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Exotic embeddings of $\#6\mathbb{R}\mathrm{P}^2$ in the 4-sphere

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Abstract. We construct an infinite sequence of smooth embeddings of $\#6\mathbb{R}\mathrm{P}^2$ in S^4, which are all ambient homeomorphic, but pairwise ambient non-diffeomorphic. The double covers of S^4 ramified along these surfaces form a family of the exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}^2}$ constructed by Park, Stipsicz and Szabó.

1. Introduction

The main goal of this paper is the following result.

Theorem 1.1. *For any $k \geq 6$ there exists an infinite family of smoothly embedded surfaces $F_i \subset S^4$, $i = 1, 2, \dots,$ homeomorphic to $F = \#k\mathbb{R}\mathrm{P}^2$ (connected sum of k copies of $\mathbb{R}\mathrm{P}^2$) and with the normal Euler number $F^2 = 2k - 4$, such that*

(1) *the pairs (S^4, F_i) are all homeomorphic; the ambient homeomorphisms can be assumed to be diffeomorphisms in some neighborhoods of F_i;*
(2) *(S^4, F_i) are all pairwise non-diffeomorphic.*

Theorem 1.1 improves the result of [FKV1]-[FKV2], where a similar family of $F_i \subset S^4$ was constructed for $k = 10$. Our construction of F_i is based on similar ideas and makes use of the examples of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ in [PSS]. More precisely, our goal can be stated as follows.

Theorem 1.2. *There exists an infinite family of smoothly embedded surfaces $F_i \subset S^4$, $i = 1, 2, \dots$ which are all homeomorphic to a connected sum $\#6\mathbb{R}\mathrm{P}^2$, such that $\pi_1(S^4 \smallsetminus F_i) = \mathbb{Z}/2$, and the double covers $X_i \to S^4$ branched along F_i provide an infinite family of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ constructed in* [PSS].

Recall the well-known diffeomorphism $\mathbb{C}\mathrm{P}^2/\mathrm{conj} = \mathrm{S}^4$. The image of $\mathbb{R}\mathrm{P}^2$ in the quotient-space S^4 represents an isotopy class of *unknotted embeddings* of $\mathbb{R}\mathrm{P}^2$ with the normal Euler number -2. Another isotopy class of unknotted embeddings (with the normal Euler class 2) is obtained by reversing the orientation of S^4. It is represented by the image of $\mathbb{R}\mathrm{P}^2$ in $S^4 = \overline{\mathbb{C}\mathrm{P}}^2/\mathrm{conj}$, which will be denoted $\overline{\mathbb{R}\mathrm{P}}^2 \subset S^4$. A non-orientable surface $F \subset S^4$ will be called *unknotted* if it splits into an ambient connected sum of such unknotted embeddings, that is $F = p\mathbb{R}\mathrm{P}^2\#\mathrm{q}\overline{\mathbb{R}\mathrm{P}}^2$.

Key words and phrases. Knotted surfaces, equivariant surgery, exotic smooth structures.

1.1. Theorem 1.2 implies Theorem 1.1

It is proven in [PSS] that X_i are pairwise non-diffeomorphic. This implies that the pairs (S^4, F_i) are non-diffeomorphic to each other. It is known moreover that for any $\ell \geq 1$, $X_i \# \ell \overline{\mathbb{CP}}^2$, $i = 1, 2, \dots$, remain pairwise non-diffeomorphic, which implies that $F_i \# \ell \overline{\mathbb{RP}}^2 \subset S^4$ are also all ambient non-diffeomorphic to each other.

The values $F_i^2 = 8$ of the normal Euler numbers for F_i in Theorem 1.2 can be obtained from the the signature formula $\sigma(X_i) = -4 = 2\sigma(S^4) - \frac{1}{2}F_i^2$. For the connected sum $F_i \# \ell \overline{\mathbb{RP}}^2$ the Euler number becomes $8 + 2\ell$, that is $2k - 4$ where $k = \ell + 6$.

The obstructions for pairwise ambient homeomorphism of surfaces F_i were analyzed in [FKV2], Proposition 3, where it was shown that they belong to some finite set (it is crucial for this proof that $\pi_1 = \mathbb{Z}/2$). This implies that we can choose an infinite subsequence of pairwise ambient homeomorphic surfaces F_i in the infinite sequence of pairwise non-diffeomorphic ones (which are covered by the corresponding exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$). □

Remark 1.1. In [K], it is shown that the finite ambiguity observed in [FKV2] for the exotic $\#10\mathbb{RP}^2$ actually vanishes. This means that all the examples of embedded $\#10\mathbb{RP}^2$ that were constructed in [FKV2] are actually homeomorphic to an unknotted $\mathbb{RP}^2 \# 9\overline{\mathbb{RP}}^2$. Formally speaking the arguments in [K] concern only the case of $\#10\mathbb{RP}^2$, and it is not clear if they can be appropriately modified for our case of $\#6\mathbb{RP}^2$. If it were possible, then all the examples of F_i in Theorem 1.2 would be actually ambient homeomorphic to an unknotted $\mathbb{RP}^2 \# 5\overline{\mathbb{RP}}^2$ (without a need to select a subsequence).

1.2. Scheme of the proof of Theorem 1.2

There are several alternative constructions of an exotic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$ in [PSS], and the one suitable for us is obtained by some surgery from a rational elliptic surface, X, with a fiber of type $\mathbb{I}_8$. The first step is a double node neighborhood knot surgery on X, which yields a 4-manifold X_K containing a nodal *pseudo-section*. Next, X_K is blown up at several points so that we obtain a suitable chain of spheres, $C = C_1 \cup \cdots \cup C_k$, which can be rationally blown down on the last step. Our aim is to perform these constructions equivariantly.

In Section 2, we construct a special example of a rational elliptic surface, X, with a fiber $\mathbb{I}_8$, which is defined over reals, and thus has an involution, c, of the complex conjugation. It is essential for the further constructions that the components of the $\mathbb{I}_8$-fiber as well as the four remaining singular fishtail fibers are all *real* (i.e., invariant under the complex conjugation). In Section 4, we observe that a certain non-singular real fiber, T, which is constructed in Section 2, can be used for an equivariant double node knot surgery. We check that the nodal pseudo-section S_K obtained after such a surgery can be chosen invariant under the involution. Following the construction in [PSS], we blowup several points, which turn out to be all real in our example of X. Finally we obtain as in [PSS] a chain of spheres $C = C_1 \cup \cdots \cup C_n$, whose components are all c-invariant.

In Section 3, we discuss equivariant blowing down such chains C. It is crucial for us that the quotient X/c turns out to be S^4 and that the quotient by the involution remains the same as we modify the 4-manifold and the involution. So, all the involved equivariant transformations of X (knot surgery, blowup at a real point and rational blowdown) just modify the fixed point set F in the quotient S^4. Another crucial fact is that the fundamental group $\pi_1(S^4 \smallsetminus F) = \mathbb{Z}/2$ is preserved under these modifications of F.

More precisely, π_1 is preserved after a rational blowdown of C if we put a certain condition on C. This condition is satisfied for two of the configurations proposed in Proposition 2.5 of [PSS]: for $C_{79,44}$ and for $C_{89,9}$, which are the chains

$$(-2,-5,-11,-2,-2,-2,-2,-2,-2,-3,-2,-2,-3), \quad \text{and}$$

$$(-10,-11,-2,-2,-2,-2,-2,-2,-3,-2,-2,-2,-2,-2,-2,-2,-2).$$

Following the construction of [PSS] in the equivariant setting, we obtain a certain 4-manifold $\widehat{X}$ homeomorphic but not diffeomorphic to $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$, with an involution, $\widehat{c} : \widehat{X} \to \widehat{X}$. In the quotient $\widehat{X}/\widehat{c} = S^4$ there is a surface $F \subset S^4$ which is the fixed point set of $\widehat{c}$. Observing that F is connected, non-orientable (because $F^2 = 8 \neq 0$), and estimating its Euler characteristic $\chi(F) = 2\chi(S^4) - \chi(\widehat{X})$, we deduce that it is $\#6\mathbb{R}\mathrm{P}^2$.

A sequence of the twist-knots K_i that can be used for the knot surgery on the first step of the construction (see Figure 6a)) yields a sequence $\widehat{X}_i$ of exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ with involutions $\widehat{c}_i$, and a sequence of surfaces $F_i \subset S^4$ required for Theorem 1.2.

In the last section we pass once more through all the steps and verify some conditions which are required to be satisfied (for instance, to apply Lemma 3.2) in the particular case of $C_{79,44}$-chain. For convenience of the reader we included in the last section a review of the construction of this chain in $\mathbb{C}\mathrm{P}^2\#18\overline{\mathbb{C}\mathrm{P}}^2$ which was given in [PSS].

2. Real rational elliptic surfaces with special singular fibers

2.1. Double planes ramified along quartics

It is well known that the double covering over $\mathbb{C}\mathrm{P}^2$ branched along a non-singular quartic, $A \subset \mathbb{C}\mathrm{P}^2$, yields a del Pezzo surface $X_A = \mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$. A pencil of lines, $L_s \subset \mathbb{C}\mathrm{P}^2$, centered at $x \in \mathbb{C}\mathrm{P}^2$, is covered by an elliptic pencil, $T_s \subset \mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$, whose singular fibers cover the lines tangent to A. Blowing up the pull-back of x in $\mathbb{C}\mathrm{P}^2\#7\overline{\mathbb{C}\mathrm{P}}^2$, we obtain an elliptic fibration, $p : X \to \mathbb{C}\mathrm{P}^1$.

Assume that the quartic A has a singular point, $y \in A$, of the type $\mathbb{A}_n$ (by definition at such a singularity A is locally defined as $z_1^{n+1} + z_2^2 = 0$), and the basepoint x is generic. Then X_A also has a singularity of type $\mathbb{A}_n$ (i.e., locally defined as $z_1^{n+1} + z_2^2 + z_3^2 = 0$) and after its resolution we obtain an elliptic fibration $p : X \to \mathbb{C}\mathrm{P}^1$ with a singular fiber T_0 of the type $\mathbb{I}_{n+1}$ (in Kodaira's classification), i.e., a cyclic chain of (-2)-spheres. The fiber T_0 is the pull-back of the line $L_0 \subset \mathbb{C}\mathrm{P}^2$ passing through x and y. If in addition L_0 is tangent to A at the basepoint $x \in A$, then the fiber T_0 is of the type $\mathbb{I}_{n+3}$. We explain it in detail for the example of $n = 5$ that we will need.

Namely, in the case of our interest we obtain $\mathbb{I}_8$-fiber T_0 if we choose a quartic $A = A_1 \cup A_3$ that splits into a cubic, A_3, and a real line, A_1, tangent to A_3 at its inflection point, y (see Figure 1a)). The corresponding line L_0 should pass through y and be tangent to A_3 at some other point, x, which will be the center of the pencil of the lines L_s. The fiber $\mathbb{I}_8$ can be seen as the double cover of the chain of spheres $(-1,-2,-2,-2,-1)$ with four points of ramification: two points on each (-1)-curve. This chain appears in $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ as we resolve $\mathbb{A}_5$-singularity at y by a triple blowup, which gives a $(-1,-2,-2)$-chain, and then blowup the intersection of L_0 with A at x twice. The line L_0 (blown up three times) gives the third (-2)-curve in the chain, and the exceptional curve of the second blowup at x is the last (-1)-curve of the chain. The exceptional curve of the first blowup at x is (-2)-curve $E_x^* \subset \mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ (since the second blowup is in its point); it is a section, which belongs to the ramification locus of the double cover of $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ by the elliptic surface $X = \mathbb{C}\mathrm{P}^2\#9\overline{\mathbb{C}\mathrm{P}}^2$. So, E_x^* lifts to a (-1)-curve, $E_x \subset X$, which is a section of our elliptic fibration. The two ramification points on the first (-1)-component of the $(-1,-2,-2,-2,-1)$-chain are intersections with the two branches of A at y. On the last (-1)-component, one of the ramification points appear from the branch of A at x, and the other is the intersection point with E_x^*.

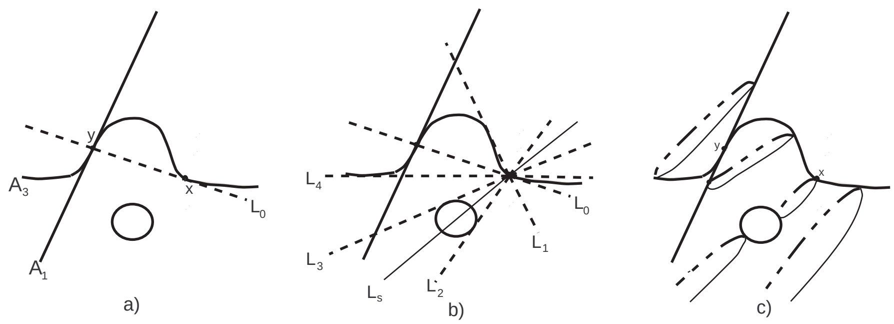

Figure 1. a) Quartic $A = A_1 \cup A_3$ and the basepoint x. b) The tangent lines L_i, $i = 1,2,3,4$ of the pencil and line L_s corresponding to the fiber $T = T_s$. c) The real locus of the double plane X_A.

2.2. Construction of a special real elliptic fibration

Consider the double covering $q : X_A \to \mathbb{C}\mathrm{P}^2$ ramified along a degree $2n$ curve $A \subset \mathbb{C}\mathrm{P}^2$ having an equation $f = 0$ with real coefficients. It is well-known (and trivial) that the complex conjugation in $\mathbb{C}\mathrm{P}^2$ can be lifted to X_A in two ways. These two liftings correspond to two real algebraic models of X_A, namely, the ones defined by a weighted homogeneous equation $y^2 = \pm f(x_0, x_1, x_2)$ in the quasi-projective space $P_{1,1,1,n}$ with the coordinates x_0, x_1, x_2, y of weights $1,1,1,n$. The corresponding involutions $c_\pm : X_A \to X_A$, induced from the complex conjugation in $P_{1,1,1,n}$, have fixed point sets $\mathrm{Fix}(c_\pm|_{X_A})$ which are projected by q to the regions $\mathbb{R}\mathrm{P}^2_{\pm\mathrm{f}} = \{\mathrm{x} \in \mathbb{R}\mathrm{P}^2 \mid \pm\,\mathrm{f}(\mathrm{x}) \geq 0\}$ bounded by the curve $A_\mathbb{R} = A \cap \mathbb{R}\mathrm{P}^2$. It is convenient to specify the choice of one of these two involution by referring to the corresponding region $\mathbb{R}\mathrm{P}^2_\mathrm{f}$ (the projection of the real locus of X_A).

In our example of a real quartic $A = A_1 \cup A_3$, we choose the region $\mathbb{R}\mathrm{P}^2_\mathrm{f}$ as is shown on Figure 1c) and the corresponding involution $c_A : X_A \to X_A$ whose fixed point set is $q^{-1}(\mathbb{R}\mathrm{P}^2_\mathrm{f})$. As it was explained, blowing up the $\mathbb{A}_5$-singularity and then the two infinitely near base-points of X_A, we obtain a *real elliptic fibration*, $p : X \to \mathbb{C}\mathrm{P}^1$, endowed with an involution $c : X \to X$ commuting with p. Let $F = \mathrm{Fix}(c)$ denote its fixed point set.

Lemma 2.1. *The real elliptic fibration $p : X \to \mathbb{C}\mathrm{P}^1$ constructed above has the following properties.*

(1) *X contains a real singular fiber $T_0 = C_1 \cup \cdots \cup C_8$ of the type $\mathbb{I}_8$, whose components C_i are c-invariant.*
(2) *X contains 4 other c-invariant singular fibers, T_i, $i = 1, 2, 3, 4$, which are ordinary fishtails.*
(3) *The elliptic fibration $p : X \to \mathbb{C}\mathrm{P}^1$ admits a c-invariant section.*
(4) *X can be blown down to $\mathbb{C}\mathrm{P}^2$, so that each of the nine consecutively contracted exceptional curves is real. This transforms the non-singular real fibers T_s to non-singular real cubics.*
(5) *$X/c = S^4$.*
(6) *If the singular fibers T_i, $i = 0, 1, 2, 3, 4$ are the pre-images of the tangent lines L_i on Figure 1b), then a non-singular real fiber $T = T_s$ chosen between T_2 and T_3 has real locus, $T \cap F$, formed by two connected components, as is shown on Figure 2c). The two vanishing curves in T, which are contracted as T is degenerated into the singular fibers T_2 and T_3, are isotopic. A vanishing curve from this isotopy class can be chosen c-invariant, and so that c reverses its orientation.*
(7) *The complement $F \smallsetminus (F \cap T)$ is connected.*

Proof. As is explained above, our fiber T_0 is of type $\mathbb{I}_8$ and is a double cover of the chain $(-1, -2, -2, -2, -1)$. All the components of the chain have real structures equivalent to that of $\mathbb{C}\mathrm{P}^1$, since we blowup only at real points. The branching points on the (-1)-curves are also real, which implies that all the components of $\mathbb{I}_8$ are c-invariant and (1) is proven.

Fibers T_i in (2) cover the tangent lines L_i shown on Figure 1b). Since L_i are real lines with ordinary tangency, the fibers T_i are c-invariant fishtails.

The section $E_x \subset X$, which was already described justifies (3). There are also other real (i.e., c-invariant) sections that we need for proving (4). For instance there is the proper transform, $A_1^* \subset X$, of the line A_1 (it has self-intersection -2 in $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ after 3 blowups and becomes (-1)-curve after lifting to X). The proper transform of a real line $L' \subset \mathbb{C}\mathrm{P}^2$ passing through y and tangent to A_3 (see Figure 2a)) splits into two components, L_1', L_2', which are both real sections of p. Another tangent line L'' shown on Figure 2a) gives similarly components L_1'' and L_2'', which are also real sections.

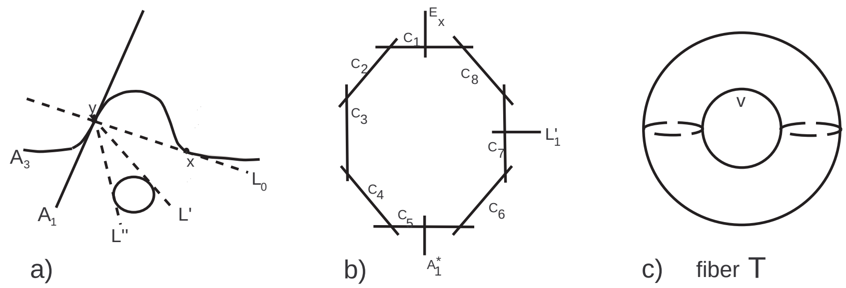

Figure 2. a) Tangent lines L' and L''. b) Fiber T_0 and 3 disjoint real sections E_x, A_1^*, and L_1'. c) The fixed point set of c dividing the fiber T, and the vanishing cycle v.

Let us numerate cyclically the components $C_1, \dots, C_8$ of the fiber T_0, so that C_1 is the double cover of the last (-1)-component in the chain $(-1,-2,-2,-2,-1)$ considered above, and thus section E_x intersects T_0 at a point of C_1. Note that the proper image of A_1 intersects this chain at the first (-1)-component, and the proper images of L' and L'' at the middle (-2)-component. This implies that A_1^* intersects C_5, and the curves L_j', L_j'', $j = 1,2$, intersect C_7 and C_3. Without loss of generality, we may suppose that L_1' intersects C_7, like it is shown on Figure 2b). We have $C_i^2 = -2$, $E_x^2 = (L_1')^2 = (A_1^*)^2 = -1$, and so we can blow down consecutively nine real exceptional curves, for example in the following order: E_x, C_1, C_2, C_3, C_4, A_1^*, L_1', C_7, and C_6. Such a blowdown must give $\mathbb{C}\mathrm{P}^2$, which proves (4). After blowing down the remaining components, C_5 and C_8, will represent a line and a conic in $\mathbb{C}\mathrm{P}^2$ (since their self-intersection indices are 1 and 4 respectively), and non-singular real fibers become non-singular real cubics.

We can deduce (5) from (4) using that $\mathbb{C}\mathrm{P}^2/\mathrm{conj} = \mathrm{S}^4$, which implies that a blowup at a real point effects to the quotient as a connected sum with $S^4 = \overline{\mathbb{C}\mathrm{P}}^2/\mathrm{conj}$, and thus, differential-topologically does not change the quotient.

Inspecting Figure 1c) we observe that the fixed point set $T \cap F$ of the complex conjugation acting on T has two connected components, as it is shown on Figure 2c). The types of the vanishing cycles on T are determined by the index of the real critical points of the projection $F = \mathrm{Fix}(c) \to \mathbb{R}\mathrm{P}^1$ (restriction of p). From Figure 1c) we can see saddle points at the fibers T_2 and T_3, and so each of the vanishing cycles have two real points (the real vanishing cycle S^0). This implies that the complex vanishing cycles at the non-singular fiber T_s for the both saddle points are isotopic to the curve v on Figure 2c). This is because up to isotopy, there is a unique c-invariant curve on a torus T_s which intersects once each component of its real locus. Namely, c should act on curve v as a reflection, and thus the homology class $[v]$ belongs to the one-dimensional (-1)-eigenspace of the c-action in $H_1(T_s)$. This proves (6).

Property (7) is clear from Figure 1c) if we take into account connectedness of the real locus of our real form of singularity $\mathbb{A}_5$ (local model $x^6 - y^2 + z^2 = 0$) after its resolution. Namely, this connectedness implies that the two branches of the surface at the singularity shown on Figure 1c) become connected after resolution. □

3. Equivariant rational blowdown

3.1. Rational blowdown surface surgery

Let X denote a smooth 4-manifold with *a chain of spheres* $C = C_1 \cup \dots \cup C_k \subset X$, which intersect each other consecutively and transversely, so that their dual weighted graph is a chain-tree shown on the bottom of Figure 3. We can (and always will) orient the spheres so that their intersection indices are positive. A regular neighborhood, $N(C)$, of C is a plumbing 4-manifold, P_C, corresponding to this weighted graph.

Certain chains C can be *rationally blown down*, i.e., we can remove $N(C) = P_C$ from X and replace it by some rational homology ball, Q_C, with the same boundary $\partial Q_C = \partial N(C)$. This gives a new 4-manifold $\widehat{X} = X' \cup Q_C$, where $X' = X \smallsetminus N(C)$.

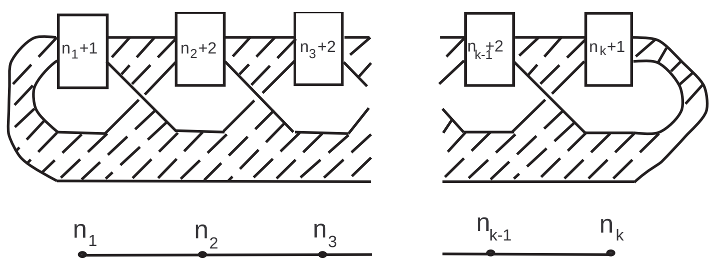

FIGURE 3. The plumbing surface F_C described by a chain-tree can be presented as the span-surface of a two-bridge link diagram (the interior of F_C is pushed inside D^4). The numbers in the boxes count the half-twists.

It is well known and easy to see that P_C can be described as the double cover over D^4 branched along a surface, F_C, obtained by plumbing of the unknotted bands, $F_{n_i} \subset D^4$, $i = 1, \dots, k$, where n_i stands for the framing of the band (number of its half-twists which is taken with sign "$-$" in the case of left-hand half-twists). Such a plumbing surface can be sketched as is shown on Figure 3.

As it is observed in [FS1], Q_C is the double cover of D^4 branched along another surface, $R_C \subset D^4$, bounded by the same link as F_C, $L_C = \partial R_C = \partial F_C$. More details about the surface R_C can be extracted from [CH], and we will only mention the idea and illustrate it by an example. We start with a plumbing surface $F_{C'}$ representing some chain $(n_1, \dots, n_k)$, which can be blown down to the chain (0), like for instance the chain $(-1, -1)$. The boundary link $L_{C'} = \partial F_{C'}$ is a 2-component unlink, as the boundary of an untwisted band F_0. For instance Figure 4c) illustrates an example of $L_{C'}$ for the chain $(-1,-1)$. There is a $(+1)$-twisted band in the middle of $F_{C'}$ on Figure 4c), and cutting it yields another plumbing surface $F_C = F_{-4}$ on Figure 4a). In general, cutting a similar band replaces a pair of consecutive numbers n_i, n_{i+1} in the chain C' by one number $n_i + n_{i+1} - 2$ in the chain C. On the other hand, $L_{C'}$ as an unlink bounds two disjoint disks $D_1 \sqcup D_2$, and the operation of smoothing the $(+1)$-crossing on Figure 4c) can be interpreted as "a ribbon move", that is gluing a ribbon to the boundary of a pair of disjoint unknotted discs, $D_1 \sqcup D_2$. Such a ribbon may intersect the discs, but as we push in the interior of the disks from S^3 inside D^4, we obtain an embedded surface $R_C \subset D^4$ bounding $L_C = \partial F_C$. If the ribbon connects ∂D_1 with ∂D_2, then L_C is a ribbon knot and R_C is a disk embedded in D^4. The ribbon may also connect the boundary of one of the discs, say D_1, to itself, then R_C is a disjoint union of the disc D_2 with a Möbius band (one can show that the band cannot be orientable, because ∂F_C cannot have 3 boundary components as a 2-bridge link). On Figure 4b) the disk D_2 bounding one component is pushed inside D^4 and so is disjoint from the Möbius band bounding another component (the band shaded on Figure 4b)).

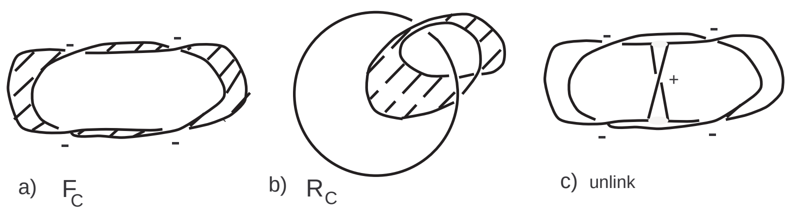

FIGURE 4. a) A band $F_C = F_{-4}$ with the four negative half-twists (marked with the signs "$-$"). b) Link $L_C = \partial F_C$ bounds also R_C (a disc and a band). c) An unlink $L_{C'}$ which yields L_C after a ribbon move.

To obtain similarly the surface R_C for $C = C_{79,44}$ we should start with a chain $C' = (-2,-5,-8,-1,-2,-2,-2,-2,-2,-2,-3,-2,-2,-3)$.

Lemma 3.1. *Consider a rational blowdown of a chain of spheres $C \subset X$, which yields $\widehat{X} = X' \cup Q_C$. Assume that X is endowed with an orientation preserving involution $c : X \to X$, which keeps each of the components, $C_i \subset C$ invariant, and reverses its orientation, so that* $\mathrm{Fix}(c) \cap C_i \neq \emptyset$. *Then, if $N(C)$ is chosen c-invariant, the rational blowdown can be made equivariantly, which yields an involution $\widehat{c} : \widehat{X} \to \widehat{X}$.*

Such a blowdown gives the same quotient $Y = X/c = \widehat{X}/\widehat{c}$. $F = \mathrm{Fix}\, c$ and $\widehat{F} = \mathrm{Fix}(\widehat{c})$ descend to Y and give the same locus $F' = F \cap Y' = \widehat{F} \cap Y'$, inside $Y' = X'/c$. The modification of the fixed point set is supported in a ball $N(C)/c = D^4$, where $F \cap D^4$

is isotopic to the plumbing surface F_C*. The piece of surface* $\widehat{F} \cap D^4$ *is isotopic to the surface* R_C*.*

Proof. Under these assumptions, $c|_{N(C)}$ is equivalent to the deck transformation of the double branched covering $N(C) \to D^4$. The involution $\widehat{c}$ just extends the involution $c|_{X \smallsetminus N(C)}$ to Q_C as the deck transformation of the branched covering $Q_C \to D^4$. □

We say that $(\widehat{X}, \widehat{c})$ is obtained by an *equivariant rational blowdown* from (X, c).

3.2. Characteristic sub-configurations

It is trivial to see that the number of components of a compact surface with non-empty boundary is greater by one than the nullity of its intersection form with $\mathbb{Z}/2$-coefficients. On the other hand, Figure 3 presents a 2-bridge link, and such links may have at most two connected components. Thus, the link $L_C = \partial F_C$ bounded by the plumbing of bands, F_C, has one component if the intersection matrix $M_C = (C_i \circ C_j)$ is non-singular modulo 2 (has odd determinant), and has two components otherwise (if the nullity over $\mathbb{Z}/2$ is 1).

We say that the union of some components C_i forms a *characteristic sub-configuration*, $W \subset C$, and call the corresponding components C_i *characteristic components* if the fundamental class $[W] \in H_2(C; \mathbb{Z}/2)$ is a Wu element of the intersection form $(C_i \circ C_j)$, that is $C_i^2 = C_i \circ W \pmod 2$ for all $i \in \{1, \dots, k\}$. If the matrix M_C is non-singular modulo 2, then a characteristic sub-configuration W does exist and is unique. Otherwise, a characteristic sub-configuration W still exists, but is not unique: it is trivial to see that another sub-configuration $W' \subset C$ is also characteristic if and only if the sum $[W] + [W'] \in H_2(C; \mathbb{Z}/2)$ belongs to the null-space of the intersection form modulo 2.

Remark 3.1. To find a characteristic sub-configuration of C (and to prove in turn its existence) one can first reduce modulo 2 the corresponding chain of integers, then blow it down (mod 2) to obtain a chain of zeros, whose characteristic sub-configuration is trivial. It is an easy exercise to prove that each blowup (mod 2) preserves all the previous characteristic components, and that the new exceptional component is also characteristic if and only if it is not a neighbor of another characteristic component. (One hint: observe first that a pair of characteristic components cannot be adjacent.)

To illustrate this method let us consider a chain $(-3, -2, -3, -4)$, which has modulo 2 reduction $(1, 0, 1, 0)$. Blowing down (mod 2) the component C_1 gives $(1, 1, 0)$, and the subsequent blowdown of C_2 gives $(0, 0)$, which implies that the remaining components C_3 and C_4 are not characteristic. As we blow down C_2 its neighbor C_3 is not characteristic, and so C_2 is characteristic. On the contrary, C_1 is not characteristic, because its neighbor C_2 is characteristic. Thus, the characteristic sub-configuration of the given chain includes only C_2. Another example illustrating this method will appear in 5.3

Remark 3.2. It is also simple to determine W using an arbitrarily chosen orientation of the link diagram of L_C shown on Figure 5. Namely, we should include component C_i in W if and only if the opposite sides of the band F_{n_i} are co-directed, as is shown on Figure 5. This description follows from the fact that $W \cap F_C$ realizes the first Stiefel-Whitney class

$w_1(F_C)$ (because w_1 is dual to the Wu element of the intersection form in $H_1(F_C;\mathbb{Z}/2)$, which is described by the matrix M_C modulo 2).

3.3. Commutativity of π_1 after rational blowdowns

Suppose that a sphere $C_0 \subset X$ extends the chain C to a longer chain $\widetilde{C} = C_0 \cup C_1 \cup \dots \cup C_n$. This means that C_0 is a c-invariant sphere (like the others C_i) which intersects C_1 transversely at a single point and does not intersect C_i, if $i > 1$.

Lemma 3.2. *Assume that*

1. *the link L_C is a knot,*
2. *a characteristic sub-configuration $W \subset C$ is not characteristic for $\widetilde{C}$.*

Then $\pi_1(Y \smallsetminus \widehat{F}) = \pi_1(Y \smallsetminus F)$.

Remark 3.3. According to the definitions, the second assumption of the Lemma means that C_1 is not included into W if $n_0 = C_0^2$ is odd, and $C_1 \subset W$ if n_0 is even.

Proof. Applying the Van Kampen theorem, we see that $\pi_1(Y \smallsetminus F) = G' *_{G_L} G_C$, where $G' = \pi_1(Y' \smallsetminus F')$, $G_C = \pi_1(D^4 \smallsetminus F_C)$, and $G_L = \pi_1(S^3 \smallsetminus L_C)$. Similarly, $\pi_1(Y \smallsetminus \widehat{F}) = G' *_{G_L} \widehat{G}$, where $\widehat{G} = \pi_1(D^4 \smallsetminus R_C)$.

The plan of the proof is to observe that the inclusion homomorphisms $G_L \to G_C$, $G_L \to \widehat{G}$ are epimorphisms, and that their kernels, K and $\widehat{K}$, vanish under the inclusion homomorphism $G_L \to G'$. This implies that the homomorphisms $G' \to G' *_{G_L} G_C$ and $G' \to G' *_{G_L} \widehat{G}$ are isomorphisms.

First of all, note that the upper Wirtinger presentation for the link L_C whose diagram is shown on Figure 3 implies that its group G_L is generated by two elements a, b, presented by the loops around the overpasses ℓ_a and ℓ_b shown in Figure 5.

The homomorphism $G_L \to G_C$ is epimorphic because G_C is a cyclic group (since F_C is a connected span-surface for L_C, whose interior is pushed out from S^3 inside D^4). Inspecting the homology, we see that $G_C = \mathbb{Z}$ is obtained from G_L by adding the relation $a = b$ in the case of orientable surface F_C. If F_C is non-orientable, then $G_C = \mathbb{Z}/2$ is obtained from G_L by adding two relations: $a = b$ and $a = b^{-1}$, which generate K.

The homomorphism $G_L \to \widehat{G}$ is also epimorphic, because R_C is a ribbon-surface. The kernel $\widehat{K}$ is contained in the commutator subgroup $[G_L, G_L]$, which is the kernel of the product map $G_L \to \widehat{G} \to H_1(D^4 \smallsetminus R_C) = H_1(S^3 \smallsetminus L_C) = \mathbb{Z}$, where the latter two equalities are due to our assumption that L_C is connected, and so R_C is a disc. Thus, $[G_L, G_L]$ is generated by the relation $a = b$ if overpasses ℓ_a and ℓ_b on Figure 5 inherit co-directed orientation from L_C, and by the relation $a = b^{-1}$ if these overpasses inherit opposite orientations.

Showing that the images of a, b under the inclusion homomorphisms $G_L \to G'$ (for which we keep the same notation a, b) satisfy both relations $a = b$ and $a = b^{-1}$ will complete the proof.

One of these relations comes from a regular neighborhood $N(D_0)$ of the disc $D_0 = C_0/c$ in Y. Note that $H = N(D_0) \cap Y'$ is a 4-ball containing an unknotted disc $F_H = F' \cap H$, so

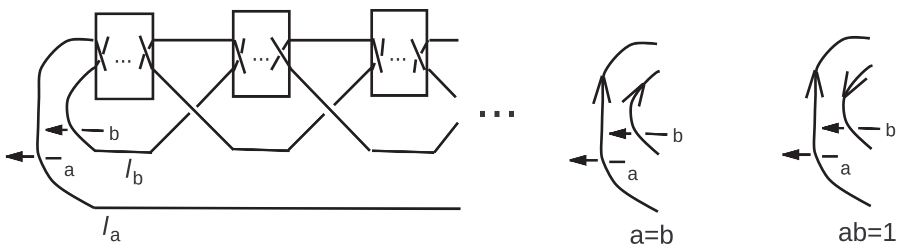

FIGURE 5. Overpasses ℓ_a, ℓ_b and the corresponding generators a and b of $\pi_1(S^3 \smallsetminus L_C)$. The case of co-directed and oppositely directed overpasses ℓ_a, ℓ_b, with the corresponding relations between a and b (after adding the commutativity relation $ab = ba$)

that $\pi_1(H \smallsetminus F_H) = \mathbb{Z}$. The common piece of the boundary of H and D^4 is a 3-ball, which intersects F along a pair of arcs, $\ell_a \cup \ell_b$. It is not difficult to see that in $\pi_1(H \smallsetminus F_H)$ we obtain the relation $a = b$ if n_0 is even, and $a = b^{-1}$ if odd. With this relation, the group G_L becomes abelian, and we obtain another relation (which depends on the orientation of ℓ_a, ℓ_b induced from L_C, as was explained). Under the second assumption of our Lemma, these two relations are different, that is both relations $a = b$ and $a = b^{-1}$ are satisfied in $\pi_1(Y \smallsetminus \widehat{F})$. □

4. Equivariant version of the Fintushel-Stern double node knot surgery

4.1. Equivariant knot surgery

The 4-dimensional knot surgery consists of removing from a 4-manifold X a trivialized tubular neighborhood $N(T) = T \times D^2$ of a torus $T \subset X$, and replacing it by $S^1 \times C(K)$, where $C(K)$ is a knot complement (see [FS1]). It is supposed that the gluing map $S^1 \times \partial C(K) \to T \times \partial D^2$ identifies a longitude $\mathrm{pt} \times \ell \subset S^1 \times C(K)$ with a meridian of T, $m_T = \mathrm{pt} \times \partial D^2 \subset \partial N(T)$, which yields a 4-manifold X_K, homologically equivalent to X.

In the equivariant version of this construction, we suppose that X is endowed with an orientation preserving involution, c, which keeps invariant the torus T as well as its neighborhood $N(T)$. We say that a trivialization $N(T) = T \times D^2$ of a tubular neighborhood, $N(T)$, of T is *equivariant* if the action of c on $N(T)$ can be presented as the direct product of $c|_T$ and the complex conjugation in $D^2 \subset \mathbb{C}$. Note that equivariant trivializability is equivalent to the existence of a projection $N(T) \to D^2$ which commutes with $c|_{N(T)}$ and the complex conjugation in D^2. In the case of our interest, T is a real non-singular fiber in a real elliptic fibration and thus admits such an equivariantly trivializable neighborhood.

Let us assume in addition that $c|_T$ reverses the orientation of T and has two-component fixed point set, $F \cap T$ (see Figure 2c)). In this case the quotient $\mathcal{A} = T/c$ is an annulus and in the coordinates defined by some diffeomorphism $T = S^1 \times S^1$ the action of c looks like $(z_0, z_1) \mapsto (z_0, \bar{z}_1)$. Thus for an appropriate diffeomorphism $N(T) = S^1 \times S^1 \times D^2$, this action looks like $(z_0, z_1, z_2) \mapsto (z_0, \bar{z}_1, \bar{z}_2)$.

From a knot $K \subset S^3$ we require that it has an axis of symmetry, and intersects this axis at a pair of points. It will be convenient to choose the complex conjugation, conj, in $S^3 \subset \mathbb{C}^2$ as such a symmetry, so that the axis is $S^1_{\mathbb{R}} = S^3 \cap \mathbb{R}^2$. It is not difficult to see that such a knot K admits an equivariant tubular neighborhood, $N(K)$, in which c acts as $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$, with respect to a trivialization $S^1 \times D^2 = N(K)$. We should choose such a trivialization to be *null-framed*, which means that a longitude $\ell_K = S^1 \times \mathrm{pt}$ is null-homologous in the knot complement $C(K) = \mathrm{Cl}(S^3 \smallsetminus N(K))$. Note that one can actually choose an equivariant trivialization with any framing: if we split $N(K) = S^1 \times D^2$ into a pair of cylinders $I \times D^2$ permuted by c and change a trivialization of one cylinder by n half-twists, then this trivialization can be c-symmetrically extended to another cylinder and will change the trivialization of $S^1 \times D^2$ by n full twists.

We can glue $S^1 \times C(K)$ to $X \smallsetminus N(T)$ via an equivariant gluing map $g : S^1 \times \partial C(K) \to \partial N(T)$. Using the coordinates (z_0, z_1, z_2) in $\partial C(K) = \partial N(K)$ and $\partial N(T)$ from the above trivializations of $N(K)$ and $N(T)$, we define the map g as $(z_0, z_1, z_2) \mapsto (z_0, z_2, z_1)$. Such an equivariant knot surgery yields a 4-manifold X_K endowed with an involution, $c_K : X_K \to X_K$.

4.2. The tangle surgery in quotient-spaces

The quotient $N(K)/\mathrm{conj}$ is a 3-ball, which can be viewed as a regular neighborhood, $N(\mathfrak{s}_K)$, of the arc $\mathfrak{s}_K = K/\mathrm{conj}$ in $S^3 = S^3/\mathrm{conj}$. Thus, $B_K = C(K)/\mathrm{conj}$ is also a 3-ball, complementary to $N(\mathfrak{s}_K)$. The unknot $S^1_{\mathbb{R}} \subset S^3/\mathrm{conj}$ splits into a trivial tangle $\mathfrak{t} = S^1_{\mathbb{R}} \cap N(\mathfrak{s}_K)$ and a non-trivial one, $\mathfrak{t}_K = S^1_{\mathbb{R}} \cap B_K$ (see Figure 6d)).

Example 4.1. The twist-knot $K = K_n$, which will be used in our construction, admits a conj-invariant presentation, as it is sketched in Figure 6a). Figures 6d)–6e) present the corresponding tangle splitting $S^1_{\mathbb{R}} = \mathfrak{t} \cup \mathfrak{t}_K \subset S^3$.

The quotient-space X_K/c_K is obtained from X/c by removing a regular neighborhood, $N = N(\mathcal{A})$, of the annulus $\mathcal{A} = T/c$, which can be viewed as $N = S^1 \times N(\mathfrak{s}_K) = S^1 \times B^3$, and replacing it by $S^1 \times B_K = S^1 \times B^3$. Such a surgery does not change the differential-topological type of a 4-manifold, so we can identify both quotients, $Y = X/c = X_K/c_K$.

The branching locus F_K of the double covering $X_K \to Y$ is obtained from F after replacing $F_N = F \cap N = S^1 \times \mathfrak{t}$ by $S^1 \times \mathfrak{t}_K$ inside $N = S^1 \times B^3$. Note that the components of $\mathfrak{t}_K$ and of $\mathfrak{t}$ connect the same pairs of their common endpoints. We denote these four endpoints $p_1^{\pm}, p_2^{\pm}$, and assume that p_i^+ is connected with p_i^-. Moreover, both tangles must have *the same framing*. This means by definition that the kernel of the inclusion homomorphism $H_1(S^2 \smallsetminus \partial\mathfrak{t}) \to H_1(D^3 \smallsetminus \mathfrak{t})$ is the same as for $H_1(S^2 \smallsetminus \partial\mathfrak{t}_K) \to H_1(D^3 \smallsetminus \mathfrak{t}_K)$.

This kind of surgery will be called *tangle surgery of $F \subset Y$ along an annulus membrane $\mathcal{A}$*. It can be applied to any surface F in a 4-manifold Y under the assumption that the annulus membrane $\mathcal{A}$ with the boundary on this surface is *null-framed*. This means

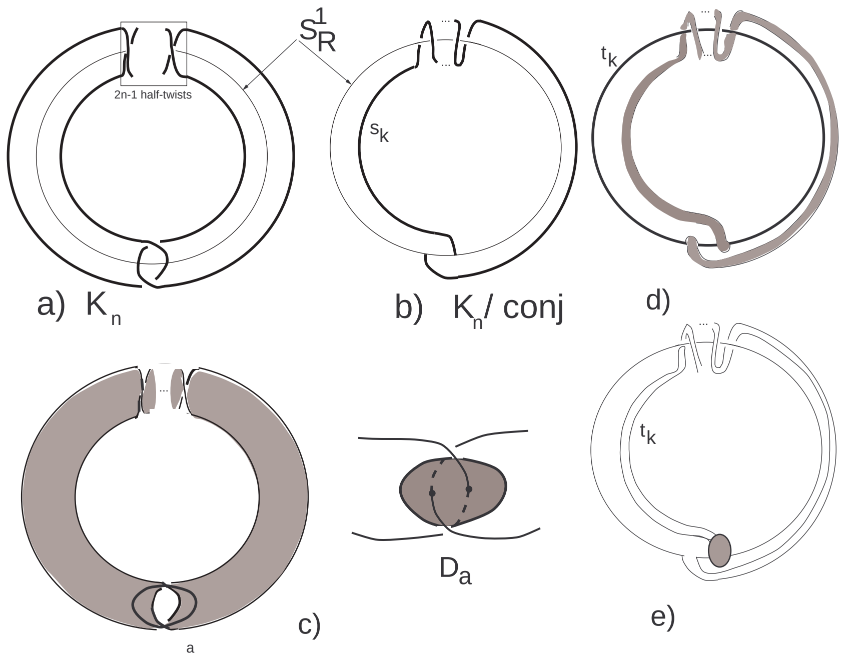

FIGURE 6. a) The twist-knot $K = K_n$ with the axis of symmetry $S^1_{\mathbb{R}}$. b) The arc $\mathfrak{s}_K = K/\text{conj}$. c) Seifert surface S°_K of genus 1 bounded by K. It contains conj-invariant curve a, which bounds a conj-invariant disc D_a. d) The ball $N(\mathfrak{s}_K)$ and tangle $\mathfrak{t}_K$ in its complement. e) $\mathfrak{t}_K$ after an isotopy of $N(\mathfrak{s}_K)$ (the ball shaded on the Figure).

by definition that for some trivialization $N = S^1 \times D^3$ of its regular neighborhood, $N = N(\mathcal{A})$, the part of surface $F \cap N$ is identified with $S^1 \times \mathfrak{t}$, and $\mathcal{A}$ is identified with $S^1 \times \mathfrak{s}$, where $\mathfrak{s}$ is a line segment connecting the midpoints of the components of $\mathfrak{t}$ (see Figure 7a)). The following Lemma summarizes our observations.

Lemma 4.1. *An equivariant knot surgery on (X, c) along a c-invariant torus $T \subset X$ gives (X_K, c_K) with the same quotient-space $Y = X_K/c_K = X/c$. The fixed point set $F_K \subset Y$ of c_K is obtained from the fixed point set F by the tangle surgery along the annulus membrane $\mathcal{A} = T/c$.* □

4.3. Commutativity of π_1 throughout the knot surgery

Lemma 4.2. *Assume that $F \subset Y$ is a surface in a 4-manifold and $\mathcal{A}$ is a null-framed annulus membrane on F such that $F \smallsetminus \partial\mathcal{A}$ is connected. Assume that F_K is obtained from F by applying the tangle surgery along $\mathcal{A}$ with respect to $\mathfrak{t}_K$, where $K = K_n$ is the* *twist-knot from Example 4.1. Assume furthermore that the group $\pi_1(Y \setminus (F \cup \mathcal{A}))$ is abelian. Then $\pi_1(Y \setminus F_K)$ is also abelian and isomorphic to $\pi_1(Y \setminus F)$.*

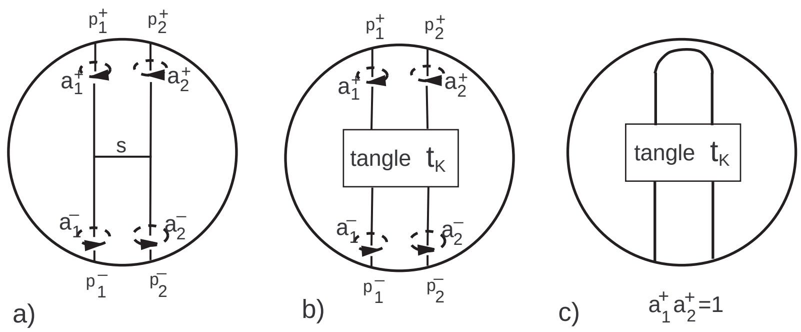

FIGURE 7. a) Trivial tangle $\mathfrak{t}$ with the connecting line segment $\mathfrak{s}$. The generators $a_i^{\pm}$ of $\pi_1(S^2 \setminus \partial\mathfrak{t})$. b) The result of a tangle surgery. c) Adding the relation $a_1^+ a_2^+ = 1$ to the group of tangle $\mathfrak{t}_K$ effects like connecting together the points p_1^+ and p_2^+.

Proof. Let $Y' = \mathrm{Cl}(Y \setminus N)$ and $F' = F \cap Y'$, $F_{N,K} = F_K \cap N$. By the Van Kampen theorem, $\pi_1(Y \setminus F_K) = G' *_H G_N$, where $G' = \pi_1(Y' \setminus F')$, $H = \pi_1(\partial N \setminus \partial F_N)$, and $G_N = \pi_1(N \setminus F_{N,K})$. Note that $N \setminus (\mathcal{A} \cup F_N)$ can be deformation retracted to its boundary $\partial N \setminus \partial F_N$, which implies that group G' is abelian, by the assumption on $\pi_1(Y \setminus (F \cup \mathcal{A}))$. Note that $G' *_H G_N = G' *_H (G_N/K)$, where K is the image in G_N of the kernel of the homomorphism $H \to G'$. We will show that G_N/K is an abelian group and the product homomorphism $H \to G_N \to G_N/K$ is epimorphic. This implies that $G' *_H G_N$ is a quotient of G' and thus is also abelian.

Note that $H = \mathbb{Z} \times \pi_1(S^2 \setminus \partial\mathfrak{t})$, where the second factor is a free group or rank 3. It is convenient to present this free group by 4 generators, $a_1^{\pm}$, $a_2^{\pm}$, satisfying the relation $a_1^+ a_1^- a_2^- a_2^+ = 1$. These generators correspond to the loops around the tangle endpoints, $p_1^{\pm}, p_2^{\pm} \in S^2$, in the positive direction, see Figure 7a).

Let us fix some element $a \in G'$ presented by a loop around $F \setminus \partial\mathcal{A}$. Commutativity of G' and connectedness of $F \setminus \partial\mathcal{A}$ imply that the inclusion homomorphism $H \to G'$ sends each of $a_i^{\pm}$ either to a, or to a^{-1} (depending on the topology of the boundary $\partial\mathcal{A}$ as an oriented curve in F). Such a relation, $a_i^{\pm} = a$ or $a_i^{\pm} = a^{-1}$, is inherited by the quotient-group G_N/K. To complete our proof of the Lemma it is enough to show that by adding this relation we transform group $\pi_1(D^3 \setminus \mathfrak{t}_K)$ into a cyclic group with a generator a (since the factor $\mathbb{Z}$ in $G_N = \mathbb{Z} \times \pi_1(D^3 \setminus \mathfrak{t}_K)$ lies in the center and comes from the corresponding factor in $H = \mathbb{Z} \times \pi_1(S^2 \setminus \{p_1^+, p_1^-, p_2^+, p_2^-\}))$.

We will present two arguments. The first one can be applied to any knot K admissible for an equivariant knot surgery, but it works only if we have a relation $a_1^+ a_2^+ = 1$, or $a_1^- a_2^- = 1$. Note that if we connect the endpoints p_1^+ and p_2^+ as shown on Figure 7c), we modify the group $\pi_1(B^3 \smallsetminus \mathfrak{t}_K)$ by adding a relation $a_1^+ a_2^+ = 1$. In the case of tangles $\mathfrak{t}_K$ constructed from conj-invariant knots K, this modification transforms the tangle into an unknotted arc in D^3. Thus, the group $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$ becomes cyclic and generated by any of the elements $a_i^\pm$. The case of adding relation $a_1^- a_2^- = 1$ is analogous.

Our second argument is specific for the twist-knot $K = K_n$, but can be applied in the case of relation $a_1^+ = a_2^+$ or $a_1^- = a_2^-$ (as well as in the case of relation $a_1^\pm a_2^\pm = 1$ considered before). First, we observe that the upper Wirtinger presentation gives 5 generators for $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$, namely $a_i^\pm$, $i = 1, 2$, and one more generator b shown on Figure 8.

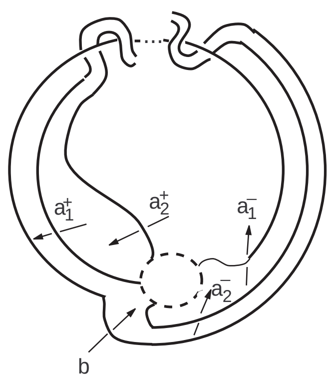

FIGURE 8. The tangle group becomes abelian after adding the relations $a_1^+ = a_2^+$, $a_1^- = (a_1^+)^{\pm 1}$, and $a_2^- = (a_1^+)^{\pm 1}$.

The two strands of the tangle $\mathfrak{t}_K$ with the origins at the points p_1^+ and p_2^+ pass together several times under $S^1_{\mathbb{R}}$. These underpasses separate the consecutive overpasses on the first strand which yield elements $a_1^+, b^{-1} a_1^+ b, \dots$, which are all conjugate to a_1^+. The similar overpasses on the second strand give elements $a_2^+, b^{-1} a_2^+ b, \dots$, which are conjugate to a_2^+ by the same sequence of elements. In the end of the sequence, we obtain elements $b^{-1} = x^{-1} a_1^+ x$ and $(a_2^-)^{-1} = x^{-1} a_2^+ x$, which are conjugate to a_1^+ and a_2^+ via the same element $x \in \pi_1(D^3 \smallsetminus \mathfrak{t}_K)$. So, a relation $a_1^+ = a_2^+$ which we add implies that $b = a_2^-$, whereas $a_1^+ a_2^+ = 1$ implies $b a_2^- = 1$. In any case, generator b can be eliminated, and after adding two more relations to $\pi_1(D^3 \smallsetminus \mathfrak{t}_K)$, namely $a_1^- = a_1^+$ (or $a_1^- = (a_1^+)^{-1}$) and $a_2^- = a_1^+$ (or $a_2^- = (a_1^+)^{-1}$), we obtain a cyclic group.

Finally, we can observe that the surface F_K is homologically equivalent to F, and thus $H_1(Y \smallsetminus F_K) = H_1(Y \smallsetminus F)$. The group $\pi_1(Y \smallsetminus F)$ is abelian due to the assumption of the Lemma, because $Y \smallsetminus F$ is obtained from $Y \smallsetminus (F \cup \mathcal{A})$ after puncturing an annulus (which may only add a relation to π_1) and then puncturing an arc (which adds a 3-cell and thus does not change π_1). Thus, we obtain an isomorphism $\pi_1(Y \smallsetminus F_K) = \pi_1(Y \smallsetminus F)$. □

Lemma 4.3. *Let (X, c) be the real elliptic surface constructed in Section 2, and T be the real fiber from Lemma 2.1(6). Then the membrane $\mathcal{A} = T/c$ satisfies the assumptions of Lemma 4.2, and thus $\pi_1(S^4 \smallsetminus F_K) = \mathbb{Z}/2$ (here $S^4 = X/c$ by Lemma 2.1(6) and $F_K \cong F \cong \#10\mathbb{R}\mathrm{P}^2$).*

Proof. Connectedness of $F \smallsetminus \mathcal{A}$ is observed in Lemma 2.1(7). The group $\pi_1(S^4 \smallsetminus (F \cup \mathcal{A}))$ was shown to be cyclic in [FKV2], §4, under the assumption that $T \subset X$ is obtained from a real non-singular cubic in $\mathbb{C}\mathrm{P}^2$ by blowing up the base-points of a real pencil of cubics. This is so in our case, as follows from property (4) of Lemma 2.1. □

4.4. The equivariant double node surgery

To justify that a pseudo-section $S_K \subset X_K$ can be chosen c-invariant we recall first its construction in [FS2]. Consider a disc $\Delta_1 \subset \mathbb{C}\mathrm{P}^1$ which contains inside precisely two critical values $s_+, s_- \subset \Delta_1$ of an elliptic Lefschetz fibration $p : X \to \mathbb{C}\mathrm{P}^1$. Assume moreover that the corresponding two vanishing cycles in a non-singular fiber, T_s, $s \in \Delta_1$, are isotopic. Let $\Delta \subset \Delta_1$ denote a smaller disc not containing points $s_\pm$, and $U = p^{-1}(\Delta)$. Consider a section $S \subset X$ of p. Its restriction over Δ is the disc $\widetilde{\Delta} = S \cap U$. Using that the gluing map in the definition of the knot surgery which yields X_K may be adjusted by an isotopy, we can make the boundary $\partial\widetilde{\Delta}$ match with the boundary of a Seifert surface $S_K^\circ \subset \mathrm{pt} \times C(K) \subset X_K$ and obtain a closed surface $S_K^* = (S \smallsetminus \widetilde{\Delta}) \cup S_K^\circ$ in X_K. If $K = K_n$ (the twist-knot on Figure 6a)), then S_K^* is a torus which has a certain disc membrane $D_a^* \subset X_K$ bounded by curve $a = \partial D_a^* = D_a^* \cap S_K^*$ and having self-intersection $(D_a^*)^2 = -1$ (relative to the boundary on the surface S_K^*). The torus S_K^* can be deformed and degenerated into a fishtail $S_K \subset X_K$, as we pinch curve $a \subset S_K^*$ along disc D_a^*. The local topology of S_K near its singular point is like near an algebraic double point, and the embedded surface $S_K \subset X_K$ is differential-topologically equivalent to a rational curve with a single node and self-intersection $(S_K)^2 = -1$.

To construct the disc D_a^*, we first take a disc $D_a \subset S^3$ bounded by a, so that D_a intersects K at a pair of points (see Figure 6c)). Disc D_a punctured at these points is embedded in $C(K) = \mathrm{pt} \times C(K) \subset S^1 \times C(K) \subset X_K$. The boundary of the punctures are two meridians, $m_\pm \subset \partial C(K)$, around K, which represent certain parallel curves in two different fibers of p after a knot surgery. We have to choose the knot surgery so that $m_\pm$ are the vanishing cycles corresponding to the singular values $s_\pm$ (this is possible because of our assumption that the vanishing curves corresponding to $s_\pm$ are isotopic). Then the two holes in the punctured disc D_a can be filled with the discs $D_{m_\pm}$ centered at the nodes of the singular fibers over $s_\pm$ and bounded by the vanishing cycles $m_\pm$. This gives D_a^*.

Lemma 4.4. *Consider the real elliptic surface (X, c) constructed in Section 2. Let $K = K_n$ be the twist-knot embedded* conj*-invariantly in S^3, as is shown on Figure 6a). Assume that (X_K, c_K) is obtained from (X, c) by an equivariant knot surgery along the* *non-singular fiber $T = T_s$ specified in Lemma 2.1(6). Then the pseudo-section S_K can be chosen c_K-invariant.*

Proof. By Lemma 2.1(3), we can suppose that section S is c-invariant. We consider a disc $\Delta \ni s$ which is invariant under the complex conjugation in $\mathbb{C}\mathrm{P}^1$ (the base of the elliptic fibration), and denote by $r_\pm$ the endpoints of the interval $\Delta \cap \mathbb{R}\mathrm{P}^1$. In our example of real elliptic surface described in Section 2, we have a pair of real critical values, $s_\pm$, whose fibers $T_2 = p^{-1}(s_-)$, $T_3 = p^{-1}(s_+)$ can be used for a double node knot surgery, as it follows from Lemma 2.1(6). The curve $S \cap \partial U$ is a conj-invariant longitude of the knot K in the boundary of $C(K) \cong \mathrm{pt} \times C(K)$. This longitude spans a conj-invariant Seifert surface $S_K^\circ \subset C(K)$, as it is shown on Figure 6c). This implies that the torus S_K^* can be chosen c_K-invariant. Furthermore, we can choose the disc D_a^* to be c_K-invariant as follows. First, one can obviously choose a conj-invariant disc D_a (see Figure 6c)), so that the two intersection points $D_a \cap K$ are both real. This intersection can be made orthogonal by an equivariant isotopy of K near the intersection points, so that the meridians $m_\pm$ become also conj-invariant. As we perform an equivariant knot surgery, meridians $m_\pm$ become c-invariant vanishing cycles at the real fibers $p^{-1}(r_\pm) \subset \partial U$ (cycle $m_\pm$ corresponds to the critical value $s_\pm$). Note that the discs $D_{m_\pm}$ can be also chosen c-invariant. Namely, such a disc can be seen as the trace of the vanishing cycle $m_\pm$ as a non-singular fiber $p^{-1}(r_\pm)$ moves towards $p^{-1}(s_\pm)$ so that $m_\pm$ collapses. More precisely, the vanishing cycle is moved by the flow of a gradient-like vector field which covers the tangent vector field along a path in $\mathbb{C}\mathrm{P}^1$ connecting $r_\pm$ with $s_\pm$. If this path goes along $\mathbb{R}\mathrm{P}^1$ and the gradient-like vector field is c-invariant (for instance, a c-symmetrization of any given gradient-like vector field), then the disc $D_{m_\pm}$ becomes c-invariant, and hence, the disc D_a^* is also c-invariant.

Finally, note that there is a c_K-equivariant deformation of S_K^*, which contracts disc D_a^* and degenerates the torus S_K^* into S_K. Its construction goes like in the non-equivariant case: we deform S_K using a flow of a vector field tangent to D_a^*. Note that the deformation is equivariant if we choose a c-invariant vector field (it can be done like before, by symmetrization of any sample vector field whose flow contracts S_K^* to S_K). □

5. Proof of Theorem 1.2

Now we can combine together all the ingredients of the proof of Theorem 1.2 and see in details how it works for the configuration $C_{79,44}$ (the case of $C_{89,9}$ is analogous).

5.1. The construction of $C_{79,44}$

We recall that the chain $C_{79,44} = (-2, -5, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -3)$ was constructed in [PSS] as follows. We start with an elliptic surface $\mathbb{C}\mathrm{P}^2 \# 9\overline{\mathbb{C}\mathrm{P}}^2$ which contains a fiber $T_0 = \mathbb{I}_8$ and four fishtails T_i, $i = 1, 2, 3, 4$, two of which, T_2 and T_3, can be used for the double node knot surgery of [FS2] producing a pseudo-section S_K. Blowing up the node of T_1 we obtain a (-4)-sphere, and blowing up at the node of T_4 twice (the second time at the intersection of T_4 with the exceptional curve) we obtain a $(-2, -5)$-chain.

Blowing up T_0 4 times gives a chain-cycle $(-6,-2,-2,-2,-2,-2,-2,-3,-2,-2,-2,-1)$ from which we drop its (-1)-component to obtain a usual chain. The points to blowup are chosen so that S_K intersects this chain at a point of (-6)-sphere. We blowup also the node of S_K to obtain a (-5)-sphere and then smooth its intersection points with (-4) and (-6)-spheres produced by the fibers T_1 and T_0. This gives a (-11)-sphere, and together with $(-2,-5)$ and the remaining part of the long chain we obtain

$$(-2,-5,-11,-2,-2,-2,-2,-2,-2,-3,-2,-2,-2)$$

in $\mathbb{C}\mathrm{P}^2\#17\overline{\mathbb{C}\mathrm{P}}^2$. After blowing up the very last (-2)-component we obtain the configuration $C_{79,44}$ in $\mathbb{C}\mathrm{P}^2\#18\overline{\mathbb{C}\mathrm{P}}^2$. Its rational blowdown yields an exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$. As a knot K for the knot surgery one can use any one from a sequence of the twist knots K_i, which gives infinitely many exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ (pairwise homeomorphic but non-diffeomorphic).

5.2. Construction of $F_i \subset S^4$ by equivariant surgery

In Section 2, Lemma 2.1, we provided a real elliptic surface suitable for the equivariant subsequent constructions, such that $\mathbb{C}\mathrm{P}^2\#9\overline{\mathbb{C}\mathrm{P}}^2/\mathrm{conj} = \mathrm{S}^4$. In Lemma 4.4 we show how to perform the double node knot surgery equivariantly, so that the pseudo-section is c-invariant. By Lemma 4.1 the quotient X/c is preserved. In what follows, all the blowups and smoothings are made at some real points and so are equivariant; real blowups do not change X/c as well. In Lemma 3.1 we described equivariant rational blowdowns and proved that they also do not change the quotient. So, in the quotient $S^4 = X/c$ of an exotic $\mathbb{C}\mathrm{P}^2\#5\overline{\mathbb{C}\mathrm{P}}^2$ we obtain a surface $F = \mathrm{Fix}(c)$. Pairwise non-diffeomorphism of an infinite family of such exotic 4-manifolds implies pairwise non-diffeomorphism of the corresponding embedded surfaces $F_i \subset S^4$.

5.3. The fundamental group $\pi_1(S^4 \smallsetminus F_i)$

In addition we justified that the fundamental group of the complement $S^4 \smallsetminus F = X/c \smallsetminus \mathrm{Fix}(c)$, where (X,c) is the corresponding 4-manifold X with an involution, is preserved abelian under all the equivariant surgery operations involved. At the first step, for $X = \mathbb{C}\mathrm{P}^2\#9\overline{\mathbb{C}\mathrm{P}}^2$, $F = \mathbb{R}\mathrm{P}^2\#9\overline{\mathbb{R}\mathrm{P}}^2$, and c being the complex conjugation, we need a bit stronger fact that the complement $S^4 \smallsetminus (F \cup \mathcal{A})$ has an abelian (and therefore cyclic) group π_1, where A is the annulus membrane on F represented by the quotient of a non-singular real elliptic fiber by the complex conjugation. Van Kampen's theorem easily implies that blowing up $\mathbb{C}\mathrm{P}^2$ at real points outside $\mathcal{A}$ preserves group $S^4 \smallsetminus (F \cup \mathcal{A})$ abelian (note that after blowing up at n points, F becomes $\mathbb{R}\mathrm{P}^2\#(\mathrm{n}+9)\overline{\mathbb{R}\mathrm{P}}^2$, while $S^4 = X/c$ and $\mathcal{A}$ remain the same). The details can be found in [FKV2], see the proof of Proposition 6 in Section 4.7. Note that these facts were also used in [F], and a brief proof was reproduced in the appendix there. In Lemmas 4.2–4.3 we prove that after an equivariant double node knot surgery applied to the real elliptic surface from Lemma 2.1, we obtain a 4-manifold with involution (X,c) such that $S^4 \smallsetminus F = X/c \smallsetminus \mathrm{Fix}(c)$ has an abelian π_1 (and thus, $\pi_1 = \mathbb{Z}/2$).

As we mentioned, the subsequent blowing up does not change π_1, and so to justify that $\pi_1(S^4 \smallsetminus F_i) = \mathbb{Z}/2$ in Theorem 1.2, it is left to prove that π_1 remains abelian after an equivariant rational blowdown of the configuration $C_{79,44}$ that we described in 5.1. Lemma 3.2 proves it under some assumptions on the chain C that we rationally blow down. Let us check that these assumptions are satisfied for $C_{79,44}$.

First, note that the last blowdown in the above construction of $C_{79,44}$ gives a (-1)-sphere which extends the chain $C = C_{79,44}$ to a longer chain $\widetilde{C}$ as required for Lemma 3.2. As we explained in Section 3.2, the condition (1) of Lemma 3.2 is equivalent to non-singularity modulo 2 of the intersection matrix $M_C = (C_i \circ C_j)$. Non-singularity can be justified for instance via modulo 2 blowdown of $C_{79,44}$ (or a reader can directly calculate the determinant). Namely, (mod 2) reduction of $C_{79,44}$ gives $(0,1,1,0,0,0,0,0,0,1,0,0,1)$, and after blowing down 9 times the first occurrence of “1” in the sequence, we obtain $(0,0,0,1)$, which can be then completely blown down. The characteristic sub-configuration of $C_{79,44}$ contains the components C_i with $i = 1,4,6,8,10,12$ (it can be either directly verified, or found using the algorithm explained in Remark 3.1). To verify the assumption (2) of Lemma 3.2 we use Remark 3.3 after this Lemma. The auxiliary (-1)-component extending our configuration C to $\widetilde{C}$ is adjacent to the last component C_{13}. Since (-1) is odd and C_{13} is not characteristic, this assumption is satisfied. $\square$

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