structures (up to isotopy) on S^3 with binding number 3, namely, the ones with d_3-invariants $-1/2$ and $3/2$. $\square$

Proof of Theorem 1.3. We will use the results of Theorem 1.1, Theorem 1.2, and Lemma 3.4. Consider the 3-sphere S^3 in Theorem 1.1 as the lens space $L(1, \pm 1)$. By Theorem 1.1, for any contact manifold (Y, η) with $sg(\eta) = 0$ and $bn(\eta) \leq 2$, we have either

(1) $(Y, \eta) \cong (S^3, \xi_{st})$ if $bn(\eta) = 1$,
(2) $(Y, \eta) \cong (L(m, -1), \eta_m)$ for some $m \geq 2$ if $bn(\eta) = 2$, and η is tight,
(3) $(Y, \eta) \cong (L(m, 1), \eta_m)$ for some $m \geq 0$ if $bn(\eta) = 2$, and η is overtwisted (for $m \neq 0$).

103

where η_m is the contact structure on the lens space $L(m,-1)$ (or $L(m,1)$) given by the contact surgery diagram consisting of a single family of Legendrian unknots (with Thurston-Benequen number -1) such that each member links all the other members of the family once, and each contact surgery coefficient is -1 (if η_m is tight) or $+1$ (if η_m is overtwisted). These are illustrated by the diagrams $(*)$ and $(\star)$ in Figure 9, respectively. Notice the exceptional cases: $m=1$ in $(*)$, and $m=0$ in $(\star)$.
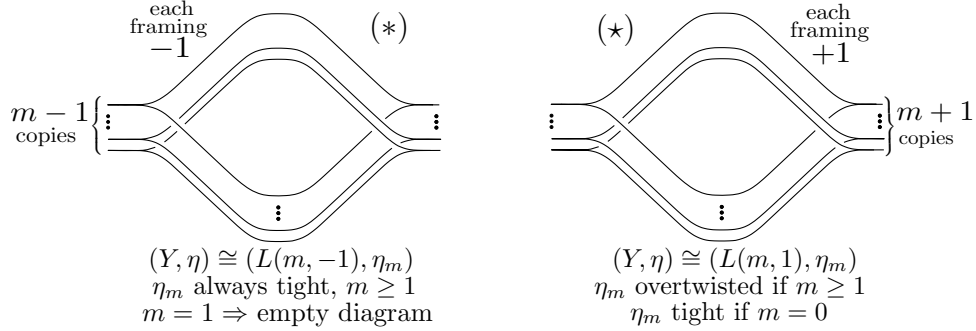

$$(Y,\eta) \cong (L(m,-1),\eta_m)$$
η_m always tight, $m \geq 1$
$m = 1 \Rightarrow$ empty diagram

$$(Y,\eta) \cong (L(m,1),\eta_m)$$
η_m overtwisted if $m \geq 1$
η_m tight if $m = 0$

FIGURE 9. Contact surgery diagrams for (Y,η).

Now, if (M,ξ) is a contact manifold with $sg(\xi) = 0$ and $bn(\xi) = 3$, then by the definitions of these invariants there exists an open book (Σ,ϕ) supporting ξ. Therefore, by Theorem 1.2, (M,ξ) is contactomorphic to $(Y(p,q,r),\xi_{p,q,r})$ for some $p,q,r \in \mathbb{Z}$, and the contact surgery diagram of ξ is given in Figure 3. However, p,q,r can not be arbitrary integers because there are several cases where the diagram in Figure 3 reduces to either $(*)$ or $(\star)$ in Figure 9 for some m. So for those values of p,q,r, (M,ξ) can not be contactomorphic to $(Y(p,q,r),\xi_{p,q,r}) \cong (Y,\eta)$ because $bn(\xi) = 3 \neq 2 \geq bn(\eta)$. Therefore, we have to determine those cases.

If $|p| \geq 2$ and $|q| \geq 2$, then the only triples (p,q,r) giving $L(m,\pm 1)'s$ are $(-2,q,1)$ and $(2,q,-1)$. Furthermore, if we assume also that $|r| > 1$, then the Seifert fibered manifolds $Y(p,q,r)$ are not homeomorphic to even a lens space $L(m,n)$ for any m,n (for instance, see Chapter 5 in [Or]). As a result, we immediately obtain $bn(\xi_{p,q,r}) = 3$ for $|p| \geq 2$ and $|q| \geq 2$ and $|r| \geq 2$. Therefore, to finish the proof of the theorem, it is enough to analyze the cases where $|p| < 2$ or $|q| < 2$, and the cases $(-2,q,1)$ and $(2,q,-1)$ for any q. As we remarked before, we do not need to list the possibilities for (p,q,r) that differ by switching p and q. We first consider $r = 0, \pm 1, \pm 2$, and then the cases $r > 2$ and $r < -2$. In Table 2 - 8, we list all possible (M,ξ) for each of these cases.

Remark 3.1. To determine the binding number $bn(\xi)$ in any row of any table below, we simply first check the topological type of the manifold under consideration. If $M \approx S^3$,

we determine the corresponding binding number using Table 1. If the topological type is not $L(m, 1)$ or $L(m, -1)$, then we immediately get that $bn(\xi) = 3$. If $M \approx L(m, 1)$ with $m > 1$, then we first compute $c_1(\xi)$. If $c_1(\xi) \neq 0$, then $bn(\xi) = 3$ as $c_1(\eta_m) = 0$ for any η_m given above. If $c_1(\xi)=0$, we compute the $d_3(\xi)$ using the 4-manifold defined by the surgery diagram in Figure 7. (Indeed, we can use the formula for d_3 given in Corollary 2.8 as long as $c_1(\xi)$ is torsion. In particular, whenever $H^2(M)$ is finite, then d_3 is computable). Then if $d_3(\xi) = d_3(\eta_m) = (-m + 3)/4$, then ξ is isotopic to η_m which implies that $bn(\xi) = 2$ by Theorem 1.1. Otherwise $bn(\xi) = 3$. In the case that $M \approx L(m, -1)$ with $m > 1$, we first ask if ξ is tight. If it is tight (which is the case if and only if $p \geq 0, q \geq 0, r \geq 0$), then $bn(\xi) = 2$ (again by Theorem 1.1) since the tight structure on $L(m, -1)$ is unique (upto isotopy). If it is overtwisted (which is the case if and only if at least one of p, q, r is negative), then $bn(\xi) = 3$ because ξ is not covered in Theorem 1.1. As a final remark, sometimes the contact structure ξ can be viewed as a positive stabilization of some η_m. For these cases we immediately obtain that $bn(\xi) = 2$ because positive stabilizations do not change the isotopy classes of contact structures.

To compute the d_3-invariant of $\xi_{p,q,r}$ (for $c_1(\xi_{p,q,r})$ torsion), we will use the $(n+1) \times (n+1)$ matrices A_n ($n \geq 1$), B_n ($n \geq 1$), and C_n ($n \geq 4$) given below. It is a standard exercise to check that

(1) $\sigma(A_n) = n - 1$ if $n \geq 1$, and $\sigma(C_n) = n - 1$ if $n \geq 4$.

(2) $\sigma(B_n) = n - 3$ if $n \geq 3$, and $\sigma(B_n) = 0$ if $n = 1, 2$.

(3) The system $A_n[\mathbf{b}]_{n+1}^T = [\mathbf{0}]_{n+1}^T$ has trivial solution $[\mathbf{b}]_{n+1}^T = [\mathbf{0}]_{n+1}^T$ where $[\mathbf{b}]_{n+1} = [b_1\ b_2 \cdots b_{n+1}]$, $[\mathbf{0}]_{n+1} = [0\ 0 \cdots 0]$ are $(n+1) \times 1$ row matrices.

$$
A_n = \begin{bmatrix} 0 & -1 & -1 & . & . & . & -1 \\ -1 & 0 & -1 & . & . & . & -1 \\ -1 & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & & \ddots & & 0 & -1 \\ -1 & -1 & \cdots & & -1 & 0 \end{bmatrix}
\quad
B_n = \begin{bmatrix} 0 & -1 & -1 & . & . & . & -1 \\ -1 & 0 & -1 & . & . & . & -1 \\ -1 & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & & \ddots & & 0 & -1 \\ -1 & -1 & \cdots & & -1 & -2 \end{bmatrix}
\quad
C_n = \begin{bmatrix} 0 & -1 & -1 & . & . & . & -1 \\ -1 & 0 & -1 & . & . & . & -1 \\ -1 & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & & \ddots & & 0 & -1 \\ -1 & -1 & \cdots & & -1 & 2 \end{bmatrix}
$$

In some cases, A_n appears (as a block matrix) in the linking matrix $\mathcal{L}_{p,q,r}$ of the framed link $\mathbb{L}_{p,q,r}$ given in Figure 7. On the other hand, B_n and C_n are very handy when we diagonalize $\mathcal{L}_{p,q,r}$ to find its signature. As we discussed before, the link $\mathbb{L}_{p,q,r}$ defines a 4-manifold $X_{p,q,r}$ with $\partial X = M$. So we have

$$
\begin{aligned}
\sigma(X_{p,q,r}) &= \sigma(\mathcal{L}_{p,q,r}), \\
\chi(X_{p,q,r}) &= 1 + (\# \text{ of components of } \mathbb{L}_{p,q,r}), \\
c^2 &= [\mathbf{b}]_k \mathcal{L}_{p,q,r} [\mathbf{b}]_k^T
\end{aligned}
$$

where $[\mathbf{b}]_k^T$ is the solution to the linear system $\mathcal{L}_{p,q,r}[\mathbf{b}]_k^T = [rot(K_1)\ rot(K_2)\cdots rot(K_k)]^T$ with $K_1, K_2, \cdots K_k$ being the components of $\mathbb{L}_{p,q,r}$.

To compute the first Chern class $c_1(\xi_{p,q,r}) \in H^2(M)$, note that in Figure 3, the rotation number of any member in the family corresponding to r is ± 1 (depending on how we orient them). We will always orient them so that their rotation numbers are all $+1$. On the other hand, the rotation number is 0 for any member in the family corresponding to p and q. Therefore, $c_1(\xi_{p,q,r}) = PD^{-1}(\mu_1 + \mu_2 + \cdots + \mu_{|r|})$ where $\mu_i \in H_1(M)$ is the class of the meridian of the Legendrian knot K_i in the family corresponding to r. Then we compute $H_1(M)$ (which is isomorphic to $H^2(M)$ by Poincaré duality) as

$$H_1(M) = \langle\ \mu_1, \mu_2, \cdots, \mu_k |\ \mathcal{L}_{p,q,r}[\mu]_k^T = [\mathbf{0}]_k^T\ \rangle$$

where $[\mu]_k = [\mu_1\ \mu_2 \cdots \mu_k]$ is the $k \times 1$ row matrix. The final step is to understand $PD(c_1(\xi_{p,q,r})) = \mu_1 + \mu_2 + \cdots + \mu_{|r|}$ in this presentation of $H_1(M)$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	-1	-1	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
0	-1	0	$S^3 \# S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z}$	1
0	-1	1	S^3	2	$(\star)\ m = 1$	$0 \in \{0\}$	$1/2$
0	-1	$q \geq 2$	$S^3 \# L(q, -1)$	3	Figure 3	$0 \in \mathbb{Z}_q$	$(q+1)/4$
0	-1	$q \leq -2$	$S^3 \# L(|q|, 1)$	3	Figure 3	$0 \in \mathbb{Z}_{|q|}$	$(-|q|+7)/4$
0	0	0	$\underset{2}{\#} S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}$	$1/2$
0	0	1	$S^1 \times S^2 \# S^3$	2	$(\star)\ m = 0$	$0 \in \mathbb{Z}$	0
0	0	$q \geq 2$	$S^1 \times S^2 \# L(q, -1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_q$	$(q-1)/4$
0	0	$q \leq -2$	$S^1 \times S^2 \# L(|q|, 1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_{|q|}$	$(-|q|+5)/4$
0	1	1	S^3	1	$(*)\ m = 1$	$0 \in \{0\}$	$-1/2$
0	1	$q \geq 2$	$S^3 \# L(q, -1)$	2	$(*)\ m = q$	$0 \in \mathbb{Z}_q$	$(q-3)/4$
0	1	$q \leq -2$	$S^3 \# L(|q|, 1)$	2	$(\star)\ m = |q|$	$0 \in \mathbb{Z}_{|q|}$	$(-|q|+3)/4$

TABLE 2. The case $r = 0$ ($|p| < 2$ or $|q| < 2$).

In **Table 2**, we need to compute the binding number $bn(\xi)$ for the rows 5, 12. For the other rows, see Remark 3.1.

- If $p = -1, q \leq -2, r = 0$, we need to compute $d_3(\xi_{-1,q,0})$ as $c_1(\xi_{-1,q,0}) = 0$: We have

$$\mathcal{L}_{-1,q,0} = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_{|q|+1} \end{pmatrix}.$$

The contact structure $\xi_{-1,q,0}$, and $\mathbb{L}_{-1,q,0}$ describing $X_{-1,q,0}$ are given in Figure 10. We compute that $s = |q|+3$, $c^2 = 0$, $\chi(X_{-1,q,0}) = |q|+4$, and $\sigma(X_{-1,q,0}) = \sigma(A_1)+\sigma(A_{|q|}) = 0+|q|-1 = |q|-1$, and so we obtain $d_3(\xi_{-1,q,0}) = (-|q|+7)/4$ by Corollary 2.8. Therefore, $\xi_{-1,q,0}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q|+3)/4$. Hence, $bn(\xi_{-1,q,0}) = 3$ for any $q \leq -2$ by Theorem 1.1.
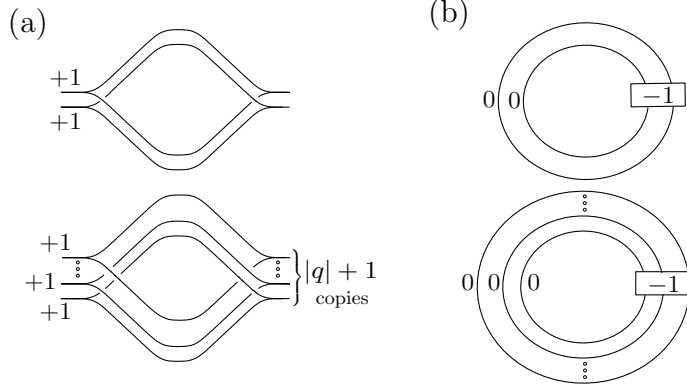
(a)   (b)

FIGURE 10. (a) The contact structure $\xi_{-1,q,0}$ on $S^3 \# L(|q|,1) \approx L(|q|,1)$, (b) The corresponding framed link $\mathbb{L}_{-1,q,0}$.

• If $p = 1, q \leq -2, r = 0$, we have $(\Sigma, \phi_{1,q,0}) = S_a^+(H^+, D_b^q)$ (recall the identification of Σ and the curves a, b, c in Figure 1). Therefore, $\xi_{1,q,0} \cong \eta_{|q|}$ since (H^+, D_b^q) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|,1)$. Hence, $bn(\xi_{1,q,0}) = 2$ for $q \leq -2$.

In **Table 3**, we need to compute the binding number $bn(\xi)$ for the rows 1 and 9. For the other rows, see Remark 3.1.

• If $p = -2, q \leq -4, r = 1$, let K_i's be the components (with the given orientations) of $\mathbb{L}_{-2,q,1}$ as in Figure 11. Then we obtain the linking matrix

$$
\mathcal{L}_{-2,q,1} = \begin{pmatrix}
-3 & -1 & -1 & -1 & -1 & \cdots & -1 \\
-1 & 0 & -1 & -1 & 0 & \cdots & 0 \\
-1 & -1 & 0 & -1 & 0 & \cdots & 0 \\
-1 & -1 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 0 & 0 & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & A_{|q|} & \\
\cdot & \cdot & \cdot & \cdot & & & \\
-1 & 0 & 0 & 0 & & &
\end{pmatrix}
$$

107

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$q \leq -4$	$L(|q+2|,1)$	3	Figure 3	$|q|-4 \in \mathbb{Z}_{|q|-2}$	$\frac{-q^2-7q-14}{-4q-8}$
1	-2	-3	S^3	3	Figure 3	$0 \in \{0\}$	$-1/2$
1	-2	-2	$S^1 \times S^2$	3	Figure 3	$-2 \in \mathbb{Z}$	$\notin Q$
1	-2	$q \geq 2$	$L(q+2,-1)$	3	Figure 3	$q \in \mathbb{Z}_{q+2}$	$\frac{q^2+q+2}{4q+8}$
1	-1	any q	S^3	2	$(\star)$ $m=1$	$0 \in \{0\}$	$1/2$
1	0	0	$S^1 \times S^2$	2	$(\star)$ $m=0$	$0 \in \mathbb{Z}$	0
1	0	1	S^3	1	$(*)$ $m=1$	$0 \in \{0\}$	$-1/2$
1	0	$q \geq 2$	$L(q,-1)$	2	$(*)$ $m=q$	$0 \in \mathbb{Z}_q$	$(q-3)/4$
1	0	$q \leq -1$	$L(|q|,1)$	2	$(\star)$ $m=|q|$	$0 \in \mathbb{Z}_{|q|}$	$(-|q|+3)/4$
1	1	-2	$L(3,-1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$1/3$
1	1	1	$L(3,1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$-1/3$
1	1	$q \leq -3$	$L(2q+1,-q-1)$	3	Figure 3	$|q|-1 \in \mathbb{Z}_{2|q|-1}$	$\frac{-q^2-4q-2}{-4q-2}$
1	1	$q \geq 2$	$L(2q+1,-q-1)$	3	Figure 3	$q+1 \in \mathbb{Z}_{2q+1}$	$\frac{q^2-2q-1}{4q+2}$

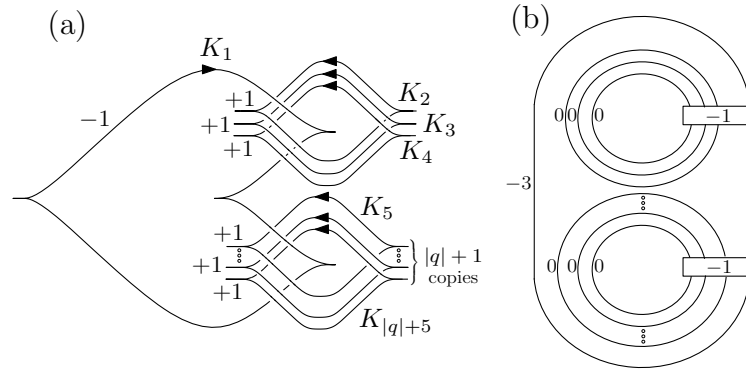TABLE 3. The case $r=1$, $|p| < 2$ or $|q| < 2$ (and the case $(p,q,r) = (-2,q,1)$).

FIGURE 11. (a) The overtwisted contact structure $\xi_{-2,q,1}$ on $L(|q+2|,1)$, (b) The corresponding framed link $\mathbb{L}_{-2,q,1}$.

It is not hard to see that

$$H_1(M) = \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+5} |\ \mathcal{L}_{-2,q,1}[\mu]_{|q|+5}^T = [\mathbf{0}]_{|q|+5}^T\,\rangle = \langle\, \mu_2 |\ (|q|-2)\mu_2 = 0\,\rangle \cong \mathbb{Z}_{|q|-2},$$

and $\mu_1 = (|q| - 4)\mu_2$. Therefore,

$$c_1(\xi_{-2,q,1}) = PD^{-1}(\mu_1) = PD^{-1}(|q| - 4)\mu_2 = |q| - 4 \in \mathbb{Z}_{|q|-2}.$$

Thus, if $q < -4$, then $\xi_{-2,q,1}$ is not isotopic to $\eta_{|q+2|}$ as $c_1(\eta_{|q+2|}) = 0$ implying that $bn(\xi_{-2,q,1}) = 3$ by Theorem 1.1. If $q = -4$, we compute that $d_3(\xi_{-2,-4,1}) = -1/4 \neq 1/4 = d_3(\eta_2)$, so $bn(\xi_{-2,-4,1}) = 3$.

$\bullet$ If $p = 0, q \leq -1, r = 1$, we have $(\Sigma, \phi_{0,q,1}) = S_c^+(H^+, D_b^q)$ (again recall the identification of Σ and the curves a, b, c in Figure 1). Therefore, $\xi_{0,q,1} \cong \eta_{|q|}$ since (H^+, D_b^q) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|, 1)$. Hence, $bn(\xi_{0,q,1}) = 2$ for $q < 0$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	2	$q \geq 4$	$L(q-2,-1)$	3	Figure 3	$q - 4 \in \mathbb{Z}_{q-2}$	$\frac{-q^2+3q-6}{-4q+8}$
-1	2	3	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
-1	2	2	$S^1 \times S^2$	3	Figure 3	$-2 \in \mathbb{Z}$	$\notin Q$
-1	2	$q \leq -2$	$L(|q-2|,1)$	3	Figure 3	$|q| \in \mathbb{Z}_{|q|+2}$	$\frac{-q^2-3q+6}{-4q+8}$
-1	1	any q	S^3	2	$(\star)\ m = 1$	$0 \in \{0\}$	$1/2$
-1	0	0	$S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z}$	1
-1	0	-1	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
-1	0	$q > 1$	$L(q,-1)$	3	Figure 3	$0 \in \mathbb{Z}_q$	$(q+1)/4$
-1	0	$q < -1$	$L(|q|,1)$	3	Figure 3	$0 \in \mathbb{Z}_{|q|}$	$(-|q|+7)/4$
-1	-1	-1	$L(3,-1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$4/3$
-1	-1	2	$L(3,1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$2/3$
-1	-1	$q \leq -2$	$L(-2q+1,-q+1)$	3	Figure 3	$|q| + 1 \in \mathbb{Z}_{2|q|+1}$	$\frac{-q^2-6q+3}{-4q+2}$
-1	-1	$q \geq 3$	$L(-2q+1,-q+1)$	3	Figure 3	$q - 1 \in \mathbb{Z}_{2q-1}$	$\frac{-q^2}{-4q+2}$

TABLE 4. The case $r = -1$, $|p| < 2$ or $|q| < 2$ (and the case $(p, q, r) = (2, q, -1)$)

In **Table 4**, we need to determine the binding number $bn(\xi)$ for the rows 4, 7, 9, and 11. For the other rows, see Remark 3.1.

• If $p = 2, q \leq -2, r = -1$, then using the corresponding matrix $\mathcal{L}_{2,q,-1}$, we have

$$H_1(M) = \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+3}\, |\ \mathcal{L}_{2,q,-1}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3}\, \rangle = \langle\, \mu_2\, |\ (|q|+2)\mu_2 = 0\, \rangle \cong \mathbb{Z}_{|q|+2},$$

and $\mu_1 = |q|\mu_2$. Therefore,

$$c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1) = PD^{-1}(|q|\mu_2) = |q| \in \mathbb{Z}_{|q|+2}.$$

Thus, if $q \leq -2$, then $\xi_{2,q,-1}$ is not isotopic to $\eta_{|q-2|}$ as $c_1(\eta_{|q-2|}) = 0$ implying that $bn(\xi_{2,q,-1}) = 3$ by Theorem 1.1.

• If $p = 0, q \leq -1, r = -1$ (the rows 7 or 9), then $c_1(\xi_{0,q,-1}) = 0$ and so we need to compute $d_3(\xi_{0,q,-1})$. Let K_i's be the components of $\mathbb{L}_{0,q,-1}$ as in Figure 12. Then

$$\mathcal{L}_{0,q,-1} = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & & & \\ . & . & & & \\ . & . & & A_{|q|} & \\ . & . & & & \\ -1 & 0 & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ . & . & & & \\ . & . & & A_{|q|} & \\ . & . & & & \\ 0 & 0 & & & \end{pmatrix}$$

By diagonalizing the first two rows of $\mathcal{L}_{0,q,-1}$, we obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,q,-1}) = \sigma(A_{|q|}) = |q| - 1$. The contact surgery diagram for $\xi_{0,q,-1}$ and the corresponding 4-manifold $X_{0,q,-1}$ (with $\partial X_{0,q,-1} = M$) are given in Figure 12.
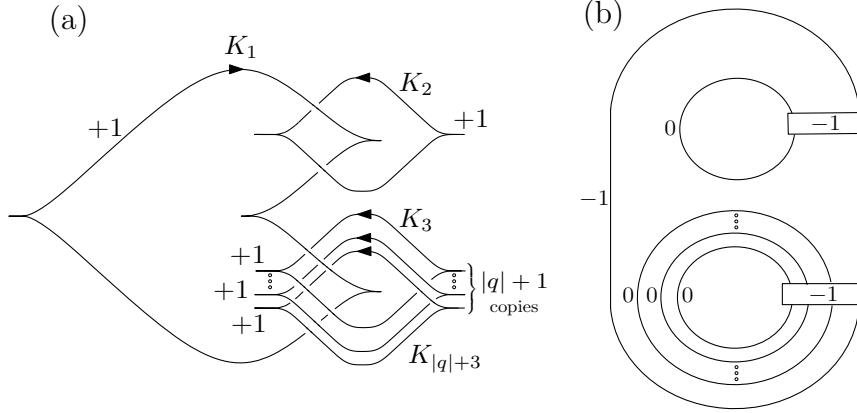

FIGURE 12. (a) The overtwisted contact structure $\xi_{0,q,-1}$ on $L(|q|,1)$, (b) The corresponding framed link $\mathbb{L}_{0,q,-1}$.

Then the system $\mathcal{L}_{0,q,-1}[\mathbf{b}]^T = [rot(K_1)\, rot(K_2) \cdots rot(K_{|q|+3})]^T = [1\ 0\ 0 \cdots 0]^T$ has the solution $[\mathbf{b}] = [0\ -1\ 0 \cdots 0]$, and so $c^2 = 0$. Moreover, $\chi(X_{0,q,-1}) = |q|+4$ and $s = |q|+3$.

Therefore, we obtain $d_3(\xi_{0,q,-1}) = (-|q| + 7)/4$ implying that $\xi_{0,q,-1}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q| + 3)/4$. Hence, $bn(\xi_{0,q,-1}) = 3$ by Theorem 1.1.

• If $p = -1, q = 2, r = -1$, we have $c_1(\xi_{-1,2,-1}) = 1$ implying that $bn(\xi_{-1,2,-1}) = 3$. To see this, note that $c_1(\xi-1, 2, -1) = PD^{-1}(\mu_1)$ where μ_1 is the meridian of the surgery curve corresponding K_1. Then using

$$\mathcal{L}_{-1,2,-1} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

we get $H_1(M) = \langle\ \mu_1, \mu_2, \mu_3, \mu_4|\ \mathcal{L}_{-1,2,-1}[\mu]_4^T = [\mathbf{0}]_4^T\ \rangle = \langle\ \mu_2|\ 3\mu_2 = 0\ \rangle \cong \mathbb{Z}_3$, and $\mu_1 = -2\mu_2$. Therefore, we compute

$$c_1(\xi_{-1,2,-1}) = PD^{-1}(\mu_1) = PD^{-1}(-2\mu_2) = -2 \in \mathbb{Z}_3 \equiv 1 \in \mathbb{Z}_3.$$

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
2	-1	$q \leq -3$	$L(|q-2|, 1)$	3	Figure 3	$2 \in \mathbb{Z}_{|q-2|}$	$\frac{-q^2-3q+6}{-4q+8}$
2	-1	$q = -1$	$L(3,1)$	3	Figure 3	$2 \in \mathbb{Z}_3$	$-5/6$
2	-1	$q = -2$	$L(4,1)$	3	Figure 3	$2 \in \mathbb{Z}_4$	$-1/4$
2	-1	1	S^3	2	$(\star)\ m = 1$	$0 \in \{0\}$	$1/2$
2	-1	2	$S^1 \times S^2$	3	Figure 3	$2 \in \mathbb{Z}$	$\notin Q$
2	-1	3	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
2	-1	$q \geq 4$	$L(q-2, -1)$	3	Figure 3	$q - 4 \in \mathbb{Z}_{q-2}$	$\frac{q^2-3q+6}{4q-8}$
2	0	0	$S^1 \times S^2 \# L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_2$	$1/4$
2	0	1	$S^3 \# L(2,-1)$	2	$(*)\ m = 2$	$0 \in \mathbb{Z}_2$	$-1/4$
2	0	$q > 1$	$L(q,-1)\#L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_2$	$(q-2)/4$
2	0	$q < 0$	$L(|q|,1)\#L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z}_{|q|} \oplus \mathbb{Z}_2$	$(-|q|+4)/4$
2	1	-2	$L(4,-1)$	3	Figure 3	$2 \in \mathbb{Z}_4$	$1/2$
2	1	1	$L(-5,2)$	3	Figure 3	$2 \in \mathbb{Z}_5$	$-1/10$
2	1	$q \leq -3$	$L(-3q-2, q+1)$	3	Figure 3	$2 \in \mathbb{Z}_{-3q-2}$	$\frac{3q^2+15q+10}{12q+8}$
2	1	$q \geq 2$	$L(-3q-2, q+1)$	3	Figure 3	$2 \in \mathbb{Z}_{3q+2}$	$\frac{3q^2-3q-2}{12q+8}$

TABLE 5. The case $r = 2$ ($|p| < 2$ or $|q| < 2$).

In **Table 5**, we need to compute the binding number $bn(\xi)$ for the rows 1, 2, and 3. For the other rows, again see Remark 3.1.

For the first three rows in Table 5, the contact structure $\xi_{-1,q,2}$ on $L(|q-2|,1)$ and the link $\mathbb{L}_{-1,q,2}$ ($q \leq -1$) are given in Figure 13. We write the linking matrix $\mathcal{L}_{-1,q,2}$ as the matrix on the left below. It is not hard to see that $c_1(\xi_{-1,q,2}) = 2 \in \mathbb{Z}_{|q-2|}$, and so $bn(\xi_{-1,q,2}) = 3$. As an illustration we will compute $d_3(\xi_{-1,q,2})$ (even though it is not necessary for the proof). The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{-1,q,2}$. So we compute $\sigma(\mathcal{L}_{-1,q,2}) = 2 + \sigma(A_1) + \sigma(B_{|q|})$ which is $|q| - 1$ if $q \leq -3$, and is equal to 2 if $q = -1, -2$ (recall $\sigma(B_n)$ is $n-3$ if $n \geq 3$, and 0 if $n = 1, 2$).

$$
\mathcal{L}_{-1,q,2} = \begin{bmatrix} -3 & -2 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -2 & -3 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & & 0 & 0 & \cdots & & 0 \\ -1 & -1 & A_1 & 0 & 0 & \cdots & & 0 \\ -1 & -1 & 0 & 0 & & & & \\ -1 & -1 & 0 & 0 & & & & \\ & & & & & A_{|q|} & & \\ -1 & -1 & 0 & 0 & & & & \end{bmatrix} \xrightarrow{\text{diagonalize}} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_1 & & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ & & & & & B_{|q|} & & \\ 0 & 0 & 0 & 0 & & & & \end{bmatrix}
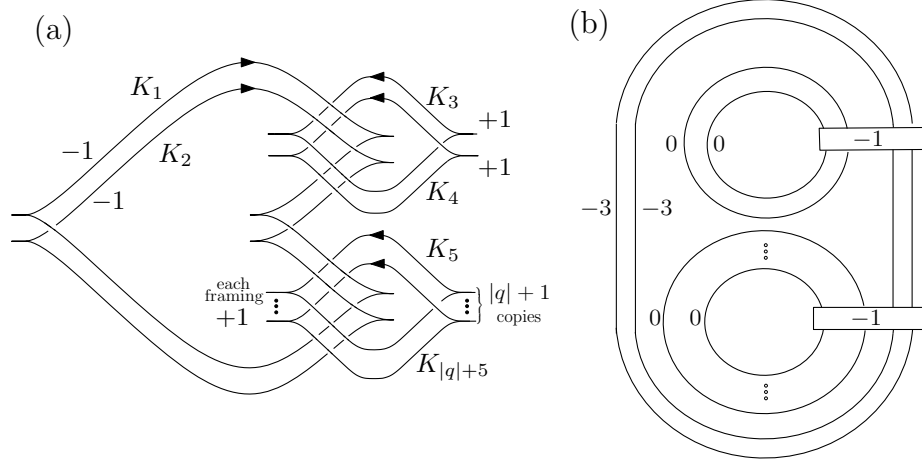$$

FIGURE 13. (a) The contact structure $\xi_{-1,q,2}$ on $L(|q-2|,1)$ for $q \leq -1$, (b) The corresponding framed link $\mathbb{L}_{-1,q,2}$.

By a standard calculation, the system

$$\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = [rot(K_1)\, rot(K_2)\cdots rot(K_{|q|+5})]^T = [1\ 1\ 0\cdots 0]^T$$

has the solution $[\mathbf{b}] = [\frac{|q|}{|q|+2}\ \ \frac{|q|}{|q|+2}\ \ \frac{-2|q|}{|q|+2}\ \ \frac{-2|q|}{|q|+2}\ \ \frac{-2}{|q|+2}\cdots\frac{-2}{|q|+2}]$ for $q \leq -1$, and so we compute

$$c^2 = [\mathbf{b}]\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = 2|q|/(|q|+2).$$

• If $p = -1, q = -1, r = 2$, then $c^2 = 2/3$, $\sigma(X_{-1,-1,2}) = 2$, $\chi(X_{-1,-1,2}) = 7$, and $s = 4$. So we get $d_3(\xi_{-1,-1,2}) = -5/6$.

• If $p = -1, q = -2, r = 2$, then $c^2 = 1$, $\sigma(X_{-1,-2,2}) = 2$, $\chi(X_{-1,-2,2}) = 8$, and $s = 5$. Therefore, we get $d_3(\xi_{-1,-2,2}) = -1/4$.

• If $p = -1, q \leq -3, r = 2$, then $c^2 = 2|q|/(|q|+2)$, $\sigma(X_{-1,q,2}) = |q| - 1$, $\chi(X_{-1,q,2}) = |q| + 6$, and $s = |q| + 3$. So we obtain

$$d_3(\xi_{-1,q,2}) = \frac{-q^2 - 3q + 6}{-4q + 8}.$$

In **Table 6**, we need to compute the binding number $bn(\xi)$ for the rows 7, 9, 10, and 13. For the other rows, see Remark 3.1.

• If $p = 1, q \leq -4, r = -2$, the contact structure $\xi_{1,q,-2}$ on $L(|q+2|,1)$ and the link $\mathbb{L}_{1,q,-2}$ are given in Figure 14.
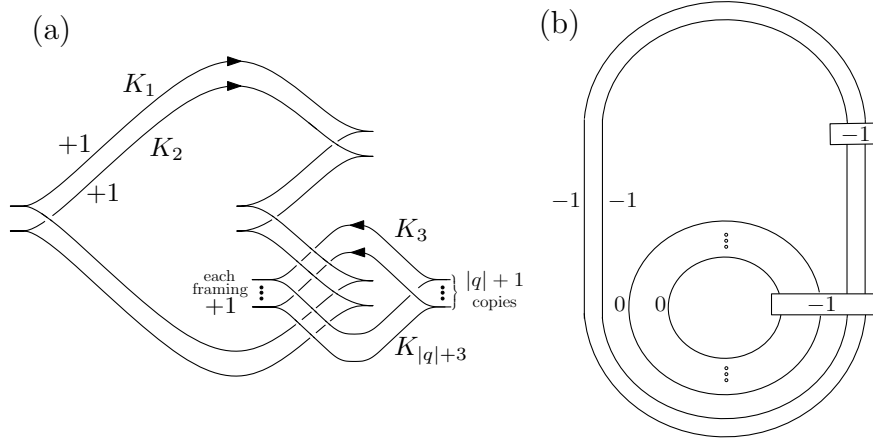

FIGURE 14. (a) The contact structure $\xi_{1,q,-2}$ on $L(|q+2|,1)$ for $q < -3$, (b) The corresponding framed link $\mathbb{L}_{1,q,-2}$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-2	1	$q \geq 3$	$L(q+2,-1)$	3	Figure 3	$2 \in \mathbb{Z}_{q+2}$	$\frac{q^2+q+2}{4q+8}$
-2	1	2	$L(4,-1)$	3	Figure 3	$0 \in \mathbb{Z}_4$	$1/2$
-2	1	1	$L(3,-1)$	3	Figure 3	$2 \in \mathbb{Z}_3$	$1/3$
-2	1	-1	S^3	2	$(\star)\ m=1$	$0 \in \{0\}$	$1/2$
-2	1	-2	$S^1 \times S^2$	3	Figure 3	$2 \in \mathbb{Z}$	$\notin Q$
-2	1	-3	S^3	3	Figure 3	$0 \in \{0\}$	$-1/2$
-2	1	$q \leq -4$	$L(|q+2|,1)$	3	Figure 3	$|q|-4 \in \mathbb{Z}_{|q|-2}$	$\frac{-q^2-7q-14}{-4q-8}$
-2	0	0	$S^1 \times S^2 \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_2$	$3/4$
-2	0	1	$S^3 \# L(2,1)$	2	$(\star)\ m=2$	$0 \in \mathbb{Z}_2$	$1/4$
-2	0	-1	$S^3 \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_2$	$5/4$
-2	0	$q \geq 2$	$L(q,-1)\# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_2$	$q/4$
-2	0	$q \leq -2$	$L(|q|,1)\# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_{|q|} \oplus \mathbb{Z}_2$	$(-|q|+6)/4$
-2	-1	2	$L(4,1)$	3	Figure 3	$0 \in \mathbb{Z}_4$	$1/2$
-2	-1	-1	$L(-5,-2)$	3	Figure 3	$2 \in \mathbb{Z}_5$	$11/10$
-2	-1	$q \leq -2$	$L(3q-2,q-1)$	3	Figure 3	$2 \in \mathbb{Z}_{3|q|+2}$	$\frac{-3q^2-15q+10}{-12q+8}$
-2	-1	$|q| \geq 3$	$L(3q-2,q-1)$	3	Figure 3	$2 \in \mathbb{Z}_{3q-2}$	$\frac{-3q^2+3q-2}{-12q+8}$

TABLE 6. The case $r = -2$ ($|p| < 2$ or $|q| < 2$).

We will first compute that $c_1(\xi_{1,q,-2}) = |q| - 4 \in \mathbb{Z}_{|q|-2}$ (so $bn(\xi_{1,q,-2}) = 3$), and then (even though it is not necessary for the proof) we will evaluate $d_3(\xi_{1,q,-2})$ as an another sample computation. Using $\mathcal{L}_{1,q,-2}$ (on the left below), we have

$$
\begin{aligned}
H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+3} \,|\, \mathcal{L}_{1,q,-2}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3} \,\rangle \\
&= \langle\, \mu_1, \mu_3 \,|\, -3\mu_1 - (|q|+1)\mu_3 = 0,\ -2\mu_1 - |q|\mu_3 = 0 \,\rangle \\
&= \langle\, \mu_3 \,|\, (|q|-2)\mu_3 = 0 \,\rangle \cong \mathbb{Z}_{|q|-2},
\end{aligned}
$$

and also we have $\mu_1 = \mu_2 = -\mu_3$. Therefore, we obtain

$$
c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1 + \mu_2) = PD^{-1}(-2\mu_3) = -2 \equiv |q| - 4 \in \mathbb{Z}_{|q|-2}.
$$

The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{1,q,-2}$. So we compute $\sigma(\mathcal{L}_{1,q,-2}) = 0 + \sigma(C_{|q|}) = |q| - 1$ (recall $\sigma(C_n) = n - 1$ if $n \geq 2$).

$$\mathcal{L}_{1,q,-2} = \begin{bmatrix} -1 & -2 & -1 & \cdots & -1 \\ -2 & -1 & -1 & \cdots & -1 \\ \hdashline -1 & -1 & & & \\ \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ -1 & -1 & & & \end{bmatrix} \xrightarrow{\text{diagonalize}} \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & -1/2 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & & & \\ \cdot & \cdot & & C_{|q|} & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & & & \end{bmatrix}$$

By a standard calculation, the system

$$\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = [rot(K_1) \ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1 \ 1 \ 0 \cdots 0]^T$$

has the solution $[\mathbf{b}] = [\frac{-|q|}{|q|-2} \ \frac{-|q|}{|q|-2} \ \frac{2}{|q|-2} \ \cdots \ \frac{2}{|q|-2}]$, and so we obtain

$$c^2 = [\mathbf{b}]\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = -2|q|/(|q|-2).$$

Moreover, $\chi(X_{1,q,-2}) = |q| + 4$, and $s = |q| + 3$. So we compute

$$d_3(\xi_{1,q,-2}) = \frac{-q^2 - 7q - 14}{-4q - 8}.$$

• If $p = 0, q = 1, r = -2$, then $\xi_{0,1,-2}$ and $\mathbb{L}_{0,1,-2}$ are given in Figure 15.
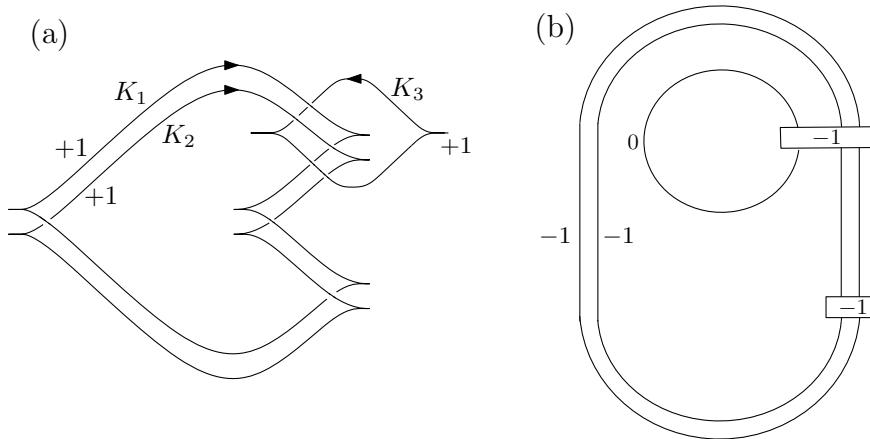

FIGURE 15. (a) The contact structure $\xi_{0,1,-2}$ on $S^3 \# L(2,1) \approx L(2,1)$, (b) The corresponding framed link $\mathbb{L}_{0,1,-2}$.

One can get $c_1(\xi_{0,1,-2}) = 0$, so we need $d_3(\xi_{0,1,-2})$. The corresponding linking matrix is

$$\mathcal{L}_{0,1,-2} = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{0,1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,1,-2}) = 1$. We find that the system $\mathcal{L}_{0,1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)]^T = [1\ 1\ 0]^T$ has the solution $[\mathbf{b}] = [0\ 0\ -1]$, and so $c^2 = 0$. Also we have $\chi(X_{0,1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,1,-2}) = 1/4 = d_3(\eta_2)$ which implies that $\xi_{0,1,-2}$ is isotopic to η_2. Thus, $bn(\xi_{0,1,-2}) = 2$ by Theorem 1.1.

• If $p = 0, q = -1, r = -2$, then the contact structure $\xi_{0,1,-2}$ on $L(2,1)$ and the link $\mathbb{L}_{0,-1,-2}$ describing $X_{0,-1,-2}$ are given in Figure 16. It is easy to check $c_1(\xi_{0,-1,-2}) = 0$, so we compute $d_3(\xi_{0,-1,-2})$:
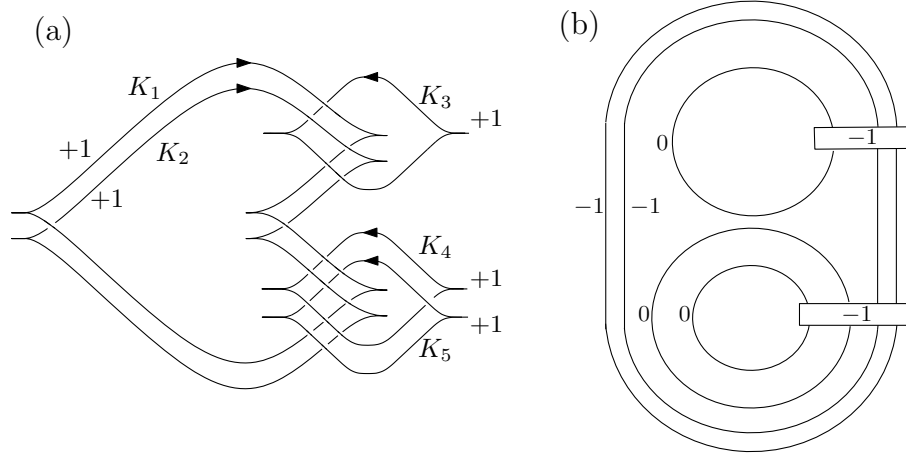

FIGURE 16. (a) The contact structure $\xi_{0,-1,-2}$ on $S^3 \# L(2,1) \approx L(2,1)$, (b) The corresponding framed link $\mathbb{L}_{0,-1,-2}$.

The corresponding linking matrix is

$$\mathcal{L}_{0,-1,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

116

We diagonalize $\mathcal{L}_{0,-1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,-1,-2}) = 1$. The system

$$\mathcal{L}_{0,-1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ 1\ 0\ 0\ 0]^T$$

has the solution $[\mathbf{b}] = [0\ 0\ -1\ 0\ 0]$ which yields $c^2 = 0$. Also we have $\chi(X_{0,-1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,-1,-2}) = 5/4 \neq 1/4 = d_3(\eta_2)$. Therefore, $\xi_{0,-1,-2}$ is not isotopic to η_2, and so $bn(\xi_{0,-1,-2}) = 3$ by Theorem 1.1.

• If $p = -1, q = 2, r = -2$, then the contact structure $\xi_{-1,2,-2}$ on $L(4,1)$ and the link $\mathbb{L}_{-1,2,-2}$ describing $X_{-1,2,-2}$ are given in Figure 17. We compute that $c_1(\xi_{-1,2,-2}) = 0$, so we need to find $d_3(\xi_{-1,2,-2})$.
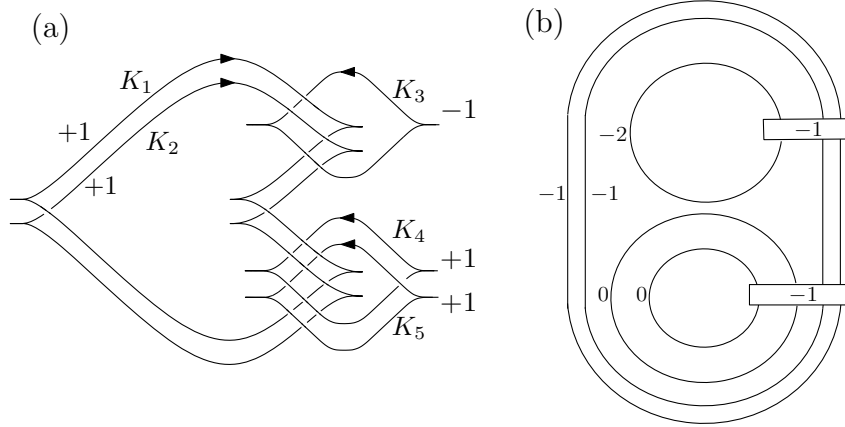

FIGURE 17. (a) The contact structure $\xi_{-1,2,-2}$ on $L(4,1)$, (b) The corresponding framed link $\mathbb{L}_{-1,2,-2}$.

The corresponding linking matrix is

$$\mathcal{L}_{-1,2,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{-1,2,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{-1,2,-2}) = 1$. The system

$$\mathcal{L}_{-1,2,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ 1\ 0\ 0\ 0]^T$$

has the solution $[\mathbf{b}] = [1/2 \ \ 1/2 \ \ -1/2 \ \ -1 \ \ -1]$, so we compute $c^2 = 1$. Moreover, $\chi(X_{-1,2,-2}) = 6$ and $s = 4$. Then we get $d_3(\xi_{-1,2,-2}) = 1/2 \neq -1/4 = d_3(\eta_4)$. Therefore, $\xi_{-1,2,-2}$ is not isotopic to η_4, and so $bn(\xi_{-1,2,-2}) = 3$ by Theorem 1.1.

p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	0	$L(r,-1)$	3	$0 \in \mathbb{Z}_r$	$(r+1)/4$
-1	1	S^3	2	$0 \in \{0\}$	$1/2$
-1	2	$L(r-2,-1)$	3	$2 \in \mathbb{Z}_{r-2}$	$(r^2-3r+6)/(4r-8)$
-1	$q \leq -1$	$L(qr-q-r,-q+1)$	3	$r \in \mathbb{Z}_{|q|r-|q|+r}$	$\frac{q^2r+qr^2-q^2-r^2-q-r}{4qr-4q-4r}$
-1	$q \geq 3$	$L(qr-q-r,-q+1)$	3	$r \in \mathbb{Z}_{qr-q-r}$	$\frac{q^2r+qr^2-q^2-r^2-6qr+5q+5r}{4qr-4q-4r}$
0	0	$S^1 \times S^2 \# L(r,-1)$	3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_r$	$(r-1)/4$
0	1	$S^3 \# L(r,-1)$	2	$0 \in \mathbb{Z}_r$	$(-r+3)/4$
0	$q \leq -2$	$L(|q|,1) \# L(r,-1)$	3	$0 \in \mathbb{Z}_{|q|} \oplus \mathbb{Z}_r$	$(q+r+2)/4$
0	$q \geq 2$	$L(q,-1) \# L(r,-1)$	3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_r$	$(q+r-4)/4$
1	-2	$L(r+2,-1)$	3	$r \in \mathbb{Z}_{r+2}$	$(r^2+r+2)/(4r+8)$
1	1	$L(2r+1,-2)$	3	$r \in \mathbb{Z}_{2r+1}$	$(r^2-2r-1)/(4r+2)$
1	$q \leq -3$	$L(qr+q+r,-q-1)$	3	$r \in \mathbb{Z}_{|q|r+|q|-r}$	$\frac{q^2r+qr^2+q^2+r^2+4qr+3q+3r}{4qr+4q+4r}$
1	$q \geq 2$	$L(qr+q+r,-q-1)$	3	$r \in \mathbb{Z}_{qr+q+r}$	$\frac{q^2r+qr^2+q^2+r^2-2qr-3q-3r}{4qr+4q+4r}$

TABLE 7. The case $r > 2$ ($|p| < 2$ or $|q| < 2$).

In **Table 7**, we do not need any computation to find $bn(\xi)$: For any row, we can use Remark 3.1. For example, in the 1st row, we have an overtwisted contact structure on the lens space $L(m,-1)$ for some $m \geq 1$. Therefore, the resulting contact manifold is not listed in Theorem 1.1, and hence we must have $bn(\xi) = 3$.

In **Table 8**, we need to compute the binding number $bn(\xi)$ for the rows 1, 3, 7, and 10. For the other rows, see Remark 3.1.

• If $p = 1, q = -2, r < -2$, $\xi_{1,-2,r}$ is an overtwisted contact structure on $L(|r+2|,1)$. It is not hard to see that $c_1(\xi_{1,-2,r}) = 2 \in \mathbb{Z}_{|r|-2}$. Therefore, we immediately get $bn(\xi_{1,q,-2}) = 3$ because $c_1(\eta_{|r+2|}) = 0$.

p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$L(|r+2|,1)$	3	$2 \in \mathbb{Z}_{|r|-2}$	$(r^2 + 7r + 14)/(4r + 8)$
1	-1	S^3	2	$0 \in \{0\}$	$1/2$
1	0	$L(|r|,1)$	2	$0 \in \mathbb{Z}_{|r|}$	$(-|r| + 3)/4$
1	$q \leq -3$	$L(qr+q+r, -q-1)$	3	$|r| \in \mathbb{Z}_{|q||r|-|q|-|r|}$	$\frac{q^2r+qr^2+q^2+r^2+10qr+9q+9r}{4qr+4q+4r}$
1	$q \geq 1$	$L(qr+q+r, -q-1)$	3	$|r| \in \mathbb{Z}_{q|r|-q+|r|}$	$\frac{q^2r+qr^2+q^2+r^2+4qr+3q+3r}{4qr+4q+4r}$
0	0	$S^1 \times S^2 \# L(|r|,1)$	3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_{|r|}$	$(-|r| + 5)/4$
0	-1	$S^3 \# L(|r|,1)$	3	$0 \in \mathbb{Z}_{|r|}$	$(-|r| + 7)/4$
0	$q \leq -2$	$L(|q|,1) \# L(|r|,1)$	3	$0 \in \mathbb{Z}_{|q|} \oplus \mathbb{Z}_{|r|}$	$(q + r + 8)/4$
0	$q \geq 2$	$L(q,-1) \# L(|r|,1)$	3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_{|r|}$	$(q + r + 2)/4$
-1	2	$L(|r-2|,1)$	3	$|r| \in \mathbb{Z}_{|r|+2}$	$(-r^2 - 3r + 6)/(-4r + 8)$
-1	-1	$L(-2r+1,2)$	3	$|r| \in \mathbb{Z}_{2|r|+1}$	$(-r^2 - 6r + 3)/(-4r + 2)$
-1	$q \leq -2$	$L(qr-q-r, -q+1)$	3	$|r| \in \mathbb{Z}_{|q||r|+|q|+|r|}$	$\frac{q^2r+qr^2-q^2-r^2+6qr-7q-7r}{4qr-4q-4r}$
-1	$q \geq 3$	$L(qr-q-r, -q+1)$	3	$|r| \in \mathbb{Z}_{q|r|+q-|r|}$	$\frac{q^2r+qr^2-q^2-r^2-q-r}{4qr-4q-4r}$

TABLE 8. The case $r < -2$ ($|p| < 2$ or $|q| < 2$).

• If $p = 1, q = 0, r < -2$, the contact structure $\xi_{1,0,r}$ on $L(|r|,1)$ and the link $\mathbb{L}_{1,0,r}$ are given in Figure 18. It is easy to see that $c_1(\xi_{1,0,r}) = 0 \in \mathbb{Z}_{|r|}$, so we need $d_3(\xi_{1,0,r})$: The corresponding linking matrix is on the left below. Diagonalize $\mathcal{L}_{1,0,r}$ to get the matrix on the right. Therefore, $\sigma(\mathcal{L}_{1,0,r}) = |r| - 1$.

$$\mathcal{L}_{1,0,r} = \begin{bmatrix} -1 & -2 & -2 & \cdots & -2 & -1 \\ -2 & -1 & -2 & \cdots & -2 & -1 \\ -2 & -2 & -1 & & \cdot & \cdot \\ \cdot & \cdot & & \ddots & \cdot & \cdot \\ \cdot & \cdot & & & -1 & -2 \\ -2 & -2 & \cdots & -2 & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{diagonalize}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & & \cdot & \cdot \\ \cdot & \cdot & & \ddots & \cdot & \cdot \\ \cdot & \cdot & & & 2 & 0 \\ 0 & 0 & \cdots & 0 & 1/2 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -3 \end{bmatrix}$$

The system

$$\mathcal{L}_{1,0,r}[\mathbf{b}]^T = [rot(K_1) \ rot(K_2) \cdots rot(K_{|q|+1})]^T = [1 \cdots 1 \ 0]^T$$
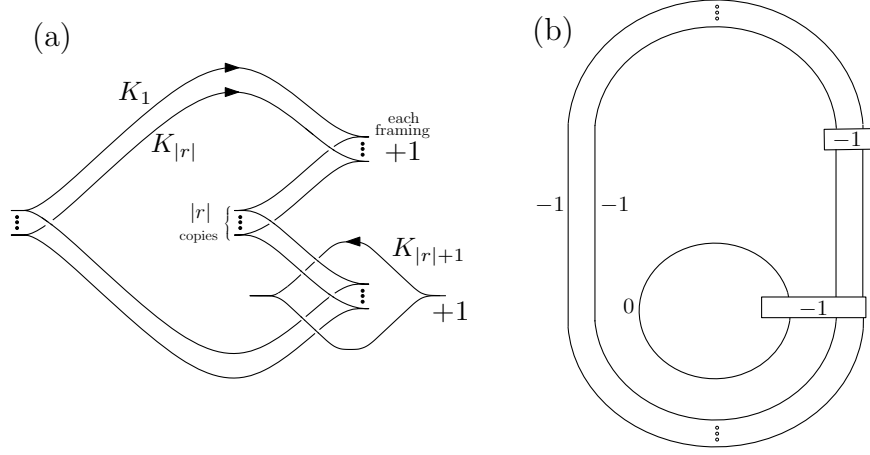
(a)

(b)

FIGURE 18. (a) The contact structure $\xi_{1,0,r}$ on $L(|r|,1)$ for $r < -2$, (b) The corresponding framed link $\mathbb{L}_{1,0,r}$.

has the solution $[\mathbf{b}] = [0 \cdots 0 \ {-1}]$, and so we obtain $c^2 = 0$. Moreover, $\chi(X_{1,0,r}) = |r|+2$, and $s = |r| + 1$. So we compute $d_3(\xi_{1,0,r}) = (-|r| + 3)/4 = d_3(\eta_{|r|})$ which implies that $\xi_{1,0,r}$ is isotopic to $\eta_{|r|}$ on $L(|r|,1)$. Thus, $bn(\xi_{1,0,r}) = 2$ by Theorem 1.1.

• If $p = 0, q = -1, r < -2$, the contact structure $\xi_{0,-1,r}$ on $L(|r|,1)$ and the link $\mathbb{L}_{0,-1,r}$ are given in Figure 19. Again we have $c_1(\xi_{0,-1,r}) = 0 \in \mathbb{Z}_{|r|}$, so we need to find $d_3(\xi_{0,-1,r})$: We diagonalize $\mathcal{L}_{0,-1,r}$ and get the matrix on the right below. So, we conclude that $\sigma(\mathcal{L}_{0,-1,r}) = |r| - 1$.

$$\mathcal{L}_{0,-1,r} = \begin{bmatrix} -1 & -2 & -2 & \cdots & -2 & -1 & -1 & -1 \\ -2 & -1 & -2 & \cdots & -2 & -1 & -1 & -1 \\ -2 & -2 & -1 & & \circ & & & \\ \circ & \circ & & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & & & -1 & -2 & & \\ -2 & -2 & \cdots & -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & \cdots & & -1 & 0 & 0 & 0 \\ -1 & -1 & \cdots & & -1 & 0 & 0 & -1 \\ -1 & -1 & \cdots & & -1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{\text{diagonalize}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & & \circ & & & \\ \circ & \circ & & \circ & & \circ & \circ & \circ \\ \circ & \circ & & & 2 & 0 & & \\ 0 & 0 & \cdots & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & \cdots & & 0 & -3 & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & 2 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

The system

$$\mathcal{L}_{0,-1,r}[\mathbf{b}]^T = [rot(K_1) \ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1 \cdots 1 \ 0 \ 0 \ 0]^T$$
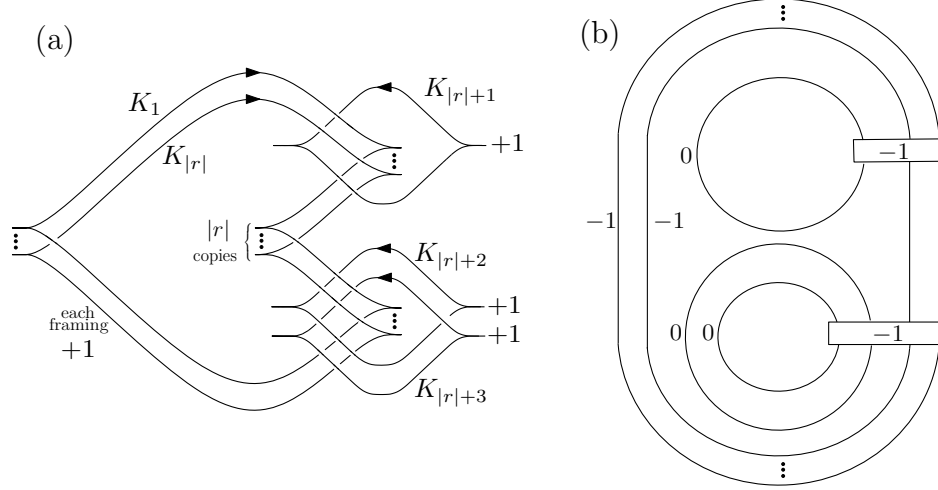
FIGURE 19. (a) The contact structure $\xi_{0,-1,r}$ on $L(|r|,1)$ for $r < -2$,
(b) The corresponding framed link $\mathbb{L}_{0,-1,r}$.

has the solution $[\mathbf{b}] = [0 \cdots 0\ 1\ 0\ 0]$, so we get $c^2 = 0$. Also $\chi(X_{1,q,-2}) = |r| + 4$, and $s = |r| + 3$. So we compute $d_3(\xi_{0,-1,r}) = (-|r| + 7)/4 \neq (-|r| + 3)/4 = d_3(\eta_{|r|})$ implying that $\xi_{0,-1,r} \not\cong \eta_{|r|}$ on $L(|r|,1)$. Hence, $bn(\xi_{0,-1,r}) = 3$ by Theorem 1.1.

• If $p = -1, q = 2, r < -2$, we have $bn(\xi_{-1,2,r}) = 3$ because $c_1(\xi_{-1,2,r}) = |r| \in \mathbb{Z}_{|r|+2}$. We compute $c_1(\xi_{-1,2,r})$ as follows: We use the linking matrix $\mathcal{L}_{-1,2,r}$ to get the representation

$$
\begin{aligned}
H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{|r|+3} |\ \mathcal{L}_{-1,2,r}[\mu]_{|r|+3}^T = [\mathbf{0}]_{|r|+3}^T \,\rangle \\
&= \langle\, \mu_1 |\ (|r| + 2)\mu_1 = 0 \,\rangle \cong \mathbb{Z}_{|r|+2}.
\end{aligned}
$$

Moreover, using the relations given by $\mathcal{L}_{-1,2,r}$ we have $\mu_1 = \mu_2 \cdots = \mu_{|r|}$ (μ_i's are the meridians as before). Therefore, we obtain

$$
c_1(\xi_{-1,2,r}) = PD^{-1}(\mu_1 + \cdots + \mu_{|r|}) = PD^{-1}(|r|\mu_1) = |r| \in \mathbb{Z}_{|r|+2}.
$$

To finish the proof, in each table above we find each particular case for (p, q, r) such that the corresponding contact structure $\xi_{p,q,r}$ has binding number 2. Note that the conditions on p, q, r given in the statement of the theorem excludes exactly these cases. This completes the proof. □

4. Remarks on the remaining cases

Assume that $r = 0, \pm 1, |p| \geq 2, |q| \geq 2$. We list all possible contact structures in Table 9. These are the only remaining cases from which we still get lens spaces or their connected

sums. Notice that we have already considered the cases $(-2, q, 1)$, and $(2, q, -1)$ in Tables 3 and 4, so we do not list them here.

r	p	q	resulting M	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	$p \geq 2$	$q \geq 2$	$L(p,-1)\#L(q,-1)$	$0 \in \mathbb{Z}_p \oplus \mathbb{Z}_q$	$(p+q-4)/4$
0	$p \geq 2$	$q \leq -2$	$L(p,-1)\#L(\lvert q\rvert,1)$	$0 \in \mathbb{Z}_p \oplus \mathbb{Z}_{\lvert q\rvert}$	$(p+q+2)/4$
0	$p \leq -2$	$q \leq -2$	$L(\lvert p\rvert,1)\#L(\lvert q\rvert,1)$	$0 \in \mathbb{Z}_{\lvert p\rvert} \oplus \mathbb{Z}_{\lvert q\rvert}$	$(p+q+8)/4$
1	$p \geq 2$	$q \geq 2$	$L(pq+p+q,-q-1)$	$-p \in \mathbb{Z}_{pq+p+q}$	$\frac{p^2q+pq^2+p^2+q^2-2pq-3p-3q}{4pq+4p+4q}$
1	$p \geq 2$	$q < -2$	$L(pq+p+q,-q-1)$	$-p \in \mathbb{Z}_{p\lvert q\rvert-p+\lvert q\rvert}$	$\frac{-p^2q+pq^2+p^2+q^2+4pq+3p+3q}{4pq+4p+4q}$
1	$p < -2$	$q < -2$	$L(pq+p+q,-q-1)$	$p \in \mathbb{Z}_{\lvert p\rvert\lvert q\rvert+\lvert p\rvert+\lvert q\rvert}$	$\frac{p^2q+pq^2+p^2+q^2+10pq+9p+9q}{4pq+4p+4q}$
-1	$p > 2$	$q > 2$	$L(pq-p-q,-q+1)$	$-p \in \mathbb{Z}_{pq-p-q}$	$\frac{p^2q+pq^2-p^2-q^2-6pq+5p+5q}{4pq-4p-4q}$
-1	$p > 2$	$q \leq -2$	$L(pq-p-q,-q+1)$	$-p \in \mathbb{Z}_{p\lvert q\rvert+p-\lvert q\rvert}$	$\frac{p^2q+pq^2-p^2-q^2-p-q}{4pq-4p-4q}$
-1	$p \leq -2$	$q \leq -2$	$L(pq-p-q,-q+1)$	$p \in \mathbb{Z}_{\lvert p\rvert\lvert q\rvert+\lvert p\rvert+\lvert q\rvert}$	$\frac{p^2q+pq^2-p^2-q^2+6pq-7p-7q}{4pq-4p-4q}$

TABLE 9. The case $r = 0, \pm 1$, $\lvert p\rvert \geq 2$, $\lvert q\rvert \geq 2$ ($bn(\xi) = 3$ in each row).

As we remarked in Section 1 (after Theorem 1.3) that one can obtain the complete list without any repetition: We first simply find all distinct homeomorphism types of the manifolds which we found in Table 2 through Table 9. Then on a fixed homeomorphism type we compare the pairs (c_1, d_3) coming from the tables to distinguish the contact structures.

Suppose now that M is a prime Seifert fibered manifold which is not a lens space. Then as we remarked before we have $\lvert p\rvert \geq 2, \lvert q\rvert \geq 2$, and $\lvert r\rvert \geq 2$. Then two such triples $(p, q, r), (p', q', r')$ give the same Seifert manifold Y if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'},$$

and (p', q', r') is a permutation of (p, q, r) (see [JN], for instance). Notice that we can drop the first condition in our case. Switching p and q does not change the contact manifold as we mentioned before. On the other hand, if we switch r and p (or r and q), we might have different contact structures on the same underlying topological manifold.

Another issue is that there are some cases where the first homology group $H_1(Y(p, q, r))$ is not finite. Indeed, consider the linking matrix $\mathcal{L}$ of the surgery diagram given on the

122

right in Figure 2 as below.

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{pmatrix}$$

The determinant $\det(\mathcal{L}) = -r(p+q) - pq = 0$ implies that $r = -\frac{pq}{p+q}$. Thus, if $r \neq -\frac{pq}{p+q}$, then $H_1(Y(p,q,r))$ is finite, and so $d_3(\xi_{p,q,r})$ is still computable since $c_1(\xi_{p,q,r})$ is torsion. For instance, if $p \geq 2, q \geq 2, r \geq 2$ or $p \leq -2, q \leq -2, r \leq -2$, than $\det(\mathcal{L}) \neq 0$, and so we can distinguish the corresponding $\xi_{p,q,r}$ by computing the pair (c_1, d_3). Whereas if the sign of the one of p, q, r is different than the others', then we might have $\det(\mathcal{L}) = 0$. For instance, for the triples $(4, 4, -2), (3, 6, -2)$ and each nonzero integer multiples of them, $\det(A) = 0$. So more care is needed for these cases.

We would like to end the article by a sample computation. Assume that $\det(\mathcal{L}) \neq 0$, and that $r \leq 2, p \geq 2, q \leq 2$ (similar calculations apply for the other cases). We compute the first homology of $M \approx Y(p,q,r)$ as

$$\begin{aligned} H_1(M) &= \langle \mu_1, \mu_2, \cdots, \mu_{p+q+|r|} | \mathcal{L}_{p,q,r}[\mu]_{p+q+|r|}^T = [\mathbf{0}]_{p+q+|r|}^T \rangle \\ &= \langle \mu_1, \mu_{|r|+1}, \mu_{p+|r|} | R_1, R_2, R_3 \rangle \end{aligned}$$

where the relations of the presentation are

$$\begin{aligned} R_1: \quad -(2|r|-1)\mu_1 &- (p-1)\mu_{|r|+1} &- (|q|+1)\mu_{p+|r|} &= 0 \\ R_2: \quad & p\,\mu_{|r|+1} &- |q|\,\mu_{p+|r|} &= 0 \\ R_3: \quad -|r|\,\mu_1 &- p\,\mu_{|r|+1} & &= 0 \end{aligned}$$

While getting these relations, we also see that $\mu_1 = \mu_2 \cdots \mu_{|r|}$ (recall μ_i's are the meridians to the surgery curves in the family corresponding to r for $i = 1, \cdots, |r|$). Then using this presentation, and knowing that $c_1(\xi_{p,q,r}) = PD^{-1}(|r|\mu_1)$, we can evaluate (understand) $c_1(\xi_{p,q,r})$ in $H^2(M) \cong H_1(M)$.

Now if $c_1(\xi_{p,q,r}) \in H^2(M)$ is a torsion class, then we can also compute $d_3(\xi_{p,q,r})$ as follows: By solving the corresponding linear system we get

$$c^2 = \frac{p|q||r|}{p|q| + p|r| - |q||r|}.$$

Moreover, we compute $\sigma(X_{p,q,r}) = \sigma(\mathcal{L}_{p,q,r}) = -p + |q| + |r|, \chi(X_{p,q,r}) = p + |q| + |r| + 1$, and $s = |q| + |r| + 1$. Hence, using Corollary 2.8, we obtain

$$d_3(\xi_{p,q,r}) = \frac{8pqr + p^2q + p^2r + 4pq^2 + 4qr^2 - pr^2 - q^2r - pq - pr - qr}{4pq + 4pr + 4qr}.$$

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