

Intersection forms, fundamental groups and 4-manifolds

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ABSTRACT. This is a short survey of some connections between the intersection form and the fundamental group for smooth and topological 4-manifolds.

1. Introduction

A classical construction of Kervaire [36] shows that any finitely-presented group can be realized as the fundamental group of a closed, oriented smooth 4-manifold M . However, much less is known about other homotopy invariants of 4-manifolds, such as the second homotopy group $\pi_2(M)$, which inherits a $\mathbb{Z}[\pi_1(M, x_0)]$ -module structure via the action of the deck transformations on the universal covering \tilde{M} .

Another basic invariant is the *equivariant intersection form* of a 4-manifold M , defined as the triple $(\pi_1(M, x_0), \pi_2(M), s_M)$, where $x_0 \in M$ is a base-point, and

$$s_M: \pi_2(M) \otimes_{\mathbb{Z}} \pi_2(M) \rightarrow \mathbb{Z}[\pi_1(M, x_0)]$$

is the form defined by counting intersections of immersed 2-spheres (see [59, Chap. 5]). This pairing is Λ -hermitian, in the sense that for all $\lambda \in \Lambda := \mathbb{Z}[\pi_1(M, x_0)]$ we have

$$s_M(\lambda \cdot x, y) = \lambda \cdot s_M(x, y) \quad \text{and} \quad s_M(y, x) = \overline{s_M(x, y)}$$

where $\lambda \mapsto \bar{\lambda}$ is the involution on Λ given by $\bar{g} = g^{-1}$ for $g \in \pi_1(M, x_0)$.

The main topics of interest for the present survey are:

- (1) To what extent does the fundamental group $\pi_1(M, x_0)$ and the equivariant intersection form s_M determine the topology of a closed, oriented 4-manifold M ?
- (2) What special properties hold for the equivariant intersection form if M is a *smooth* 4-manifold ?

The material will be divided into sections according to the complexity of the fundamental group. From now on, all manifolds considered will be closed, connected and oriented.

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2. Simply-connected 4-manifolds

Wall [58], [57] showed in the 1960's that homotopy equivalent, simply-connected smooth 4-manifolds M_1, M_2 are smoothly h -cobordant, and hence are stably diffeomorphic

$$M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$$

for some integer $r \geq 0$. It is still not known whether the existence of such a stable diffeomorphism actually requires more than one copy of $S^2 \times S^2$ (see [46], [37] for other aspects of smooth h -cobordisms).

Spectacular results concerning 4-manifolds were proved in the 1980's by S. Donaldson and M. Freedman, building on work of Atiyah, Casson, Hitchin, Taubes and Uhlenbeck. If M is simply-connected, then $\pi_2(M) \cong \mathbb{Z}^r$ is a free abelian group and the ordinary intersection form

$$q_M: H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

is a symmetric, unimodular bilinear form. The signature of this form, denoted $\text{sign}(M)$, is the difference between the number of positive and negative eigenvalues of a matrix representing q_M .

Freedman [16], [17] proved that any such form is realized by one or two *topological* 4-manifolds. Moreover, M is classified up to homeomorphism by q_M and the Kirby-Siebenmann invariant $KS(M) \in \mathbb{Z}/2$ (see [38] for the definition). Donaldson [6], [7], [8] showed using gauge theory that if q_M is a positive definite form then

$$q_M \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$$

is standard, and that the h -cobordant, smooth, simply-connected 4-manifolds are not necessarily diffeomorphic.

These results show a striking difference between smooth and topological 4-manifolds. By combining them, it follows that a smooth, non-spin, simply-connected 4-manifold M is homeomorphic to a connected sum of copies of $\pm \mathbf{CP}^2$. If M is smooth, simply-connected and spin, then M is homeomorphic to a connected sum of copies of $S^2 \times S^2$ and $\pm K3$ surfaces, provided that

$$b_2(M) \geq \frac{11}{8} |\text{sign}(M)|,$$

where $b_2(M) = \text{rank}(H_2(M; \mathbb{Z}))$. The well-known $\frac{11}{8}$ -conjecture, still unresolved, states that this inequality always holds for smooth, spin 4-manifolds: the best partial result to date is $b_2(M) \geq \frac{5}{4} |\text{sign}(M)| + 2$, if q_M is indefinite, proved by Furuta [20]. The exciting subsequent developments in the study of smooth, simply-connected 4-manifolds are outside the scope of this survey (there is a large and growing literature: for example, the work of Fintushel-Stern [13], [14], Gompf [21], Friedman-Morgan [19], Kronheimer-Mrowka [41], Ozsváth-Szabó [48], [47], Jongil Park [50], Taubes [53], and Seiberg-Witten [61]).

3. Infinite cyclic fundamental groups

If $\pi_1(M) = \mathbb{Z}$ and $\Lambda = \mathbb{Z}[\mathbb{Z}]$, then $\pi_2(M)$ is a finitely-generated, free Λ -module of rank $b_2(M)$ and s_M is a non-singular hermitian form. The classification theorem for these manifolds uses the full equivariant intersection form.

Theorem 3.1 (Freedman-Quinn [17]). *A closed, oriented topological 4-manifold M with $\pi_1(M) = \mathbb{Z}$ is classified up to homeomorphism by s_M and $KS(M)$. Any non-singular hermitian form on a finitely-generated free Λ -module can be realized by one or two manifolds.*

More precisely, such forms are *even* and realized by a unique spin manifold, or *odd* and realized by two non-spin manifolds with different Kirby-Siebenmann invariants. The equivariant intersection form of a connected sum $M = (S^1 \times S^3) \# N$, with a 1-connected manifold N , is said to be *extended from the integers*. In other words, $s_M = q_N \otimes_{\mathbb{Z}} \Lambda$. Conversely, by the classification theorem, any manifold whose equivariant intersection form is extended from the integers must be homeomorphic to a connected sum with $S^1 \times S^3$.

Fintushel and Stern [12] constructed a smooth 4-manifold M , which was homeomorphic but not diffeomorphic to a connected sum with $S^1 \times S^3$. The existence of indecomposable topological 4-manifolds with $\pi_1 = \mathbb{Z}$ and $\chi(M) > 0$ was settled later.

Theorem 3.2 ([23]). *There exists a closed, oriented topological 4-manifold M with $\pi_1(M) = \mathbb{Z}$ and $\chi(M) = 4$, and M is not homotopy equivalent to a connected sum $(S^1 \times S^3) \# N$ for any 1-connected N .*

The main step in the proof was the construction of a non-extended hermitian form L on a free Λ -module (using a certain odd, definite, rank 4 form over $\mathbb{Z}[t]$ found by Quebbemann [51, §6]). We also showed that any 4-manifold M with $\pi_1(M) = \mathbb{Z}$ and $b_2(M) - |\text{sign}(M)| \geq 6$ splits off $S^1 \times S^3$ and is determined up to homeomorphism by the explicit invariants b_2 , sign , w_2 and KS .

Question. If M is a smooth, closed, oriented 4-manifold with $\pi_1(M) = \mathbb{Z}$, then is s_M extended from the integers?

This is a natural question after comparing the example $M = M_L$ in Theorem 3.2 with the Fintushel-Stern example.

Theorem 3.3 ([18]). *The manifold M_L is not smoothable.*

The idea of the proof is to consider the n -fold cyclic coverings $M_n \rightarrow M_L$. Since q_{M_L} is standard of rank 4, and both Euler characteristic and signature multiply by the index of a finite covering, the forms q_{M_n} of rank $= 4n$ are all definite, odd, unimodular forms over \mathbb{Z} . This seems to be an interesting series of definite forms: we showed that for $n \geq 3$ they were all non-standard, and for $n = 3, 4$ they were the unique indecomposable odd lattices in dimension 12 and 16 respectively. In any case, by Donaldson's theorem M_n is non-smoothable for $n \geq 3$ and hence M_L is non-smoothable. In [18] we found many more

examples of non-extended forms, and manifolds realizing these forms with a wide variety of other infinite fundamental groups.

4. The quadratic 2-type and surgery

In the non simply-connected case, the obvious homotopy invariants are the equivariant intersection form s_M and the first k -invariant

$$k_M \in H^3(\pi; \pi_2(M)),$$

which together with $\pi := \pi_1(M, x_0)$ and π_2 specifies the algebraic 2-type $B = B(M)$ as introduced by MacLane and Whitehead [44]. The space B is a fibration over $K(\pi, 1)$, classified by k_M , with fibre $K(\pi_2(M), 2)$ and there is a 3-connected reference map $\tilde{c}: M \rightarrow B(M)$ lifting the classifying map $c: M \rightarrow K(\pi, 1)$ for the universal covering $\tilde{M} \rightarrow M$. In [22] we introduced the *quadratic 2-type* of M as the quadruple

$$[\pi_1(M, x_0), \pi_2(M), k_M, s_M] .$$

An *isometry* of two such quadruples is an isomorphism on π_1, π_2 inducing an isometry of the equivariant intersection forms, and respecting the k -invariants.

In general, not much is known about these homotopy invariants, but they are related by an exact sequence

$$0 \rightarrow H^2(\pi; \Lambda) \rightarrow H^2(M; \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda) \rightarrow H^3(\pi; \Lambda) \rightarrow 0 \quad (1)$$

arising from the universal coefficient spectral sequence. In this sequence, $H^2(M; \Lambda) \cong H_2(M; \Lambda) \cong \pi_2(M)$ by Poincaré duality, and the middle map

$$H^2(M; \Lambda) \rightarrow \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda)$$

is the adjoint of s_M . The radical $R(s_M)$ of the intersection form s_M is isomorphic to the π -module $R(\pi) := H^2(\pi; \Lambda)$, and $\pi_2(M)$ is a finitely-generated Λ -module.

If $\pi := \pi_1(M, x_0)$ is a non-trivial finite group, then $\pi_2(M)$ is a finitely-generated free abelian group with a $\Lambda := \mathbb{Z}\pi$ -module structure, as studied in integral representation theory. In general, there are infinitely many non-isomorphic indecomposable integral representations (e.g. for $\pi = \mathbb{Z}/p \times \mathbb{Z}/p$), and there is no known classification. If $\pi_1(M)$ is infinite, the precise structure of $\pi_2(M)$ is unknown except in very special cases, such as $\pi_1(M) = \mathbb{Z}$ mentioned in Section 3.

The study of these modules can be simplified somewhat by considering stable equivalence classes: two modules L_1, L_2 are *stably isomorphic*, denoted $L_1 \simeq_s L_2$, if there exists a free module Λ^r such that $L_1 \oplus \Lambda^r \cong L_2 \oplus \Lambda^r$. For example, the kernel

$$0 \rightarrow \Omega^{n+1}\mathbb{Z} \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0$$

after n -steps in a free resolution $\{F_*\}$ of the trivial module \mathbb{Z} is stably unique by Schanuel's Lemma. For $n = 3$, such modules arise as $\pi_2(K) = H_2(K; \Lambda)$, where K is a finite 2-complex with $\pi_1(K, x_0) = \pi$, and the resolution is obtained from the chain complex $C_*(\tilde{K})$ of the universal covering. Finite 2-complexes K provide examples of smooth

4-manifolds by taking the boundary of a thickening (i.e. a regular neighbourhood) of K in \mathbb{R}^5 .

The stabilization operation in algebra has analogues in topology. For 2-complexes, $K \mapsto K \vee S^2$ gives the stabilization $\pi_2(K) \mapsto \pi_2(K) \oplus \Lambda$. Whitehead [60, Theorem 19] showed that any two finite 2-complexes K, K' with isomorphic fundamental groups are stably simple-homotopy equivalent, but the problem of finding the minimal Euler characteristic realized by a 2-complex with given π_1 is still unsolved. This is a cancellation problem. Note that Whitehead's Theorem implies that the stable isomorphism type of the Λ -modules $H_2(K; \Lambda)$ and $H^2(K; \Lambda)$ depend only on the fundamental group, and not on the choice of finite 2-complex K .

It turns out that the stable structure of $\pi_2(M)$ for a 4-manifold is very special.

Theorem 4.2. *Let M be a closed, oriented 4-manifold with fundamental group π . Then $\pi_2(M)$ is stably isomorphic as a Λ -module to a certain extension*

$$\mathcal{E}_M : 0 \rightarrow H_2(K; \Lambda) \rightarrow E \rightarrow H^2(K; \Lambda) \rightarrow 0$$

where K is any finite 2-complex with $\pi_1(K, x_0) = \pi$.

Remark 4.3. The boundaries of thickenings of 2-complexes yield trivial extensions. The finite fundamental group case was done in [22, 2.4], and in that case the extension class of \mathcal{E}_M corresponds to the image of the fundamental class $c_*[M] \in H_4(\pi; \mathbb{Z})$ under a natural isomorphism $\theta: H_4(\pi; \mathbb{Z}) \cong \text{Ext}_\Lambda^1(H^2(K; \Lambda), H_2(K; \Lambda))$.

Proof. We will use a chain complex argument. By stabilizing $K \mapsto K \vee rS^2$ and $M \mapsto M \# t(S^2 \times S^2)$ if necessary, we may assume that K is the sub-complex of 2-cells of M . Consider the cellular chain complex $C_* = C_*(\widehat{M})$ of finitely-generated free Λ -modules. We have the exact sequences

$$0 \rightarrow \mathcal{Z}_2 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0$$

and

$$0 \rightarrow \mathfrak{B}_3^* \rightarrow C_3^* \rightarrow C_4^* \rightarrow \mathbb{Z} \rightarrow 0$$

showing that $\mathfrak{B}_3^* = \text{Hom}_\Lambda(\mathfrak{B}_3, \Lambda)$ is stably isomorphic to the 2-boundaries \mathfrak{B}_2 . The details here depend on whether π is finite or infinite: in the latter case note that

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$\text{Ext}_\Lambda^1(\mathfrak{B}_3, \Lambda) \cong H^4(M; \Lambda) = \mathbb{Z}$. We now form the pull-back diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & H^2(M; \Lambda) & \longrightarrow & H^2(K; \Lambda) & \longrightarrow & \mathfrak{B}_2 \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & H^2(M; \Lambda) & \longrightarrow & E & \longrightarrow & C_2 \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & H_2(K; \Lambda) & \equiv & \mathcal{Z}_2 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array} \tag{4}$$

where $\mathcal{Z}_2 = H_2(K; \Lambda)$ since K is the 2-skeleton of M . The middle horizontal sequence splits since C_2 is a free Λ -module. The middle vertical sequence is \mathcal{E}_M , and $E \cong \pi_2(M) \oplus C_2$ is a stabilization of $\pi_2(M)$. \square

Surgery theory as developed by Browder, Novikov, Sullivan and Wall [59] provides a powerful framework for classifying manifolds of dimension ≥ 5 within a fixed homotopy type. However, in dimension 4 there are serious obstacles arising from the failure of the Whitney trick. One approach, developed by Cappell and Shaneson [3], is based on Wall’s idea of studying smooth 4-manifolds after stabilization with copies of $S^2 \times S^2$. The drawback is that information about the original (unstabilized) homotopy type is lost in the process.

Freedman’s work [16] fully established 4-dimensional surgery theory for topological manifolds whose fundamental groups do not “grow” too quickly. This class includes the poly-cyclic by finite groups, but it is not known at present if 4-dimensional topological surgery theory works for (non-cyclic) free fundamental groups. Note that Donaldson’s results show that smooth surgery theory definitely does not work in dimension 4, and there are s -cobordant smooth 4-manifolds which are not diffeomorphic.

The modified surgery theory of M. Kreck [39] requires less initial information about the homotopy type: for example, one can try to classify smooth 4-manifolds which have the same algebraic 2-type (up to smooth s -cobordism). In this theory, the key step is to compute certain bordism groups $\Omega_4(B, \xi)$, where ξ is a bundle over B whose pullback $c^*(\xi) \cong \nu_M$ is the stable normal bundle of M . For such computations there are a variety of methods available, including the Atiyah-Hirzebruch and Adams spectral sequences. If two manifolds $[M_1, \tilde{c}_1], [M_2, \tilde{c}_2]$ are bordant over the type (B, ξ) , then the triviality of an algebraically defined invariant implies that M_1 and M_2 are smoothly s -cobordant (see [39, Theorem B]).

One possible way of analysing the final step is to note that the relation $[M_1, \tilde{c}_1] = [M_2, \tilde{c}_2] \in \Omega_4(B, \xi)$ implies that

$$M_1 \# r_1(S^2 \times S^2) \cong M_2 \# r_2(S^2 \times S^2)$$

are stably diffeomorphic [39, Cor. 3], with control on the reference maps to B (see [40], [5] for further applications of stabilization). The *cancellation* problem is to find techniques for removing $S^2 \times S^2$ factors from both sides. In algebra, cancellation theorems for modules and quadratic forms over noetherian rings were proved by Bak [1], Bass [2], Stafford [52] and Vaserstein [56]. In [23], we realized that these algebraic results could be combined with the constructions by [3, 1.5] of self-diffeomorphisms of 4-manifolds to prove cancellation theorems for certain 4-manifolds. For example, the integral group rings $\mathbb{Z}[\pi]$ of poly-cyclic by finite groups are noetherian rings, but the group ring of a free group on 2 generators is not noetherian. This theme has recently been taken up again in [4].

5. Finite fundamental group

In early joint work with M. Kreck [23] we studied the topology of 4-manifolds with finite fundamental groups, and obtained a good description of the homotopy types within a prescribed algebraic 2-type. We also showed that there are only finitely many homeomorphism types of closed, oriented 4-manifolds with given finite π_1 , and given Euler characteristic (see [22, p. 87]). To obtain precise classification results up to homeomorphism we needed to restrict to special fundamental groups. We say that M has w_2 -type (I) if $w_2(\widetilde{M}) \neq 0$, w_2 -type (II) if $w_2(M) = 0$, and w_2 -type (III) if $w_2(M) \neq 0$ but $w_2(\widetilde{M}) = 0$.

Theorem 5.1 ([22], [24]). *Closed, oriented topological 4-manifolds with finite cyclic fundamental groups are classified up to homeomorphism by $\pi_1(M)$, q_M , the w_2 -type, and $KS(M)$.*

This is a generalization of Freedman's Theorem in the simply-connected case (where the w_2 -type is determined by the intersection form). One interesting consequence is that certain automorphisms of the cohomology ring of a smooth 4-manifold are induced by self-homeomorphisms but not by a self-diffeomorphism (Donaldson's work is used to rule out a self-diffeomorphism, see the example [22, p.87]).

However, for more complicated fundamental groups we can not expect a classification in terms of the ordinary intersection form q_M on $H_2(M; \mathbb{Z})$ (see [54], [55]). Here is a sample result involving the quadratic 2-type.

Theorem 5.2. *Closed, oriented, topological 4-manifolds with $w_2(M) = 0$ and odd order finite fundamental groups are classified up to homeomorphism by the simple isometry class of the quadratic 2-type $[\pi_1(M), \pi_2(M), k_M, s_M]$.*

Proof. This is essentially an exercise in the methods of [39] and [25], and the information needed to use the odd order assumption is provided by [26, Section 4]. The definition of *simple isometry class* will be explained below.

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Here are some details of the steps in the argument. We first notice that the normal 2-type for such a spin manifold M is $B \times BTOPSPIN$, where $B = B(M)$ is the algebraic 2-type. Since $\pi := \pi_1(M)$ has odd order,

$$\Omega_4^{TOPSPIN}(K(\pi, 1)) = \mathbb{Z} \oplus H_4(\pi; \mathbb{Z}),$$

and the stable homeomorphism class of M is determined by its signature and the image of the fundamental class $c_*[M] \in H_4(\pi; \mathbb{Z})$. If M and M' have isometric quadratic 2-types, then there exists an isometry $\alpha: s_M \cong s_{M'}$ respecting the k -invariants. We use α to identify the 2-types $B(M) \cong B(M')$, and conclude that the images of their fundamental classes agree by [26, p. 168]. Hence M and M' are spin bordant over B , and there exists a stable homeomorphism

$$h: M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2)$$

respecting the reference maps to $K(\pi, 1)$. If α is a simple isometry of the quadratic 2-types, then we claim that this data will allow us to construct another stable homeomorphism h' between these manifolds, with the additional property that the induced isometry h'_* of equivariant intersection forms induces the identity on the hyperbolic summands (in fact, we will obtain $h'_* = (\alpha \oplus 1)$). With that additional property, one can attach handles to both domain and range to obtain an s -cobordism between M and M' (see [39, §4]).

To modify the homeomorphism h we proceed as follows. Let $M_r := M \# r(S^2 \times S^2)$ denote the r -fold stabilization of M . By composition, we obtain an element

$$\beta := (\alpha \oplus 1)^{-1} \circ h_* \in \text{Isom}[\pi_1(M_r), \pi_2(M_r), k_{M_r}, s_{M_r}].$$

The braid diagram of [26, p. 168], combined with [26, Theorem B], now shows that $\beta = \phi_*$ for some $\phi \in \text{Aut}_\bullet(M_r)$, such that ϕ is induced by an inertial s -cobordism $(W; M_r, -M_r)$. We have further assumed that α is a *simple* isometry of the quadratic 2-types. By definition, this means that the Whitehead torsion $\tau(\phi) = 0 \in \text{Wh}(\mathbb{Z}\pi)$, and hence $\tau(W, M_r) = u \in \text{Wh}(\mathbb{Z}\pi)$ is self-dual ($\bar{u} = u$). Note that this definition of a simple isometry is independent of the choice of h -cobordism inducing ϕ . From the exact sequences in the proof of [26, 4.1], and the fact that the discriminant map $L_6^h(\mathbb{Z}\pi) \rightarrow \widehat{H}^0(\mathbb{Z}/2; \text{Wh}(\mathbb{Z}\pi))$ is surjective (since π has odd order), we can realize the self-equivalence ϕ by an s -cobordism W' : if necessary, we modify our first choice by the action of $L_6^h(\mathbb{Z}\pi)$ on $\mathcal{H}(M)$. It follows that the homotopy self-equivalence induced by W' is realized by a self-homeomorphism $f: M_r \rightarrow M_r$. We now define $h' := h \circ f^{-1}$ and notice that $h'_* = \alpha \oplus 1$, as required. \square

Stabilization and cancellation techniques can also be used effectively for manifolds with arbitrary finite fundamental groups (see [25]). For 4-manifolds, the connected sum operation gives the stabilization

$$\pi_2(M \# (S^2 \times S^2)) = \pi_2(M) \oplus \Lambda \oplus \Lambda$$

where the equivariant intersection form is stabilized by adding a hyperbolic plane

$$H(\Lambda) = (\Lambda \oplus \Lambda, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

The cancellation problem for 4-manifolds with finite fundamental group has the following optimal solution:

Theorem 5.3 ([25]). *Let M, M' be closed, oriented topological 4-manifolds with finite fundamental group. If $M = M_0 \# (S^2 \times S^2)$, and*

$$M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2)$$

then $M \cong M'$.

Note that even in the simply-connected case, non-isomorphic forms can become isomorphic after adding a hyperbolic plane, so the statement is best possible.

6. Fundamental groups of aspherical 2-complexes

A finitely-presented group π is *geometrically 2-dimensional* ($\text{g-dim } \pi \leq 2$) if there exists a finite aspherical 2-complex with fundamental group π . Examples of geometrically 2-dimensional groups include free groups, 1-relator groups (e.g. surface groups) and small cancellation groups [43], provided they are torsion-free, as well as many word-hyperbolic groups, see also [31, 2.3], [34, §10].

Recall that the radical $R(s_M)$ of the equivariant intersection form s_M is isomorphic to the π -module $R(\pi) := H^2(\pi; \Lambda)$. A closed oriented 4-manifold M will be called *minimal* if the equivariant intersection form on $\pi_2(M)$ vanishes, or equivalently, if $\pi_2(M) = R(s_M) \cong R(\pi)$. It turns out that a thickening of an aspherical 2-complex for π gives a minimal, smooth 4-manifold M_0 with fundamental group π , whenever $\text{g-dim } \pi \leq 2$ (see [27, Lemma 3.7]). For example, the manifolds $\#_r(S^1 \times S^3)$ and $S^2 \times \Sigma$ are minimal, where Σ denotes an oriented surface of genus ≥ 1 .

In a series of papers [31], [33], [32], [34], [35], J. Hillman investigated the homotopy classification of Poincaré 4-complexes under various fundamental group assumptions. In the case of $\text{g-dim } \leq 2$, the problem was reduced to the minimal case, where the homotopy classification was completed for free or surface fundamental groups (see also [49] where these cases were studied from a different viewpoint).

In recent joint work with Matthias Kreck and Peter Teichner (described below), we used the modified surgery approach to obtain classification results for topological 4-manifolds with geometrically 2-dimensional fundamental groups, up to homeomorphism (in favourable cases) or s -cobordism.

A particular nice family of examples is provided by the solvable Baumslag-Solitar groups

$$BS(k) := \{a, b \mid aba^{-1} = b^k\}, \quad k \in \mathbb{Z}.$$

The groups $BS(k)$ have geometrical dimension ≤ 2 because the 2-complex corresponding to the above presentation is aspherical. The easiest cases are

$$BS(0) = \mathbb{Z}, \quad BS(1) = \mathbb{Z} \times \mathbb{Z}, \quad \text{and } BS(-1) = \mathbb{Z} \rtimes \mathbb{Z},$$

and these are the only Poincaré duality groups in this family. Each $BS(k)$ is solvable, so is a “good” fundamental group for topological 4-manifolds [16]. This implies that Freedman’s s -cobordism theorem is available to complete the homeomorphism classification. This had been done previously only for the three special cases above, see [17] for $BS(0)$, and [33] for $BS(\pm 1)$, using a more classical surgery approach.

Theorem 6.1 ([27, Theorem A]). *For closed oriented 4-manifolds with solvable Baumslag-Solitar fundamental groups, and given w_2 -type and Kirby-Siebenmann invariant, any isometry between equivariant intersection forms can be realized by a homeomorphism.*

In particular, we showed that a minimal 4-manifold is unique up to homeomorphism and established some relations between the invariants in general (based in part on [55]). For fundamental groups π with $H_4(\pi; \mathbb{Z}) = 0$ we showed that the signature is determined by s_M via the formula $\text{sign}(M) = \text{sign}(s_M \otimes_{\Lambda} \mathbb{Z})$. This formula does not hold for arbitrary 4-manifolds, as one can see from examples of surface bundles over surfaces with nontrivial signature (but vanishing π_2).

For $\pi_1(M) = BS(k)$, type (III) can only occur if k is odd. In this case, we gave a generalization of Rochlin’s formula (see [27, Corollary 6.10]):

$$KS(M) \equiv \text{sign}(M)/8 + \text{Arf}(M) \pmod{2}$$

where $\text{Arf}(M) \in \mathbb{Z}/2$ is a codimension 2 Arf invariant. In contrast, for spin manifolds $KS(M) \equiv \text{sign}(M)/8 \pmod{2}$.

We also proved a realization theorem for hermitian forms in this setting. If M has fundamental group $BS(k)$, then the quotient module $\pi_2(M)^\dagger := \pi_2(M)/R(s_M)$ is a finitely-generated, stably-free Λ -module, and the equivariant intersection form s_M is non-singular on this quotient. It turns out that any such hermitian form can be realized by one or two 4-manifolds.

A close inspection of the arguments shows that we used a number of special facts about the Baumslag-Solitar groups. For more general fundamental groups π , we need to assume the corresponding properties for its algebraic K -theory and L -theory.

Definition 6.2. A group π satisfies properties (W-AA) whenever

- (1) The Whitehead group $\text{Wh}(\pi)$ vanishes,
- (2) The assembly map $A_5: H_5(\pi; \mathbb{L}_0) \rightarrow L_5(\mathbb{Z}\pi)$ is surjective.
- (3) The assembly map $A_4: H_4(\pi; \mathbb{L}_0) \rightarrow L_4(\mathbb{Z}\pi)$ is injective.

Note that these properties (and more) do hold whenever the group π satisfies the Farrell-Jones isomorphism conjectures [11] (see [42] for a survey of results on these conjectures).

Theorem 6.3 ([27, Theorem C]). *Let π be a geometrically 2-dimensional group satisfying properties (W-AA). For closed oriented 4-manifolds with fundamental group π , and given Kirby-Siebenmann invariant, any isometry between equivariant intersection forms inducing an isomorphism of w_2 -types can be realized by an s -cobordism.*

The w_2 -type mentioned in this statement is actually a refinement of the notion defined in Section 4, in which we now keep track of the class $w \in H^2(\pi; \mathbb{Z}/2)$ determining $w_2(M)$ in type (III).

7. Some questions

Here are a few questions and problems concerning smooth and topological 4-manifolds with non-trivial fundamental group.

- (1) For a smooth 4-manifold M with geometrically 2-dimensional fundamental group, is M homeomorphic to $M_0 \# N$, where M_0 is minimal and N is simply-connected? In other words, is the equivariant intersection form s_M always extended from the integers?
- (2) Construct distinct smooth structures on indecomposable, non-simply connected 4-manifolds. Is there a minimal 4-manifold with more than one smooth structure?
- (3) For a given group π , there exist 4-manifolds $M(\alpha)$ with $\pi_1(M) = \pi$ and $c_*[M]$ a given element $\alpha \in H_4(\pi; \mathbb{Z})$. How does the minimal possible Euler characteristic and signature of $M(\alpha)$ depend on the class α ?
- (4) For a given group π , does there exist a stable range constant $c(\pi)$, with the property that a stable homeomorphism or diffeomorphism $M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$ between manifolds with fundamental group π admits cancellation of at least one copy of $S^2 \times S^2$ (up to s -cobordism) whenever $r > c(\pi)$?
- (5) Compare the actions of $\text{Diff}(M)$ and $\text{Homeo}(M)$ on the equivariant intersection form of a smooth 4-manifold.

Remark 7.1. There are many interesting problems related to the study the existence and uniqueness of non-free finite group actions on smooth or topological 4-manifolds. One may ask, for example, which equivariant intersection forms are realized by smooth actions of finite cyclic groups on simply-connected 4-manifolds. For topological actions there is a satisfactory picture, particularly for cyclic groups of prime order (see Edmonds [9], [10], Edmonds-Ewing [9], and McCooey [45]). For smooth actions, there are restrictions detected by equivariant gauge theory [28], [29] and the answer is interesting even for the permutation representations which arise for actions on connected sums of \mathbf{CP}^2 's (see [30, 1.18]). A striking contrast between smooth and topological actions is shown by the recent paper of Finstushel, Stern and Sunukjian [15], where infinite families of topologically equivalent but smoothly distinct cyclic group actions are constructed on 4-manifolds with non-trivial Seiberg-Witten invariants.

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