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Twisting 4-manifolds along $\mathbb{RP}^2$

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Abstract. We prove that the Dolgachev surface $E(1)_{2,3}$ (which is an exotic copy of the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\bar{\mathbb{CP}}^2$) can be obtained from $E(1)$ by twisting along a simple "plug", in particular it can be obtained from $E(1)$ by twisting along $\mathbb{RP}^2$.

1. Introduction

Given a smooth 4-manifold M^4, what is the minimal genus g of an imbedded surface $\Sigma_g \subset M^4$, such that twisting M along Σ produces an exotic copy of M? Here twisting means cutting out a tubular neighborhood of Σ and regluing back by a nontrivial diffeomorphism. When $g > 1$ we don't get anything new (because by ([O], page 133)[1] any diffeomorphism of a circle bundle over Σ_g can be isotoped to preserve the fiber, and hence it extends to the corresponding disk bundle). The case $g = 1$ is the well known "logarithmic transform" operation, which can change the smooth structure in some cases; in fact the first example of a closed exotic manifold found by Donaldson [D] was the Dolgachev surface $E(1)_{2,3}$ which is obtained from $E(1) = \mathbb{CP}^2 \# 9\bar{\mathbb{CP}}^2$ by two log transforms. The $g = 0$ case is not well understood, twisting along S^2 is usually called "Gluck construction" and we don't know if this operation changes the smooth structure of any orientable manifold, but there is an example of non-orientable manifold which the Gluck construction changes its smooth structure [A1]. The interesting case of $\Sigma = \mathbb{RP}^2$ was studied indirectly in [AY1] under the guise of *plugs*, which are more general objects. Recall that Figure 1 describes the tubular neighborhood W of $\mathbb{RP}^2$ in S^4 as a disc bundle over $\mathbb{RP}^2$ (e.g. [A2]):

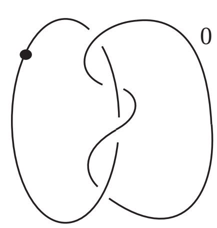

Figure 1. W

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[1]We thank Cameron Gordon for pointing out this reference.

If we attach a 2-handle to W as in Figure 2 we obtain an interesting manifold, which is the $W_{1,2}$ "plug" of [AY1]. Recall [AY1], a *plug* (P, f) of M^4 is a codimension zero Stein submanifold $P \subset M$ with an involution $f : \partial P \to \partial P$, such that f does not extend to a homemorphism inside; and the operation $N \cup_{id} P \mapsto N \cup_f P$ of removing P from M and regluing it to its complement N by f, changes the smooth structure of M (this operation is called a "*plug twisting*"). For example the involution $f : \partial W_{1,2} \to \partial W_{1,2}$ is induced from 180^0 rotation of the Figure 2 , e.g. it maps the (red and blue) loops to each other $\alpha \leftrightarrow \beta$.

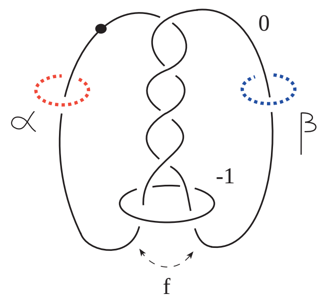

FIGURE 2. $W_{1,2}$

Notice that the twisting along $W_{1,2}$ is induced by twisting along $\mathbb{RP}^2$ inside (i.e. cutting out W and regluing by the involution induced by the rotation). In [AY1] some examples of changing smooth structures via plug twisting were given, including twisting the $W_{1,2}$ plug. Here we prove that by twisting along a $W_{1,2}$ plug (in particular twisting along $\mathbb{RP}^2$) we can completely decompose the Dolgachev surface $E(1)_{2,3}$. The following theorem should be considered as a structure theorem for the Dolgachev surface complementing Theorem 1 of [A3], where it was shown that a "*cork twisting*" also completely decomposes $E(1)_{2,3}$.

Theorem 1.1. *$E(1)_{2,3}$ is obtained by plug twisting $E(1)$ along $W_{1,2}$, i.e. we have a decomposition $E(1) = N \cup_{id} W_{1,2}$, so that $E(1)_{2,3} = N \cup_f W_{1,2}$.*

Proof. By cancelling the 1- and 2-handle pair of Figure 2 we obtain Figure 3, which is an alternative picture of $W_{1,2}$. By inspecting the diffeomorphism Figure 2 $\mapsto$ Figure 3 we see that the involution f twists the tubular neighborhood of α once, while mapping to β.

By attaching a chain of eight 2-handles to $-W_{1,2}$ (the mirror image of Figure 3) and a $+1$ framed 2-handle to α, we obtain Figure 4, which is a handlebody of $E(1)$ given in [A3]. In Figure 4 performing $W_{1,2}$ plug twist to E(1) has the effect of replacing the $+1$-framed 2-handle attached to α, with a zero framed 2-handle attached to β. Here the complement of $W_{1,2}$ in $E(1)$ is the submanifold N consisting of the zero framed 2-handle (the cusp)

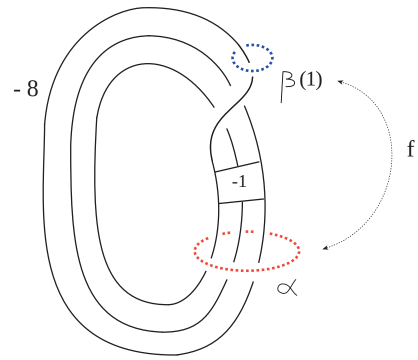

FIGURE 3. $W_{1,2}$

and the chain of eight 2-handles, and the plug twisting is the operation: $N \cup \alpha^{+1} \mapsto N \cup \beta^0$ (as seen from N).

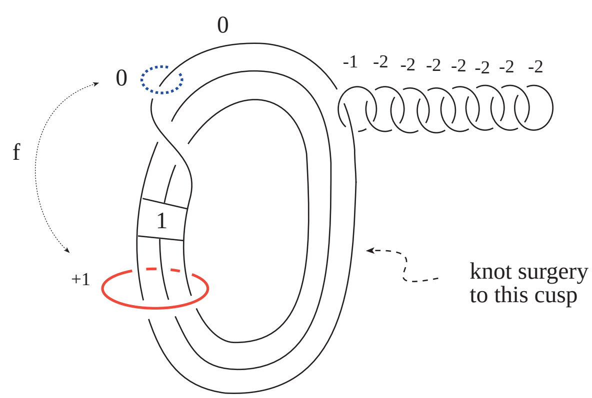

FIGURE 4. E(1)

Therefore the plug twisting of $E(1)$ along $W_{1,2}$ gives Figure 5. After sliding over β, the chain of eight 2-handles become free from the rest of the figure, giving a splitting: $Q\#8\bar{\mathbb{CP}}^2$, where Q is the cusp with the trivially linking zero framed cicle, hence we get $Q = S^2 \times S^2$. So the Figure 5 is just $S^2 \times S^2\#8\bar{\mathbb{CP}}^2 = E(1)$.

Next notice that if we first perform a "knot surgery" operation $E(1) \mapsto E(1)_K$ by a knot K, along the cusp inside of Figure 4, and then do the plug twist along $W_{1,2}$ (notice

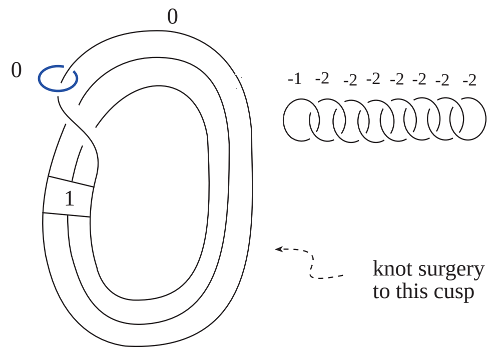

FIGURE 5

the cusp is disjoint from the plug since it lies in N) we get the similar splitting except this time resulting: $Q_K \# 8\bar{\mathbf{CP}}^2$, where Q_K is the knot surgered Q. Notice the manifold $Q = S^2 \times S^2$ is obtained by doubling the cusp, and Q_K is obtained by doing knot surgery to one of these cusps. In Theorem 4.1 of [A4] it was shown that when K is the trefoil knot then $Q_K = S^2 \times S^2$. Also recall that when K is the trefoil knot we have the identification with the Dolgachev surface $E(1)_K = E(1)_{2,3}$ (e.g. [A3]). □

Remark 1.1. If we could identify Q_K with $S^2 \times S^2$ for infinitely many knots K with distinct Alexander polynomials, we would have infinitely many transforms $E(1) \mapsto E(1)_K$ obtained by plug twistings along $W_{1,2}$. This would give infinitely many non-isotopic imbeddings $W_{1,2} \subset E(1)$, similar to the examples in [AY2]. In the absence of such identification we can only conclude that $W_{1,2}$ is a plug of infinitely many distinct exotic copies $E(1)_K$ of $E(1)$.

Remark 1.2. Recall that ∂W is the quaternionic 3-manifold, which is the quotient of S^3 by the free action of the quaternionic group of order eight (e.g. [A2]):

$$G =< i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j > .$$

This manifold is a positively curved space-form and an L-space (Heegaard Floer homology groups are trivial). Hence the change of smooth structure of $E(1)$ by twisting W is due to the change of $Spin^c$ structures, rather than permuting the Floer homology by the involution as in [A3], [AD].

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