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An introduction to reduced volume

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Abstract. In these lecture notes we give an introduction to Perelman's theory of reduced distance and reduced volume. We begin with a quick introduction to some classical results about Riemannian manifolds with nonnegative Ricci curvature, and we end with two applications of the monotonicity of reduced volume.

1. Introduction

In Riemannian geometry we study the geometric properties of Riemannian manifolds, such as the relation between curvature and topology, the existence of metrics satisfying certain curvature conditions, and properties of various other geometric quantities. Such a study often requires us to consider collections of Riemannian manifolds, one such example is the Cheeger-Gromov compactness theorem and its applications. A special case of collections of Riemannian manifolds is a one-parameter family of Riemannian metrics $g(s)$ on a given manifold M, this is used by S.T. Yau in his celebrated proof of the existence of Ricci flat Kähler metrics using the continuity method. Ricci flow, introduced by R. Hamilton, is a one-parameter family of metrics $\tilde{g}(t)$, $t \in (\alpha, T)$, which satisfies the flow equation $\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Rc}(\tilde{g}(t))$ where $\mathrm{Rc}(\tilde{g}(t))$ stands for the Ricci curvature of $\tilde{g}(t)$. We hasten to add that Ricci flow is a very special one-parameter family of metrics. Through the work of G. Perelman on smooth Ricci flow we clearly see that there is an organic relation between the time variable t and the space variables in Ricci flow, which suggests some kind of space-time geometry. In these notes of a mini-course we give an introduction to two key notions which support the space-time viewpoint: reduced distance and reduced volume.

Because of the analogy between Ricci flow and the geometry of Riemannian manifolds with nonnegative Ricci curvature, we begin the lectures with a quick review of some classical global results which hold on any Riemannian manifolds with nonnegative Ricci curvature. The purpose of §2 is to help the reader to understand the results and calculations that appear later in §3 and §4 about the reduced distance and the reduced volume, respectively. In §5 we give two applications of reduced volume, in particular, the no local collapsing theorem.

Key words and phrases. Ricci flow, reduced distance, reduced volume, no local collapsing.

As always going back to the source (original article(s)) fundamentally helps one's understanding of math, for reduced distance and reduced volume we strongly suggest that the reader consults Perelman's paper [Pe02], in particular, §7, §8, and §6.

We assume that the reader has some exposure to Riemannian geometry and Ricci flow before.

We end this introduction with a **warning** and a list. In these notes we have swept under the rug the issue about how to do calculus at cut-locus points where distance function $r(\cdot)$ is not smooth. There are standard procedures to handle it: Calabi trick or barrier functions. In the calculation below we pretend that $r(\cdot)$ is smooth. Similar issue exists for the reduced distance (e.g., §3.2A) and the issue can be solved using barrier functions, again in the calculation below we pretend that the reduced distance is smooth.

Conventions and notations:
We adopt Einstein summation convention
$g(x,t)$: the inner product on the tangent space at point x defined by metric $g(t)$
R: scalar curvature
$R(x,t)$: scalar curvature of metric $g(t)$ at x
Rc: Ricci curvature. $R_{ij} = R_{ikkj} = R^k_{kij}$
$\mathrm{Rc}(x,t)$: Ricci curvature of metric $g(t)$ at x
Rm: Riemann curvature tensor; $\mathrm{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l_{ijk}\frac{\partial}{\partial x^l}$ and $R_{ijkl} = g_{lp}R^p_{ijk}$
$\mathrm{Rm}(x,t)$: Riemann curvature tensor of metric $g(t)$ at x
Distance function: $r(x) = d(p,x)$ for some point p
Lapalace-Beltrami operator: $\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right)$ where $|g| = \det(g_{ij})$
Volume form: $d\mu_g = \sqrt{|g|}dx^1 \cdots dx^n$

2. Riemannian manifolds with nonnegative Ricci curvature

Throughout this section (M^n, g) denotes a smooth n dimensional complete connected oriented Riemannian manifold with $\mathrm{Rc} \geq 0$. Mainly this section is about the distance function.

2.1. First variation of length

2.1A. Length. Given a smooth path $\gamma : [a,b] \to M$, its **length** is defined by

$$\mathrm{L}(\gamma) = \int_a^b |\dot{\gamma}(u)| du \tag{2.1}$$

where $|\dot{\gamma}(u)|$ is the length of the tangent vector $\dot{\gamma}(u) = \frac{d\gamma}{du}$. Given two points $x, y \in M$ the **distance** between them is defined by

$$d(x,y) = \inf_{\{\gamma:\gamma(a)=x,\gamma(b)=y\}} \mathrm{L}(\gamma)$$

where γ runs through smooth paths.

2.1B. First variation of length. Now we compute the **first variation of length**. Let $\gamma_s(u)$, $s \in (-\epsilon, \epsilon)$ be a smooth family of smooth paths with $\gamma_0 = \gamma$ parametrized by arc-length parameter u (i.e., $|\dot\gamma(u)| = 1$). Denote $Y = \frac{\partial \gamma_s}{\partial s}\Big|_{s=0}$ to be the variation vector field. Then using $\nabla_Y \dot\gamma_s - \nabla_{\dot\gamma_s} Y = [Y, \dot\gamma_s] = 0$ and integrating by parts we get

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} \mathrm{L}(\gamma_s) &= \int_a^b \frac{1}{2} \langle \dot\gamma_s(u), \dot\gamma_s(u)\rangle^{-1/2} \cdot Y \langle \dot\gamma_s, \dot\gamma_s\rangle \Big|_{s=0} du \\ &= \int_a^b \langle \dot\gamma, \nabla_{\dot\gamma} Y\rangle \, du = \int_a^b (\dot\gamma(\langle \dot\gamma, Y\rangle) - \langle \nabla_{\dot\gamma}\dot\gamma, Y\rangle)\, du \\ &= \langle \dot\gamma, Y\rangle (b) - \langle \dot\gamma, Y\rangle (a) - \int_a^b \langle \nabla_{\dot\gamma}\dot\gamma, Y\rangle \, du.\end{aligned}$$

Hence the critical point equation of the length functional L on the space of smooth paths with fixed end points is

$$\nabla_{\dot\gamma}\dot\gamma = 0. \tag{2.2}$$

This is the **equation of geodesics**. Any path that satisfies (2.2) is called a **geodesic**. In local coordinates $x = (x^i)$ the equation is the following system of ordinary differential equations (ode)

$$\frac{d^2 x^i}{du^2} + \Gamma^i_{jk}(x(u)) \frac{dx^j}{du} \cdot \frac{dx^k}{du} = 0, \tag{2.3}$$

where $\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$ is the **Christoffel symbol**.

2.1C. Jacobi field. Consider the set of all geodesics defined on $[a, b]$ and call it the moduli space of geodesics, the tangent directions of this space satisfy a second order linear ode. Let $\gamma_s(u)$, $s \in (-\epsilon, \epsilon)$ be a smooth family of geodesics with $\gamma_0 = \gamma$. Let $Y = \frac{\partial \gamma_s}{\partial s}\Big|_{s=0}$ be the variation vector field of γ_s. Taking the ∇_Y-derivative of the geodesic equation $\nabla_{\dot\gamma_s}\dot\gamma_s \equiv 0$, we have

$$0 = \nabla_Y \nabla_{\dot\gamma_s} \dot\gamma_s = \nabla_{\dot\gamma_s} \nabla_Y \dot\gamma_s + \mathrm{Rm}\,(Y, \dot\gamma_s)\, \dot\gamma_s.$$

Using $[Y, \dot\gamma_s] = 0$ and evaluating at $s = 0$, we get the **Jacobi equation**

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} Y + \mathrm{Rm}\,(Y, \dot\gamma)\, \dot\gamma = 0. \tag{2.4}$$

Any vector field Y along a geodesic γ which satisfies (2.4) is called a **Jacobi field**.

2.1D. Exponential map and Jacobian under geodesic spherical coordinates. Consider the initial value problem of second order ode (2.2) by choosing a point $p \in M$ (corresponding to the value of $\{x^i(0)\}$ in (2.3)) and a unit vector $V \in T_p M$ (corresponding to the value of $\{\frac{dx^i}{du}(0)\}$ in (2.3)). This defines the so-called **exponential map** $\exp_p(uV) = \gamma(u)$. There is a star-shaped open connected subset $0 \in \Omega_p \subset T_p M$ such that $\exp_p : \Omega_p \to M \setminus \mathrm{Cut}_p$ is a diffeomorphism. Here Cut_p is the so-called **cut locus** of p in M.

Let $r(x) = d(x,p)$ and let $(\theta^1, \cdots, \theta^{n-1}, r)$ be local spherical coordinates on $T_pM - \{0\}$. Then the inverse map $\exp_p^{-1} : M \backslash (\{p\} \cup \mathrm{Cut}_p) \to \Omega_p \backslash \{0\}$ defines the **geodesic spherical coordinates** of M. By Gauss lemma $g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 1$ and $g(\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial r}) = 0$, $a = 1, \cdots, n-1$, and hence the metric can be written as

$$g = dr^2 + g_{ab} d\theta^a d\theta^b. \tag{2.5}$$

The volume form can be written as

$$d\mu = \sqrt{\det(g_{ab})}\, d\theta^1 \cdots d\theta^{n-1} dr.$$

$J = \sqrt{\det(g_{ab})}$ is called the **Jacobian of the exponential map**. The **area form** on the sphere $S(p, r_0)$ of radius r_0 and centered at p can be written as

$$d\sigma = J d\theta^1 \wedge \cdots d\theta^{n-1}. \tag{2.6}$$

Recall **Hessian** $\mathrm{Hess}\, f$ of a smooth function f on M is defined by $(\mathrm{Hess}\, f)(X, Y) = X(Yf) - (\nabla_X Y) f$ for any $X, Y \in T_xM$. It follows from (2.5) that $|\frac{\partial}{\partial r}| = 1$, $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$, and $(\mathrm{Hess}\, r)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$. Hence $|\mathrm{Hess}\, r|^2 \geq \frac{1}{n-1} (\Delta r)^2$.

2.1E. Mean curvature of spheres. We continue to use geodesic spherical coordinates $(\theta^1, \cdots, \theta^{n-1}, r)$. Let h denote the **second fundamental form** of $S(p, r_0)$. Note $\frac{\partial}{\partial r}$ is the unit outward normal vector field to $S(p, r_0)$. From equation (2.5) we have

$$\begin{aligned} h_{ab} = h\left(\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\right) &= \left\langle \nabla_{\frac{\partial}{\partial \theta^a}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^b} \right\rangle \\ &= -\left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^a}} \frac{\partial}{\partial \theta^b} \right\rangle = -\Gamma_{ab}^n = \frac{1}{2} \frac{\partial}{\partial r} g_{ab}. \end{aligned}$$

Hence the **mean curvature** H of $S(p, r_0)$ is given by

$$H = g^{ab} h_{ab} = \frac{1}{2} g^{ab} \frac{\partial}{\partial r} g_{ab} = \frac{\partial}{\partial r} \log \sqrt{\det(g_{ab})} = \frac{\partial}{\partial r} \log \mathrm{J}\,. \tag{2.7}$$

On the other hand the mean curvature H can be computed as

$$H = \left\langle \nabla_{\frac{\partial}{\partial \theta_a}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_a} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial \theta_a}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_a} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = \Delta r. \tag{2.8}$$

2.2. Bochner formula for functions and its consequence

2.2A. Bochner formula. Let f be a smooth function on M. We compute the commutator $[\Delta, \nabla_j] f$ as

$$\Delta \nabla_j f = \nabla_i \nabla_i \nabla_j f = \nabla_i \nabla_j \nabla_i f = \nabla_j \nabla_i \nabla_i f - R_{ijil} \nabla_l f = \nabla_j \Delta f + R_{jl} \nabla_l f. \tag{2.9}$$

We also compute

$$\Delta |\nabla f|^2 = 2 \nabla_i (\nabla_j f \cdot \nabla_i \nabla_j f) = 2 \nabla_i \nabla_j f \cdot \nabla_i \nabla_j f + 2 \nabla_j f \cdot \nabla_i \nabla_i \nabla_j f.$$

Hence we get the **Bochner formula** for f

$$\Delta |\nabla f|^2 = 2 |\nabla \nabla f|^2 + 2 R_{ij} \nabla_i f \nabla_j f + 2 \nabla_i f \nabla_i (\Delta f). \tag{2.10}$$

2.2B. Laplacian comparison theorem. In (2.10) let $f(x) = r(x)$ be the distance function. Using $\mathrm{Rc} \geq 0$, $|\nabla r| = 1$ and $|\mathrm{Hess}\, r|^2 \geq \frac{1}{n-1}(\Delta r)^2$ we get

$$\frac{\partial}{\partial r}(\Delta r) + \frac{1}{n-1}(\Delta r)^2 \leq 0.$$

Since $\Delta r \to \frac{n-1}{r}$ as $r \to 0^+$ and $\frac{\partial}{\partial r}(\frac{n-1}{r}) + \frac{1}{n-1}(\frac{n-1}{r})^2 = 0$, from the differential inequality above and the ode comparison property we conclude the **Laplacian comparison theorem** on complete manifolds with $\mathrm{Rc} \geq 0$

$$\Delta r \leq \frac{n-1}{r}. \tag{2.11}$$

2.2C. Bishop-Gromov volume comparison theorem. Let $A(r_0)$ be the **area** of sphere $S(p, r_0)$. Then by (2.6), (2.7), (2.8), and (2.11) we have

$$\frac{d}{dr}A(r) = \int_{S^{n-1}} \frac{dJ}{dr} d\theta^1 \wedge \cdots d\theta^{n-1} = \int_{S(p,r)} HJ d\theta^1 \wedge \cdots d\theta^{n-1} \leq \frac{n-1}{r} A(r).$$

Integrating this differential inequality, we see that

$$A(r_2)\, r_1^{n-1} \leq A(r_1)\, r_2^{n-1} \qquad \text{for } r_2 \geq r_1 > 0.$$

Integrating again we get

$$n\omega_n \int_0^{r_2} dr_2 \int_0^{r_1} A(r_2)\, r_1^{n-1} dr_1 \leq n\omega_n \int_0^{r_2} dr_2 \int_0^{r_1} A(r_1)\, r_2^{n-1} dr_1,$$

where ω_n is the volume of unit Euclidean ball.

Let $B(p, r_0)$ be the ball of radius r_0 and centered at p, and let $\mathrm{Vol}\, B(p, r_0)$ denote the volume of the ball. Using $\mathrm{Vol}\, B(p, r_0) = \int_0^{r_0} A(r)\, dr$ the inequality above can be written as

$$\frac{\mathrm{Vol}\, B(p, r_2)}{\omega_n r_2^n} \leq \frac{\mathrm{Vol}\, B(p, r_1)}{\omega_n r_1^n} \qquad \text{for } r_2 \geq r_1 > 0, \tag{2.12}$$

i.e., $\frac{\mathrm{Vol}\, B(p,r)}{\omega_n r^n}$ is a monotone decreasing function (compare with (4.7)). (2.12) is the **Bishop-Gromov volume comparison theorem** for complete manifolds with $\mathrm{Rc} \geq 0$.

2.3. Differential Harnack inequality

2.3A. Li-Yau Harnack estimate. Now we discuss Li-Yau differential Harnack estimate for positive solutions of heat equation $\frac{\partial u}{\partial t} = \Delta u$ defined on (M^n, g). Let $L = \log u$ and $Q = \Delta L$. We compute the evolution equation of L as

$$\frac{\partial L}{\partial t} = \frac{1}{u} \triangle u = \triangle L + |\nabla L|^2.$$

We compute the evolution equation of Q as

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \triangle\left(\frac{\partial}{\partial t}L\right) = \triangle\left(\triangle L + |\nabla L|^2\right)\\ &= \triangle Q + 2\langle \triangle \nabla L, \nabla L\rangle + 2|\nabla\nabla L|^2\\ &= \triangle Q + 2\langle \nabla \triangle L, \nabla L\rangle + 2\,\mathrm{Rc}\,(\nabla L, \nabla L) + 2|\nabla\nabla L|^2,\end{aligned}$$

where we have used (2.9) to get the last equality. Hence, from $\mathrm{Rc} \geq 0$ and $|\nabla\nabla L|^2 \geq \frac{1}{n}(\Delta L)^2$ we have

$$\frac{\partial Q}{\partial t} \geq \triangle Q + 2\langle \nabla L, \nabla Q\rangle + \frac{2}{n}Q^2.$$

From this inequality and the maximum principle we may deduce the following **Li-Yau differential Harnack estimate** for positive solutions of heat equation on complete manifolds with $\mathrm{Rc} \geq 0$

$$Q = \Delta \log u \geq -\frac{n}{2t}. \tag{2.13}$$

Note that the equality above is satisfied by the heat kernel $u = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ on Euclidean space $\mathbb{R}^n$.

2.3B. Harnack estimate for heat kernel. The following estimate is proved by Lei Ni in 2004 (J. Geom. Anal. **14**, 87–100). Let $H = \frac{1}{(4\pi t)^{n/2}} e^{-f}$ be a heat kernel of (M^n, g) with $\mathrm{Rc} \geq 0$. Then for $t > 0$

$$t(2\Delta f - |\nabla f|^2) + f - n \leq 0. \tag{2.14}$$

This estimate is closely related to Perelman's Harnack inequality ([Pe02], Corollary 9.3).

2.4. Second variation of length

2.4A. Second variation formula. Let $\gamma_s(u)$, $u \in [a,b]$, $s \in (-\epsilon, \epsilon)$ be a smooth family of paths. Assume that $\gamma_0(u) = \gamma(u)$ is a geodesic of unit speed (i.e., $|\dot\gamma(u)| = 1$). Denote $Y = \left.\frac{\partial \gamma_s}{\partial s}\right|_{s=0}$ to be the variation field. We compute

$$\begin{aligned}\left.\frac{d^2}{ds^2}\right|_{s=0} \mathrm{L}(\gamma_s) &= \left.\frac{d}{ds}\right|_{s=0} \int_a^b |\dot\gamma_s(u)|^{-1}\cdot\langle \dot\gamma_s, \nabla_Y \dot\gamma_s\rangle\, du\\ &= \int_a^b \left(-\langle \dot\gamma, \nabla_Y\dot\gamma\rangle^2 + \langle \nabla_Y \dot\gamma, \nabla_{\dot\gamma} Y\rangle + \langle \dot\gamma, \nabla_Y\nabla_{\dot\gamma} Y\rangle\right) du\\ &= \int_a^b \left(|\nabla_{\dot\gamma} Y|^2 - \langle \nabla_{\dot\gamma} Y, \dot\gamma\rangle^2 + \langle \dot\gamma, \mathrm{Rm}\,(Y, \dot\gamma)\, Y\rangle + \langle \dot\gamma, \nabla_{\dot\gamma}\nabla_Y Y\rangle\right) du.\end{aligned}$$

Hence the **second variation of length** at a unit speed geodesic γ is given by

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \mathrm{L}\left(\gamma_s\right)$$
$$= \int_a^b \left(\left| \nabla_{\dot{\gamma}} Y \right|^2 - \langle \nabla_{\dot{\gamma}} Y, \dot{\gamma} \rangle^2 - \langle \mathrm{Rm}\left(Y, \dot{\gamma}\right) \dot{\gamma}, Y \rangle \right) ds + \left. \langle \nabla_Y Y, \dot{\gamma} \rangle \right|_a^b . \tag{2.15}$$

2.4B. Index form. Let $\gamma : [a, b] \to M$ be a geodesic. The **index form** is defined by

$$I(V, W) = \int_a^b \left(\langle \nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} W \rangle - \langle \mathrm{Rm}(V, \dot{\gamma}) \dot{\gamma}, W \rangle \right) du, \tag{2.16}$$

where V and W are vector fields along γ vanishing at $\gamma(a)$, $\gamma(b)$ and perpendicular to $\dot{\gamma}$. Note that $\langle \nabla_{\dot{\gamma}} Y, \dot{\gamma} \rangle = 0$ when Y in (2.15) is perpendicular to $\dot{\gamma}$, hence as a quadratic form of Y the right side of (2.15) gives rise to the bilinear form (2.16).

The **Index lemma** says the following. Suppose there is not any pair of conjugate points on geodesic $\gamma(u)$, $u \in [a, b]$. Given $A \in T_{\gamma(a)} M, B \in T_{\gamma(b)} M$ with $\langle A, \dot{\gamma}(a) \rangle = \langle B, \dot{\gamma}(b) \rangle = 0$, the unique Jacobi field J along γ with $J(a) = A$, $J(b) = B$, satisfies

$$I(J, J) \leq I(W, W). \tag{2.17}$$

Here W is any the vector field along γ which is perpendicular to $\dot{\gamma}$ and satisfies $W(a) = A$, $W(b) = B$.

2.5. Some other results

There are other theorems that hold for all complete manifolds with $\mathrm{Rc} \geq 0$. Here we give one example. Recall that a geodesic $\gamma : \mathbb{R} \to (M, g)$ is called a **line** if $\mathrm{L}(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b))$. The **Cheeger-Gromoll splitting theorem** says the following. Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Rc} \geq 0$. Suppose there is a line in M, then (M, g) is isometric to $\mathbb{R} \times \left(N^{n-1}, h\right)$ with the product metric. Here (N, h) is a complete Riemannian manifold with $\mathrm{Rc} \geq 0$. The proof uses **Busemann** function, an important function in the study of noncompact Riemannian manifolds.

3. The reduced distance

The reduced distance and reduced volume are formally introduced by Perelman in [Pe02], §7. The motivation he gives in §6 is both interesting and mysterious. From the work of Perelman and others it is evident that the reduced distance and reduced volume are fundamental tools in Ricci flow. It is desirable to find more applications of them.

In this section we establish various identities and inequalities about reduced distance. Some of these inequalities will be used in next section to show the finiteness and monotonicity of reduced volume. Since formulae in this section come out of relatively lengthy calculation, here we only provide glimpses of these calculation, readers can either figure out the detail themselves or find the detailed calculation in the literature.

Throughout this section N^n is a n-dimensional connected oriented manifold, and $(N^n, g(\tau))$, $\tau \in [0,T]$, is a solution to the **backward Ricci flow** $\frac{\partial}{\partial\tau} g(\tau) = 2\operatorname{Rc}(g(\tau))$ with bounded Riemann curvature $\sup_{M\times[0,T]} |\operatorname{Rm}(x,\tau)| < \infty$. We assume that $g(\tau)$ is complete for each $\tau \in [0,T]$ (called **complete solution**). Below, the notation ∇, Δ, R, Rc, Rm stand for connection, Lapalace-Beltrami operator and curvatures defined by $g(\tau)$.

3.1. First variation formula of $\mathcal{L}$-length

3.1A. Definition of reduced distance. Let $\gamma : [0,\tau] \to N$ be a smooth path with $\tau \leq T$. The **$\mathcal{L}$-length** of γ is defined by

$$\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\tilde{\tau}} \left(R(\gamma(\tilde{\tau}), \tilde{\tau}) + \left| \frac{d\gamma}{d\tilde{\tau}}(\tilde{\tau}) \right|^2_{g(\tilde{\tau})} \right) d\tilde{\tau}. \tag{3.1}$$

Fix a point $p \in N$, the **L-distance** from $(p,0)$ to $(x,\tau) \in N \times (0,T]$ is defined by

$$L(x,\tau) = \inf_{\{\gamma:\gamma(0)=p,\gamma(\tau)=x\}} \mathcal{L}(\gamma).$$

We call $(p,0)$ the **basepoint**. A minimizing path in the definition of L-distance is called a **minimal L-geodesic**. The **reduced distance** is defined by

$$\ell(x,\tau) = \frac{1}{2\sqrt{\tau}} L(x,\tau).$$

3.1B. First variation of $\mathcal{L}$-length. Let $\gamma_s(\tilde{\tau})$, $\tilde{\tau} \in [0,\tau]$, $s \in (-\epsilon,\epsilon)$, be a smooth family of smooth paths with $\gamma_0 = \gamma$. Let $X = \dot{\gamma}$, and let $Y = \frac{\partial \gamma_s}{\partial s}\Big|_{s=0}$ be the variation field. The **first variation formula** for $\mathcal{L}$-length is given by

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \mathcal{L}(\gamma_s) \\ &= \sqrt{\tilde{\tau}}\langle Y, X\rangle\Big|_0^\tau + \int_0^\tau \sqrt{\tilde{\tau}}\, Y \cdot \left(\frac{1}{2}\nabla R - \frac{1}{2\tilde{\tau}} X - \nabla_X X - 2\operatorname{Rc}(X) \right) d\tilde{\tau}, \end{aligned} \tag{3.2}$$

where the covariant derivative $\nabla = \nabla_{g(\tilde{\tau})}$.

If γ is a critical point of the $\mathcal{L}$-length functional (3.1) among smooth paths with fixed endpoints, then γ is called an **$\mathcal{L}$-geodesic**. From (3.2) we get the **$\mathcal{L}$-geodesic equation**:

$$\nabla_X X - \frac{1}{2}\nabla R + 2\operatorname{Rc}(X) + \frac{1}{2\tilde{\tau}} X = 0. \tag{3.3}$$

3.1C. Calculating (3.2). We compute in a way similar to the deduction of the first variation formula for length (§2.1B).

$$\begin{aligned} \frac{d}{ds}\mathcal{L}(\gamma_s) &= \int_0^\tau \sqrt{\tilde{\tau}} \left(\frac{\partial}{\partial s} R(\gamma_s(\tilde{\tau}), \tilde{\tau}) + \frac{\partial}{\partial s} \left| \frac{\partial \gamma_s}{\partial \tilde{\tau}}(\tilde{\tau}) \right|^2_{g(\tilde{\tau})} \right) d\tilde{\tau} \\ &= \int_0^\tau \sqrt{\tilde{\tau}} \left(\langle \nabla R, Y\rangle + 2\langle \nabla_Y X, X\rangle \right) d\tilde{\tau}. \end{aligned} \tag{3.4}$$

Using

$$\langle \nabla_Y X, X\rangle = \langle \nabla_X Y, X\rangle = \frac{d}{d\tilde{\tau}}\left[g\left(Y,X\right)\right] - \langle Y, \nabla_X X\rangle - 2\,\mathrm{Rc}\left(Y,X\right),$$

we get

$$\frac{1}{2}\frac{d}{ds}\mathcal{L}\left(\gamma_s\right) = \int_0^\tau \sqrt{\tilde{\tau}}\left(\frac{1}{2}\left\langle \nabla R, Y\right\rangle + \frac{d}{d\tilde{\tau}}\left\langle Y, X\right\rangle - \left\langle Y, \nabla_X X\right\rangle - 2\,\mathrm{Rc}\left(Y,X\right)\right) d\tilde{\tau}.$$

(3.2) follows from integration by parts

$$\int_0^\tau \sqrt{\tilde{\tau}}\frac{d}{d\tilde{\tau}}\left\langle Y, X\right\rangle d\tilde{\tau} = -\frac{1}{2}\int_0^\tau \frac{1}{\sqrt{\tilde{\tau}}}\left\langle Y, X\right\rangle d\tilde{\tau} + \sqrt{\tilde{\tau}}\left\langle Y, X\right\rangle\Big|_0^\tau.$$

3.1D. Other form of (3.1) and (3.3). Sometimes we need to use the following parametrization:

$$\tilde{\sigma} = 2\sqrt{\tilde{\tau}} \quad \text{and} \quad \beta\left(\tilde{\sigma}\right) = \gamma\left(\tilde{\sigma}^2/4\right). \tag{3.5}$$

Then we may rewrite $\mathcal{L}$-length as

$$\mathcal{L}\left(\gamma\right) = \int_0^{\sigma = 2\sqrt{\tau}} \left(\frac{\tilde{\sigma}^2}{4} R\left(\beta\left(\tilde{\sigma}\right), \tilde{\sigma}^2/4\right) + \left|\frac{d\beta}{d\tilde{\sigma}}\left(\tilde{\sigma}\right)\right|^2_{g\left(\tilde{\sigma}^2/4\right)}\right) d\tilde{\sigma}, \tag{3.6}$$

and $\mathcal{L}$-geodesic equation as

$$\nabla_Z Z - \frac{\tilde{\sigma}^2}{8}\nabla R + \tilde{\sigma}\,\mathrm{Rc}\left(Z\right) = 0, \tag{3.7}$$

where $Z\left(\tilde{\sigma}\right) = \frac{d\beta(\tilde{\sigma})}{d\tilde{\sigma}} = \sqrt{\tilde{\tau}}X(\tilde{\tau})$. A simple consequence of (3.7) is that solutions to the initial value problem for β exist. Consequently the minimal $\mathcal{L}$-geodesic between any two points exists.

3.2. First order derivatives of L-distance

In this subsection we give some consequences of the first variation formula of $\mathcal{L}$-length. Below L-distance is defined using the basepoint $(p, 0)$.

3.2A. Spatial derivative ∇L. Given $Y \in T_x N$, by choosing a family of minimal $\mathcal{L}$-geodesics $\gamma_s(\tilde{\tau})$, $\tilde{\tau} \in [0, \tau]$, $s \in (-\epsilon, \epsilon)$ such that $\gamma_s(0) = p$, $\frac{d}{ds}\big|_{s=0}\gamma_s(\tau) = Y$, and $\mathcal{L}(\gamma_s) = L(\gamma_s(\tau), \tau)$, we get from the first variation formula (3.2)

$$\left\langle \nabla L\left(x,\tau\right), Y\right\rangle = \frac{d}{ds}\bigg|_{s=0}\mathcal{L}(\gamma_s) = \left\langle 2\sqrt{\tau}X\left(\tau\right), Y\right\rangle.$$

Hence the spatial derivative of the L-distance function

$$\nabla L\left(x,\tau\right) = 2\sqrt{\tau}X\left(\tau\right), \tag{3.8}$$

where X is the tangent vector field of the minimal $\mathcal{L}$-geodesic $\gamma = \gamma_0$ from $(p, 0)$ to (x, τ).

3.2B. Time derivative $\frac{\partial L}{\partial \tau}$. We compute, using the chain rule and (3.8),

$$\begin{aligned}\frac{\partial L}{\partial \tau}(x,\tau) &= \frac{\partial L(\gamma(\tau),\tau)}{\partial \tau} = \frac{d}{d\tau}\left(L(\gamma(\tau),\tau)\right) - \nabla L(x,\tau)\cdot X(\tau) \\ &= \frac{d}{d\tau}\left[\int_0^\tau \sqrt{\tilde\tau}\left(R(\gamma(\tilde\tau),\tilde\tau) + \left|\frac{d\gamma}{d\tilde\tau}(\tilde\tau)\right|^2\right)d\tilde\tau\right] - 2\sqrt{\tau}\left|X(\tau)\right|^2 \\ &= \sqrt{\tau}\left(R(x,\tau) + |X(\tau)|^2\right) - 2\sqrt{\tau}\left|X(\tau)\right|^2.\end{aligned}$$

Hence the time-derivative of the L-distance function

$$\frac{\partial L}{\partial \tau}(x,\tau) = -\sqrt{\tau}\left(R(x,\tau) + |X(\tau)|^2\right) + 2\sqrt{\tau}R(x,\tau), \tag{3.9}$$

where X is the tangent vector field of the minimal $\mathcal{L}$-geodesic γ from $(p,0)$ to (x,τ).

3.3. Second variation formula of $\mathcal{L}$-length

For the purpose of getting information about the second order derivatives of L-distance, in this subsection we compute the second variation of $\mathcal{L}$-length.

3.3A. Second variation formula of $\mathcal{L}$-length. Using (3.4) and the notation in calculating the first variation formula we have

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} = \int_0^\tau \sqrt{\tilde\tau}\left(Y(Y(R)) + 2\langle \nabla_Y\nabla_Y X, X\rangle + 2\left|\nabla_Y X\right|^2\right)d\tilde\tau.$$

From

$$\langle \nabla_Y\nabla_Y X, X\rangle = \langle \nabla_Y\nabla_X Y, X\rangle = \langle \mathrm{Rm}(Y,X)Y, X\rangle + \langle \nabla_X\nabla_Y Y, X\rangle,$$

it follows

$$\begin{aligned}&\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} \\ &= \int_0^\tau \sqrt{\tilde\tau}\left(Y(Y(R)) + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2\langle \nabla_X\nabla_Y Y, X\rangle + 2\left|\nabla_Y X\right|^2\right)d\tilde\tau.\end{aligned}$$

To get a better form for the term $2\langle \nabla_X\nabla_Y Y, X\rangle$, we have

$$\frac{d}{d\tilde\tau}\langle \nabla_Y Y, X\rangle = \langle \nabla_X\nabla_Y Y, X\rangle + \langle \nabla_Y Y, \nabla_X X\rangle + \frac{\partial g}{\partial \tilde\tau}(\nabla_Y Y, X) + \left\langle \left(\frac{\partial}{\partial\tilde\tau}\nabla\right)_Y Y, X\right\rangle,$$

this is because both the inner product and the connection in $\langle \nabla_Y Y, X\rangle$ depend on $\tilde\tau$. Using

$$\left\langle \left(\frac{\partial}{\partial\tilde\tau}\nabla\right)_Y Y, X\right\rangle = 2\left(\nabla_Y \mathrm{Rc}\right)(Y,X) - \left(\nabla_X \mathrm{Rc}\right)(Y,Y)$$

(derived from the formula for Christoffel symbols), we get

$$\begin{aligned}\frac{d}{d\tilde\tau}\langle \nabla_Y Y, X\rangle =& \langle \nabla_X\nabla_Y Y, X\rangle + \langle \nabla_Y Y, \nabla_X X\rangle + 2\,\mathrm{Rc}(\nabla_Y Y, X) \\ &+ 2\left(\nabla_Y \mathrm{Rc}\right)(Y,X) - \left(\nabla_X \mathrm{Rc}\right)(Y,Y).\end{aligned} \tag{3.10}$$

Hence

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} = \int_0^\tau \sqrt{\tilde{\tau}}\left(Y(Y(R)) + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2|\nabla_Y X|^2\right)d\tilde{\tau}$$

$$+ 2\int_0^\tau \sqrt{\tilde{\tau}}\begin{pmatrix} \frac{d}{d\tilde{\tau}}\langle \nabla_Y Y, X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - 2\,\mathrm{Rc}(\nabla_Y Y, X) \\ -2(\nabla_Y \mathrm{Rc})(Y,X) + (\nabla_X \mathrm{Rc})(Y,Y) \end{pmatrix} d\tilde{\tau}$$

$$= \int_0^\tau \sqrt{\tilde{\tau}}\left(Y(Y(R)) + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2|\nabla_Y X|^2\right)d\tilde{\tau}$$

$$+ 2\int_0^\tau \sqrt{\tilde{\tau}}\begin{pmatrix} -\langle \nabla_Y Y, \nabla_X X\rangle - 2\,\mathrm{Rc}(\nabla_Y Y, X) \\ -2(\nabla_Y \mathrm{Rc})(Y,X) + (\nabla_X \mathrm{Rc})(Y,Y) \end{pmatrix} d\tilde{\tau}$$

$$+ 2\sqrt{\tilde{\tau}}\langle \nabla_Y Y, X\rangle\Big|_0^\tau - \int_0^\tau \frac{1}{\sqrt{\tilde{\tau}}}\langle \nabla_Y Y, X\rangle\, d\tilde{\tau}.$$

Assume γ_0 is an $\mathcal{L}$-geodesic and

$$Y(0) = 0, \tag{3.11}$$

we compute using integration by parts,

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} = 2\sqrt{\tau}\langle \nabla_Y Y, X\rangle + \int_0^\tau \sqrt{\tilde{\tau}}\begin{pmatrix} Y(Y(R)) - \nabla_Y Y\cdot \nabla R \\ +2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2|\nabla_Y X|^2 \end{pmatrix} d\tilde{\tau}$$

$$+ 2\int_0^\tau \sqrt{\tilde{\tau}}\begin{pmatrix} -\left\langle \nabla_Y Y, \left[\nabla_X X + 2\,\mathrm{Rc}(X) - \frac{1}{2}\nabla R + \frac{1}{2\tilde{\tau}}X\right]\right\rangle \\ -2(\nabla_Y \mathrm{Rc})(Y,X) + (\nabla_X \mathrm{Rc})(Y,Y) \end{pmatrix} d\tilde{\tau}$$

$$= 2\sqrt{\tau}\langle \nabla_Y Y, X\rangle + \int_0^\tau \sqrt{\tilde{\tau}}\left(\nabla^2_{Y,Y}R + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2|\nabla_Y X|^2\right)d\tilde{\tau}$$

$$+ \int_0^\tau \sqrt{\tilde{\tau}}\left(-4(\nabla_Y \mathrm{Rc})(Y,X) + 2(\nabla_X \mathrm{Rc})(Y,Y)\right)d\tilde{\tau},$$

where $\nabla^2_{Y,Y}R$ denotes the Hessian $(\mathrm{Hess}\,R)(Y,Y)$.

We have derived formula (7.7) in [Pe02].

Lemma 3.1. *Let $\gamma : [0,\tau] \to N$ be an $\mathcal{L}$-geodesic and let γ_s be a smooth variation of $\gamma = \gamma_0$. Assume the variation field $Y = \frac{\partial}{\partial s}\gamma_s$ satisfies $Y(0) = 0$. The second variation of $\mathcal{L}$-length is given by*

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} = 2\sqrt{\tau}\langle \nabla_Y Y, X\rangle(\tau)$$

$$+ \int_0^\tau \sqrt{\tilde{\tau}}\begin{pmatrix} \nabla^2_{Y,Y}R + 2\langle \mathrm{Rm}(Y,X)Y,X\rangle + 2|\nabla_Y X|^2 \\ -4(\nabla_Y \mathrm{Rc})(Y,X) + 2(\nabla_X \mathrm{Rc})(Y,Y) \end{pmatrix} d\tilde{\tau}. \tag{3.12}$$

3.3B. Another form of the second variation formula of $\mathcal{L}$-length. To write the second variation formula of $\mathcal{L}$-length in a better form (relating to Hamilton's matrix

Harnack quantity), we need to introduce a term $\left(\frac{\partial}{\partial\tilde{\tau}}\operatorname{Rc}\right)(Y,Y)$ into (3.12). Note that $\operatorname{Rc}(Y,Y)=\operatorname{Rc}(\gamma(\tilde{\tau}),\tilde{\tau})\left(Y\left(\tilde{\tau}\right),Y\left(\tilde{\tau}\right)\right)$, we have

$$\frac{d}{d\tilde{\tau}}\left[\operatorname{Rc}(Y,Y)\right]=\left(\frac{\partial}{\partial\tilde{\tau}}\operatorname{Rc}\right)(Y,Y)+\left(\nabla_X\operatorname{Rc}\right)(Y,Y)+2\operatorname{Rc}\left(\nabla_X Y,Y\right).$$

It follows

$$\begin{aligned}&-\int_0^\tau\sqrt{\tilde{\tau}}\left(\frac{\partial}{\partial\tilde{\tau}}\operatorname{Rc}\right)(Y,Y)\,d\tilde{\tau}=-\sqrt{\tilde{\tau}}\operatorname{Rc}(Y,Y)\Big|_0^\tau+\\&+\int_0^\tau\sqrt{\tilde{\tau}}\left(\frac{1}{2\tilde{\tau}}\operatorname{Rc}(Y,Y)+\left(\nabla_X\operatorname{Rc}\right)(Y,Y)+2\operatorname{Rc}\left(\nabla_X Y,Y\right)\right)d\tilde{\tau}.\end{aligned}$$

Hence we can rewrite (3.12) as

$$\begin{aligned}&\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0}-2\sqrt{\tau}\left\langle\nabla_Y Y,X\right\rangle+2\sqrt{\tau}\operatorname{Rc}(Y,Y)\\&=\int_0^\tau\sqrt{\tilde{\tau}}\left(\left(2\frac{\partial}{\partial\tilde{\tau}}\operatorname{Rc}+\frac{1}{\tilde{\tau}}\operatorname{Rc}\right)(Y,Y)+\nabla^2_{Y,Y}R-2\left|\operatorname{Rc}(Y)\right|^2\right)d\tilde{\tau}\\&\quad+\int_0^\tau\sqrt{\tilde{\tau}}\left(2\left\langle\operatorname{Rm}(Y,X)Y,X\right\rangle-4\left(\nabla_Y\operatorname{Rc}\right)(Y,X)+4\left(\nabla_X\operatorname{Rc}\right)(Y,Y)\right)d\tilde{\tau}\\&\quad+\int_0^\tau 2\sqrt{\tilde{\tau}}\left|\nabla_X Y+\operatorname{Rc}(Y)\right|^2d\tilde{\tau}.\end{aligned}$$

In the above formula by substituting Hamilton's **matrix Harnack quantity**

$$\begin{aligned}H(X,Y)=&-2\left(\frac{\partial}{\partial\tilde{\tau}}\operatorname{Rc}\right)(Y,Y)-\nabla^2_{Y,Y}R+2\left|\operatorname{Rc}(Y)\right|^2-\frac{1}{\tilde{\tau}}\operatorname{Rc}(Y,Y)\\&-2\left\langle\operatorname{Rm}(Y,X)Y,X\right\rangle-4\left(\nabla_X\operatorname{Rc}\right)(Y,Y)+4\left(\nabla_Y\operatorname{Rc}\right)(Y,X),\end{aligned}\tag{3.13}$$

we obtain

$$\begin{aligned}&\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0}-2\sqrt{\tau}\left\langle\nabla_Y Y,X\right\rangle+2\sqrt{\tau}\operatorname{Rc}(Y,Y)\\&=-\int_0^\tau\sqrt{\tilde{\tau}}H(X,Y)\,d\tilde{\tau}+\int_0^\tau 2\sqrt{\tilde{\tau}}\left|\nabla_X Y+\operatorname{Rc}(Y)\right|^2d\tilde{\tau}.\end{aligned}\tag{3.14}$$

By a little calculation we have

$$\int_0^\tau 2\sqrt{\tilde{\tau}}\left|\nabla_X Y+\operatorname{Rc}(Y)\right|^2d\tilde{\tau}=\int_0^\tau 2\sqrt{\tilde{\tau}}\left|\nabla_X Y+\operatorname{Rc}(Y)-\frac{1}{2\tilde{\tau}}Y\right|^2d\tilde{\tau}+\frac{\left|Y(\tau)\right|^2}{\sqrt{\tau}}.$$

Hence we have proved

Lemma 3.2. *Let $\gamma : [0, \tau] \to N$ be an $\mathcal{L}$-geodesic and let γ_s be a smooth variation of $\gamma = \gamma_0$. Assume the variation field $Y = \frac{\partial}{\partial s}\gamma_s$ satisfies that $Y(0) = 0$ when $s = 0$. Then*

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} - 2\sqrt{\tau}\,\langle\nabla_Y Y, X\rangle(\tau) + 2\sqrt{\tau}\,\mathrm{Rc}(Y,Y)(\tau)$$

$$= -\int_0^\tau \sqrt{\tilde{\tau}} H(X,Y)\,d\tilde{\tau} + \int_0^\tau 2\sqrt{\tilde{\tau}}\left|\nabla_X Y + \mathrm{Rc}(Y) - \frac{1}{2\tilde{\tau}}Y\right|^2 d\tilde{\tau} + \frac{|Y(\tau)|^2}{\sqrt{\tau}}, \tag{3.15}$$

where $H(X,Y)$ is defined by (3.13).

3.4. Estimate of second derivatives of L-distance

In this subsection we give some consequences of the second variation formula of $\mathcal{L}$-length. Below L-distance is defined using the basepoint $(p, 0)$.

3.4A. Estimate of Hessian of L-distance. Given $(x, \tau) \in N \times (0, T]$, let $\gamma : [0, \tau] \to N$ be a minimal $\mathcal{L}$-geodesic from p to x. Fix a vector $Y \in T_x N$ and define a vector field $\tilde{Y}(\tilde{\tau})$ along γ by solving the following ode along γ:

$$\nabla_X \tilde{Y} = -\mathrm{Rc}\left(\tilde{Y}\right) + \frac{1}{2\tilde{\tau}}\tilde{Y}, \quad \tilde{\tau} \in [0, \tau], \tag{3.16a}$$

$$\tilde{Y}(\tau) = Y. \tag{3.16b}$$

A direct computation gives $\frac{d}{d\tilde{\tau}}|\tilde{Y}|^2 = \frac{1}{\tilde{\tau}}|\tilde{Y}|^2$, which implies that

$$|\tilde{Y}(\tilde{\tau})|^2_{g(\tilde{\tau})} = |\tilde{Y}|^2 = \frac{\tilde{\tau}}{\tau}|Y|^2. \tag{3.17}$$

Hence from (3.15) we have

$$\left.\frac{d^2\mathcal{L}(\gamma_s)}{ds^2}\right|_{s=0} - 2\sqrt{\tau}\,\langle\nabla_Y Y, X\rangle + 2\sqrt{\tau}\,\mathrm{Rc}(Y,Y)$$

$$= -\int_0^\tau \sqrt{\tilde{\tau}} H\left(X, \tilde{Y}\right) d\tilde{\tau} + \frac{|Y|^2}{\sqrt{\tau}} \tag{3.18}$$

for any γ_s with variation field being $\tilde{Y}(\tilde{\tau})$.

Let $\gamma_s : [0, \tau] \to N$, $s \in (-\varepsilon, \varepsilon)$, be a smooth family of paths with

$$\left.\frac{\partial\gamma_s}{\partial s}\right|_{s=0}(\tilde{\tau}) = \tilde{Y}(\tilde{\tau}) \quad \text{and} \quad \left.\left(\nabla_{\frac{\partial\gamma_s}{\partial s}}\frac{\partial\gamma_s}{\partial s}\right)\right|_{s=0}(\tau) = 0.$$

Since $\mathcal{L}(\gamma_s)$ is an upper barrier function for the L-distance function $L(\gamma_s(\tau), \tau)$ at $s = 0$, we have

$$\left(\mathrm{Hess}_{(x,\tau)} L\right)(Y,Y) \le \left.\frac{d^2}{ds^2}\right|_{s=0} \mathcal{L}(\gamma_s).$$

From $(\nabla_Y Y)(\tau) = 0$ and (3.18) we get

Theorem 3.3 (**Hessian Comparison for L-distance**). *Let γ be a minimal $\mathcal{L}$-geodesic from $(p,0)$ to (x,τ). Given $Y \in T_xN$, let $\tilde{Y}(\tilde{\tau})$ be a solution of (3.16a) and (3.16b). Then*

$$\left(\mathrm{Hess}_{(x,\tau)} L\right)(Y,Y) \leq -\int_0^\tau \sqrt{\tilde{\tau}} H\left(X, \tilde{Y}\right) d\tilde{\tau} + \frac{|Y|^2}{\sqrt{\tau}} - 2\sqrt{\tau}\,\mathrm{Rc}\,(Y,Y)\,, \tag{3.19}$$

where X is the tangent vector field of γ and $H(X,\tilde{Y})$ is defined by (3.13). Equality in (3.19) *holds when $L(\cdot,\tau)$ is C^2 at x and $\tilde{Y}(\tilde{\tau})$ is the variation vector field of a family of minimal $\mathcal{L}$-geodesics γ_s satisfying $\left(\nabla_{\frac{\partial\gamma_s}{\partial s}} \frac{\partial\gamma_s}{\partial s}\right)\Big|_{s=0}(\tau) = 0$.*

3.4B. Laplacian comparison theorem for L-distance. Given $(x,\tau) \in N \times (0,T]$, let $\gamma : [0,\tau] \to N$ be a minimal $\mathcal{L}$-geodesic from p to x and let $X = \dot{\gamma}$. Fix an orthonormal basis $\{E_i\}_{i=1}^n$ of $(T_xN, g(x,\tau))$. For each i we define the vector field $\tilde{E}_i(\tilde{\tau})$ along γ to be the solution of (3.16a) with $\tilde{E}_i(\tau) = E_i$. A direct computation gives $\frac{d}{d\tilde{\tau}}\left\langle \tilde{E}_i, \tilde{E}_j\right\rangle = \frac{1}{\tilde{\tau}}\left\langle \tilde{E}_i, \tilde{E}_j\right\rangle$, which implies that

$$\left\langle \tilde{E}_i, \tilde{E}_j\right\rangle(\tilde{\tau}) = \frac{\tilde{\tau}}{\tau}\left\langle E_i, E_j\right\rangle = \frac{\tilde{\tau}}{\tau}\delta_{ij}. \tag{3.20}$$

Hence the matrix Harnack expression

$$\sum_{i=1}^n H\left(X(\tilde{\tau}), \tilde{E}_i(\tilde{\tau})\right) = \frac{\tilde{\tau}}{\tau}\sum_{i=1}^n H\left(X(\tilde{\tau}), \sqrt{\frac{\tau}{\tilde{\tau}}}\tilde{E}_i(\tilde{\tau})\right) = \frac{\tilde{\tau}}{\tau} H(X)\,, \tag{3.21}$$

where $H(X)$ is the **trace Harnack quantity**

$$H(X) = -\frac{\partial R}{\partial \tilde{\tau}} - 2\nabla R \cdot X + 2\,\mathrm{Rc}\,(X,X) - \frac{R}{\tilde{\tau}}. \tag{3.22}$$

Taking $Y = E_i$ in (3.19) and summing over i, we have

$$\begin{aligned}
\Delta L(x,\tau) &= \sum_{i=1}^n \left(\mathrm{Hess}_{(x,\tau)}\right) L(E_i, E_i) \\
&\leq -\int_0^\tau \sqrt{\tilde{\tau}} \sum_{i=1}^n H\left(X(\tilde{\tau}), \tilde{E}_i(\tilde{\tau})\right) d\tilde{\tau} + \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}\sum_{i=1}^n \mathrm{Rc}\,(E_i,E_i) \\
&= -\int_0^\tau \frac{\tilde{\tau}^{3/2}}{\tau} H(X)\, d\tilde{\tau} + \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau} R(x,\tau) \\
&= -\frac{1}{\tau}K + \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}R(x,\tau)\,,
\end{aligned}$$

where K is the **trace Harnack integral**

$$K = K(\gamma,\tau) = \int_0^\tau \tilde{\tau}^{3/2} H(X)\, d\tilde{\tau}. \tag{3.23}$$

We have proven

Theorem 3.4 (**Laplacian comparison for L-distance**). *For any $(x,\tau) \in N \times (0,T]$ the L-distance satisfies*

$$\Delta L(x,\tau) \leq -\frac{1}{\tau}K + \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}R(x,\tau), \tag{3.24}$$

where K is defined by (3.23) using a minimal $\mathcal{L}$-geodesic $\gamma : [0,\tau] \to N$ from p to x.

3.5. Equalities and inequalities satisfied by L and ℓ

In this subsection L-distance and reduced distance ℓ are defined using basepoint $(p,0)$.

3.5A. A formula for K in (3.23). We need a better formula for K in (3.23). Let $\gamma : [0,\tau] \to N$ be an $\mathcal{L}$-geodesic from p to x and let $X = \dot{\gamma}$. Using the $\mathcal{L}$-geodesic equation (3.3), we compute the evolution of the $\mathcal{L}$-length integrand for γ

$$\begin{aligned}
&\frac{d}{d\tilde{\tau}}\left(R(\gamma(\tilde{\tau}),\tilde{\tau}) + |X(\tilde{\tau})|^2_{g(\tilde{\tau})}\right) \\
&= \frac{\partial R}{\partial \tilde{\tau}} + \nabla R \cdot X + 2\operatorname{Rc}(X,X) + 2\langle \nabla_X X, X\rangle \\
&= \frac{\partial R}{\partial \tilde{\tau}} + \nabla R \cdot X + 2\operatorname{Rc}(X,X) + \left\langle \nabla R - 4\operatorname{Rc}(X) - \frac{1}{\tilde{\tau}}X, X\right\rangle \\
&= \frac{\partial R}{\partial \tilde{\tau}} + 2\nabla R \cdot X - 2\operatorname{Rc}(X,X) - \frac{1}{\tilde{\tau}}|X|^2.
\end{aligned}$$

Hence

$$\frac{d}{d\tilde{\tau}}\left(R + |X|^2\right) = -H(X) - \frac{1}{\tilde{\tau}}\left(R + |X|^2\right).$$

Multiplying the above equation by $\tilde{\tau}^{3/2}$ and integrating by parts, we get

$$\begin{aligned}
-K(\gamma,\tau) &= \int_0^\tau \left[\tilde{\tau}^{3/2}\frac{d}{d\tilde{\tau}}\left(R + |X|^2\right) + \tilde{\tau}^{1/2}\left(R + |X|^2\right)\right] d\tilde{\tau} \\
&= \tau^{3/2}\left(R(\gamma(\tau),\tau) + |X(\tau)|^2\right) - \frac{1}{2}\int_0^\tau \tilde{\tau}^{1/2}\left(R + |X|^2\right) d\tilde{\tau} \\
&= \tau^{3/2}\left(R(x,\tau) + |X(\tau)|^2\right) - \frac{1}{2}\mathcal{L}(\gamma).
\end{aligned}$$

We have proved that for any minimal $\mathcal{L}$-geodesic $\gamma : [0,\tau] \to N$ from p to x, we have

$$\tau^{3/2}\left(R(x,\tau) + |X(\tau)|^2\right) = -K(\gamma,\tau) + \frac{1}{2}L(x,\tau). \tag{3.25}$$

3.5B. Equalities and inequalities satisfied by L-distance. Using (3.25), we can rewrite (3.9) and (3.8), and (3.24) as the following: At (x,τ),

$$\frac{\partial L}{\partial \tau} = \frac{1}{\tau}K - \frac{1}{2\tau}L + 2\sqrt{\tau}R, \tag{3.26a}$$

$$|\nabla L|^2 = -4\tau R - \frac{4}{\sqrt{\tau}}K + \frac{2}{\sqrt{\tau}}L, \tag{3.26b}$$

$$\Delta L \le -\frac{1}{\tau}K + \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau}R, \tag{3.26c}$$

where $K = K(\gamma,\tau)$ is given by *(3.23)* and $\gamma : [0,\tau] \to N$ is a minimal $\mathcal{L}$-geodesic from p to x.

3.5C. Equalities and inequalities satisfied by reduced distance ℓ. Recall that the reduced distance $\ell(x,\tau) = \frac{1}{2\sqrt{\tau}}L(x,\tau)$. We have at (x,τ),

$$\frac{\partial \ell}{\partial \tau} = \frac{1}{2\tau^{3/2}}K - \frac{\ell}{\tau} + R, \tag{3.27a}$$

$$|\nabla \ell|^2 = -R - \frac{1}{\tau^{3/2}}K + \frac{\ell}{\tau}, \tag{3.27b}$$

$$\Delta \ell \le -\frac{1}{2\tau^{3/2}}K + \frac{n}{2\tau} - R, \tag{3.27c}$$

where $K = K(\gamma,\tau)$ is given by *(3.23)* and $\gamma : [0,\tau] \to N$ is a minimal $\mathcal{L}$-geodesic from p to x.

Note that in (3.27a), (3.27b) and (3.27c) the trace Harnack integral K depends on the path γ which is not favorable. However, from (3.27a), (3.27b) and (3.27c) we have the following four partial differential inequalities or equality which do not involve K.

Lemma 3.5. *At (x,τ) the reduced distance ℓ satisfies*

$$\frac{\partial \ell}{\partial \tau} - \Delta\ell + |\nabla \ell|^2 - R + \frac{n}{2\tau} \ge 0, \tag{3.28a}$$

$$2\Delta\ell - |\nabla\ell|^2 + R + \frac{\ell - n}{\tau} \le 0, \tag{3.28b}$$

$$\frac{\partial \ell}{\partial \tau} + \Delta\ell + \frac{\ell}{\tau} - \frac{n}{2\tau} \le 0, \tag{3.28c}$$

$$2\frac{\partial \ell}{\partial \tau} + |\nabla\ell|^2 - R + \frac{\ell}{\tau} = 0, \tag{3.28d}$$

$$\lim_{\tau\to 0_+} \frac{\ell(x,\tau)}{\left(d_{g(0)}(p,x)\right)^2 \big/ 4\tau} = 1, \tag{3.28e}$$

$$\inf_{x\in N} \ell(x,\tau) \le \frac{n}{2}. \tag{3.28f}$$

We skip the proof of (3.28e) but use it to give a proof of (3.28f). It follows from (3.28c) that

$$\left(\frac{\partial}{\partial(-\tau)} - \Delta\right)(4\tau\ell(x,\tau) - 2n\tau) \geq 0.$$

It follows from the maximum principle that $\inf_{x\in N}(4\tau\ell(x,\tau) - 2n\tau)$ is a nondecreasing function of $-\tau$. Note that (3.28e) implies $\lim_{\tau\to 0_+} 4\tau\ell(x,\tau) = \left(d_{g(0)}(p,x)\right)^2$ and hence $\lim_{\tau\to 0_+}\inf_{x\in N}(4\tau\ell(x,\tau) - 2n\tau) = 0$. We get $\inf_{x\in N}(4\tau\ell(x,\tau) - 2n\tau) \leq 0$ and (3.28f) follows.

3.6. $\mathcal{L}$-Jacobi field

In this subsection we discuss the $\mathcal{L}$-Jacobi field associated with $\mathcal{L}$-length, which is analogous to the Jacobi field associated with length.

3.6A. $\mathcal{L}$-Jacobi field. Now we consider the moduli space of $\mathcal{L}$-geodesics, the tangent direction to this space satisfies a second order linear ode. Let $\gamma_s : [0,\tau] \to N$, $s \in (-\varepsilon, \varepsilon)$, be a smooth family of $\mathcal{L}$-geodesics. Denote $\gamma_0 = \gamma$, $X_s = \dot{\gamma}_s$, and $Y_s = \frac{d}{ds}\gamma_s$. Taking ∇_{Y_s}-derivative of the $\mathcal{L}$-geodesic equation (3.3) for γ_s, we compute

$$\begin{aligned}\nabla_{X_s}(\nabla_{X_s}Y_s) &= \nabla_{X_s}(\nabla_{Y_s}X_s) = \mathrm{Rm}(X_s,Y_s)X_s + \nabla_{Y_s}(\nabla_{X_s}X_s)\\ &= \mathrm{Rm}(X_s,Y_s)X_s + \nabla_{Y_s}\left(\frac{1}{2}\nabla R - 2\,\mathrm{Rc}(X_s) - \frac{1}{2\tilde{\tau}}X_s\right).\end{aligned}$$

Set $s = 0$, then $Y(\tilde{\tau}) = Y_0(\tilde{\tau})$ satisfies the following ode called the **$\mathcal{L}$-Jacobi equation**:

$$\begin{aligned}&\nabla_X(\nabla_X Y)\\ &= -2\,\mathrm{Rc}(\nabla_X Y) - \frac{1}{2\tilde{\tau}}\nabla_X Y + \mathrm{Rm}(X,Y)X + \frac{1}{2}\nabla_Y(\nabla R) - 2(\nabla_Y \mathrm{Rc})(X). \qquad (3.29)\end{aligned}$$

We call any solution of the above equation an **$\mathcal{L}$-Jacobi field**.

Using the parametrization defined in (3.5) and $Z(\tilde{\sigma}) = \frac{d\beta}{d\tilde{\sigma}} = \sqrt{\tilde{\tau}}X(\tilde{\tau})$, we can rewrite the $\mathcal{L}$-Jacobi equation of $\hat{Y}(\tilde{\sigma}) = Y(\tilde{\tau})$ as

$$\begin{aligned}&\nabla_Z\left(\nabla_Z\hat{Y}\right)\\ &= -2\tilde{\sigma}\,\mathrm{Rc}\left(\nabla_Z\hat{Y}\right) + \mathrm{Rm}\left(Z,\hat{Y}\right)Z + \frac{\tilde{\sigma}^2}{2}\nabla_{\hat{Y}}(\nabla R) - 2\tilde{\sigma}\left(\nabla_{\hat{Y}}\mathrm{Rc}\right)(Z). \qquad (3.30)\end{aligned}$$

Hence the initial value problem of (3.30) is solvable.

3.6B. Estimate of $\mathcal{L}$-Jacobi field. Let γ_s be as in §3.6A. By the first variation formula for the $\mathcal{L}$-length,

$$\left.\frac{d}{ds}\right|_{s=0}\mathcal{L}(\gamma_s) = 2\sqrt{\tau}\langle X_s, Y_s\rangle(\tau).$$

We differentiate this again to get

$$\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\gamma_s) = 2\sqrt{\tau}\langle\nabla_X Y, Y\rangle(\tau) + 2\sqrt{\tau}\left\langle X, \nabla_{Y_s}Y_s|_{s=0}\right\rangle(\tau).$$

Now we compute the derivative of the norm squared of the $\mathcal{L}$-Jacobi field

$$\begin{aligned}
&\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} |Y(\tilde{\tau})|^2_{g(\tilde{\tau})} = 2\operatorname{Rc}(Y,Y)(\tau) + 2\langle \nabla_X Y, Y\rangle(\tau) \\
&= 2\operatorname{Rc}(Y,Y)(\tau) + \frac{1}{\sqrt{\tau}}\left(\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\gamma_s)\right) - 2\left\langle X, \nabla_{Y_s} Y_s|_{s=0}\right\rangle(\tau), \qquad (3.31)
\end{aligned}$$

Let $\tilde{Y}$ be a vector field along γ which satisfies (3.16a) with $\tilde{Y}(\tau) = Y(\tau)$, here we need to assume $Y(0)=0$. Let $\tilde{\gamma}_s : [0,\tau] \to N$ be a variation of γ with

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\tilde{\gamma}_s = \tilde{Y}, \quad \tilde{\gamma}_s(\tau) = \gamma_s(\tau), \quad \text{and} \quad \tilde{\gamma}_s(0) = \gamma_s(0).$$

Note that the choice of $\tilde{\gamma}_s$ implies $\nabla_{Y_s} Y_s|_{s=0}(\tau) = \left.\left(\nabla_{\frac{\partial\tilde{\gamma}_s}{\partial s}}\frac{\partial\tilde{\gamma}_s}{\partial s}\right)\right|_{s=0}(\tau)$. If we assume that the γ_s are all minimal $\mathcal{L}$-geodesics, then $\mathcal{L}(\gamma_s) \le \mathcal{L}(\tilde{\gamma}_s)$ for all s, and equality holds at $s=0$. Hence

$$\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\gamma_s) \le \left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\tilde{\gamma}_s),$$

where equality holds if $\tilde{Y}$ is an $\mathcal{L}$-Jacobi field. Combining this with (3.31), we get

$$\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau}|Y|^2 \le 2\operatorname{Rc}(Y,Y)(\tau) + \frac{1}{\sqrt{\tau}}\left(\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\tilde{\gamma}_s)\right) - 2\langle X, \nabla_Y Y\rangle(\tau).$$

By (3.18), we have

$$\begin{aligned}
&\left(\left.\frac{d^2}{ds^2}\right|_{s=0}\mathcal{L}(\tilde{\gamma}_s)\right) - 2\sqrt{\tau}\left\langle X, \left.\left(\nabla_{\frac{\partial\tilde{\gamma}_s}{\partial s}}\frac{\partial\tilde{\gamma}_s}{\partial s}\right)\right|_{s=0}\right\rangle(\tau) \\
&= -\int_0^\tau \sqrt{\tilde{\tau}} H\left(X,\tilde{Y}\right)d\tilde{\tau} + \frac{|Y(\tau)|^2}{\sqrt{\tau}} - 2\sqrt{\tau}\operatorname{Rc}(Y,Y)(\tau).
\end{aligned}$$

Hence we have proved

Lemma 3.6. *Let $\gamma_s : [0,\bar{\tau}] \to N$ be a smooth family of minimal $\mathcal{L}$-geodesics with $\gamma_s(0) = p$. Then for any $\tau \in (0,\bar{\tau}]$ the $\mathcal{L}$-Jacobi field $Y(\tau) = \left.\frac{d\gamma_s}{ds}\right|_{s=0}(\tau)$ satisfies the estimate*

$$\frac{d}{d\tau}|Y(\tau)|^2_{g(\tau)} \le -\frac{1}{\sqrt{\tau}}\int_0^\tau \sqrt{\tilde{\tau}} H\left(X,\tilde{Y}\right)d\tilde{\tau} + \frac{|Y(\tau)|^2}{\tau}, \qquad (3.32)$$

where X is the tangent vector field of $\gamma = \gamma_0$, $\tilde{Y}(\tilde{\tau})$ is a solution of (3.16a) on $[0,\tau]$ with $\tilde{Y}(\tau) = Y(\tau)$, and $H(X,\tilde{Y})$ is defined by (3.13).

3.7. $\mathcal{L}$-exponential map and $\mathcal{L}$-Jacobian

In §2 we have seen the role played by the exponential map in considering the volume of balls, in this subsection we consider the $\mathcal{L}$-exponential map, in next section we will see that it plays a similar role in reduced volume. In this subsection we use the basepoint $(p, 0)$ to define the L-distance.

3.7A. $\mathcal{L}$-exponential map. Given τ, define the **$\mathcal{L}$-exponential map at time** τ

$$\mathcal{L}_\tau \exp : T_pN \to N, \qquad \mathcal{L}_\tau \exp (V) = \gamma_V(\tau),$$

where $\gamma_V(\tilde{\tau}) = \beta(\tilde{\sigma})$ is the $\mathcal{L}$-geodesic obtained by solving (3.7) with $\frac{d\beta}{d\tilde{\sigma}}(0) = V \in T_pN$ and $\beta(0) = p$.

3.7B. $\mathcal{L}$-Jacobian. If we want to compute the tangent map $D(\mathcal{L}_\tau \exp(V))$ of $\mathcal{L}_\tau \exp$ at $V \in T_pN$, we need to consider a family of $\mathcal{L}$-geodesics $\gamma_{V_s}(\tilde{\tau})$ where V_s is a variation of V, and hence we need to consider the corresponding $\mathcal{L}$-Jacobi field.

Given an orthonormal basis $\left\{E_i^0\right\}_{i=1}^n$ of $(T_pN, g(p,0))$, let $J_i^V(\tilde{\tau}) = \hat{J}_i^V(\tilde{\sigma})$, $i = 1, \dots, n$, be $\mathcal{L}$-Jacobi fields along γ_V where $\tilde{\sigma} = 2\sqrt{\tilde{\tau}}$ and $\hat{J}_i^V(\tilde{\sigma})$ is defined by solving (3.30) with initial value

$$\hat{J}_i^V(0) = 0 \quad \text{and} \quad \left(\nabla_Z \hat{J}_i^V\right)(0) = E_i^0.$$

Then $D(\mathcal{L}_\tau \exp(V))(E_i^0) = J_i^V(\tau)$, and the Jacobian of the $\mathcal{L}$-exponential map $\mathcal{L}_\tau \mathrm{J}_V \in \mathbb{R}$ (called the **$\mathcal{L}$-Jacobian**) is given by

$$\mathcal{L}_\tau \mathrm{J}_V = \sqrt{\det\left(\left\langle J_i^V(\tau), J_j^V(\tau)\right\rangle_{g(\mathcal{L}_\tau \exp(V),\tau)}\right)_{n\times n}}. \tag{3.33}$$

Note that the pull-back volume form is

$$(\mathcal{L}_\tau \exp(V))^* \, d\mu_{g(\mathcal{L}_\tau \exp(V),\tau)} = \mathcal{L}_\tau \mathrm{J}_V \, dy$$

where dy is the standard Euclidean volume form on $(T_pN, g(p,0))$.

Let $\bar{E}_i(\tau)$ be the parallel translation of E_i^0 along $\gamma_V(\tilde{\tau})$ with respect to $g(0)$. From the definition of $\hat{J}_i^V(\tilde{\sigma})$ we have $|J_i^V(\tau) - 2\sqrt{\tau}\bar{E}_i(\tau)|_{g(0)} = o(2\sqrt{\tau})$ and hence we get the following asymptotic behavior of the $\mathcal{L}$-Jacobian

$$\lim_{\tau\to 0_+} \frac{\mathcal{L}_\tau \mathrm{J}_V}{\tau^{n/2}} = \lim_{\tau\to 0_+} \tau^{-n/2}\sqrt{\det\left(\langle 2\sqrt{\tau}\bar{E}_i(\tau), 2\sqrt{\tau}\bar{E}_j(\tau)\rangle_{g(0)}\right)} = 2^n. \tag{3.34}$$

3.7C. Estimate of $\mathcal{L}$-Jacobian. We have the following estimate of the time-derivative of the $\mathcal{L}$-Jacobian, which follows from the estimate of Jacobi fields.

Proposition 3.7. *Fix a $V \in T_pN$, let $\gamma_V(\tilde{\tau})$, $\tilde{\tau} \in [0, \bar{\tau}]$, be a minimal $\mathcal{L}$-geodesic with $\gamma_V(0) = p$ and $\lim_{\tilde{\tau}\to 0_+} \sqrt{\tilde{\tau}}\frac{d\gamma_V}{d\tilde{\tau}} = V$. For any $\tau \in (0, \bar{\tau})$ the $\mathcal{L}$-Jacobian $\mathcal{L}_\tau \mathrm{J}_V$ satisfies*

$$\left(\frac{d}{d\tau}\log \mathcal{L}_\tau \mathrm{J}_V\right) \le \frac{n}{2\tau} - \frac{1}{2\tau^{\frac{3}{2}}}K, \tag{3.35}$$

where $K = K(\gamma_V, \tau)$ is defined by (3.23).

Proof. Choose an orthonormal basis $\{E_i(\tau)\}$ of $(T_{\gamma_V(\tau)}N, g(\gamma_V(\tau), \tau))$. We can extend $E_i(\tau)$ to an $\mathcal{L}$-Jacobi field $E_i(\tilde{\tau})$ along γ_V for $\tilde{\tau} \in [0, \tau]$ with $E_i(0) = 0$. Then there is a matrix $\left(A_i^j\right) \in \mathrm{GL}(n, \mathbb{R})$, such that

$$J_i^V(\tilde{\tau}) = \sum_{j=1}^n A_i^j E_j(\tilde{\tau})$$

for all $\tilde{\tau} \in [0, \tau]$. The reason for the existence of $E_i(\tilde{\tau})$ and $\left(A_i^j\right)$ is that there is no nontrivial $\mathcal{L}$-Jacobi field along $\gamma_V(\tilde{\tau})$ which vanishes at the endpoints $\tilde{\tau} = 0, \tau$.

Now we compute the evolution of the $\mathcal{L}$-Jacobian along γ_V using (3.33) and (3.32):

$$\begin{aligned}
&\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \log \mathcal{L}_{\tilde{\tau}} \mathrm{J}_V = \left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \log \sqrt{\det\left(\left\langle \sum_{k=1}^n A_i^k E_k(\tilde{\tau}), \sum_{\ell=1}^n A_j^\ell E_\ell(\tilde{\tau}) \right\rangle_{g(\gamma_V(\tilde{\tau}),\tilde{\tau})}\right)} \\
&= \frac{1}{2}\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \log\det\left(\langle E_k, E_l\rangle(\tilde{\tau})\right) + \frac{1}{2}\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \log\det\left(A_i^k\right) + \frac{1}{2}\left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \log\det\left(A_j^\ell\right) \\
&= \frac{1}{2}\sum_{i=1}^n \left.\frac{d}{d\tilde{\tau}}\right|_{\tilde{\tau}=\tau} \langle E_i, E_i\rangle(\tilde{\tau}) \\
&\leq -\frac{1}{2}\frac{1}{\sqrt{\tau}}\int_0^\tau \sqrt{\tilde{\tau}} \sum_{i=1}^n H\left(X, \tilde{E}_i\right) d\tilde{\tau} + \frac{1}{2}\sum_{i=1}^n \frac{|E_i(\tau)|^2}{\tau} \\
&= -\frac{1}{2\tau^{3/2}}\int_0^\tau \tilde{\tau}^{3/2} H(X)\, d\tilde{\tau} + \frac{n}{2\tau}.
\end{aligned}$$

Here $\det\left(\langle E_k, E_l\rangle(\tilde{\tau})\right)$ denotes the determinant of $n \times n$ matrix $\left(\langle E_k, E_l\rangle(\tilde{\tau})\right)$, the $\tilde{E}_i(\tilde{\tau})$ are the vector fields along γ_V satisfying (3.16a) and $\tilde{E}_i(\tau) = E_i(\tau)$, and in the last equality above we have used (3.21). ☐

4. Reduced volume

In this section we use the properties of reduced distance to prove the monotonicity of reduced volume, in the next section we will see that this monotonicity has fundamental consequences about the properties of Ricci flow.

Throughout this section N^n is an n-dimensional connected oriented manifold, and $(N^n, g(\tau))$, $\tau \in [0, T]$, is a complete solution to the **backward Ricci flow** $\frac{\partial}{\partial\tau} g(\tau) = 2\,\mathrm{Rc}(g(\tau))$ with bounded Riemann curvature $\sup_{M\times[0,T]} |\mathrm{Rm}(x, \tau)| < \infty$. The reduced distance ℓ is defined using the basepoint $(p, 0)$ for some $p \in N$. Several claims in this section will not be proved.

4.1. Definition of reduced volume

The **reduced volume** function $\tilde{V} : (0, T] \to \mathbb{R}_{>0}$ is defined by

$$\tilde{V}(\tau) = \int_N (4\pi\tau)^{-n/2} \exp\left[-\ell(x,\tau)\right] d\mu_{g(\tau)}(x), \tag{4.1}$$

and $\tilde{V}(0)$ is defined to be 1. We will see that the integral in (4.1) is finite in §4.2A.

Comment. 1) A simple exercise shows that on the Euclidean space where $g(\tau) = g_{\text{Euc}}$ we have $\ell(x,\tau) = \frac{(d_{\text{Euc}}(x,p))^2}{4\tau}$ and $\tilde{V}(\tau) = 1$ for all τ.
2) There is some similarity between the reduced volume and the volume ratio $\frac{\operatorname{Vol} B(p,r)}{\omega_n r^n}$ which appeared in the Bishop-Gromov volume comparison theorem for complete manifolds with $\operatorname{Rc} \geq 0$.

In the rest of this subsection we give another formula for the integral in (4.1). The **$\mathcal{L}$-cut-locus** $\mathcal{L}_\tau \operatorname{Cut}_p$ of the map $\mathcal{L}_\tau \exp$ consists of points $\mathcal{L}_\tau \exp(V)$ where either $V \in T_pN$ is a critical point of $\mathcal{L}_\tau \exp$ or there is $\hat{V} \neq V$ such that $\gamma_{\hat{V}}$ (as defined above (3.35)) is a minimal $\mathcal{L}$-geodesic over $[0,\tau]$ joins p and $\mathcal{L}_\tau \exp(V)$. Note that $\mathcal{L}_\tau \operatorname{Cut}_p$ has measure zero in $(N, g(\tau))$ (claim).

Given $V \in T_pN$ there is a unique $\tau_V \in (0,T]$ such that the $\mathcal{L}$-geodesic $\gamma_V|_{[0,\tau]}$ is minimal when $\tau < \tau_V$ and is not minimal when $\tau > \tau_V$ (claim). We define $\Omega_p(\tau) = \{V \in T_pN, \tau < \tau_V\}$. Then

$$\mathcal{L}_\tau \exp : \Omega_p(\tau) \to N \setminus \mathcal{L}_\tau \operatorname{Cut}_p$$

is a diffeomorphism (claim).

Let dy be the standard Euclidean volume form on $(T_pN, g(p,0))$. Using $\mathcal{L}_\tau \exp$ and $\Omega_p(\tau)$ we can rewrite the reduced volume as

$$\begin{aligned}
\tilde{V}(\tau) &= \int_{N \setminus \mathcal{L}_\tau \operatorname{Cut}_p} (4\pi\tau)^{-n/2} \exp\left[-\ell(x,\tau)\right] d\mu_{g(\tau)}(x) \\
&= \int_{\Omega_p(\tau)} (4\pi\tau)^{-n/2} e^{-\ell(\mathcal{L}_\tau \exp(V),\tau)} \mathcal{L}_\tau \mathrm{J}_V \, dy(V), && (4.2) \\
&= \int_{T_pN} (4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V \, dy(V), && (4.3)
\end{aligned}$$

where we have used the formula after (3.33) to get (4.2) and the convention $\mathcal{L}_\tau \mathrm{J}_V = 0$ for $V \notin \Omega_p(\tau)$ to get (4.3).

4.2. The monotonicity of reduced volume

In this subsection we give two proofs of the monotonicity.

4.2A. Monotonicity of reduced volume using $\mathcal{L}$-Jacobian. For any $V \in \Omega_p(\tau)$, from (3.8) we have $\dot{\gamma}_V(\tau) = (\nabla \ell)(\gamma_V(\tau), \tau)$. We compute

$$\begin{aligned}
&\frac{d}{d\tau}\left[(4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V\right] \\
=&\left[(4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V\right]\left(-\frac{n}{2\tau} - \nabla\ell \cdot \dot{\gamma}_V - \frac{\partial \ell}{\partial \tau} + \frac{d}{d\tau}\log \mathcal{L}_\tau \mathrm{J}_V\right) \\
=&\left[(4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V\right]\left(-\frac{n}{2\tau} - |\nabla\ell|^2 - \frac{\partial \ell}{\partial \tau} + \frac{d}{d\tau}\log \mathcal{L}_\tau \mathrm{J}_V\right) \\
\leq& 0,
\end{aligned}$$

where the last inequality follows from (3.27a), (3.27b) and (3.35). Hence we have proved (i) of the following.

Lemma 4.1. *Using the notation from §4.1 we have*
(i) for any $V \in \Omega_p(\tau)$

$$\frac{d}{d\tau}\left[(4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V\right] \leq 0. \tag{4.4}$$

(ii) For any $V \in T_p\mathcal{M}$ *and* $0 \leq \tau \leq T$,

$$(4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V \leq \pi^{-n/2} e^{-|V|^2_{g(p,0)}}. \tag{4.5}$$

(iii) The reduced volume is well defined and takes values in $(0, 1]$.

Proof. (ii) Similar to (3.28e) we have (claim)

$$\lim_{\tau \to 0_+} \ell(\gamma_V(\tau), \tau) = |V|^2_{g(p,0)}.$$

Hence from (3.34) we have

$$\lim_{\tau \to 0_+} (4\pi\tau)^{-n/2} e^{-\ell(\gamma_V(\tau),\tau)} \mathcal{L}_\tau \mathrm{J}_V = \pi^{-n/2} e^{-|V|^2_{g(p,0)}}. \tag{4.6}$$

(ii) then follows from (i).

(iii) This follows from (4.3), (ii) and

$$\int_{T_pN} \pi^{-n/2} e^{-|V|^2_{g(p,0)}} dy(V) = 1$$

where dy is the standard Euclidean volume form on $(T_pN, g(p, 0))$. The above equality and (4.6) indicates that defining $\tilde{V}(0) = 1$ is reasonable. □

The next theorem follows from (4.3) and (4.4).

Theorem 4.2 (**Monotonicity of reduced volume**). *We have* $\tilde{V}(\tau_2) \leq \tilde{V}(\tau_1)$ *for* $\tau_2 \geq \tau_1$, *i.e.,*

$$\frac{d}{d\tau}\tilde{V}(\tau) \leq 0. \tag{4.7}$$

4.2B. Monotonicity of reduced volume without using $\mathcal{L}$-Jacobian. We compute formally

$$\begin{aligned}\frac{d}{d\tau}\tilde{V}(\tau) &= \int_N \frac{\partial}{\partial\tau}\left((4\pi\tau)^{-n/2}e^{-\ell(\cdot,\tau)}d\mu_{g(\tau)}\right) \qquad (4.8)\\ &= \int_N \left(-\frac{n}{2\tau}-\frac{\partial\ell}{\partial\tau}+R\right)(4\pi\tau)^{-n/2}e^{-\ell(\cdot,\tau)}d\mu_{g(\tau)}\\ &\le \int_N \left(|\nabla\ell|^2-\Delta\ell\right)(4\pi\tau)^{-n/2}e^{-\ell(\cdot,\tau)}d\mu_{g(\tau)} \qquad (4.9)\\ &\le 0. \qquad (4.10)\end{aligned}$$

Here (4.9) follows from (3.28a). To justify the switch of the order of differentiation and integration to get (4.8) and the integration by parts to get (4.10), one need certain estimates on the growth of reduced distance $\ell(x,\tau)$ and its derivatives. These estimates are not established in §3, we encourage the reader to find the details of proving (4.8) and (4.10) in the literature.

5. Applications of monotonicity of reduced volume

In this section we give two applications of reduced distance and reduced volume to justify their importance in the study of Ricci flow. The reader can find their other applications in the literature.

5.1. No local collapsing theorem

Recall that if a Ricci flow solution does not exist up to time $+\infty$, we say that it develops a **singularity** in finite time. The no local collapsing theorem is used in the singularity analysis of Ricci flow. When combined with the **Hamilton's Cheeger-Gromov-type compactness theorem** it implies the existence of **singularity models** for Ricci flow developing singularities in finite time.

Throughout this subsection M^n is a n-dimensional connected oriented manifold, and $(M^n,\tilde{g}(t))$, $t\in[0,T)$, is a complete solution to the Ricci flow with $T<\infty$, and we assume $\sup_{M\times[0,T_0]}|\mathrm{Rm}_{\tilde{g}}(x,t)|<\infty$ for all $T_0<T$.

5.1A. The statement of no local collapsing theorem. Before we state the theorem we need a

Definition 5.1 (**Strongly κ-collapsed**)**.** Let $\kappa>0$ be a constant. We say that Ricci flow $(M^n,\tilde{g}(t))$, $t\in[0,T)$, is **strongly κ-collapsed at** $(x_0,T_0)\in M\times(0,T)$ **at scale** $r>0$ if

(i) *(curvature bound in a parabolic cylinder)* $|\mathrm{Rm}_{\tilde{g}}(x,t)|\le\frac{1}{r^2}$ for all $x\in B_{\tilde{g}(T_0)}(x_0,r)$ and $t\in[\max\{T_0-r^2,0\},T_0]$, and

(ii) *(volume of ball is κ-collapsed)*

$$\frac{\mathrm{Vol}_{\tilde{g}(T_0)}B_{\tilde{g}(T_0)}(x_0,r)}{r^n}<\kappa.$$

Given an $r>0$, if for any $T_0 \in [r^2, T)$ and any $x_0 \in M$ the solution $\tilde{g}(t)$ is not strongly κ-collapsed at (x_0, T_0) at scale r, then we say that $(M, \tilde{g}(t))$ is **weakly κ-noncollapsed at scale r**.

The following is the so-called **weakened no local collapsing theorem** ([Pe02], §7.3).

Theorem 5.1. *Let $(M^n, \tilde{g}(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with $T < \infty$. Suppose there exist $r_1 > 0$ and $v_1 > 0$ such that*

$$\operatorname{Vol}_{\tilde{g}(0)} B_{\tilde{g}(0)}(x, r_1) \geq v_1 \text{ for all } x \in M.$$

Then there exists $\kappa > 0$ depending only on r_1, v_1, n, T, and $\sup_{M\times[0,T/2]} \operatorname{Rc}_{\tilde{g}}(x,t)$ such that $\tilde{g}(t)$ is weakly κ-noncollapsed at any point $(p, T_0) \in M \times (T/2, T)$ at any scale $r < \sqrt{T/2}$. Here $\sup \operatorname{Rc}_{\tilde{g}}(x,t)$ stands for the largest eigenvalue of $\operatorname{Rc}_{\tilde{g}}(x,t)$.

5.1B. Sketch of the proof of Theorem 5.1. Given a time $T_0 \in \left(\frac{T}{2}, T\right)$, let $g(\tau) = \tilde{g}(T_0 - \tau)$. Then $(M^n, g(\tau))$, $\tau \in [0, T_0]$, is a complete solution to the backward Ricci flow with initial metric $g(0) = \tilde{g}(T_0)$ and with bounded Riemann curvature tensor. Given a point $p \in M$, then we can define reduced distance $\ell(x,\tau)$ and reduced volume $\tilde{V}(\tau)$ using the basepoint $(p, 0)$. The theorem follows easily from the following two lemmas.

On one hand, we have

Lemma 5.2. *There exist $c_1(n) > 0$ depending only on n and a function $\phi(\epsilon, n)$ satisfying $\lim_{\epsilon\to 0_+} \phi(\varepsilon, n) = 0$ such that if for some κ satisfying $\kappa^{1/n} \leq c_1(n)$, the solution $\tilde{g}(t)$ is strongly κ-collapsed at some (p, T_0) at scale r, where $T_0 \in (\frac{T}{2}, T)$ and $r < \sqrt{T_0}$, then the reduced volume $\tilde{V}$ as defined above has the upper bound*

$$\tilde{V}\left(\varepsilon r^2\right) \leq \phi(\varepsilon, n),$$

where $\varepsilon = \kappa^{1/n}$.

Sketch of the proof of Lemma 5.2. From (4.3) we can write the reduced volume integral over T_pM as $\tilde{V}\left(\varepsilon r^2\right) = \tilde{V}_1\left(\varepsilon r^2\right) + \tilde{V}_2\left(\varepsilon r^2\right)$ where $\tilde{V}_1\left(\varepsilon r^2\right)$ and $\tilde{V}_2\left(\varepsilon r^2\right)$ are the integrals over $\{V \in T_pM, |V|_{g(p,0)} \leq \epsilon^{-1/4}\}$ and $\{V \in T_pM, |V|_{g(p,0)} > \epsilon^{-1/4}\}$, respectively. The lemma is proved by bounding $\tilde{V}_1\left(\varepsilon r^2\right)$ and $\tilde{V}_2\left(\varepsilon r^2\right)$ from above separately.

We have

$$\tilde{V}_1\left(\varepsilon r^2\right) \leq C_1(n)\epsilon^{n/2} \qquad \text{for } \epsilon < c_1(n)$$

where $C_1(n)$ and $c_1(n)$ are positive constants depending only on n. Using the assumption that the solution is strongly κ-collapsed at (p, T_0) at scale r, the upper bound estimate is proved by showing the following two estimates: the $\mathcal{L}$-geodesic $\gamma_V(\tau)$ (as defined in §3.7A) is contained in $B_{\tilde{g}(T_0)}(p, r/2)$ for some choice of $c_1(n)$; and when $|V|_{g(p,0)} \leq \epsilon^{-1/4}$ $\ell(\gamma_V(\varepsilon r^2), \varepsilon r^2)$ is bounded from below by a constant independent of ϵ.

We have

$$\tilde{V}_2\left(\varepsilon r^2\right) \leq C_2(n) e^{-\frac{1}{2\sqrt{\epsilon}}}$$

where $C_2(n)$ is a positive constant depending only on n. To see this estimate, by (4.5) we have

$$\left(4\pi\varepsilon r^2\right)^{-n/2} e^{-\ell\left(\gamma_V\left(\varepsilon r^2\right),\varepsilon r^2\right)}\mathcal{L}_{\varepsilon r^2}\,\mathrm{J}_V \leq \pi^{-n/2}e^{-|V|^2_{g(p,0)}}.$$

Then

$$\tilde{V}_2\left(\varepsilon r^2\right) \leq \int_{|V|_{g(p,0)}>\epsilon^{-1/4}} \pi^{-n/2}e^{-|V|^2_{g(p,0)}}dy$$

where dy is the standard Euclidean volume form on $(T_pN, g(p,0))$. The estimate follows.

On the other hand, we have

Lemma 5.3. *(i) Fix an arbitrary $r_0 > 0$. There exists a constant $C_3 > 0$, depending only on r_0, n, T, and $\sup_{M\times[0,T/2]}\mathrm{Rc}_{\tilde{g}}\,(x,t)$, and there exists $x_0 \in M$ such that reduced distance*

$$\ell\,(x,T_0) \leq C_3 \quad \text{for all } x \in B_{\tilde{g}(0)}\,(x_0,r_0)\,.$$

(ii) Suppose there exist $r_1 > 0$ and $v_1 > 0$ such that

$$\mathrm{Vol}_{\tilde{g}(0)}\,B_{\tilde{g}(0)}\,(x,r_1) \geq v_1$$

for all $x \in M$. Then there exists a constant $C_4 > 0$, depending only on r_1, v_1, n, T, and $\sup_{M\times[0,T/2]}\mathrm{Rc}_{\,\tilde{g}}\,(x,t)$, such that reduced volume

$$\tilde{V}\,(T_0) \geq C_4.$$

Sketch of the proof of Lemma 5.3. (i) By (3.28f), there is $x_0 \in M$ and a minimal $\mathcal{L}$-geodesic $\gamma_1 : [0, T_0 - \frac{T}{2}] \to M$ joining p and x_0 such that $\frac{1}{2\sqrt{T_0-\frac{T}{2}}}\mathcal{L}(\gamma_1) = \ell(x_0, T_0 - \frac{T}{2}) \leq \frac{n}{2}$. Let $\beta : [T_0 - \frac{T}{2}, T_0] \to (M, \tilde{g}(0))$ be the constant speed path joining x_0 and $x \in B_{\tilde{g}(0)}\,(x_0, r_0)$. Since γ_1 followed by β is a path joining $(p, 0)$ and (x, T_0),

$$\ell\,(x,T_0) \leq \frac{1}{2\sqrt{T_0}}\left(\mathcal{L}(\gamma_1) + \int_{T_0-\frac{T}{2}}^{T_0}\sqrt{\tilde{\tau}}\left(R\left(\beta\left(\tilde{\tau}\right),\tilde{\tau}\right) + \left|\frac{d\beta}{d\tilde{\tau}}\left(\tilde{\tau}\right)\right|^2_{g(\tilde{\tau})}\right)d\tilde{\tau}\right).$$

Note that the metric $g\,(\tilde{\tau})\,,\,\tilde{\tau} \in [T_0 - \frac{T}{2}, T_0]$, corresponds to the metric $\tilde{g}\,(t)\,,\,t \in [0, \frac{T}{2}]$, we can estimate the integral above to get (i).

(ii) We compute using x_0 in (i)

$$\begin{aligned}\tilde{V}\,(T_0) &\geq \int_{B_{\tilde{g}(0)}(x_0,r_1)} (4\pi T_0)^{-\frac{n}{2}}\,e^{-\ell(x,T_0)}d\mu_{\tilde{g}(0)}(x)\\ &\geq (4\pi T)^{-\frac{n}{2}}\,e^{-C_3}v_1.\end{aligned}$$

Now we finish the proof of Theorem 5.1. Suppose the solution $\tilde{g}\,(t)$ is strongly κ-collapsed at some (p, T_0) at scale r, where $\kappa^{1/n} \leq c_1\,(n)$, $T_0 \in (\frac{T}{2}, T)$ and $r < \sqrt{T_0}$. Combining the two lemmas above about the upper and lower bound of reduced volume and the monotonicity of reduced volume, we have

$$C_4 \leq \tilde{V}\,(T_0) \leq \tilde{V}\left(\varepsilon r^2\right) \leq \phi\,(\varepsilon, n)\,.$$

This forces $\varepsilon = \kappa^{1/n}$ not going to zero. Hence the theorem is proved.

5.1C. Type I solution and Lemma 5.3. Let $(M^n, \tilde{g}(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with $T < \infty$. Recall that $\tilde{g}(t)$ is called a **type-I** solution if there is a constant C_0 such that $(T-t)|\operatorname{Rm}_{\tilde{g}}(x,t)| \leq C_0$ for all $(x,t) \in M \times [0,T)$. For type-I solutions we have the following modification of Lemma 5.3.

Lemma 5.4. *Let $(M^n, \tilde{g}(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with $T < \infty$. Suppose for some constants C_0 and $\alpha \in [1, \frac{3}{2})$ we have*

$$(T-t)^{\alpha} |\operatorname{Rc}_{\tilde{g}}(x,t)| \leq C_0 \qquad \text{for all } (x,t) \in M \times [0,T),$$

and suppose there exist $x_1 \in M$, $r_1 > 0$, and $v_1 > 0$ such that

$$\operatorname{Vol}_{\tilde{g}(0)} B_{\tilde{g}(0)}(x_1, r_1) \geq v_1.$$

Then for any $A > 0$ there exist two positive constants C_5 and C_6, both depending only on $A, \alpha, r_1, v_1, n, T$, and C_0, such that for any $p \in B_{\tilde{g}(0)}(x_1, A)$ and $T_0 \in (T/2, T)$ the reduced distance, defined by backward solution $g(\tau) = \tilde{g}(T_0 - \tau)$ and basepoint $(p, 0)$, satisfies

$$\ell(x, T_0) \leq C_5 \quad \text{for all } x \in B_{\tilde{g}(0)}(x_1, r_1),$$

and the reduced volume $\tilde{V}(T_0) \geq C_6$.

Sketch of the proof of Lemma 5.4. Define $\gamma : [0, T_0] \to M$ to be a path joining p and $x \in B_{\tilde{g}(0)}(x_1, r_1)$ such that $\gamma(\tilde{\tau}) = x$ for $\tilde{\tau} \in [0, T_0 - T/2]$ and $\gamma|_{[T_0 - T/2, T_0]}$ is a constant speed minimal geodesic with respect to metric $\tilde{g}(0)$. By the Ricci curvature bound assumption we have for any $\tilde{\tau} \in [T_0 - T/2, T_0]$

$$|\dot{\gamma}(\tilde{\tau})|^2_{g(\tilde{\tau})} \leq e^{2^{\alpha} C_0 T^{1-\alpha}} |\dot{\gamma}(\tilde{\tau})|^2_{\tilde{g}(0)} = e^{2^{\alpha} C_0 T^{1-\alpha}} \cdot \frac{4(A + r_1)^2}{T^2}.$$

Since $|\operatorname{Rc}_g(x,\tau)| \leq C_0 \tau^{-\alpha}$, we compute

$$\begin{aligned} \ell(x, T_0) \leq & \frac{1}{2\sqrt{T_0}} \mathcal{L}(\gamma) \\ = & \frac{1}{2\sqrt{T_0}} \left(\int_0^{T_0} 2\sqrt{\tilde{\tau}} R_g(\gamma(\tilde{\tau}), \tilde{\tau}) d\tilde{\tau} + \int_{T_0 - T/2}^{T_0} 2\sqrt{\tilde{\tau}} |\dot{\gamma}(\tilde{\tau})|^2_{g(\tilde{\tau})} d\tilde{\tau} \right) \leq C_5. \end{aligned}$$

Hence

$$\tilde{V}(T_0) \geq \int_{B_{\tilde{g}(0)}(x_1, r_1)} (4\pi T_0)^{-\frac{n}{2}} e^{-\ell(x, T_0)} d\mu_{\tilde{g}(0)}(x) \geq (4\pi T)^{-\frac{n}{2}} e^{-C_5} v_1.$$

Following the same idea of the proof of Theorem 5.1 (see the paragraph above §5.1C), we have

Lemma 5.5. *Let $(M^n, \tilde{g}(t))$, $t \in [0,T)$, be a complete solution to the Ricci flow with $T < \infty$. Suppose for some constants C_0 and $\alpha \in [1, \frac{3}{2})$ we have*

$$(T-t)^{\alpha} |\operatorname{Rc}_{\tilde{g}}(x,t)| \leq C_0 \qquad \text{for all } (x,t) \in M \times [0,T),$$

and suppose there exist $x_1 \in M$, $r_1 > 0$, *and* $v_1 > 0$ *such that* $\mathrm{Vol}_{\tilde{g}(0)} B_{\tilde{g}(0)}(x_1, r_1) \geq v_1$. *Then for any* $A > 0$ *there exists a positive constant* κ, *depending only on* $A, \alpha, r_1, v_1, n, T$, *and* C_0, *such that at any scale* $r < \sqrt{T/2}$, $\tilde{g}(t)$ *is weakly* κ*-noncollapsed at any point* $(p, T_0) \in B_{\tilde{g}(0)}(x_1, A) \times (T/2, T)$.

Note that this lemma has a weaker volume assumption than that of Theorem 5.1.

5.2. Backward limits of κ-solutions are shrinkers

A special family of singularity models is the so-called κ**-solutions**. The main theorem of this subsection shows that some blow-down limits of the solutions are even more special: shrinking gradient Ricci solitons. This opens the door for possible classification of singularity models in lower dimensions. A near classification knowledge about κ-solutions in dimension 3 enables us to perform surgeries on Ricci flow and eventually leads to the longtime existence of the so-called surgical Ricci flow.

5.2A. κ-solutions and the theorem. First we give a

Definition 5.2. Let κ be a positive constant. A complete ancient solution $(M^n, \tilde{g}(t))$, $t \in (-\infty, 0]$, of the Ricci flow is called a κ**-solution** if it satisfies

(i) $\tilde{g}(t)$ is nonflat and has nonnegative curvature operator for each $t \in (-\infty, 0]$.

(ii) Scalar curvature satisfies $\sup_{M \times (-\infty, 0]} R_{\tilde{g}}(x, t) < \infty$.

(iii) $\tilde{g}(t)$ is κ**-noncollapsed on all scales** for all $t \in (-\infty, 0]$; i.e., for any $r > 0$ and for any $(p, t) \in M \times (-\infty, 0]$, if $|\mathrm{Rm}_{\tilde{g}}(x, t)| \leq r^{-2}$ for all $x \in B_{\tilde{g}(t)}(p, r)$, then

$$\frac{\mathrm{Vol}_{\tilde{g}(t)} B_{\tilde{g}(t)}(p, r)}{r^n} \geq \kappa.$$

Given a κ-solution $(\mathcal{M}^n, \tilde{g}(t))$, $t \in (-\infty, 0]$, we define a solution to the backward Ricci flow $(M^n, g(\tau))$, $\tau \in [0, \infty)$, by

$$g(\tau) = \tilde{g}(-\tau).$$

Given a point $p \in M$, we can define the reduced distance $\ell(x, \tau)$ and reduced volume $\tilde{V}(\tau)$ using basepoint $(p, 0)$. Let $q_\tau \in M$ be a point such that $\ell(q_\tau, \tau) \leq \frac{n}{2}$. The existence of q_τ is guaranteed by (3.28f). For any $\tau > 0$, we define solutions to the backward Ricci flow by parabolic scaling:

$$g_\tau(\theta) = \tau^{-1} \cdot g(\tau\theta), \quad \text{for } \theta \in [0, \infty). \tag{5.1}$$

The following is Proposition 11.2 in [Pe02].

Theorem 5.6. *For any sequence* $\tau_i \to \infty$, *there exists a subsequence still denoted by* τ_i, *such that* $(M^n, g_{\tau_i}(\theta), (q_{\tau_i}, 1))$, $\theta \in (0, \infty)$, *converges in the Cheeger-Gromov sense to a complete nonflat shrinking gradient Ricci soliton* $(M^n_\infty, g_\infty(\theta), (q_\infty, 1))$.

Below we give a **sketch** of the proof of Theorem 5.6.

5.2B. Estimating reduced distance associated to κ-solutions. The **Hamilton's trace Harnack inequality** says that for a backward Ricci flow solution on $[0,T)$ with nonegative curvature operator, the trace Harnack quantity as defined in (3.22) satisfies

$$H(X)(x,\tilde{\tau}) \geq -\left(\frac{1}{\tilde{\tau}}+\frac{1}{T-\tilde{\tau}}\right)R(x,\tilde{\tau}). \tag{5.2}$$

Here for $g(\tau)$ we can take $T=\infty$. By (3.23) we have

$$K \geq -\int_0^\tau \tilde{\tau}^{3/2}\cdot\tilde{\tau}^{-1}R(\gamma(\tilde{\tau}),\tilde{\tau})d\tilde{\tau} \geq -\mathcal{L}(\gamma) = -2\sqrt{\tau}\ell(x,\tau).$$

Plug this into (3.27b), we have the following estimate

$$|\nabla\ell(x,\tau)|^2 + R(x,\tau) \leq \frac{3\ell(x,\tau)}{\tau} \tag{5.3}$$

for $g(\tau)$ coming out of the κ-solution $\tilde{g}(t)$. From (3.27a) and (3.27b) we have

$$\frac{\partial\ell}{\partial\tau} = -\frac{1}{2}|\nabla\ell|^2 - \frac{\ell}{2\tau} + \frac{1}{2}R,$$

hence from (5.3) it follows

$$\left|\frac{\partial\ell}{\partial\tau}\right| \leq \frac{2\ell}{\tau}. \tag{5.4}$$

Lemma 5.7. *Given any $\varepsilon>0$ and $A>1$, there exists $\delta(n,\varepsilon,A)>0$ such that for any $\tau>0$,*

$$\ell(x,\tilde{\tau}) \leq \delta(n,\varepsilon,A)^{-1} \quad \text{and} \quad \tilde{\tau}R(x,\tilde{\tau}) \leq \delta(n,\varepsilon,A)^{-1}$$

for all $(x,\tilde{\tau}) \in B_{g(\tau)}\left(q_\tau,\sqrt{\varepsilon^{-1}\tau}\right)\times\left[A^{-1}\tau, A\tau\right]$.

Sketch of the proof of Lemma 5.7. Note that the estimate of $\tilde{\tau}R(x,\tilde{\tau})$ follows from the estimate of $\ell(x,\tilde{\tau})$ and (5.3). To see the estimate of $\ell(x,\tilde{\tau})$, by (5.3) we have $|\nabla\sqrt{\ell(x,\tau)}|_{g(\tau)} \leq \frac{\sqrt{3}}{2}\tau^{-1/2}$, combining with $\ell(q_\tau,\tau)\leq\frac{n}{2}$ we get an estimate of $\ell(x,\tau)$. The estimate of $\ell(x,\tilde{\tau})$ then follows from (5.4).

5.2C. The existence of the limit in Theorem 5.6. Fix an $A>1$, for the sequence $\tau_i\to\infty$ in Theorem 5.6, we consider the sequence of pointed backward Ricci flow solutions

$$\left(M^n, g_{\tau_i}(\theta), (q_{\tau_i},1)\right), \quad \theta\in\left[A^{-1},A\right].$$

For any $\varepsilon>0$, after parabolic scaling of $g(\tau)$ by τ_i, Lemma 5.7 yields the curvature bound

$$\left|\mathrm{Rm}_{g_{\tau_i}}(x,\theta)\right| \leq R_{g_{\tau_i}}(x,\theta) \leq \theta^{-1}\delta(n,\varepsilon,A)^{-1} \leq A\delta(n,\varepsilon,A)^{-1} \tag{5.5}$$

on $B_{g_{\tau_i}(1)}\left(q_{\tau_i},\sqrt{\varepsilon^{-1}}\right)\times\left[A^{-1},A\right]$, here we have used that $g(\tau)$ has nonnegative curvature operator. In particular taking $\varepsilon=1$ and $A=2$, we obtain that for some $\delta(n,1,2)<1$

$$\left|\mathrm{Rm}_{g_{\tau_i}}(x,1)\right| \leq 2\delta(n,1,2)^{-1} \quad \text{for } x\in B_{g_{\tau_i}(1)}(q_{\tau_i},1).$$

Since $g(\theta)$ is κ-noncollapsed on all scales, we have $g_{\tau_i}(\theta)$ is κ-noncollapsed on $B_{g_{\tau_i}(1)}(q_{\tau_i}, \sqrt{\delta(n,1,2)/2})$ and hence

$$\mathrm{Vol}_{g_{\tau_i}(1)} B_{g_{\tau_i}(1)}\left(q_{\tau_i}, \sqrt{\delta(n,1,2)/2}\right) \geq \kappa \left(\sqrt{\delta(n,1,2)/2}\right)^n .$$

By a theorem of Cheeger, Gromov and Taylor we have the injectivity radius estimate

$$\mathrm{inj}_{g_{\tau_i}(1)}(q_{\tau_i}) \geq \delta_1(n,\kappa) \tag{5.6}$$

for some positive constant $\delta_1(n,\kappa)$ depending only on n and κ.

(5.5) and (5.6) enable us to apply Hamilton's Cheeger-Gromov-type compactness theorem to the sequence of solutions $g_{\tau_i}(\theta)$ of the backward Ricci flow to get a convergent subsequence

$$(M^n, g_{\tau_i}(\theta), (q_{\tau_i}, 1)) \longrightarrow (M^n_\infty, g_\infty(\theta), (q_\infty, 1)) \quad \text{for } \theta \in \left[A^{-1}, A\right]. \tag{5.7}$$

The limit $g_\infty(\theta)$ is a complete solution to the backward Ricci flow. Since each $g_{\tau_i}(\theta)$ satisfies the trace Harnack inequality, $g_\infty(\theta)$ satisfies the trace Harnack inequality (5.2) with $T = \infty$. Also $g_\infty(\theta)$ is κ-noncollapsed on all scales, has nonnegative curvature operator, and satisfies $\mathrm{inj}_{\, g_\infty(1)}(q_\infty) \geq \delta_1(n,\kappa)$. However because the curvature bound $A\delta(n,\varepsilon,A)^{-1}$ in (5.5) depends on ε and hence on the radius $\sqrt{\varepsilon^{-1}}$, we may not have curvature bound $\sup_{M_\infty} |\mathrm{Rm}_{g_\infty}(x,\theta)| < \infty$ for each $\theta \in [A^{-1}, A]$.

By choosing a sequence of $A_k \to \infty$ and using a diagonalization argument, we may assume that $(M^n_\infty, g_\infty(\theta))$ exists for $\theta \in (0,\infty)$ and that the convergence in (5.7) holds for $\theta \in (0,\infty)$.

5.2D. Finishing the proof of Theorem 5.6. To finish the proof of Theorem 5.6, we need to show that for each θ, $g_\infty(\theta)$ is a nonflat shrinking gradient Ricci soliton. Let $\ell_i(x,\theta)$ denote the reduced distance of the solution $g_{\tau_i}(\theta)$ with respect to the basepoint $(p,0)$. After scaling, (5.3) and (5.4) give the derivative estimates of $\ell_i(x,\theta)$, by Arzela-Ascoli theorem some subsequence $\ell_i(x,\theta)$ converges to a Lipschitz function $\ell_\infty(x,\theta)$ on M_∞ in the Cheeger-Gromov sense. Similar to the definition of reduced volume (4.1), we use $\ell_\infty(x,\theta)$ to define

$$\hat{V}_\infty(\theta) = \int_{M_\infty} (4\pi\theta)^{-n/2} \exp\left[-\ell_\infty(x,\theta)\right] d\mu_{g_\infty(\theta)}(x), \qquad \theta \in (0,\infty). \tag{5.8}$$

By certain estimates on the growth of reduced distance $\ell(x,\tau)$ (not covered in §3), one can prove the convergence of the reduced volume (defined above (5.1))

$$\lim_{i\to\infty} \tilde{V}(\tau_i\theta) = \hat{V}_\infty(\theta) \qquad \text{for each } \theta > 0.$$

The monotonicity of reduced volume then implies that $\hat{V}_\infty(\theta)$ is a constant function. Combining this and (3.28a) for $\ell_i(x,\theta)$, one can argue that

$$\frac{\partial \ell_\infty}{\partial \theta} - \Delta_{g_\infty}\ell_\infty + |\nabla_{g_\infty}\ell_\infty|^2 - R_{g_\infty} + \frac{n}{2\theta} = 0 \tag{5.9}$$

in weak sense (we suggest the reader to find the details of the argument in the literature). Regularity theory of parabolic partial differential equations implies that ℓ_∞ is a smooth function.

Define two functions on $M_\infty \times (0,\infty)$ by $u_\infty(x,\theta) = (4\pi\theta)^{-\frac{n}{2}} e^{-\ell_\infty(x,\theta)}$ and

$$v_\infty = \left(\theta(2\Delta_{g_\infty}\ell_\infty - |\nabla_{g_\infty}\ell_\infty|^2 + R_{g_\infty}) + \ell_\infty - n\right) u_\infty.$$

Let operator $\Box^* = \frac{\partial}{\partial\theta} - \Delta_{g_\infty} + R_{g_\infty}$. Equation (5.9) implies that $\Box^* u_\infty = 0$. By some calculation one can show

$$\Box^* v_\infty = -2\theta \left| \mathrm{Rc}\,(g_\infty) + \nabla_{g_\infty}\nabla_{g_\infty}\ell_\infty - \frac{1}{2\theta} g_\infty \right|^2 u_\infty. \tag{5.10}$$

Applying (3.28d) to $l_i(x,\theta)$ we have

$$2\frac{\partial \ell_i}{\partial\theta} + \left|\nabla_{g_{\tau_i}}\ell_i\right|^2 - R_{g_{\tau_i}} + \frac{\ell_i}{\theta} = 0.$$

It can be argued that when $i \to \infty$ the above equality implies

$$2\frac{\partial \ell_\infty}{\partial\theta} + |\nabla_{g_\infty}\ell_\infty|^2 - R_{g_\infty} + \frac{\ell_\infty}{\theta} = 0.$$

Combining this with (5.9) we get $v_\infty = 0$, and hence it follows from (5.10) that

$$\mathrm{Rc}\,(g_\infty) + \nabla_{g_\infty}\nabla_{g_\infty}\ell_\infty - \frac{1}{2\theta} g_\infty = 0. \tag{5.11}$$

We have proved that $g_\infty(\theta)$ is a shrinking gradient Ricci soliton.

The last part that $g_\infty(\theta)$ is nonflat, is argued by contradiction. If it is flat, then the soliton equation (5.11) gives enough information of g_∞ and ℓ_∞ (Euclidean shrinking solution) to conclude that $\hat{V}_\infty(\theta) = 1$. But the equation above (5.9) implies that $\hat{V}_\infty(\theta) = \lim_{\tau\to\infty} \tilde{V}(\tau) < 1$. We get a contradiction. Now we have finished the sketch of the proof of Theorem 5.6.

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