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Intersection forms, fundamental groups and 4-manifolds

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Abstract. This is a short survey of some connections between the intersection form and the fundamental group for smooth and topological 4-manifolds.

1. Introduction

A classical construction of Kervaire [36] shows that any finitely-presented group can be realized as the fundamental group of a closed, oriented smooth 4-manifold \( M \). However, much less is known about other homotopy invariants of 4-manifolds, such as the second homotopy group \( \pi_2(M) \), which inherits a \( \mathbb{Z}[\pi_1(M, x_0)] \)-module structure via the action of the deck transformations on the universal covering \( \tilde{M} \).

Another basic invariant is the equivariant intersection form of a 4-manifold \( M \), defined as the triple \((\pi_1(M, x_0), \pi_2(M), s_M)\), where \( x_0 \in M \) is a base-point, and

\[ s_M : \pi_2(M) \otimes \mathbb{Z} \pi_2(M) \rightarrow \mathbb{Z}[\pi_1(M, x_0)] \]

is the form defined by counting intersections of immersed 2-spheres (see [59, Chap. 5]). This pairing is \( \Lambda \)-hermitian, in the sense that for all \( \lambda \in \Lambda := \mathbb{Z}[\pi_1(M, x_0)] \) we have

\[ s_M(\lambda \cdot x, y) = \lambda \cdot s_M(x, y) \quad \text{and} \quad s_M(y, x) = s_M(x, y) \]

where \( \lambda \mapsto \bar{\lambda} \) is the involution on \( \Lambda \) given by \( \bar{g} = g^{-1} \) for \( g \in \pi_1(M, x_0) \).

The main topics of interest for the present survey are:

(1) To what extent does the fundamental group \( \pi_1(M, x_0) \) and the equivariant intersection form \( s_M \) determine the topology of a closed, oriented 4-manifold \( M \) ?

(2) What special properties hold for the equivariant intersection form if \( M \) is a smooth 4-manifold ?

The material will be divided into sections according to the complexity of the fundamental group. From now on, all manifolds considered will be closed, connected and oriented.

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2. Simply-connected 4-manifolds

Wall [58], [57] showed in the 1960’s that homotopy equivalent, simply-connected smooth 4-manifolds $M_1$, $M_2$ are smoothly $h$-cobordant, and hence are stably diffeomorphic

$$M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$$

for some integer $r \geq 0$. It is still not known whether the existence of such a stable diffeomorphism actually requires more than one copy of $S^2 \times S^2$ (see [46], [37] for other aspects of smooth $h$-cobordisms).

Spectacular results concerning 4-manifolds were proved in the 1980’s by S. Donaldson and M. Freedman, building on work of Atiyah, Casson, Hitchin, Taubes and Uhlenbeck. If $M$ is simply-connected, then $\pi_2(M) \cong \mathbb{Z}^r$ is a free abelian group and the ordinary intersection form

$$q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$$

is a symmetric, unimodular bilinear form. The signature of this form, denoted $\text{sign}(M)$, is the difference between the number of positive and negative eigenvalues of a matrix representing $q_M$.

Freedman [16], [17] proved that any such form is realized by one or two topological 4-manifolds. Moreover, $M$ is classified up to homeomorphism by $q_M$ and the Kirby-Siebenmann invariant $KS(M) \in \mathbb{Z}/2$ (see [38] for the definition). Donaldson [6], [7], [8] showed using gauge theory that if $q_M$ is a positive definite form then

$$q_M \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$$

is standard, and that the $h$-cobordant, smooth, simply-connected 4-manifolds are not necessarily diffeomorphic.

These results show a striking difference between smooth and topological 4-manifolds. By combining them, it follows that a smooth, non-spin, simply-connected 4-manifold $M$ is homeomorphic to a connected sum of copies of $\pm \mathbb{C}P^2$. If $M$ is smooth, simply-connected and spin, then $M$ is homeomorphic to a connected sum of copies of $S^2 \times S^2$ and $\pm K3$ surfaces, provided that

$$b_2(M) \geq \frac{11}{8} |\text{sign}(M)|,$$

where $b_2(M) = \text{rank}(H_2(M;\mathbb{Z}))$. The well-known $\frac{11}{8}$-conjecture, still unresolved, states that this inequality always holds for smooth, spin 4-manifolds: the best partial result to date is $b_2(M) \geq \frac{1}{2} |\text{sign}(M)| + 2$, if $q_M$ is indefinite, proved by Furuta [20]. The exciting subsequent developments in the study of smooth, simply-connected 4-manifolds are outside the scope of this survey (there is a large and growing literature: for example, the work of Fintushel-Stern [13], [14], Gompf [21], Friedman-Morgan [19], Kronheimer-Mrowka [41], Ozsváth-Szabó [48], [47], Jongil Park [50], Taubes [53], and Seiberg-Witten [61]).
3. Infinite cyclic fundamental groups

If \( \pi_1(M) = \mathbb{Z} \) and \( \Lambda = \mathbb{Z}[\mathbb{Z}] \), then \( \pi_2(M) \) is a finitely-generated, free \( \Lambda \)-module of rank \( b_2(M) \) and \( s_M \) is a non-singular hermitian form. The classification theorem for these manifolds uses the full equivariant intersection form.

**Theorem 3.1** (Freedman-Quinn [17]). A closed, oriented topological 4-manifold \( M \) with \( \pi_1(M) = \mathbb{Z} \) is classified up to homeomorphism by \( s_M \) and \( KS(M) \). Any non-singular hermitian form on a finitely-generated free \( \Lambda \)-module can be realized by one or two manifolds.

More precisely, such forms are even and realized by a unique spin manifold, or odd and realized by two non-spin manifolds with different Kirby-Siebenmann invariants. The equivariant intersection form of a connected sum \( M = (S^1 \times S^3) \# N \), with a 1-connected manifold \( N \), is said to be extended from the integers. In other words, \( s_M = q_N \otimes_{\mathbb{Z}} \Lambda \).

Conversely, by the classification theorem, any manifold whose equivariant intersection form is extended from the integers must be homeomorphic to a connected sum with \( S^1 \times S^3 \).

Fintushel and Stern [12] constructed a smooth 4-manifold \( M \), which was homeomorphic but not diffeomorphic to a connected sum with \( S^1 \times S^3 \). The existence of indecomposable topological 4-manifolds with \( \pi_1 = \mathbb{Z} \) and \( \chi(M) > 0 \) was settled later.

**Theorem 3.2** ([23]). There exists a closed, oriented topological 4-manifold \( M \) with \( \pi_1(M) = \mathbb{Z} \) and \( \chi(M) = 4 \), and \( M \) is not homotopy equivalent to a connected sum \( (S^1 \times S^3) \# N \) for any 1-connected \( N \).

The main step in the proof was the construction of a non-extended hermitian form \( L \) on a free \( \Lambda \)-module (using a certain odd, definite, rank 4 form over \( \mathbb{Z}[t] \) found by Quebbemann [51, §6]). We also showed that any 4-manifold \( M \) with \( \pi_1(M) = \mathbb{Z} \) and \( b_2(M) - |\text{sign}(M)| \geq 6 \) splits off \( S^1 \times S^3 \) and is determined up to homeomorphism by the explicit invariants \( b_2 \), \( \text{sign} \), \( w_2 \) and \( KS \).

**Question.** If \( M \) is a smooth, closed, oriented 4-manifold with \( \pi_1(M) = \mathbb{Z} \), then is \( s_M \) extended from the integers ?

This is a natural question after comparing the example \( M = M_L \) in Theorem 3.2 with the Fintushel-Stern example.

**Theorem 3.3** ([18]). The manifold \( M_L \) is not smoothable.

The idea of the proof is to consider the \( n \)-fold cyclic coverings \( M_n \to M_L \). Since \( q_{M_L} \) is standard of rank 4, and both Euler characteristic and signature multiply by the index of a finite covering, the forms \( q_{M_n} \) of rank = 4n are all definite, odd, unimodular forms over \( \mathbb{Z} \). This seems to be an interesting series of definite forms: we showed that for \( n \geq 3 \) they were all non-standard, and for \( n = 3, 4 \) they were the unique indecomposable odd lattices in dimension 12 and 16 respectively. In any case, by Donaldson’s theorem \( M_n \) is non-smoothable for \( n \geq 3 \) and hence \( M_L \) is non-smoothable. In [18] we found many more...
examples of non-extended forms, and manifolds realizing these forms with a wide variety of other infinite fundamental groups.

4. The quadratic 2-type and surgery

In the non simply-connected case, the obvious homotopy invariants are the equivariant intersection form \( s_M \) and the first \( k \)-invariant

\[
k_M \in H^3(\pi; \pi_2(M)),
\]

which together with \( \pi := \pi_1(M, x_0) \) and \( \pi_2 \) specifies the algebraic 2-type \( B = B(M) \) as introduced by MacLane and Whitehead [44]. The space \( B \) is a fibration over \( K(\pi, 1) \), classified by \( k_M \), with fibre \( K(\pi_2(M), 2) \) and there is a 3-connected reference map \( c: M \to B(M) \) lifting the classifying map \( c: M \to K(\pi, 1) \) for the universal covering \( \tilde{M} \to M \). In [22] we introduced the quadratic 2-type of \( M \) as the quadruple

\[
[\pi_1(M, x_0), \pi_2(M), k_M, s_M].
\]

An isometry of two such quadruples is an isomorphism on \( \pi_1, \pi_2 \) inducing an isometry of the equivariant intersection forms, and respecting the \( k \)-invariants.

In general, not much is known about these homotopy invariants, but they are related by an exact sequence

\[
0 \to H^2(\pi; \Lambda) \to H^2(M; \Lambda) \to \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda) \to H^3(\pi; \Lambda) \to 0 \tag{1}
\]

arising from the universal coefficient spectral sequence. In this sequence, \( H^2(M; \Lambda) \cong H_2(M; \Lambda) \cong \pi_2(M) \) by Poincaré duality, and the middle map

\[
H^2(M; \Lambda) \to \text{Hom}_\Lambda(H_2(M; \Lambda), \Lambda)
\]

is the adjoint of \( s_M \). The radical \( R(s_M) \) of the intersection form \( s_M \) is isomorphic to the \( \pi \)-module \( R(\pi) := H^2(\pi; \Lambda) \), and \( \pi_2(M) \) is a finitely-generated \( \Lambda \)-module.

If \( \pi := \pi_1(M, x_0) \) is a non-trivial finite group, then \( \pi_2(M) \) is a finitely-generated free abelian group with a \( \Lambda := \mathbb{Z}[\pi] \)-module structure, as studied in integral representation theory. In general, there are infinitely many non-isomorphic indecomposable integral representations (e.g. for \( \pi = \mathbb{Z}/p \times \mathbb{Z}/p \)), and there is no known classification. If \( \pi_1(M) \) is infinite, the precise structure of \( \pi_2(M) \) is unknown except in very special cases, such as \( \pi_1(M) = \mathbb{Z} \) mentioned in Section 3.

The study of these modules can be simplified somewhat by considering stable equivalence classes: two modules \( L_1, L_2 \) are stably isomorphic, denoted \( L_1 \simeq_s L_2 \), if there exists a free module \( \Lambda' \) such that \( L_1 \oplus \Lambda' \cong L_2 \oplus \Lambda' \). For example, the kernel

\[
0 \to \Omega^{n+1} \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0
\]

after \( n \)-steps in a free resolution \( \{F_i\} \) of the trivial module \( \mathbb{Z} \) is stably unique by Schanuel’s Lemma. For \( n = 3 \), such modules arise as \( \pi_2(K) = H_2(K; \Lambda) \), where \( K \) is a finite 2-complex with \( \pi_1(K, x_0) = \pi \), and the resolution is obtained from the chain complex \( C_*(\tilde{K}) \) of the universal covering. Finite 2-complexes \( K \) provide examples of smooth
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4-manifolds by taking the boundary of a thickening (i.e. a regular neigbourhood) of \( K \) in \( \mathbb{R}^5 \).

The stabilization operation in algebra has analogues in topology. For 2-complexes, \( K \mapsto K \vee S^2 \) gives the stabilization \( \pi_2(K) \mapsto \pi_2(K) \oplus \Lambda \). Whitehead [60, Theorem 19] showed that any two finite 2-complexes \( K, K' \) with isomorphic fundamental groups are stably simple-homotopy equivalent, but the problem of finding the minimal Euler characteristic realized by a 2-complex with given \( \pi_1 \) is still unsolved. This is a cancellation problem. Note that Whitehead’s Theorem implies that the stable isomorphism type of the \( \Lambda \)-modules \( H_2(K; \Lambda) \) and \( H^2(K; \Lambda) \) depend only on the fundamental group, and not on the choice of finite 2-complex \( K \).

It turns out that the stable structure of \( \pi_2(M) \) for a 4-manifold is very special.

**Theorem 4.2.** Let \( M \) be a closed, oriented 4-manifold with fundamental group \( \pi \). Then \( \pi_2(M) \) is stably isomorphic as a \( \Lambda \)-module to a certain extension

\[
\mathcal{E}_M : 0 \to H_2(K; \Lambda) \to E \to H^2(K; \Lambda) \to 0
\]

where \( K \) is any finite 2-complex with \( \pi_1(K, x_0) = \pi \).

**Remark 4.3.** The boundaries of thickenings of 2-complexes yield trivial extensions. The finite fundamental group case was done in [22, 2.4], and in that case the extension class of \( \mathcal{E}_M \) corresponds to the image of the fundamental class \( c_*[M] \in H_4(\pi; \mathbb{Z}) \) under a natural isomorphism \( \theta : H_4(\pi; \mathbb{Z}) \cong \text{Ext}^1_\Lambda(H^2(K; \Lambda), H_2(K; \Lambda)) \).

**Proof.** We will use a chain complex argument. By stabilizing \( K \mapsto K \vee rS^2 \) and \( M \mapsto M \# t(S^2 \times S^2) \) if necessary, we may assume that \( K \) is the sub-complex of 2-cells of \( M \). Consider the cellular chain complex \( C_* = C_*(\tilde{M}) \) of finitely-generated free \( \Lambda \)-modules. We have the exact sequences

\[
0 \to \mathbb{Z}_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0
\]

and

\[
0 \to \mathcal{B}_3^* \to C_3^* \to C_4^* \to \mathbb{Z} \to 0
\]

showing that \( \mathcal{B}_3^* = \text{Hom}_{\Lambda}(\mathcal{B}_3, \Lambda) \) is stably isomorphic to the 2-boundaries \( \mathcal{B}_2 \). The details here depend on whether \( \pi \) is finite or infinite: in the latter case note that
Ext^1_\Lambda(\mathfrak{M}, \Lambda) \cong H^4(M; \Lambda) = \mathbb{Z}. We now form the pull-back diagram

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow & \\
0 & \to & H^2(M; \Lambda) & \to & H^2(K; \Lambda) & \to \mathfrak{B}_2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^2(M; \Lambda) & \to & E & \to C_2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(K; \Lambda) & \to & Z_2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & & & & & 
\end{array}
\]

where \( Z_2 = H_2(K; \Lambda) \) since \( K \) is the 2-skeleton of \( M \). The middle horizontal sequence splits since \( C_2 \) is a free \( \Lambda \)-module. The middle vertical sequence is \( E_M \), and \( E \cong \pi_2(M) \oplus C_2 \) is a stabilization of \( \pi_2(M) \). □

Surgery theory as developed by Browder, Novikov, Sullivan and Wall [59] provides a powerful framework for classifying manifolds of dimension \( \geq 5 \) within a fixed homotopy type. However, in dimension 4 there are serious obstacles arising from the failure of the Whitney trick. One approach, developed by Cappell and Shaneson [3], is based on Wall’s idea of studying smooth 4-manifolds after stabilization with copies of \( S^2 \times S^2 \). The drawback is that information about the original (unstabilized) homotopy type is lost in the process.

Freedman’s work [16] fully established 4-dimensional surgery theory for topological manifolds whose fundamental groups do not “grow” too quickly. This class includes the poly-cyclic by finite groups, but it is not known at present if 4-dimensional topological surgery theory works for (non-cyclic) free fundamental groups. Note that Donaldson’s results show that smooth surgery theory definitely does not work in dimension 4, and there are \( s \)-cobordant smooth 4-manifolds which are not diffeomorphic.

The modified surgery theory of M. Kreck [39] requires less initial information about the homotopy type: for example, one can try to classify smooth 4-manifolds which have the same algebraic 2-type (up to smooth \( s \)-cobordism). In this theory, the key step is to compute certain bordism groups \( \Omega_4(B, \xi) \), where \( \xi \) is a bundle over \( B \) whose pullback \( c^*(\xi) \cong \nu_M \) is the stable normal bundle of \( M \). For such computations there are a variety of methods available, including the Atiyah-Hirzebruch and Adams spectral sequences. If two manifolds \([M_1, \hat{c}_1] \), \([M_2, \hat{c}_2] \) are bordant over the type \((B, \xi)\), then the triviality of an algebraically defined invariant implies that \( M_1 \) and \( M_2 \) are smoothly \( s \)-cobordant (see [39, Theorem B]).
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One possible way of analysing the final step is to note that the relation \([M_1, \tilde{c}_1] = [M_2, \tilde{c}_2] \in \Omega_4(B, \xi)\) implies that

\[ M_1 \# r_1(S^2 \times S^2) \cong M_2 \# r_2(S^2 \times S^2) \]

are stably diffeomorphic [39, Cor. 3], with control on the reference maps to \(B\) (see [40], [5] for further applications of stabilization). The cancellation problem is to find techniques for removing \(S^2 \times S^2\) factors from both sides. In algebra, cancellation theorems for modules and quadratic forms over noetherian rings were proved by Bak [1], Bass [2], Stafford [52] and Vaserstein [56]. In [23], we realized that these algebraic results could be combined with the constructions by [3, 1.5] of self-diffeomorphisms of 4-manifolds to prove cancellation theorems for certain 4-manifolds. For example, the integral group rings \(\mathbb{Z}[\pi]\) of poly-cyclic by finite groups are noetherian rings, but the group ring of a free group on 2 generators is not noetherian. This theme has recently been taken up again in [4].

5. Finite fundamental group

In early joint work with M. Kreck [23] we studied the topology of 4-manifolds with finite fundamental groups, and obtained a good description of the homotopy types within a prescribed algebraic 2-type. We also showed that there are only finitely many homeomorphism types of closed, oriented 4-manifolds with given finite \(\pi_1\), and given Euler characteristic (see [22, p. 87]). To obtain precise classification results up to homeomorphism we needed to restrict to special fundamental groups. We say that \(M\) has \(w_2\)-type (I) if \(w_2(M) \neq 0\), \(w_2\)-type (II) if \(w_2(M) = 0\), and \(w_2\)-type (III) if \(w_2(M) \neq 0\) but \(w_2(M) = 0\).

**Theorem 5.1** ([22], [24]). Closed, oriented topological 4-manifolds with finite cyclic fundamental groups are classified up to homeomorphism by \(\pi_1(M), q_M, w_2\)-type, and \(KS(M)\).

This is a generalization of Freedman’s Theorem in the simply-connected case (where the \(w_2\)-type is determined by the intersection form). One interesting consequence is that certain automorphisms of the cohomology ring of a smooth 4-manifold are induced by self-homeomorphisms but not by a self-diffeomorphism (Donaldson’s work is used to rule out a self-diffeomorphism, see the example [22, p. 87]).

However, for more complicated fundamental groups we can not expect a classification in terms of the ordinary intersection form \(q_M\) on \(H_2(M; \mathbb{Z})\) (see [54], [55]). Here is a sample result involving the quadratic 2-type.

**Theorem 5.2.** Closed, oriented, topological 4-manifolds with \(w_2(M) = 0\) and odd order finite fundamental groups are classified up to homeomorphism by the simple isometry class of the quadratic 2-type \([\pi_1(M), \pi_2(M), k_M, s_M]\).

**Proof.** This is essentially an exercise in the methods of [39] and [25], and the information needed to use the odd order assumption is provided by [26, Section 4]. The definition of simple isometry class will be explained below.
Here are some details of the steps in the argument. We first notice that the normal 2-type for such a spin manifold $M$ is $B \times BTOPSPIN$, where $B = B(M)$ is the algebraic 2-type. Since $\pi := \pi_1(M)$ has odd order,

$$\Omega^T_{4}(K(\pi, 1)) = \mathbb{Z} \oplus H_4(\pi; \mathbb{Z}),$$

and the stable homeomorphism class of $M$ is determined by its signature and the image of the fundamental class $c_*[M] \in H_4(\pi; \mathbb{Z})$. If $M$ and $M'$ have isometric quadratic 2-types, then there exists an isometry $\alpha: s_M \cong s_{M'}$ respecting the reference maps to both domain and range to obtain an equivariant intersection forms induces the identity on the hyperbolic summands (in fact, we will obtain $h' = (\alpha \oplus 1)$). With that additional property, one can attach handles to both domain and range to obtain an $s$-cobordism between $M$ and $M'$ (see [39, §4]).

To modify the homeomorphism $h$ we proceed as follows. Let $M_r := M \# r(S^2 \times S^2)$ denote the $r$-fold stabilization of $M$. By composition, we obtain an element

$$\beta := (\alpha \oplus 1)^{-1} \circ h_r \in \text{Isom}[\pi_1(M_r), \pi_2(M_r), k_{M_r}, s_{M_r}].$$

The braid diagram of [26, p. 168], combined with [26, Theorem B], now shows that $\beta = \phi_*$ for some $\phi \in \text{Aut}_*(M_r)$, such that $\phi$ is induced by an inertial $s$-cobordism $(W; M_r, \bar{M}_r)$. We have further assumed that $\alpha$ is a simple isometry of the quadratic 2-types. By definition, this means that the Whitehead torsion $\tau(\phi) = 0 \in \text{Wh}(\mathbb{Z}\pi)$, and hence $\tau(W, M_r) = \bar{u} \in \text{Wh}(\mathbb{Z}\pi)$ is self-dual ($\bar{u} = u$). Note that this definition of a simple isometry is independent of the choice of $h$-cobordism inducing $\phi$. From the exact sequences in the proof of [26, 4.1], and the fact that the discriminant map $L_0^h(Z\pi) \to H^0(\mathbb{Z}/2; \text{Wh}(\mathbb{Z}\pi))$ is surjective (since $\pi$ has odd order), we can realize the self-equivalence $\phi$ by an $s$-cobordism $W'$; if necessary, we modify our first choice by the action of $L_0^h(Z\pi)$ on $H(M)$. It follows that the homotopy self-equivalence induced by $W'$ is realized by a self-homeomorphism $f: M \to M_r$. We now define $h' := h \circ f^{-1}$ and notice that $h'_* = \alpha \oplus 1$, as required. \hfill \Box

Stabilization and cancellation techniques can also be used effectively for manifolds with arbitrary finite fundamental groups (see [25]). For 4-manifolds, the connected sum operation gives the stabilization

$$\pi_2(M \# (S^2 \times S^2)) = \pi_2(M) \oplus \Lambda \oplus \Lambda$$

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where the equivariant intersection form is stabilized by adding a hyperbolic plane

\[ H(\Lambda) = (\Lambda \oplus \Lambda, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \]

The cancellation problem for 4-manifolds with finite fundamental group has the following optimal solution:

**Theorem 5.3 ([25]).** Let \( M, M' \) be closed, oriented topological 4-manifolds with finite fundamental group. If \( M = M_0 \# (S^2 \times S^2) \), and

\[ M \# r(S^2 \times S^2) \cong M' \# r(S^2 \times S^2) \]

then \( M \cong M' \).

Note that even in the simply-connected case, non-isomorphic forms can become isomorphic after adding a hyperbolic plane, so the statement is best possible.

6. Fundamental groups of aspherical 2-complexes

A finitely-presented group \( \pi \) is *geometrically 2-dimensional* (\( g\dim \pi \leq 2 \)) if there exists a finite aspherical 2-complex with fundamental group \( \pi \). Examples of geometrically 2-dimensional groups include free groups, 1-relator groups (e.g. surface groups) and small cancellation groups [43], provided they are torsion-free, as well as many word-hyperbolic groups, see also [31, 2.3], [34, §10].

Recall that the radical \( R(s_M) \) of the equivariant intersection form \( s_M \) is isomorphic to the \( \pi \)-module \( R(\pi) := H^2(\pi; \Lambda) \). A closed oriented 4-manifold \( M \) will be called *minimal* if the equivariant intersection form on \( \pi_2(M) \) vanishes, or equivalently, if \( \pi_2(M) = R(s_M) \cong R(\pi) \). It turns out that a thickening of an aspherical 2-complex for \( \pi \) gives a minimal, smooth 4-manifold \( M_0 \) with fundamental group \( \pi \), whenever \( g\dim \pi \leq 2 \) (see [27, Lemma 3.7]). For example, the manifolds \( \#_r(S^1 \times S^3) \) and \( S^2 \times \Sigma \) are minimal, where \( \Sigma \) denotes an oriented surface of genus \( \geq 1 \).

In a series of papers [31], [33], [32], [34], [35], J. Hillman investigated the homotopy classification of Poincaré 4-complexes under various fundamental group assumptions. In the case of \( g\dim \leq 2 \), the problem was reduced to the minimal case, where the homotopy classification was completed for free or surface fundamental groups (see also [49] where these cases were studied from a different viewpoint).

In recent joint work with Matthias Kreck and Peter Teichner (described below), we used the modified surgery approach to obtain classification results for topological 4-manifolds with geometrically 2-dimensional fundamental groups, up to homeomorphism (in favourable cases) or \( s \)-cobordism.

A particular nice family of examples if provided by the solvable Baumslag-Solitar groups

\[ BS(k) := \{ a, b \mid aba^{-1} = b^k \}, \quad k \in \mathbb{Z}. \]
The groups $BS(k)$ have geometrical dimension $\leq 2$ because the 2-complex corresponding to the above presentation is aspherical. The easiest cases are

$$BS(0) = \mathbb{Z}, \quad BS(1) = \mathbb{Z} \times \mathbb{Z}, \quad \text{and} \quad BS(-1) = \mathbb{Z} \rtimes \mathbb{Z},$$

and these are the only Poincaré duality groups in this family. Each $BS(k)$ is solvable, so is a “good” fundamental group for topological 4-manifolds [16]. This implies that Freedman’s $s$-cobordism theorem is available to complete the homeomorphism classification. This had been done previously only for the three special cases above, see [17] for $BS(0)$, and [33] for $BS(\pm 1)$, using a more classical surgery approach.

**Theorem 6.1** ([27, Theorem A]). For closed oriented 4-manifolds with solvable Baumslag-Solitar fundamental groups, and given $w_2$-type and Kirby-Siebenmann invariant, any isometry between equivariant intersection forms can be realized by a homeomorphism.

In particular, we showed that a minimal 4-manifold is unique up to homeomorphism and established some relations between the invariants in general (based in part on [55]). For fundamental groups $\pi$ with $H_4(\pi; \mathbb{Z}) = 0$ we showed that the signature is determined by $s_M$ via the formula

$$\text{sign}(M) = \frac{\text{sign}(s_M \otimes \Lambda \mathbb{Z})}{8}.$$ 

This formula does not hold for arbitrary 4-manifolds, as one can see from examples of surface bundles over surfaces with nontrivial signature (but vanishing $\pi_2$).

For $\pi_1(M) = BS(k)$, type (III) can only occur if $k$ is odd. In this case, we gave a generalization of Rochlin’s formula (see [27, Corollary 6.10]):

$$KS(M) \equiv \text{sign}(M)/8 + \text{Arf}(M) \pmod{2}$$

where $\text{Arf}(M) \in \mathbb{Z}/2$ is a codimension 2 Arf invariant. In contrast, for spin manifolds $KS(M) \equiv \text{sign}(M)/8 \pmod{2}$.

We also proved a realization theorem for hermitian forms in this setting. If $M$ has fundamental group $BS(k)$, then the quotient module $\pi_2(M)/R(s_M)$ is a finitely-generated, stably-free $\Lambda$-module, and the equivariant intersection form $s_M$ is non-singular on this quotient. It turns out that any such hermitian form can be realized by one or two 4-manifolds.

A close inspection of the arguments shows that we used a number of special facts about the Baumslag-Solitar groups. For more general fundamental groups $\pi$, we need to assume the corresponding properties for its algebraic $K$-theory and $L$-theory.

**Definition 6.2.** A group $\pi$ satisfies properties (W-AA) whenever

1. The Whitehead group $\text{Wh}(\pi)$ vanishes,
2. The assembly map $A_5: H_5(\pi; L_0) \to L_5(\mathbb{Z}\pi)$ is surjective.
3. The assembly map $A_4: H_4(\pi; L_0) \to L_4(\mathbb{Z}\pi)$ is injective.

Note that these properties (and more) do hold whenever the group $\pi$ satisfies the Farrell-Jones isomorphism conjectures [11] (see [42] for a survey of results on these conjectures).
**Theorem 6.3 ([27, Theorem C]).** Let $\pi$ be a geometrically 2-dimensional group satisfying properties (W-AA). For closed oriented 4-manifolds with fundamental group $\pi$, and given Kirby-Siebenmann invariant, any isometry between equivariant intersection forms inducing an isomorphism of $w_2$-types can be realized by an $s$-cobordism.

The $w_2$-type mentioned in this statement is actually a refinement of the notion defined in Section 4, in which we now keep track of the class $w \in H^2(\pi; \mathbb{Z}/2)$ determining $w_2(M)$ in type (III).

**7. Some questions**

Here are a few questions and problems concerning smooth and topological 4-manifolds with non-trivial fundamental group.

(1) For a smooth 4-manifold $M$ with geometrically 2-dimensional fundamental group, is $M$ homeomorphic to $M_0 \# N$, where $M_0$ is minimal and $N$ is simply-connected? In other words, is the equivariant intersection form $s_M$ always extended from the integers?

(2) Construct distinct smooth structures on indecomposable, non-simply connected 4-manifolds. Is there a minimal 4-manifold with more than one smooth structure?

(3) For a given group $\pi$, there exist 4-manifolds $M(\alpha)$ with $\pi_1(M) = \pi$ and $c_*[M]$ a given element $\alpha \in H_4(\pi; \mathbb{Z})$. How does the minimal possible Euler characteristic and signature of $M(\alpha)$ depend on the class $\alpha$?

(4) For a given group $\pi$, does there exists a stable range constant $c(\pi)$, with the property that a stable homeomorphism or diffeomorphism $M_1 \# r(S^2 \times S^2) \cong M_2 \# r(S^2 \times S^2)$ between manifolds with fundamental group $\pi$ admits cancellation of at least one copy of $S^2 \times S^2$ (up to $s$-cobordism) whenever $r > c(\pi)$?

(5) Compare the actions of Diff($M$) and Homeo($M$) on the equivariant intersection form of a smooth 4-manifold.

**Remark 7.1.** There are many interesting problems related to the study the existence and uniqueness of non-free finite group actions on smooth or topological 4-manifolds. One may ask, for example, which equivariant intersection forms are realized by smooth actions of finite cyclic groups on simply-connected 4-manifolds. For topological actions there is a satisfactory picture, particularly for cyclic groups of prime order (see Edmonds [9], [10], Edmonds-Ewing [9], and McCooey [45]). For smooth actions, there are restrictions detected by equivariant gauge theory [28], [29] and the answer is interesting even for the permutation representations which arise for actions on connected sums of $\mathbb{CP}^2$'s (see [30, 1.18]). A striking contrast between smooth and topological actions is shown by the recent paper of Finstushel, Stern and Sunukjian [15], where infinite families of topologically equivalent but smoothly distinct cyclic group actions are constructed on 4-manifolds with non-trivial Seiberg-Witten invariants.
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