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Tight contact structures on the Weeks manifold

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Abstract. We construct tight contact structures on the Weeks manifold W, the closed hyperbolic 3–manifold with smallest volume. The contact structures are constructed through contact surgery diagrams involving contact (± 1)–surgeries. The tightness of these contact structures are verified by showing that the contact Ozsváth–Szabó invariants of the structures are nonzero.

1. Introduction

Suppose that Y is a closed oriented 3–manifold and ξ is a 2–plane field on Y which can be given as the kernel of a 1–form α satisfying $\alpha \wedge d\alpha > 0$. Such ξ is called a (positive, cooriented) *contact structure* on Y. A contact structure ξ is *tight* if Y does not contain any embedded disk D with the property that ξ is tangent to D along ∂D. Such disks are called *overtwisted disks*, and a contact structure containing such a disk is called overtwisted. According to the famous result of Eliashberg [5], overtwisted contact structures are determined by the homotopy type of the 2–plane field, hence do not capture geometric information of the underlying 3–manifold. Tight contact structures, however, are more geometric objects, and play prominent role in the study of low dimensional manifolds.

It is known that on a connected sum $Y_1 \# Y_2$ tight structures decompose uniquely as tight structures on the factors Y_1, Y_2. Therefore in understanding existence and classification questions we can restrict our attention to prime 3–manifolds. According to the solution of the Geometrization Conjecture, a prime 3–manifold Y either admits a hyperbolic metric, is Seifert fibered, or contains an essential torus (i.e., a 2–torus $T^2 \subset Y$ such that the embedding induces an injective homomorphism on the fundamental groups). According to [2, 8] a *toroidal* 3–manifold (i.e., a 3–manifold containing an essential torus) admits infinitely many different tight contact structures. (Moreover, by [2] an atoroidal manifold admits at most finitely many isotopy classes of tight contact structures.) The existence problem for Seifert fibered 3–manifolds was recently answered:

Theorem 1.1 ([14]). *A Seifert fibered 3–manifold admits a tight contact structure if and only if it is not orientation preserving diffeomorphic to the result of* $(2n-1)$*–surgery along the* $(2,2n+1)$*–torus knot* $T_{2,2n+1}\subset S^3$ *for some* $n\in\mathbb{N}$. □

The proof of this theorem utilized the explicite surgery presentation of the Seifert fibered 3–manifolds. Using a surgery presentation, we were able to modify it to a contact surgery presentation of the same 3–manifold, where contact (±1)–surgeries were performed along Legendrian knots in the standard contact 3–sphere. (For more on contact surgery see [3, 11].) Tigthness of the resulting contact structures were verified by the computation of the corresponding contact Ozsváth–Szabó invariants — their nonvanishing implied that the surgery diagrams gave rise to tight structures.

This approach cannot be applied for hyperbolic 3–manifolds in general. In this note we show that the Weeks manifold W, which is known to be the closed hyperbolic 3–manifold with smallest hyperbolic volume [6], admits tight contact structures. Recall that W is defined as surgery along the Whitehead link, with surgery coefficients $(5,\frac{5}{2})$. Following the convention of [1] we take the clasp at the Whitehead link in such a way that $(+1)$–surgery on one of its components turns the other one into the right–handed trefoil knot, while (-1)–surgery on the same component turns the other into the Figure–8 knot. For the surgery presentation of W see Figure 1, cf. also [4, page 249].

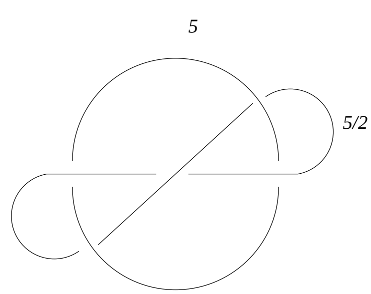

FIGURE 1. The Weeks manifold, presented as surgery along the Whitehead link

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2. Contact structures on the Weeks manifold W

In the following, through a sequence of surgery diagrams we describe contact structures on W with both of its orientations. Let us start with $-W$ (where the orientation of W is given by the diagram of Figure 1). A diagram for $-W$ can be given by considering the mirror image of the link of Figure 1, with the framings multiplied by (-1), as it is given by Figure 2. After an inverse slam dunk (cf. [7]) and an isotopy of the projection we can put these knots in Legendrian position in a way that a sequence of contact (-1)–surgeries is needed (when surgery coefficient is measured with respect to the contact framing), cf. Figure 2. Since on the

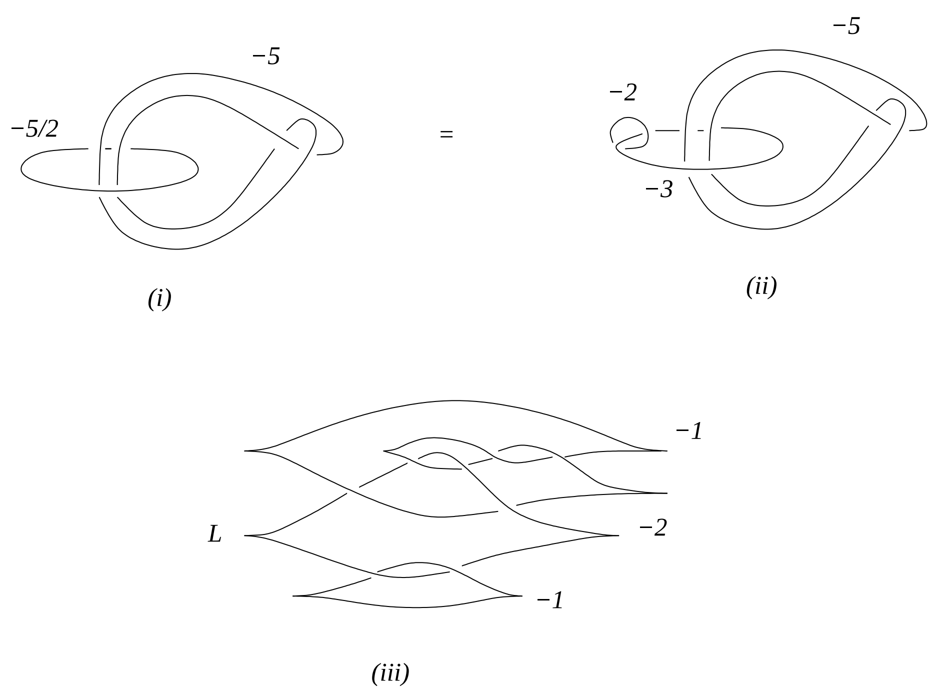

FIGURE 2. Stein fillable contact structures on $-W$

Legendrian knot L of Figure 2(iii) we are free to choose the stabilization either on the right or on the left, we constructed two nonisotopic Stein fillable contact structures, which can be distinguished by the first Chern classes of their Stein fillings, cf. [9].

Now we turn our attention to the Weeks manifold W with the orientation given by Figure 1. A sequence of inverse slam dunks and blow–downs (cf. [7]) turns this diagram into the one given by Figure 3(iii). Isotoping the link to Legendrian

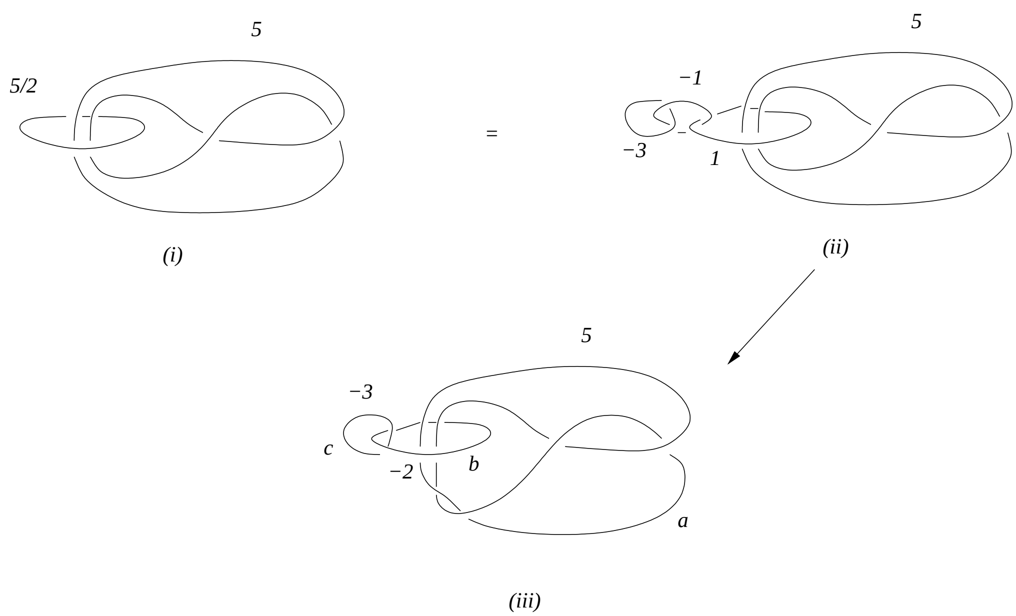

Figure 3. Another surgery diagrams for the Weeks manifold

position we get Figure 4. Now the standrad algorithm for turning a positive integer contact surgery into a sequence of contact (-1)–surgeries and a contact $(+1)$–surgery (as it is described in detail in [3]) yields contact surgery diagrams for W:

- stabilize the Legendrian unknot of Figure 4 with contact surgery coefficient (-2) once (either on the left or on the right side) and perform contact (-1)–surgery on the result, and

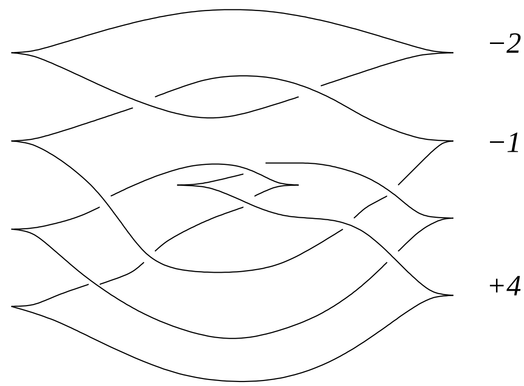

FIGURE 4. Contact surgery diagram for W

- consider the Legendian push-off of the $(+4)$–framed curve, stabilize it once (either on the right or on the left), and then push off the result two more times; perform contact $(+1)$–surgery on the orginial knot (which we will call L), while on the push–offs perform contact (-1)–surgeries.

Notice that in this algorithm we have two choices on whether we put the stabilizations to the left or to the right on the two curves, so in fact we produce four diagrams and hence four contact structures $\xi_{i,j}$ where $i, j \in \{l, r\}$ (corresponding to left or right stabilizations).

3. Contact Ozsváth–Szabó invariants

In [15, 16] Ozsváth and Szabó defined an invariant of a spin^c 3–manifold $(Y, \mathbf{t})$, which is now called the *Ozsváth–Szabó homology group* $\widehat{HF}(Y, \mathbf{t})$ of $(Y, \mathbf{t})$. The theory (which, for the sake of simplicity we consider here with $\mathbb{Z}_2$–coefficients) fits into the framework of a Topological Quantum Field Theory, in the sense that 4–dimensional spin^c cobordisms between spin^c 3–manifolds induce homomorphisms between the corresponding Ozsváth–Szabó homology groups. Moreover, these groups and maps provide exact triangles for surgery triples, as it is explained in [16], cf. also [10]. In addition, in [17] an invariant $c(Y, \xi)$ of a contact 3–manifold (Y, ξ) as an element of the group $\widehat{HF}(-Y, \mathbf{t}_\xi)$ is defined. (Here $\mathbf{t}_\xi$ denotes the spin^c structure induced by the contact structure ξ.) This invariant has the remarkable property that it vanishes for overtwisted contact strucutres (hence can be used to detect tightness), it is nonzero for Stein fillable contact structures and satisfies the following simple transformation rule: If (Y_L, ξ_L) is given as contact $(+1)$–surgery along the Legendrian knot $L \subset (Y, \xi)$, and the surgery induces the oriented cobordism X then

$$c(Y_L, \xi_L) = F_{-X}(c(Y, \xi)),$$

where F_{-X} is the map induced by the cobordism X, with its orientation reversed.

These invariants can be effectively computed in many cases when the contact structure is given through a contact surgery presentation, cf. [10, 11, 12]. Next we will determine the contact Ozsváth–Szabó invariants for the contact structures defined by the surgery diagrams we found on the Weeks manifold.

Theorem 3.1. *The contact structures $\xi_{i,j}$ $(i, j \in \{l, r\})$ defined by Figure 4 have nonvanishing contact Ozsváth–Szabó invariants, hence are all tight. In addition, the four structures are nonisotopic.*

Proof. Let us consider the single curve L of the contact surgery diagram defining ξ_{ij} on which contact $(+1)$–surgery is performed. Recall that by the recipe of [3], this curve results from the $(+4)$–framed Legendrian knot of Figure 4. It is not hard see that L is smoothly isotopic to the right–handed trefoil knot, and that $tb(L) = 1$. Therefore the main result of [10] (cf. also [13]) applies and shows that the contact Ozsváth–Szabó invariant of the resulting contact structure $(S^3_2(L), \xi_L)$ is nonzero. More explicitely, consider the cobordism X given by the handle attachment induced by the single surgery along L. The map F_{-X} fits in the exact triangle

$$\begin{array}{ccc} \widehat{HF}(S^3) & \xrightarrow{F_{-X}} & \widehat{HF}(S^3_{-2}(\overline{L})) \\ & \nwarrow \quad \swarrow & \\ & \widehat{HF}(S^3_{-1}(\overline{L})) & \end{array}$$

where $\overline{L}$ is the mirror image of L (hence smoothly it is the left–handed trefoil knot). It is not hard to see that $\widehat{HF}(S^3_{-n}(\overline{L}) = \widehat{HF}(S^3_n(L)) = \mathbb{Z}_2^n$ for all $n > 0$, and it is known that $\widehat{HF}(S^3) = \mathbb{Z}_2$. Since the exactness of the triangle

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{F_{-X}} & \mathbb{Z}_2^2 \\ & \nwarrow \quad \swarrow & \\ & \mathbb{Z}_2 & \end{array}$$

implies the injectivity of F_{-X}, the nonvanishing of the contact invariant $c(S^3, \xi_{st})$ of the standard contact 3–sphere and the identity $F_{-X}(c(S^3, \xi_{st})) = c(S^3_2(L), \xi_L)$ verifies the nonvanishing of $c(S^3_2(L), \xi_L)$.

Since the contact structures ξ_{ij} constructed on W are all contact (-1)–surgeries on ξ_L, the nonvanishing follows from [11, Corollary 3.6]. In fact, since a contact (-1)–surgery can be cancelled by a contact $(+1)$–surgery along the Legendrian push–off of the curve at hand [3, 18], we see that $(S^3_2(L), \xi_L)$ can be given as a sequence of contact $(+1)$–surgeries on $(W, \xi_{i,j})$. Therefore, for the corresponding cobordism U (induced by the surgeries viewed as handle attachments) we get

$$F_{-U}(c(W, \xi_{i,j})) = c(S^3_2(L), \xi_L),$$

and since the image is nonzero, we conclude that $c(W, \xi_{i,j}) \neq 0$, implying that these structures are all tight.

Finally, we show that the structures $\xi_{i,j}$ represent different spin^c structures, hence are nonisotopic. To this end, consider Figure 3(iii). Let the normal circle to the surgery curves a, b, c be denoted by μ_a, μ_b, μ_c. Then it is easy to see (cf. [7]) that $H_1(W; \mathbb{Z})$ can be presented as

$$\begin{aligned} H_1(W; \mathbb{Z}) &= \langle \mu_1, \mu_b, \mu_c \mid 5\mu_a = 0, -2\mu_b + \mu_c = 0, -3\mu_c + \mu_b = 0\rangle \\ &= \langle \mu_a, \mu_c \mid 5\mu_a = 5\mu_c = 0\rangle. \end{aligned}$$

From the surgery presentation of $\xi_{i,j}$ we get a 4–manifold X and a cohomology class $K \in H^2(X; \mathbb{Z})$, evaluating on the generators of the second homology corresponding to the surgery curves as their rotation numbers. As it is explained in [3], this characteristic cohomology class determines a spin^c structure $\mathbf{s}$ on X, with the property that the restriction of $\mathbf{s}$ to the boundary is exactly the spin^c structure of the contact structure given by the diagram. Suppose now that a choice of stabilizations is fixed, in other words, we picked one of $\xi_{i,j}$ $(i, j \in \{l, r\})$. After elementary Kirby calculus we can arrive to the diagram of Figure 3(iii), with the corresponding cohomology class K' evaluating on c as ± 1 (depending on the choice of the zig-zag), on b as 0, and on a as ± 3. Consequently the Poincaré dual of $c_1(\xi_{i,j})$, which is now an element of $H_1(W; \mathbb{Z})$ can be given in the presentation discussed above as $\pm 3a \pm c$. Since these elements are all different in $H_1(W; \mathbb{Z})$ we get that the contact structures $\xi_{i,j}$ induce different spin^c structures, hence are nonisotopic. $\square$

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