Proceedings of 14th Gökova
Geometry-Topology Conference
pp. 1 – 14

A question analogous to the flux conjecture concerning Lagrangian submanifolds

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Abstract. We discuss a question analogous to the flux conjecture concerning the space of Lagrangian submanifolds. Under suitable conditions, we prove C^1-closedness of orbits of the action of Hamiltonian diffeomorphism group on the space of Lagrangian submanifolds.

1. Introduction

In this note, we discuss a question analogous to the flux conjecture for the quotient space of Lagrangian submanifolds modulo the Hamiltonian diffeomorphism group of the ambient symplectic manifold and related questions. We briefly recall the flux conjecture, see section 2 for more details. Let (M, ω) be a closed symplectic manifold and $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω). It is obviously a subgroup in the identity component $\mathrm{Symp}_0(M, \omega)$ of the group of symplectomorphisms. The C^1-flux conjecture states that $\mathrm{Ham}(M, \omega)$ is closed $\mathrm{Symp}_0(M, \omega)$ with respect to the C^1-topology. This conjecture is equivalent to that the so-called flux group Γ_ω is discrete in $\mathrm{H}^1(M; \mathbf{R})$. From now on, we simply call it the flux conjecture. We proved the flux conjecture in [5] using Floer cohomology for symplectic isotopies (Floer-Novikov theory). Note that it implies that the space $\mathrm{Symp}_0(M, \omega)/\mathrm{Ham}(M, \omega) \cong \mathrm{H}^1(M; \mathbf{R})/\Gamma_\omega$ is Hausdorff.

For a symplectomorphism ϕ of (M, ω), its graph $Gr_\phi \subset M \times M$ is a Lagrangian submanifold with respect to the symplectic form $-\pi_1^*\omega + \pi_2^*\omega$. Hence the statement on symplectomorphisms can be rephrased in terms of associated Lagrangian submanifolds, namely their graphs. We can also consider a Lagrangian submanifold L in a general symplectic manifold (P, Ω). This leads to the following question.

Let L be a closed embedded Lagrangian submanifold in a closed symplectic submanifold (P, Ω). Denote by $\mathrm{Lag}(L)$ the space of all embedded Lagrangian submanifolds, which are Lagrangian isotopic to L. The group $\mathrm{Ham}(P, \Omega)$ of Hamiltonian diffeomorphisms acts on $\mathrm{Lag}(L)$ in a natural way.

Question 1.1. Is $\mathrm{Lag}(L)/\mathrm{Ham}(P, \Omega)$ Hausdorff with respect to the C^1-topology?

Key words and phrases. Lagrangian submanifolds, Floer cohomology, flux conjecture.
The author is partially supported by Grant-in-Aid for Scientific Research Nos. 17654009 and 18340014, JSPS.

In general, the answer is negative. Chekanov [1] studied a certain class of embedded Lagrangian tori in symplectic vector spaces. The Hausdorffness property fails at monotone Lagrangian tori in his examples. Although the question is known to be false in general, we would still like to discuss the question under appropriate conditions on the Lagrangian submanifold L. In this note, we mainly discuss the following weaker question.

Question 1.2. Are $\mathrm{Ham}(P,\Omega)$-orbits closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology?

It is natural to expect the relation between the stability for Lagrangian submanifolds and the above question, cf. [8], [9] as well as [3]. Our main result is the following:

Theorem 1.1. *Let (P,Ω) be a closed symplectic manifold and $L \subset P$ a closed embedded Lagrangian submanifold with vanishing Maslov class. If L is unobstructed in the sense of* [4]*, $\mathrm{Ham}(P,\Omega)\cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology.*

Corollary 1.2. *Let (P,Ω) be a closed symplectic manifold and $L \subset P$ a closed embedded Lagrangian submanifold with vanishing Maslov class. If $\mathrm{H}^2(L;\mathbb{Q}) = 0$, $\mathrm{Ham}(P,\Omega)\cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology.*

Under a restrictive condition, we also show the following:

Theorem 1.3. *In addition, suppose that $\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L;\mathbb{R})$ is surjective. Then $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff.*

The contents of this note are as follows. In section 2, we review some facts related to the flux conjecture and formulate our question. In section 3, we collect necessary results in [4]. We will prove our main theorem in section 4.

2. Preliminaries

From now on, we assume that Lagrangian submanifolds are closed and embedded. We recall the flux homomorphism and summarize some facts on the flux conjecture. For a one-parameter family $H = \{h_t\}$ of smooth functions on M, we have the Hamiltonian isotopy $\{\phi_t^H\}$, which is obtained by integrating the one-parameter family $\{X_{h_t}\}$ of Hamiltonian vector fields, i.e., $i(X_{h_t})\omega = dh_t$. A diffeomorphism ϕ is called a Hamiltonian diffeomorphism, if ϕ is expressed as the time-one map ϕ_1^H for some $H = \{h_t\}$. Denote by $\mathrm{Ham}(M,\omega)$ the group of all Hamiltonian diffeomorphisms of (M,ω), which is a significant subgroup of the group $\mathrm{Symp}(M,\omega)$ of all symplectomorphisms of (M,ω). Clearly, $\mathrm{Ham}(M,\omega)$ is contained in the identity component $\mathrm{Symp}_0(M,\omega)$.

We introduce some notation. Let θ be a closed 1-form on M. Denote by X_θ the symplectic vector field such that $i(X_\theta)\omega = \theta$. Let $\Theta = \{\theta_t\}$ be a one-parameter family of closed 1-forms. Denote by $\{\psi_t^\Theta\}$ the symplectic isotopy generated by $\{X_{\theta_t}\}$. Any symplectic isotopy $\{\psi_t\}$ such that $\psi_0 = id$ is described in this way. When Θ is not specified, we denote by $\{X_t\}$ the vector field generating $\{\psi_t\}$.

Let $\widetilde{\mathrm{Symp}}_0(M,\omega)$ be the universal covering group of $\mathrm{Symp}_0(M,\omega)$. Set

$$\widetilde{F}(\{\psi_t\}) = \int_0^1 [i(X_t)\omega]dt \in \mathrm{H}^1(M;\mathbb{R}).$$

Then it descends to a homomorphism

$$\widetilde{F} : \widetilde{\mathrm{Symp}_0}(M,\omega) \to \mathrm{H}^1(M;\mathbb{R}).$$

Denote by Γ_ω, which is called the flux group, the image of the kernel of the covering projection $\widetilde{\mathrm{Symp}_0}(M,\omega) \to \mathrm{Symp}_0(M,\omega)$, which is isomorphic to $\pi_1(\mathrm{Symp}_0(M,\omega))$, under $\widetilde{F}$. Then $\widetilde{F}$ induces the flux homomorphism

$$F : \mathrm{Symp}_0(M,\omega) \to \mathrm{H}^1(M;\mathbb{R})/\Gamma_\omega,$$

which is continuous with respect to the C^1-topology on the domain. It is known that $\mathrm{Ham}(M,\omega)$ coincides with the kernel of the flux homomorphism F. The flux conjecture stated in the introduction is equivalent to that Γ_ω is discrete in $\mathrm{H}^1(M;\mathbb{R})$ in the usual topology. In fact, this is the statement, which we proved in [5]. Hence, we have

$$\mathrm{Symp}_0(M,\omega)/\mathrm{Ham}(M,\omega) \cong \mathrm{H}^1(M;\mathbb{R})/\Gamma_\omega$$

is a Hausdorff space.

Consider the graph Gr_ψ of a symplectomorphism ψ as a Lagrangian submanifold in $(M \times M, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$. The diagonal $\Delta \subset M \times M$ is the graph of the identity. When $\psi \in \mathrm{Symp}_0(M,\omega)$, Gr_ψ is Lagrangian isotopic to Δ, i.e., $Gr_\psi \in \mathrm{Lag}(\Delta)$. Thus we can identify the space $\mathrm{Symp}_0(M,\omega)/\mathrm{Ham}(M,\omega)$ with $\mathrm{Lag}^{\pitchfork}(\Delta)/(\{id\} \times \mathrm{Ham}(M,\omega))$, where $\mathrm{Lag}^{\pitchfork}(\Delta)$ is the subspace of $\mathrm{Lag}(\Delta)$ consisting of L transverse to all fibers of the first and second factor projections. We may ask whether $\mathrm{Lag}(\Delta)/\mathrm{Ham}(M \times M, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$ is Hausdorff. This is a special case of Question 1.1. We come back to this question in the end of section 4.

Before proceeding further, we explain the C^1-topology on $\mathrm{Lag}(L)$ for the sake of reader's convenience. For $L' \in \mathrm{Lag}(L)$, we pick a tubular neighborhood U of L', which is certainly diffeomorphic to L, such that U is symplectomorphic to a neighborhood V of the zero section $O_{L'}$in T^*L' (Weinstein neighborhood). Let $f : U \to V$ be a symplectomorphism such that $f(L') = O_{L'}$. Denote by S_η the cross section of T^*L' corresponding to a 1-form on L'. Pick and fix a Riemannian metric on L'. Note that the C^1-norm of differential forms depends on the choice of Riemannian metrics and is not preserved by self-diffeomorphisms. However, the C^1-topology is certainly well-defined, when the manifold is compact. For a sufficiently small $\epsilon > 0$, we set

$$\mathcal{U}_\epsilon(L') = \{f^{-1}(S_\eta) | \eta \text{ is a closed 1-form on } L',\ \| \eta \|_{C^1} < \epsilon\}.$$

The family of such sets gives a basis of the C^1-topology on $\mathrm{Lag}(L)$. As a warm-up to Question 1.1, we discuss the case that the ambient symplectic manifold is exact, e.g., the total space of the cotangent bundle equipped with the standard symplectic form. Firstly, we have the following:

Proposition 2.1. *Let $(X, d\lambda)$ be an exact symplectic manifold and $i : L \to X$ a Lagrangian embedding. For any $\phi \in \mathrm{Ham}(X, d\lambda)$, we have*

$$[i^*\lambda] = [(\phi \circ i)^*\lambda] \in \mathrm{H}^1(L;\mathbb{R}).$$

In particular, if $i : L \to X$ is an exact Lagrangian embedding, i.e., $i^*\lambda$ is an exact 1-form, then $\phi \circ i$ is also an exact Lagrangian embedding.

Proof. Let $\{\phi_t^H\}$ be a Hamiltonian isotopy generated by $H = \{h_t\}$. Then we have

$$\begin{aligned} \frac{d}{dt}(\phi_t^H \circ i)^*\lambda &= i^* \circ \frac{d}{dt}\phi_t^{H*}\lambda \\ &= i^* \circ \phi_t^{H*} L_{X_{h_t}}\lambda \\ &= d\left(i^* \circ \phi_t^{H*}(i(X_{h_t})\lambda + h_t)\right). \end{aligned}$$

Thus $i^*\lambda$ and $(\phi \circ i)^*\lambda$ are cohomologous.

If $i^*\lambda$ is an exact 1-form, $(\phi_t^H \circ i)^*\lambda$ is also exact. Namely, $\phi_1^H \circ i$ is an exact Lagrangian embedding. □

Remark 2.1. If a symplectic manifold (P, Ω) satisfies $[\Omega] \in \mathrm{Im}\left(\mathrm{H}^2(P;\mathbb{Z}) \to \mathrm{H}^2(P;\mathbb{R})\right)$, there exists a so-called prequantum line bundle $L \to P$. A Lagrangian submanifold $L \subset P$ is called exact (a Bohr-Sommerfeld orbit), if L can be lifted to a Legendrian submanifold in the unit circle bundle $S(L)$. The above argument is easily extended to show that Hamiltonian deformations of exact Lagrangian submanifolds are exact.

Let $i_t : L \to X$ be a one-parameter family of exact Lagrangian embeddings. Then we can find a Hamiltonian isotopy $\{\phi_t^H\}$ of the ambient exact symplectic manifold such that $i_t = \phi_t^H \circ i$. In general, a one-parameter family of Lagrangian embeddings may not extend to a symplectic isotopy of the ambient symplectic manifold. We have the following:

Lemma 2.2. *Let (P, Ω) be a symplectic manifold and L an embedded closed connected Lagrangian submanifold. If the embedding $i : L \to P$ induces a surjective homomorphism $i^* : \mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L;\mathbb{R})$, then any one-parameter deformation of Lagrangian embedding i extends to an ambient symplectic isotopy of (P, Ω) starting from id. In other words, the action of $\mathrm{Symp}_0(P, \Omega)$ on $\mathrm{Lag}(L)$ is transitive.*

In the case of the zero section O_X in the cotangent bundle T^*X equipped with the standard symplectic form $d\lambda$, we have the following:

Proposition 2.3. *There is a canonical continuous bijection*

$$\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda) \cong \mathrm{H}^1(L;\mathbb{R}),$$

hence $\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda)$ is Hausdorff.

Proof. Firstly, we define $G : \mathrm{Lag}(O_X) \to \mathrm{H}^1(L;\mathbb{R})$ as follows. Let $L \in \mathrm{Lag}(O_X)$. For a loop γ in O_X, we pick a loop γ' in L such that γ and γ' are homologous in T^*X. Pick a 2-chain C such that $\partial C = \gamma' - \gamma$. Then we set

$$\begin{aligned} \langle G(L), [\gamma] \rangle &= \int_C d\lambda \\ &= \int_{\gamma'} \lambda - \int_\gamma \lambda. \end{aligned}$$

Let θ be a closed 1-form and $\{\psi_1^\theta\}$ the corresponding symplectic isotopy, i.e., the addition of $t\theta$ fiberwisely. Then $G\left(\psi_1^\theta(O_X)\right) = [\theta]$, so G is surjective.

Let $L', L'' \in \mathrm{Lag}(O_X)$ such that $G(L') = G(L'')$. Choose $\Theta' = \{\theta'_t\}, \Theta'' = \{\theta''_t\}$ such that $L' = \psi^{\Theta'}(O_X)$ and $L'' = \psi^{\Theta''}(O_X)$. Note that

$$G(L') = \int_0^1 [\theta'_t]dt \ \text{ and } \ G(L'') = \int_0^1 [\theta''_t]dt,$$

which are equal to each other. Consider $\Theta(s) = \{(1-s)\theta'_t + s\theta''_t\}$. Then the time-one maps $\psi_1^{\Theta(s)}$ are Hamiltonian isotopic. Therefore L' and L'' are Hamiltonian isotopic. Hence G induces a bijection $\overline{G} : \mathrm{Lag}(O_X) \to \mathrm{H}^1(L;\mathbb{R})$. Since G is continuous, $\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda)$ is Hausdorff. □

3. Floer theory for Lagrangian submanifolds

We summarize some results from [4], which are necessary for our later argument. In this section, a Lagrangian submanifold L, resp. a pair $(L^{(0)}, L^{(1)})$ of closed embedded Lagrangian submanifolds, is equipped with a relative spin structure (see §44 in [4]). For such an L, we constructed a filtered A_∞-algebra associated to L, which satisfies the gapped condition. Although it depends on various auxiliary choices, its homotopy type is uniquely determined (Theorems 10.11 and 14.1 in [4]). By the canonical model theorem (Theorem 23.2 in [4]), we can reduce the filtered A_∞-structure to the $\overline{\mathfrak{m}}_1$-cohomology, namely we have the following:

Theorem 3.1 (see Theorem A in [4])**.** *To each relatively spin Lagrangian submanifold L we can associate a structure of filtered A_∞-algebra $\{\mathfrak{m}_k\}$ on $\mathrm{H}^*(L;\Lambda_{0,nov})$, which is well-defined up to isomorphism. A symplectomorphism $\psi : (P, L) \to (P', L')$ induces an isomorphism of filtered A_∞-algebras, whose homotopy class depends only on ψ.*

Here $\Lambda_{0,nov}$ is a graded ring called the universal Novikov ring (see Definition 6.2 in [4]). We introduce two generators e and T such that $\deg e = 0$ and $\deg T = 0$.

$$\begin{aligned}
\Lambda_{nov} &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty (i \to +\infty)\} \\
\Lambda_{0,nov} &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i \geq 0\} \\
\Lambda_{0,nov}^+ &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i > 0\}
\end{aligned}$$

We will also use the subring consisting of elements of degree zero:

$$\Lambda_{nov}^{\deg=0} = \{\sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty (i \to +\infty)\}.$$

Let $(L^{(0)}, L^{(1)})$ be a relative spin pair of Lagrangian submanifolds, which is of clean intersection. We constructed a filtered A_∞-bimodule over the filtered A_∞-algebras associated to $L^{(1)}$ and $L^{(0)}$ (see Theorem F in [4]). However, in general, the Floer complex of

the pair $(L^{(0)}, L^{(1)})$ may not be constructed, since the bubbling-off of pseudo-holomorphic discs can happen as codimension one phenomena. We developed obstruction theory in order to define the Floer complex for a pair of Lagrangian submanifolds.

Theorem 3.2 (see Theorems B and G in [4])**.** *To each relatively spin Lagrangian submanifold $L \subset P$, we can associate a set $\mathcal{M}_{\mathrm{weak,def}}(L)$, which may be empty, and maps*

$$\begin{aligned} \pi_{\mathrm{amb}} &: \mathcal{M}_{\mathrm{weak,def}}(L) \to \mathrm{H}^2(P; \Lambda_{0,nov}) \\ \mathfrak{PO} &: \mathcal{M}_{\mathrm{weak,def}}(L) \to \Lambda_{0,nov}^+ \end{aligned}$$

with the following properties.
(1) A symplectomorphism $\psi : (P, L) \to (P', L')$ induces a map

$$\psi_* : \mathcal{M}_{\mathrm{weak,def}}(L) \to \mathcal{M}_{\mathrm{weak,def}}(L')$$

such that $\pi_{\mathrm{amb}} \circ \psi_* = (\psi^{-1})^* \circ \pi_{\mathrm{amb}}$ and $\mathfrak{PO} \circ \psi_* = \mathfrak{PO}$.
(2) For $(\mathbf{b}_1, \mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{\pi_{\mathrm{amb}}, \mathfrak{PO}} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)})$, we can associate the Floer cohomology $\mathrm{HF}^*\left((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{0,nov}\right)$. After extending the coefficient ring from $\Lambda_{0,nov}$ to Λ_{nov}, we have $\mathrm{HF}^*\left((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\right)$.
(3) Let $\{\phi_t^H\}$ be a Hamiltonian isotopy generated by H and

$$(\mathbf{b}_1, \mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{\pi_{\mathrm{amb}}, \mathfrak{PO}} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)}).$$

Then $\{\phi_t^H\}$ induces an isomorphism

$$(\{\phi_t^H\}, id)_* : \mathrm{HF}^*\left((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\right) \to \mathrm{HF}^*\left((\phi_1^H(L^{(1)}), \phi_{1*}^H \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\right).$$

Definition 3.1. A relatively spin Lagrangian submanifold L is said to be unobstructed after infinitesimal deformation, or bulk/boundary deformation, if $\mathcal{M}_{\mathrm{weak,def}}(L) \neq \emptyset$. The element $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ is called a solution of the Maurer-Cartan equation in the filtered A_∞-algebra associated to L. If there is $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ such that $\pi_{\mathrm{amb}}(\mathbf{b}) = 0$ and $\mathfrak{PO}(\mathbf{b}) = 0$, then we simply call L unobstructed.

Remark 3.1. In the case that the Maslov class vanishes on $\pi_2(P, L)$, the potential function $\mathfrak{PO}$ is automatically zero.

We have not yet found satisfactory sufficient condition for unobstructedness (after infinitesimal deformation), but have the following:

Theorem 3.3 (Theorem C in [4])**.** *There exist a series of positive integers m_k and classes*

$$[o_k^{2m_k}(L; \mathrm{weak, def})] \in \frac{\mathrm{H}^{2m_k}(L; \mathbb{Q})}{\mathrm{Im}\,(i^* \mathrm{H}^{2m_k}(P; \mathbb{Q}) \to \mathrm{H}^{2m_k}(L; \mathbb{Q}))},$$

$k = 1, 2, \ldots,$ (obstruction classes) such that if $[o_k^{2m_k}(L; \mathrm{weak, def})]$ are all zero, then $\mathcal{M}_{\mathrm{weak,def}}(L)$ is non-empty. The number $2 - 2m_k$ is a sum of the Maslov indices of a finite collection of homotopy classes in $\pi_2(P, L)$ realized by pseudo-holomorphic discs with respect to a compatible almost complex structure on P.

In particular, we find the following:

Corollary 3.4. *If* $i^* : \mathrm{H}^{2m_k}(P;\mathbb{Q}) \to \mathrm{H}^{2m_k}(L;\mathbb{Q})$ *is surjective,* $\mathcal{M}_{\mathrm{weak,def}}(L) \neq \emptyset$.

Corollary 3.5. *If the Maslov class* $\mu_L : \pi_2(P,L) \to \mathbb{Z}$ *is zero, all obstruction classes* $[o_k^{2m_k}(L;\mathrm{weak},\mathrm{def})]$ *lie in* $\mathrm{H}^2(L;\mathbb{Q})$.

We have the following non-vanishing result.

Theorem 3.6 (see Theorem E, Theorem 24.12 in [4])**.** *Suppose that* $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ *and the Maslov indices of all pseudo-holomorphic discs bounding* L *are non-positive. Then the Poincare dual* $PD[pt]$ *of the point class gives non-trivial element in the Bott-Morse Floer cohomology* $\mathrm{HF}^*\left((L,\mathbf{b})(L,\mathbf{b});\Lambda_{nov}\right)$. *The Poincare dual* $PD[L]$ *of the fundamental class of* L *can be deformed to a non-trivial element* $PD[L]'$ *of* $\mathrm{HF}^*\left((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{nov}\right)$ *such that* $PD[L]' \equiv PD[L]$ *mod* $\Lambda_{0,nov}^+$.

If the Maslov class μ_L vanishes on $\pi_2(P,L)$, we can equip L with a grading structure (a graded Lagrangian submanifold) in the sense of Kontsevich and Seidel [7]. We denote a graded Lagrangian submanifold by (L,σ). Changing the grading structure shifts the degree by a constant. For a relatively spin pair $(L^{(0)},L^{(1)})$ of graded Lagrangian submanifolds with $\mathbf{b}_i \in \mathcal{M}_{\mathrm{weak,def}}(L^{(i)})$, we can construct the Floer cohomology $\mathrm{HF}^*\left((L^{(1)},\mathbf{b}_1,\sigma_1),(L^{(0)},\mathbf{b}_0,\sigma_0);\Lambda_{nov}^{\deg=0}\right)$ over $\Lambda_{nov}^{\deg=0}$. We also have the Floer cohomology over $\Lambda_{0,nov}^{\deg=0} = \Lambda_{0,nov} \cap \Lambda_{nov}^{\deg=0}$. A Lagrangian isotopy $\{L_t\}$ induces the one-to-one correspondence between grading structures on $L = L_0$ to those on $L' = L_1$. All results above hold for these Floer cohomologies for graded Lagrangian submanifolds. Then the degree of $PD[pt]$, resp. $PD[L]'$, is n, resp. 0. Here $n = \dim L$.

4. Proof of Theorem 1.1

Let $L \subset P$ be a relatively spin Lagrangian submanifold and $L_* \in \mathrm{Lag}(L)$. It is obvious that L_* is diffeomorphic to L. Pick a tubular neighborhood U of L_*, which is symplectomorphic to a tubular neighborhood V of the zero section O_{L_*} of T^*L_*, i.e., there exists a symplectomorphism $f : (U,L_*) \to (V,O_{L_*})$. Then a Lagrangian submanifold L', which is sufficiently close to L_* in the C^1-topology, is described by a closed 1-form on L_*. If two Lagrangian submanifolds L_1, L_2 are sufficiently close to L_* in the C^1-topology and correspond to cohomologous closed 1-forms, L_2 can be realized by a Hamiltonian deformation of L_1. Hence small Lagrangian deformations of L' up to Hamiltonian deformations are parametrized by a small neighborhood $W \subset \mathrm{H}^1(L;\mathbb{R})$ of the origin. We announced the following statement in the case that $L_* = L$ in [6].

Proposition 4.1. *Let* $L \subset P$ *be a relatively spin Lagrangian submanifold such that the Maslov class* μ_L *vanishes on* $\pi_2(P,L)$ *and* $L_* \in \mathrm{Lag}(L)$. *Suppose that* L *is unobstructed after infinitesimal deformation, i.e., there exists* $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$. *Let* $\{\phi_t^H\}$ *be a Hamiltonian isotopy generated by* H. *Suppose that* L *and* $\phi_1^H(L)$ *are sufficiently close to* L_* *in the* C^1*-topology, hence* $f(L)$ *and* $f \circ \phi_1^H(L)$ *are the graphs of closed* 1*-forms* η *and* η' *on* L. *Then* η' *is cohomologous to* η.

Before giving the proof, we prepare some notation. Let θ be a closed 1-form on L_*. By abuse of notation, we denote by θ the closed 1-form $f^*\circ \mathrm{pr}^*(\theta)$ on the tubular neighborhood U of L_*, where pr is the projection of the cotangent bundle of L_*. Then denote by X^θ the vector field on U such that $i(X^\theta)\Omega = \theta$ and by $\{\psi_t^\theta\}$ the symplectic local flow generated by X^θ. Once a Riemannian metric is chosen on L_*, we have a one-to-one correspondence between vector fields and 1-forms. We denote by $\theta^\#$ the vector field such that

$$\langle \theta^\#, \cdot \rangle = \theta(\cdot).$$

Proof. Firstly, we note that $L' = \phi_1^H(L)$ is also unobstructed after infinitesimal deformation, i.e., $\mathbf{b}' = \phi_{1*}^H(\mathbf{b}) \in \mathcal{M}_{\mathrm{weak,def}}(L')$. We also have $\pi_{amb}(\mathbf{b}) = \pi_{amb}(\mathbf{b}')$ and $\mathfrak{PO}(\mathbf{b}) = \mathfrak{PO}(\mathbf{b}')$. (Under the assumption that the Maslov class vanishes on $\pi_2(P, L)$, the latter is zero.)

Since μ_L vanishes on $\pi_2(P, L)$, there is a grading structure σ on L. A Lagrangian deformation $\{L_t\}$ such that $L' = L_1$ induces the grading structure $\sigma^{\{L_t\}}$ on L'. In particular, the Hamiltonian isotopy $\{\phi_t^H\}$ induces the grading structure $\sigma^H = \sigma^{\{\phi_t^H(L)\}}$.

If L and L' are sufficiently C^1-close to L_*, we can regard L, resp. L', as the graph of a closed 1-form η, resp. η' in T^*L_*. Clearly $L' = \psi_1^{\eta'-\eta}(L)$. The Lagrangian deformation $\{\psi_t^{\eta'-\eta}(L)\}$ induces the grading structure $\sigma^{\eta,\eta'} = \sigma^{\{\psi_t^{\eta'-\eta}(L)\}}$ on L'. Two grading structures σ^H and $\sigma^{\eta,\eta'}$ may not coincide, hence the degrees in Floer theory may be shifted by a constant. Namely, there is an integer c such that $\mathrm{HF}^*\left((L', \mathbf{b}', \sigma^H), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right)$ is isomorphic to $\mathrm{HF}^{*+c}\left((L', \mathbf{b}', \sigma^{\eta,\eta'}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right)$ as graded $\Lambda_{nov}^{\deg=0}$-modules.

Now we explain, more precisely, the condition how close L and L' are to L_* in the C^1-topology. Namely, $\| \eta \|_{C^1}, \| \eta' \|_{C^1} < \epsilon$ for some $\epsilon > 0$, which we will specify later. Pick and fix a Riemannian metric on L_*. We may assume that $V \subset T^*L_*$ is the δ-neighborhood of the zero section O_{L_*} with respect to the induced metric. Denote by V' the $2\delta/3$-neighborhood of O_{L_*} and set $U' = f^{-1}(V')$. Pick an almost complex structure J compatible with Ω. By the monotonicity lemma for J-holomorphic curves, there exists a constant $e > 0$ such that $\int_S \Omega > e$ for any non-constant compact J-holomorphic curve in $P \setminus U'$ with $\partial S \subset \partial U'$. We will choose $\delta > 0$ sufficiently small later. Note that the constant $e > 0$ does not depend on the choice of δ, whenver $\partial U'$ is of contact type. (By the maximal principle, such a J-holomorphic curve is not contained in any Weinstein neighborhood of L_*.)

Choose a Morse function h on L_* such that $|dh| < \delta/3$. We may assume, without loss of generality, that h has a unique maximum and a unique minimum. We also assume that $\max h - \min h < e/4$. Next we pick contractible neighborhoods $W' \Subset W$ of the set of critial points of h and a smooth cut off function ρ such that the diameter of W is bounded by $1/\ell$ and

$$\rho = 1 \text{ on } W', \quad \rho = 0 \text{ on } L_* \setminus W, \quad |d\rho| < 2\ell,$$

where K is a positive number.

The first condition for $\epsilon > 0$ is

$$\epsilon < \frac{1}{100} \min_{L_* \setminus W'} |dh|. \tag{1}$$

In particular, this condition implies that $\epsilon < \delta/300$.

Suppose that the closed 1-forms η and η' satisfy that $\| \eta \|_{C^0}, \| \eta' \|_{C^0} < \epsilon$. Since W is contractible, we find that $(\eta' - \eta)|_W = dg$ for some smooth function g on W. We may assume that $|g| < 2\epsilon/\ell$, since $|dg| < 2\epsilon$ and the diameter of W is at most $1/\ell$. We regard $\rho \cdot g$ as a smooth function on L_*. Then $\eta'' = dh + \eta' - d(\rho \cdot g)$ is also a closed 1-form cohomologous to η'.

It is easy to see that

$$|\eta''| \leq |dh| + |\eta'| + |d(\rho \cdot g)| < |dh| + 7\epsilon,$$

which is smaller than $2\delta/3$. Thus $S_{\eta''}$ is contained in V'. It guarantees a Hamiltonian isotopy $\{\phi_t^K\}$, which maps $f^{-1}(S_{\eta'})$ to $f^{-1}(S_{\eta''})$, such that the support of K is contained in $U' \cong V'$. Note also that

$$|dh - (\eta'' - \eta)| < 6\epsilon.$$

By the construction, we find that S_η and $S_{\eta''}$ intersect transversally and their intersection points are identified with the critical points of h.

Since (1) holds on $L_* \setminus W'$ and $\eta'' - \eta = dh$ on W',

$$|dh - (\eta'' - \eta)| < \frac{1}{10}|dh|, \tag{2}$$

we find that

$$dh\left((\eta'' - \eta)^\#\right) \geq \frac{9}{10} dh\left((dh)^\#\right).$$

Hence the function h increases along integral curves of $(\eta'' - \eta)^\#$, i.e., h is a Liapunov function.

Now we give the second condition for ϵ through δ. Note that $\| \eta \|_{C^1}, \| \eta'' \|_{C^1} < 2\delta/3$. Pick $\delta > 0$ sufficiently small, there is a diffeomorphism Ψ supported in V' such that $\Psi(O_{L_*}) = S_\eta$ and $\Psi(S_{dh}) = S_{\eta''}$ with the following property.

- Let $\{J_t^h\}$ be the t-dependent family of almost complex structures on $V \subset T^*L_*$ associated to h, which Floer introduced in [2]. Ψ is C^1-close enough to the identity such that the family $\widehat{J} = \{\Psi_* J_t^h\}$ of almost complex structures is tamed by the canonical symplectic structure on T^*L_*.

We arrange the almost complex structure J on P, which may depend on $t \in [0,1]$, so that J coincides with $\widehat{J}$ on U'. Then all Floer connecting trajectories, i.e., J-holomorphic strip u bounded by $L = f^{-1}(S_\eta)$ and $L'' = f^{-1}(S_{\eta''})$, contained in U correspond to the bounded solution of

$$\dot{\gamma}(\tau) = (\eta'' - \eta)^\#\left(\gamma(\tau)\right).$$

Then the energy $E(u)$ of the Floer connecting trajectory u satisfies

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times[0,1]} u^*\omega_{\mathrm{can}} \\ &= \int_{-\infty}^{+\infty} \gamma^*(\eta''-\eta), \end{aligned}$$

where ω_{can} is the standard symplectic form on T^*L_*. By (2), we have

$$(\eta''-\eta)\left((\eta''-\eta)^{\#}\right) = |(\eta''-\eta)^{\#}|^2 < \frac{9}{8}dh\left((\eta''-\eta)^{\#}\right).$$

Thus we find that

$$E(u) < \frac{9}{8}\left(h(p^+)-h(p^-)\right) < \frac{e}{2}.$$

Here $p^{\pm} = \lim_{\tau\to\pm\infty}\gamma(\tau)$, which are critical points of h. Thus all Floer connecting trajectories for (L, L'') contained in U have symplectic area less than $e/2$. Note also that ∂U is a boundary of contact type.

We fix $\epsilon > 0$ satisfying the above conditions. Note that, if a Floer connecting trajectory is not contained in U, its symplectic area must be more than e.

Denote by $H\#K$ the time-dependent Hamiltonian, which generates the concatenation of $\{\phi_t^H\}$ and $\{\phi_t^K\}$. Write $(L'', \mathbf{b}'', \sigma'') = (\phi_1^{H\#K}(L), \phi_{1*}^{H\#K}\mathbf{b}, \phi_{1*}^K\sigma^{\eta,\eta'})$. Now we consider the Floer complex $\mathrm{CF}^*\left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right)$. The Floer coboundary operator $\delta^{\mathbf{b}'',\mathbf{b}}$ decomposes into

$$\delta^{\mathbf{b}'',\mathbf{b}} = \delta_{<e}^{\mathbf{b}'',\mathbf{b}} + \delta_{\geq e}^{\mathbf{b}'',\mathbf{b}}.$$

Here $\delta_{<e}^{\mathbf{b}'',\mathbf{b}}$ counts Floer connecting trajectories of relative index 1 and with energy less than e, i.e, those contained in U, while $\delta_{\geq e}^{\mathbf{b}'',\mathbf{b}}$ is other contributions to $\delta^{\mathbf{b}'',\mathbf{b}}$.

The operator $\delta_{<e}^{\mathbf{b}'',\mathbf{b}}$ can be identified with the coboundary operator of the Novikov complex for the closed 1-form $\eta''-\eta$.

Denote by p_{min} and p_{max} the unique minimum and maximum of h, hence, the unique zero of $\eta''-\eta$ of index 0, n, respectively. If $\eta''-\eta$, hence $\eta'-\eta$, is not exact, the Novikov cohomology $\mathrm{HN}^*(\eta''-\eta)$ vanishes in degrees $* = 0, n$. Let $\overline{p}_{min}, \overline{p}_{max}$ be lifts of p_{min}, p_{max} to the abelian covering space of L_* associated to the de Rham cohomology class $[\eta''-\eta] = [\eta'-\eta]$. Hence $\delta_{<e}^{\mathbf{b}'',\mathbf{b}}\overline{p}_{min} \neq 0$, and there are $\overline{q}_i$ of index n such that $\delta_{<e}^{\mathbf{b}'',\mathbf{b}}(\sum_i a_i\overline{q}_i) = \overline{p}_{max}$.

Since p_{min} is the unique zero of $\eta''-\eta$ with index 0 and the Floer coboundary operator $\delta^{\mathbf{b}'',\mathbf{b}}$ is linear over the field $\Lambda_{nov}^{\deg=0}$, there does not exist a non-trivial Floer cocycle of degree 0.

Note also that p_{max} is the unique zero of $\eta''-\eta$ with index n. Using the energy filtration, we can find that any Floer cocycle of degree n is a Floer coboundary. Therefore we find that

$$\mathrm{HF}^*\left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right) = 0$$

for $* = 0, n$. Because the Maslov class μ_L vanishes on $\pi_2(P, L)$, the degree of $L \cap L''$ is determined by the index of zeros of $\eta'' - \eta$. Hence the degrees are between 0 and n. Combining these, we find that if

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \neq 0$$

for $* = k, \ell$, then $|k - \ell| < n - 1$.

By the hypothesis that μ_L vanishes on $\pi_2(P, L)$, the Maslov indices of all pseudo-holomorphic discs are zero. Thus Theorem 3.6 guarantees existence of non-trivial classes $PD[pt]$ and $PD[L]'$ in $\mathrm{HF}^* \left((L, \mathbf{b}, \sigma), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right)$ with $* = n, 0$, respectively.

By the graded Lagrangian version of Theorem 3.2 (3), we find that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \cong \mathrm{HF}^* \left((L, \mathbf{b}, \sigma), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right).$$

Therefore there are k and ℓ such that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \neq 0$$

for $* = k, \ell$ and $k - \ell = n$.

Although the grading structures $\sigma^{H\#K}$ and σ'' may be different, the Floer connecting trajectories do not depend on the choice of a grading structure and the resulting cohomology groups as graded modules are isomorphic up to a degree shift. Namely, there is an integer c such that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{\ H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \cong \mathrm{HF}^{*+c} \left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right).$$

Hence we obtain a contradiction. □

Now we are ready to finish

Proof of Theorem 1.1. Recall that $f : U \to V$ is a Weinstein neighborhood of L_*. Set

$$\mathcal{U}_\epsilon = \{ f^{-1}(S_\theta) \mid \theta \text{ is a closed 1-form on } L_*, \quad \| \theta \|_{C^1} < \epsilon \},$$

which is a neighborhood of L_* in $\mathrm{Lag}(L)$. Then Proposition 4.1 implies that

$$\mathrm{Ham}(P, \Omega) \cdot L \cap \mathcal{U}_\epsilon$$

consists of $f^{-1}(S_\theta)$ such that $\| \eta \|_{C^1} < \epsilon$ and θ belongs to a unique cohomology class $a \in \mathrm{H}^1(L_*; \mathbb{R})$.

If L and L_* belong to different $\mathrm{Ham}(P, \Omega)$-orbits, then the class a is non-zero. Then there exists a constant $d > 0$ such that $\| \theta \|_{C^0} > d$, hence $\| \theta \|_{C^1} > d$, for any θ representing the class a. If $\epsilon < d$, then

$$\mathrm{Ham}(P, L) \cdot L \cap \mathcal{U}_\epsilon = \emptyset.$$

Therefore $\mathrm{Ham}(P, L) \cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology. □

Remark 4.1. The above proof is similar to the proof of Theorem 4.3 in [9].

By Corollary 3.5, if a relatively spin Lagrangian submanifold L satisfies that

$$\mathrm{H}^2(L;\mathbb{Q}) = 0$$

and the Maslov class vanishes on $\pi_2(P, L)$, L is unobstructed. Hence Corollary 1.2 follows from Theorem 1.1.

Restrict ourselves to unobstructed Lagrangian submanifolds and denote by $\mathrm{Lag}^{\mathrm{unob}}(L)$ the subspace of unobstructed Lagrangian submanifolds in $\mathrm{Lag}(L)$. Note that this space may no longer connected. It may be more natural to ask the following:

Question 4.1. Is $\mathrm{Lag}^{\mathrm{unob}}(L)/\mathrm{Ham}(P,\Omega)$ Hausdorff?

Proof of Theorem 1.3. Let $L_1, L_2 \in \mathrm{Lag}(L)$ such that $\phi(L_1) \neq L_2$ for any $\phi \in \mathrm{Ham}(P,\Omega)$. We will find neighborhoods $\mathcal{U}_i$ of L_i in the C^1-topology such that $\mathrm{Ham}(P,\Omega)\cdot\mathcal{U}_i$, $i = 1, 2$, are disjoint, which is equivalent to that

$$\mathcal{U}_1 \cap \mathrm{Ham}(P,\Omega)\cdot\mathcal{U}_2 = \emptyset. \tag{3}$$

Let $f_i : U(L_i) \to V(O_{L_i})$ be Weinstein's standard form around the Lagrangian submanifold L_i. We assume that f_i, $i = 1, 2$, extend to slightly larger Weinstein neighborhoods. (This condition makes the smooth extension of θ_i to $\widetilde{\theta}_i$ exist in a later argument.) Pick and fix a Riemannian metric on L_i. We set

$$\mathcal{V}_{i,\epsilon} = \{S_\theta \mid \theta \text{ is a closed 1-form on } L_i,\ \parallel \theta \parallel_{C^1(\Omega^1(L_i))} < \epsilon\}.$$

Pick $\overline{\epsilon}_i > 0$ such that $V(O_{L_i})$ contains the $\overline{\epsilon}_i$-neighborhood of O_{L_i} with respect to the induced metric. By Theorem 1.1, there is small $\epsilon_1 > 0$ such that $2\epsilon_1 < \overline{\epsilon}_1$ and

$$\mathrm{Ham}(P,\Omega)\cdot L_2 \cap \widetilde{\mathcal{U}}_1 = \emptyset,$$

where

$$\widetilde{\mathcal{U}}_1 = \{f^{-1}(S) \mid S \in \mathcal{V}_{1,2\epsilon_1}\}.$$

Since L_1 and L_2 are the images of isotopic embeddings, we find that

$$\ker\left(\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_1;\mathbb{R})\right) = \ker\left(\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_2;\mathbb{R})\right).$$

We denote by $\mathcal{K}$ this submodule of $\mathrm{H}^1(P;\mathbb{R})$. Although there are no canonical choice of parametrizing L_i, $i = 1, 2$, we can identify $\mathrm{H}^1(L_1;\mathbb{R})$ and $\mathrm{H}^1(L_2;\mathbb{R})$ through $\mathrm{H}^1(P;\mathbb{R})/\mathcal{K}$.

Choose a neighborhood $W \subset \mathrm{H}^1(P;\mathbb{R})/\mathcal{K}$, which we identify with $\mathrm{H}^1(L_i,\mathbb{R})$, $i = 1, 2$, of the origin such that any cohomology class in W is represented by some θ on L_1 with $\parallel \theta \parallel_{C^1(\Omega^1(L_1))} < \epsilon_1$. Since the de Rham cohomology class is determined by the integration on cycles generating the first homology group, we can also find a sufficiently small $\epsilon_2 > 0$ such that $\epsilon_2 < \overline{\epsilon}_2$ and $[\theta] \in W$ for any closed 1-form θ on L_2 with $\parallel \theta \parallel_{C^1(\Omega^1(L_2))} < \epsilon_2$.

Write

$$\mathcal{U}_i = \{f^{-1}(S) \mid S \in \mathcal{V}_{i,\epsilon_i}\},$$

and prove (3). Suppose that there exist $L' \in \mathcal{U}_1$, $L'' \in \mathcal{U}_2$ and $\phi \in \mathrm{Ham}(P,\Omega)$ such that $L' = \phi(L'')$. Let θ_i be closed 1-form on L_i such that $f_1(L') = S_{\theta_1}$ and $f_2(L'') = S_{\theta_2}$, in particular $\parallel \theta_i \parallel_{C^1(\Omega^1(L_i))} < \epsilon_i$. We extend θ_i to the tubular neighborhoods $U(L_i)$

by $f_i^* \circ \pi_i^*(\theta_i)$, where π_i is the projection of the cotangent bundle of L_i. Using the hypothesis that $\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_i;\mathbb{R})$ is surjective, we extend them to closed 1-forms $\widetilde{\theta}_i$ on P. Note that $\psi_1^{\widetilde{\theta}_2}(L_2) = L''$. By the choice of ϵ_2, the cohomology class $[\theta_2]$ belongs to W. Since ϕ is isotopic to the identity, $[\phi^*(\widetilde{\theta}_2)] \mod \mathcal{K}$ belongs to W. Pick a closed 1-form θ_2' on L_1 in this class such that $\| \theta_2' \|_{C^1(\Omega^1(L_1))} < \epsilon_1$. Then extend it to $\widetilde{\theta}_2'$ on P such that $[\widetilde{\theta}_2'] = [\widetilde{\theta}_2] \in \mathrm{H}^1(P;\mathbb{R})$ and $\widetilde{\theta}_2'$ coincides with $f_1^* \circ \pi_1^*(\theta_2')$ on $U(L_1)$. Note that $\| \theta_1 - \theta_2' \|_{C^1(\Omega(L_1))} < 2\epsilon$. Since $\phi \in \mathrm{Ham}(P,\Omega)$ and $[\widetilde{\theta}_2'] = [\widetilde{\theta}_2]$, $f_1^{-1}(S_{\theta_1 - \theta_2'}) = \psi_1^{\widetilde{\theta}_1 - \widetilde{\theta}_2'}(L_1)$ is Hamiltonian isotopic to L_2. It implies that there is a Hamiltonian isotopy $\{\phi_t^H\}$ such that $\phi_1^H(L_2) \in \widetilde{\mathcal{U}}_1$, which is a contradiction. □

Under the hypothesis that $\mathrm{H}^*(P;\mathbb{R}) \to \mathrm{H}^*(L;\mathbb{R})$ is surjective, all $L' \in \mathrm{Lag}(L)$ are unobstructed, i.e., $\mathrm{Lag}(L) = \mathrm{Lag}^{\mathrm{unob}}(L)$. Namely, for any $L' \in \mathrm{Lag}(L)$, Lemma 2.2 states that $\psi \in \mathrm{Symp}_0(P,\Omega)$ such that $L' = \psi(L)$. Let $\mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L)$. Then $\psi_* \mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L')$, i.e., L' is unobstructed. Combining the argument in [5], we obtain the following:

Theorem 4.2. *Let $i : L \subset P$ be a relatively spin Lagrangian submanifold such that $i^* : \mathrm{H}^*(P;\mathbb{R}) \to \mathrm{H}^*(L;\mathbb{R})$ is surjective. Then $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff with respect to the C^1-topology.*

Remark 4.2. In the case that $\mathrm{H}^2(P;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(L;\mathbb{Z}/2\mathbb{Z})$ is surjective, L is a relatively spin submanifold. In particular, the diagonal set $\Delta \subset M \times M$ satisfies the assumption in Theorem 4.1. Hence $\mathrm{Lag}(\Delta)/\mathrm{Ham}(M^{\times 2}, \tilde{\omega})$ is Hausdorff.

Sketch of the proof. We will show that the $\mathrm{Ham}(P,\Omega)$-orbits are closed in the C^1-topology. Then the Hausdorff property follows as in the proof of Theorem 1.2.

Firstly, we note that L is unobstructed after infinitesimal deformation by Corollary 3.4. Pick $\mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L)$. Suppose that there exists $\phi \in \mathrm{Ham}(P,\Omega)$ such that L and $L' = \phi(L)$ are sufficiently close to $L_* \in \mathrm{Lag}(L)$. Let $\mathcal{U}$ be as in the proof of Theorem 1.1. Then L, resp. L', correspond to a closed 1-form η, resp., η' on L_* such that $\| \eta \|_{C^1(\Omega^1(L_*))}, \| \eta' \|_{C^1(\Omega^1(L_*))} < \epsilon$. For a closed 1-form θ on L_* denote by $H_\theta \to L$ a flat line bundle corresponding to $[\theta] : \pi_1(L) \to \mathbb{R}$.

Theorem 3.2 also holds in the case of twisted by flat bundles on $L^{(i)}$. We use the flat line bundle $H_{\epsilon(\eta'-\eta)}$ on L and the trivial bundle $\underline{\mathbb{R}}$ on L'. (We identify L and L_* by the projection $L \subset U \to L_*$ and regard η, η' as closed 1-forms on L.) Thus we have

$$\mathrm{HF}^*\left((L', \mathbf{b}', \underline{\mathbb{R}}), (L, \mathbf{b}, H_{\epsilon(\eta'-\eta)}); \Lambda_{nov}\right) \cong \mathrm{HF}^*\left((L, \mathbf{b}, \underline{\mathbb{R}}), (L, \mathbf{b}, H_{\epsilon(\eta'-\eta)}); \Lambda_{nov}\right).$$

While the right hand side is described by ordinary cohomology of L with coefficients in $H_{\epsilon(\eta'-\eta)}$, the left hand side is described by Novikov cohomology with coefficients in $H_{\epsilon(\eta'-\eta)}$, cf. Theorem H, Theorem 24.5 in [4]. Then we can derive a contradiction in a similar way as in [5]. □

Acknowledgements: The author wishes to thank his collaborators K. Fukaya, Y.-G. Oh and H. Ohta of [4], on which the argument in this note heavily based. In particular, he thanks K. Fukaya for stimulating discussion.

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