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Deformations of asymptotically cylindrical special Lagrangian submanifolds

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Abstract. Given an asymptotically cylindrical special Lagrangian submanifold L with fixed boundary in an asymptotically cylindrical Calabi-Yau 3-fold X, we determine conditions on a decay rate γ which make the moduli space of (local) special Lagrangian deformations of L in X a smooth manifold and show that it has dimension equal to the dimension of the image of $H^1_{cs}(L,\mathbb{R})$ in $H^1(L,\mathbb{R})$ under the natural inclusion map, $[\chi] \mapsto [\chi]$. Then we prove the analogous result for asymptotically cylindrical special Lagrangian submanifolds with moving boundary.

1. Introduction

McLean [14] proved that the moduli space of special Lagrangian deformations of a compact special Lagrangian submanifold L of a Calabi-Yau n-fold is a smooth manifold of dimension $b^1(L)$, the first Betti number of L. The goal of this paper is to prove analogous results for an asymptotically cylindrical special Lagrangian submanifold of an asymptotically cylindrical Calabi-Yau 3-fold.

Examples of complete Calabi-Yau n-folds are constructed by Tian and Yau in [19, 20]. In general, they construct complete Kähler metrics with a prescribed Ricci curvature on quasi-projective manifolds M, that is, $M = \overline{M} \setminus D$ where $\overline{M}$ is a projective manifold and D a smooth, ample divisor in $\overline{M}$. Then, under further assumptions that D is anticanonical and $K^{-1}_{\overline{M}}$ is ample, $K_{\overline{M}}$ the canonical line bundle of $\overline{M}$, M admits a complete Ricci-flat Kähler metric. Building on this theory, Kovalev [8] constructs asymptotically cylindrical Calabi-Yau 3-folds by $W = \overline{W} \setminus D$, where $\overline{W}$ is a compact, simply-connected Kähler manifold and D, a $K3$ surface in $\overline{W}$, is an anticanonical divisor with trivial self-intersection class $D \cdot D = 0$ in the second integral homology class of $\overline{W}$. Given two asymptotically cylindrical Calabi-Yau 3-folds W_1, W_2, one can obtain asymptotically cylindrical 7-manifolds with holonomy in $SU(3)$ by taking $W_1 \times S^1$ and $W_2 \times S^1$. In his paper, Kovalev constructs examples of compact G_2-manifolds by gluing two such noncompact manifolds together.

We begin in Section 2 with elementary definitions from Calabi-Yau and special Lagrangian geometry then give the definitions of asymptotically cylindrical Calabi-Yau 3-folds and their asymptotically cylindrical special Lagrangian submanifolds. Included is a sketch of McLean's result in the compact case. The next section gives the material on weighted Sobolev spaces and elliptic operators on asymptotically cylindrical special

Lagrangian 3-folds developed by Lockhart and McOwen in [10, 11]. We use this theory in Section 4 to study the operator $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$ on special Lagrangian 3-folds. In particular, we are able to determine necessary and sufficient conditions on the decay rate γ that ensures this operator is Fredholm which then allows us to calculate its kernel and cokernel. Further, using a bit of linear algebra, we can calculate the dimension of the kernel explicitly. The final two sections prove the following results:

Theorem 1.1. *Assume that (X,ω,g,Ω) is an asymptotically cylindrical Calabi-Yau 3-fold, asymptotic to $M\times S^1\times(R,\infty)$ with decay rate $\alpha<0$, where M is a compact connected Calabi-Yau 2-fold, and that L is an asymptotically cylindrical special Lagrangian 3-submanifold in X asymptotic to $N\times\{p\}\times(R',\infty)$ for $R'>R$ with decay rate β for $\alpha\le\beta<0$ where N is a compact special Lagrangian 2-fold in M and $p\in S^1$.*

If $\gamma<0$ is such that $\beta<\gamma$ and $(0,\gamma^2]$ contains no eigenvalues of the Laplacian on functions on N, then the moduli space $\mathcal{M}^\gamma_L$ of asymptotically cylindrical special Lagrangian submanifolds in X near L, and asymptotic to $N\times\{p\}\times(R',\infty)$ with decay rate γ, is a smooth manifold of dimension $\dim V=b^2(L)-b^0(N)+b^0(L)$ where V is the image of the natural inclusion map of $H^1_{cs}(L,\mathbb{R})\hookrightarrow H^1(L,\mathbb{R})$.

Theorem 1.2. *Assume that (X,ω,g,Ω) is an asymptotically cylindrical Calabi-Yau 3-fold, asymptotic to $M\times S^1\times(R,\infty)$ with decay rate $\alpha<0$, where M is a compact connected Calabi-Yau 2-fold, and that L is an asymptotically cylindrical special Lagrangian 3-submanifold in X asymptotic to $N\times\{p\}\times(R',\infty)$ for $R'>R$ with decay rate β for $\alpha\le\beta<0$ where N is a compact special Lagrangian 2-fold in M and $p\in S^1$. Let $\gamma<0$ be such that $\beta<\gamma$ and such that $(0,\gamma^2]$ contains no eigenvalues of the Laplacian on functions on N.*

Let $\Upsilon: H^1(N,\mathbb{R})\to H^2_{cs}(L,\mathbb{R})$ be the linear map coming from the long exact sequence in cohomology (so that $\ker\Upsilon$ is a vector subspace of $H^1(N)$), let $\mathcal{U}$ be a small open neighborhood of 0 in $\ker\Upsilon$ and let N_s denote the special Lagrangian submanifolds of M near N for $s\in\mathcal{U}$. Let also $N^1,\ldots,N^{b^0(N)}$ be the connected components of N, and let N^i_s denote the special Lagrangian submanifolds of M near N^i for $i=1,\ldots,b^0(N)$. Then the moduli space of asymptotically cylindrical special Lagrangian 3-folds in X close to L, and asymptotic to $\coprod_{i=1}^{b^0(N)}N^i_s\times\{q^i\}\times(R',\infty)$ for $q^1,\ldots,q^{b^0(N)}$ in S^1 close to p, with decay rate γ, is a smooth manifold of dimension $\dim V+\dim\ker\Upsilon+b^0(N)=(b^2(L)-b^0(N)+b^0(L))+(b^2(L)-b^1(L)+b^0(L)-b^2(N)+b^1(N))+b^0(N)$, where V is the image of $H^1_{cs}(L)$ in $H^1(L)$.

Remark 1.1. In [7] and [17], similar results are proven for asymptotically cylindrical coassociative submanifolds with fixed and moving boundaries in an asymptotically cylindrical G_2 manifold. The arguments presented and the techniques used in this paper parallel the arguments and techniques therein.

To prove the first theorem, we construct a deformation map using the identification of the normal bundle of a special Lagrangian submanifold with its cotangent bundle. We then prove that this map extends to a smooth map of weighted Sobolev space, calculate its linearization and show that the image lies in spaces of exact forms. The map as defined will not be elliptic, so we extend it over a space of three forms to gain ellipticity; the image of this map is then shown to be in the image of the map $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$. This allows us to use the Implicit Mapping Theorem for Banach Spaces to conclude that the kernel of our elliptic operator is smooth, finite-dimensional and locally isomorphic to the kernel of $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$. We get the theorem then by showing that the moduli space of asymptotically cylindrical special Lagrangian deformations is homeomorphic to the kernel of our elliptic operator.

The proof of the second theorem is similar. We first prove rigidity results regarding cylindrical and asymptotically cylindrical special Lagrangians; this allows us to give precise meaning to "moving boundary at infinity." We then proceed as in the proof of Theorem 1.1; to calculate the dimension of the moduli space in this situation, we will use results obtained by Lockhart and McOwen regarding the Fredholm index of elliptic operators.

2. Calabi-Yau and special Lagrangian geometry

We begin with the definitions of a Calabi-Yau n-fold and special Lagrangian n-submanifold, and we give the definitions of (asymptotically) cylindrical Calabi-Yau 3-folds and (asymptotically) cylindrical special Lagrangian 3-submanifolds. References for this section include: Harvey and Lawson, [4]; Joyce, [6]; Kovalev, [8]; and McLean, [14].

Definition 2.1. A *complex* n-dimensional *Calabi-Yau* manifold (X,ω,g,Ω) is a Kähler manifold with zero first Chern class, that is $c_1(X)=0$. In this case, (X,ω,g,Ω) is also called a *Calabi-Yau n-fold.*

In [4, Section 3.1], it is proved that $\operatorname{Re}\Omega$ is a calibration form. The calibrated submanifolds with respect to $\operatorname{Re}\Omega$ are called special Lagrangian; that is, $\operatorname{Re}\Omega$ restricts to be the volume form with respect to the induced metric on special Lagrangian submanifolds. Equivalent to this condition, and what we will use as our definition, is the following:

Definition 2.2. A *real* n-dimensional submanifold $L\subseteq X$ is *special Lagrangian* if L is Lagrangian (i.e. $\omega|_L\equiv 0$) and $\operatorname{Im}\Omega$ restricted to L is zero. In this case, we will also call L a *special Lagrangian n-submanifold.*

We now define cylindrical and asymptotically cylindrical Calabi-Yau 3-manifolds. See [8] for more information on the definitions in this section.

Definition 2.3. A Calabi-Yau 3-fold $(X_0,\omega_0,g_0,\Omega_0)$ is called *cylindrical* if X_0 is $M\times S^1\times\mathbb{R}$ where M is a compact connected Calabi-Yau 2-fold with Kähler form κ_I and holomorphic $(2,0)$-form $\kappa_J+i\kappa_K$, and (ω_0,g_0,Ω_0) is compatible with the product structure $M\times S^1\times\mathbb{R}$, that is, $\omega_0=\kappa_I+\mathrm{d}\theta\wedge\mathrm{d}t$, $\Omega_0=(\kappa_J+i\kappa_K)\wedge(\mathrm{d}t+i\mathrm{d}\theta)$ and $g_0=g_{(M\times S^1)}+\mathrm{d}t^2$.

Calabi-Yau 2-folds are also hyperkähler since they have holonomy $SU(2)=Sp(1)$. The indices on the 2-forms κ_I, κ_J and κ_K on M are then meant to reflect the hyperkähler structure of M.

Definition 2.4. A connected, complete Calabi-Yau 3-fold (X, ω, g, Ω) is called *asymptotically cylindrical with decay rate* $\alpha < 0$ if there exists a cylindrical Calabi-Yau 3-fold $(X_0, \omega_0, g_0, \Omega_0)$ as in Definition 2.3, a compact subset $K \subset X$, a real number $R > 0$, and a diffeomorphism $\Psi : M \times S^1 \times (R, \infty) \to X \setminus K$ such that $\Psi^*(\omega) = \omega_0 + \mathrm{d}\xi_1$ for some 1-form ξ_1 with $\left|\nabla^k \xi_1\right| = O(e^{\alpha t})$ and $\Psi^*(\Omega) = \Omega_0 + \mathrm{d}\xi_2$ for some complex 2-form ξ_2 with $\left|\nabla^k \xi_2\right| = O(e^{\alpha t})$ on $M \times S^1 \times \mathbb{R}$ for all $k \geqslant 0$, where ∇ is the Levi-Cività connection of the cylindrical metric $g_0 = g_{(M \times S^1)} + dt^2$.

This definition implies that the restriction of the Kähler form ω and holomorphic $(3,0)$-form Ω equals to the above for large t, up to a possible error of order $O(e^{\alpha t})$ for some parameter $\alpha < 0$. Notice also in this definition that we are assuming X and M are connected, so that X only has one end; since we are working with Ricci-flat manifolds, this is not a restrictive assumption by a result obtained by the first author in [16].

We now define cylindrical and asymptotically cylindrical special Lagrangian submanifolds similarly.

Definition 2.5. Let $(X_0, \omega_0, g_0, \Omega_0)$ be a cylindrical Calabi-Yau 3-fold as in Definition 2.3. A 3-dimensional submanifold L_0 of X_0 is called *cylindrical special Lagrangian* if $L_0 = N \times \{p\} \times \mathbb{R}$ for some compact special Lagrangian submanifold N in M, a point p in S^1 and the restrictions of ω_0 and $\operatorname{Im} \Omega_0$ to L_0 are zero, that is, $\omega_0|_{L_0} = (\kappa_I + \mathrm{d}\theta \wedge \mathrm{d}t)|_{L_0} = 0$ and $\operatorname{Im}(\Omega_0)|_{L_0} = \operatorname{Im}((\kappa_J + i\kappa_K) \wedge (\mathrm{d}t + i\mathrm{d}\theta))|_{L_0} = 0$.

Definition 2.6. Let $(X_0, \omega_0, g_0, \Omega_0)$, $M \times S^1 \times \mathbb{R}$, (X, ω, g, Ω), K, R, Ψ and α be as in Definitions 2.3 and 2.4, and let $L_0 = N \times \{p\} \times \mathbb{R}$ be a cylindrical special Lagrangian 3-fold in X_0 as in Definition 2.5.

A connected, complete special Lagrangian 3-fold L in (X, ω, g, Ω) is called *asymptotically cylindrical with decay rate* β for $\alpha \leq \beta < 0$ if there exists a compact subset $K' \subset L$, a normal vector field v on $N \times \{p\} \times (R', \infty)$ for some $R' > R$, and a diffeomorphism $\Phi : N \times \{p\} \times (R', \infty) \to L \setminus K'$ such that the following diagram commutes:

$$\begin{tikzcd}
M \times S^1 \times (R', \infty) \arrow[d, "\subset"] & N \times \{p\} \times (R', \infty) \arrow[l, "\exp_v"'] \arrow[r, "\Phi"'] & (L \setminus K') \arrow[d, "\subset"] \\
M \times S^1 \times (R, \infty) \arrow[rr, "\Psi"] & & (X \setminus K),
\end{tikzcd} \tag{1}$$

and $\left|\nabla^k v\right| = O(e^{\beta t})$ on $N \times \{p\} \times (R', \infty)$ for all $k \geq 0$.

Notice that here again, we require L to be connected; however, unlike the case of the ambient manifold, it may be that N is *not* connected, so that L may have multiple ends. This will not present a problem in the analysis since we are assuming the ends have the same decay rate.

We now sketch a proof of McLean's result on the moduli space of special Lagrangian deformations in the case of compact special Lagrangian submanifolds of a Calabi-Yau n-fold.

Theorem 2.7. *The moduli space of deformations of a smooth, compact, orientable special Lagrangian submanifold L in a Calabi-Yau manifold X within the class of special Lagrangian submanifolds is a smooth manifold of dimension equal to dim$(H^1(L))$.*

Sketch of proof. Define the deformation map $F : \nu_L \to \Lambda^2 T^*(L) \oplus \Lambda^n T^*(L)$ from the space of smooth sections of the normal bundle on L to the spaces of differential 2-forms and differential n-forms on L as follows:

$$F(V) = ((\exp_V)^*(-\omega), (\exp_V)^*(\operatorname{Im}\Omega)).$$

Note that F is just the restrictions of $-\omega$ and $\operatorname{Im}\Omega$ to L_V which are then pulled back to L via $(\exp_V)^*$ where $\exp_V$ is the exponential map giving a diffeomorphism of L onto its image L_V in a neighborhood of 0.

Recall that the normal bundle ν_L of a special Lagrangian submanifold is isomorphic to the cotangent bundle $T^*(L)$. Thus there is a natural identification of normal vector fields to L with differential 1-forms on L; moreover, since L is compact, these normal vector fields can be identified with nearby submanifolds, so that, under these identifications, the kernel of F corresponds to the special Lagrangian deformations.

The linearization of F at 0

$$\mathrm{d}F(0) : \nu_L \to \Lambda^2 T^*(L) \oplus \Lambda^n T^*(L)$$

is given by

$$\begin{aligned}\mathrm{d}F(0)(V) =& \frac{\partial}{\partial t} F(tV)|_{t=0} = \frac{\partial}{\partial t}(\exp^*_{tV}(-\omega), \exp^*_{tV}(\operatorname{Im}\Omega))|_{t=0} \\ =&(-\mathcal{L}_V\omega|_L, \mathcal{L}_V(\operatorname{Im}\Omega)|_L).\end{aligned}$$

Using the Cartan Formula for the Lie derivative $\mathcal{L}_V$, we find

$$\begin{aligned}\mathrm{d}F(0)(V) &= (-(i_V\mathrm{d}\omega + \mathrm{d}(i_V\omega))|_L, (i_V\mathrm{d}(\operatorname{Im}\Omega) + \mathrm{d}(i_V(\operatorname{Im}\Omega)))|_L) \\ &= (-\mathrm{d}(i_V\omega)|_L, \mathrm{d}(i_V(\operatorname{Im}\Omega))|_L) = (\mathrm{d}v, \mathrm{d} * v),\end{aligned}$$

where i_V represents the interior derivative and v is the dual 1-form to the vector field V with respect to the induced metric. Hence we have

$$\mathrm{d}F(0)(V) = (\mathrm{d}v, \mathrm{d} * v) = (\mathrm{d}v, *\mathrm{d}^*v).$$

The next step is to show that $\mathrm{d}F(0)(V) = (\mathrm{d}v, \mathrm{d} * v) = (\mathrm{d}v, *\mathrm{d}^*v)$ is onto when considered as a map from ν_L to *exact* 2-forms and *exact* n-forms. McLean shows this by first proving that F is a map from ν_L to exact 2-forms and exact n-forms as follows:

Note that the image of F lies in closed 2-forms and closed n-forms since F is the pullback of the closed forms ω and $\operatorname{Im}\Omega$. Now, by replacing V with tV, we see that $exp : L \to X$ is homotopic to the inclusion $i : L \to X$, so that exp^*_V and i^* induce the same map on the level of cohomology. Since L is special Lagrangian, we get $[exp^*_V(\omega)] = [i^*(\omega)] = [\omega|_L] = 0$ and $[exp^*_V(\operatorname{Im}\Omega)] = [i^*(\operatorname{Im}\Omega)] = [\operatorname{Im}\Omega|_L] = 0$, so the forms in the image of F are cohomologous to zero, that is, they are exact forms.

One can now easily show that, given any exact 2-form a and exact n-form b, there exists a 1-form v satisfying the equations $\mathrm{d}v = a$ and $\mathrm{d} * v = b$; hence, $\mathrm{d}F(0)(V)$ is surjective as claimed. Finally, after completing the spaces of differential forms with appropriate norms, we can use the Implicit Function Theorem for Banach spaces and an elliptic regularity result to conclude that $F^{-1}(0,0)$ is a smooth manifold with tangent space at 0 isomorphic to $H^1(L)$. $\square$

3. Analysis on asymptotically cylindrical special Lagrangian 3-manifolds

Lockhart and McOwen [10, 11] set up a general framework for elliptic operators on noncompact manifolds where the basic tools are weighted Sobolev spaces. Using these spaces and the notion of asymptotically cylindrical linear elliptic partial differential operators, they get weighted Sobolev embedding theorems and recover elliptic regularity and Fredholm results.

Remark 3.1. For an alternate approach to these types of results on noncompact manifolds, see Melrose [12, 13].

Let L be an asymptotically cylindrical special Lagrangian 3-manifold with data as in Definition 2.6.

Definition 3.1. Let A be a vector bundle on L with smooth metrics h on the fibers and a connection ∇_A on A compatible with h; let A_0 be a vector bundle on $N \times \{p\} \times \mathbb{R}$ that is invariant under translations in $\mathbb{R}$, that is a *cylindrical* vector bundle, with metrics h_0 on the fibers and ∇_{A_0} a connection on A_0 compatible with h_0 where h_0 and ∇_{A_0} are also invariant under translations in $\mathbb{R}$.

A, h and ∇_A are said to be *asymptotically cylindrical*, asymptotic to A_0, h_0 and ∇_{A_0} respectively, if $\Phi^*(A) \cong A_0$ on $N \times \{p\} \times (R', \infty)$ with $|\Phi^*(h) - h_0| = O(e^{\beta t})$ and $|\Phi^*(\nabla_A) - \nabla_{A_0}| = O(e^{\beta t})$ as $t \to \infty$.

Recall that for $k \geq 0$,

$$L^p_k(A) = \left\{ s \in \Gamma(A) : s \text{ is } k\text{-times weakly differentiable and } \|s\|_{L^p_k} < \infty \right\},$$

where $\Gamma(A)$ is the space of *all* sections of A,

$$\|s\|_{L^p_k} = \left(\sum_{j=0}^{k} \int_L \left| \nabla^j_A s \right|^p \, dvol_L \right)^{1/p}$$

and $dvol_L$ is the volume element of L, and

$$L^p_{k,\mathrm{loc}}(A) = \{ s \in \Gamma(A) : \phi s \in L^p_k(A) \text{ for all } \phi \in C^\infty_0(L) \},$$

where $C^\infty_0(L)$ is the set of all compactly-supported smooth functions on L.

Definition 3.2. Choose a real valued smooth function $\rho : L \to \mathbb{R}$ such that $\Phi^*(\rho) \equiv t$ on $N \times \{p\} \times (R', \infty)$. By Definition 2.6, this condition prescribes ρ on $L \setminus K'$, so it is only necessary to smoothly extend ρ over K'. For $p \geq 1$, $k \geq 0$ and $\gamma \in \mathbb{R}$, the *weighted Sobolev space* $L^p_{k,\gamma}(A)$ is then the set of sections s of A such that $s \in L^p_{\mathrm{loc}}(A)$, s is k-times weakly differentiable and

$$\|s\|_{L^p_{k,\gamma}} = \left(\sum_{j=0}^{k} \int_L e^{-\gamma\rho} \left| \nabla^j_A s \right|^p \, dvol_L \right)^{1/p} < \infty.$$

In particular, $L^p_{k,\gamma}(A)$ is a Banach space; further, note that different choices of ρ give the same space $L^p_{k,\gamma}(A)$ with equivalent norms since ρ is uniquely determined outside of K.

Define the Banach space $C^k_\gamma(A)$ of continuous sections s of A with k continuous derivatives such that $e^{-\gamma\rho}|\nabla^j_A s|$ is bounded for each $j = 0, 1, \ldots, k$ where the norm is given by

$$\|s\|_{C^k_\gamma} = \sum_{j=0}^{k} \sup_L e^{-\gamma\rho} \left| \nabla^j_A s \right|.$$

In general, from [2, Theorem 1.2], [10, Theorem 3.10] and [11, Lemma 7.2] there is the following Weighted Sobolev Embedding Theorem (adapted to our case):

Theorem 3.3 (Weighted Sobolev Embedding Theorem)**.** *Suppose that k, l are integers with $k \geq l \geq 0$ and that $p, q, \gamma, \overline{\gamma}$ are real numbers with $p, q > 1$. Then*

(1) *If $\frac{k-l}{3} \geq \frac{1}{p} - \frac{1}{q}$ and $\overline{\gamma} \geq \gamma$ there is a continuous embedding of $L^p_{k,\gamma}(A) \hookrightarrow L^q_{l,\overline{\gamma}}(A)$;*
(2) *If $\frac{k-l}{3} > \frac{1}{p}$ and $\overline{\gamma} \geq \gamma$ there is a continuous embedding of $L^p_{k,\gamma}(A) \hookrightarrow C^l_{\overline{\gamma}}(A)$;*

Now suppose that A, B are two asymptotically cylindrical vector bundles on L which are asymptotic to the cylindrical vector bundles A_0 and B_0 on $N \times \{p\} \times \mathbb{R}$. Let $F_0 : C^\infty(A_0) \to C^\infty(B_0)$ be a cylindrical linear partial differential operator of order k, that is, a linear partial differential operator which is invariant under translations in $\mathbb{R}$, from *smooth* sections $C^\infty(A_0)$ of A_0 to *smooth* sections $C^\infty(B_0)$ of B_0. Let $F : C^\infty(A) \to C^\infty(B)$ be a linear partial differential operator of order k on L.

Definition 3.4. F is said to be an *asymptotically cylindrical operator*, asymptotic to F_0, if, under the identifications $\Phi^*(A) \cong A_0$, $\Phi^*(B) \cong B_0$ on $N \times \{p\} \times \mathbb{R}$, $|\Phi^*(F) - F_0| = O(e^{\beta t})$ as $t \to \infty$.

From these definitions F extends to a bounded linear operator

$$F^p_{k+l,\gamma} : L^p_{k+l,\gamma}(A) \to L^p_{l,\gamma}(B)$$

for all $p > 1$, $l \geq 0$ and $\gamma \in \mathbb{R}$. We then have the following elliptic regularity result [10, Theorem 3.7.2]:

Theorem 3.5. *Let F and F_0 be as above. If $1 < p < \infty$, $l \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, then for all $s \in L^p_{k+l,loc}(A)$,*

$$\|s\|_{L^p_{k+l,\gamma}} \leq C(\|Fs\|_{L^p_{l,\gamma}} + \|s\|_{L^p_{l,\gamma}})$$

for some C independent of s.

Definition 3.6. Assume now that F, F_0 are also elliptic on L and $N \times \{p\} \times \mathbb{R}$ respectively. Extend F_0 to $F_0 : C^\infty(A_0 \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(B_0 \otimes_{\mathbb{R}} \mathbb{C})$, and define $\mathcal{D}_{F_0}$ as the set of $\gamma \in \mathbb{R}$ such that for some $\delta \in \mathbb{R}$ there exists a nonzero section $s \in C^\infty(A_0 \otimes_{\mathbb{R}} \mathbb{C})$, invariant under translations in $\mathbb{R}$, such that $F_0(e^{(\gamma+i\delta)t}s) = 0$.

The conditions for $F^p_{k+l,\gamma}$ to be Fredholm are now given by [11, Theorem 1.1].

Theorem 3.7. *Using the setup of this section, $\mathcal{D}_{F_0}$ is a discrete subset of $\mathbb{R}$, and for $p > 1$, $l \geq 0$ and $\gamma \in \mathbb{R}$, $F^p_{k+l,\gamma} : L^p_{k+l,\gamma}(A) \to L^p_{l,\gamma}(B)$ is Fredholm if and only if $\gamma \notin \mathcal{D}_{F_0}$.*

This theorem and Theorem 3.5 imply that $\ker(F^p_{k+l,\gamma})$ is a finite-dimensional vector space of smooth sections of A whenever $\gamma \notin \mathcal{D}_{F_0}$, and from the Weighted Sobolev Embedding Theorem and the fact that $\ker(F^p_{k+l,\gamma})$ is invariant under small changes in γ we have the following:

Lemma 3.8. *If $\gamma \notin \mathcal{D}_{F_0}$, then the kernel $\ker(F^p_{k+l,\gamma})$ is the same for any choices of $p > 1$ and $l \geq 0$ and is a finite-dimensional vector space consisting of smooth sections of A.*

Let $F^* : C^\infty(B) \to C^\infty(A)$ be the formal adjoint of F; that is, F^* is the asymptotically cylindrical linear elliptic partial differential operator of order k on L such that

$$\langle Fs, \tilde{s} \rangle_{L^2(B)} = \langle s, F^* \tilde{s} \rangle_{L^2(A)}$$

for all compactly-supported $s \in C^\infty(A)$, $\tilde{s} \in C^\infty(B)$.

Then for $p > 1$, $l \geq 0$ and $\gamma \notin \mathcal{D}_{F_0}$, $(F^*)^q_{-l,-\gamma} : L^q_{-l,-\gamma}(B) \to L^q_{-k-l,-\gamma}(A)$ is the dual operator of $F^p_{k+l,\gamma}$ where $q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$, $L^q_{-l,-\gamma}(B)$ and $L^q_{-k-l,-\gamma}(A)$ are isomorphic as Banach spaces to the dual spaces $(L^p_{l,\gamma}(B))^*$ and $(L^p_{k+l,\gamma}(A))^*$ respectively and these isomorphisms identify $(F^*)^q_{-l,-\gamma}$ with $(F^p_{k+l,\gamma})^*$. Further, since Theorem 3.5 also holds for negative differentiability, $\ker((F^*)^q_{-l,-\gamma}) = \ker((F^*)^q_{k+m,-\gamma})$ for all $m \in \mathbb{Z}$. This allows us to identify $\operatorname{coker}(F^p_{k+l,\gamma})$ with $\ker((F^*)^q_{k+m,-\gamma})^*$ for $\gamma \notin \mathcal{D}_{F_0}$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $l, m \geq 0$; moreover, the index of $F^p_{k+l,\gamma}$ is then given by

$$\operatorname{ind}(F^p_{k+l,\gamma}) = \dim \ker(F^p_{k+l,\gamma}) - \dim \ker((F^*)^q_{k+m,-\gamma}). \tag{2}$$

4. $\mathrm{d} + \mathrm{d}^*$ on asymptotically cylindrical special Lagrangian 3-folds

Let L be an asymptotically cylindrical special Lagrangian 3-manifold with data as in Definition 2.6, and consider the asymptotically cylindrical linear elliptic operator

$$\mathrm{d} + \mathrm{d}^* : C^\infty(T^*L \oplus \Lambda^3 T^*L) \to C^\infty(\Lambda^0 T^*L \oplus \Lambda^2 T^*L),$$

with formal adjoint given by

$$\mathrm{d}^* + \mathrm{d} : C^\infty(\Lambda^0 T^*L \oplus \Lambda^2 T^*L) \to C^\infty(T^*L \oplus \Lambda^3 T^*L).$$

We study the extension

$$(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma} : L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3 T^*L) \to L^p_{1+l,\gamma}(\Lambda^0 T^*L \oplus \Lambda^2 T^*L) \tag{3}$$

for $p > 1$, $l \geq 0$ and $\gamma \in \mathbb{R}$. Suppose further that L has no compact, connected components, so that $H^3(L,\mathbb{R}) = H^0_{cs}(L,\mathbb{R}) = 0$; then N is a compact, oriented 2-manifold, and L is the interior of a compact, oriented 3-manifold $\overline{L}$ with boundary $\partial\overline{L} = N$.

From this we have the following long exact sequence in cohomology:

$$\begin{array}{c} 0 \to H^0(L) \to H^0(N) \to H^1_{cs}(L) \to H^1(L) \to H^1(N) \\ \downarrow \\ 0 \leftarrow H^3_{cs}(L) \leftarrow H^2(N) \leftarrow H^2(L) \leftarrow H^2_{cs}(L) \end{array}$$

where $H^k(L) = H^k(L,\mathbb{R})$ and $H^k(N) = H^k(N,\mathbb{R})$ are the de Rham cohomology groups, $H^k_{cs}(L,\mathbb{R})$ are compactly-supported de Rham cohomology groups and $b^k(L)$, $b^k(N)$ and $b^k_{cs}(L)$ the corresponding Betti numbers. If $V \subseteq H^1(L,\mathbb{R})$ denotes the image of the natural map $H^1_{cs}(L,\mathbb{R}) \hookrightarrow H^1(L,\mathbb{R})$, $[\chi] \mapsto [\chi]$, then, from the long exact sequence, $\dim(V) = b^1_{cs}(L) - b^0(N) + b^0(L) = b^2(L) - b^0(N) + b^0(L)$.

We now summarize a number of results from the previous section applied specifically to $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$ as well as identify the kernel and cokernel of this operator for small $\gamma < 0$.

Theorem 4.1. *Suppose* $\max\{\mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0} \cap (-\infty, 0)\} < \gamma < 0$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* l *and* m *non-negative. Then the operator* $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$ *is Fredholm with* $\operatorname{coker}((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \cong (\ker((\mathrm{d}^*+\mathrm{d})^q_{2+m,-\gamma}))^*$. *The kernel* $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$ *is a vector space of smooth, closed, coclosed* 1*-forms, and the map* $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \to H^1(L,\mathbb{R})$, $\chi \mapsto [\chi]$, *induces an isomorphism of* $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$ *with the image,* V, *of the natural inclusion map* $H^1_{cs}(L,\mathbb{R}) \hookrightarrow H^1(L,\mathbb{R})$*; hence,* $\dim\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) = \dim V$. *Finally, the kernel* $\ker(\mathrm{d}^*+\mathrm{d})^q_{2+m,-\gamma})$ *is a vector space of smooth, closed, coclosed* 0*- and* 2*-forms.*

Proof. Since $\gamma \notin \mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}$, $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$ is Fredholm with cokernel isomorphic to $\ker((\mathrm{d}^*+\mathrm{d})^q_{2+m,-\gamma})^*$ from Lemma 3.7 and the remarks following Lemma 3.8.

For $(\eta_1, \eta_3) \in \ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$, $(0,0) \equiv (\mathrm{d}+\mathrm{d}^*)(\eta_1,\eta_3) = (\mathrm{d}^*\eta_1, \mathrm{d}\eta_1 + \mathrm{d}^*\eta_3)$, from which it follows that η_1 is coclosed. $\mathrm{d}\eta_1 = -\mathrm{d}^*\eta_3$ implies that $(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\eta_1 = 0$ and so we get $\eta_1 \in \ker((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{2+l,\gamma})$.By [7, Proposition 3.8], $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) = \ker((\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)^p_{2+l,\gamma})$, and both kernels consist of smooth closed and coclosed forms. Hence η_1 is closed, and η_3 is coclosed; smoothness follows from Theorem 3.5. Using the injectivity of the map in [7, Proposition 3.9], $\eta_3 = 0$ since $H^3(L) = 0$. Thus $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$ consists of smooth, closed, coclosed 1-forms on L. Also, by [7, Proposition 3.9], the map $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \to H^1(L,\mathbb{R})$ is injective with image that of the natural inclusion map $H^1_{cs}(L,\mathbb{R}) \to H^1(L,\mathbb{R})$, which is V in the notation above.

Finally, let $(\eta_0, \eta_2) \in \ker((\mathrm{d}^* + \mathrm{d})^q_{2+m,-\gamma})$, so that $(0,0) = (\mathrm{d}\eta_0 + \mathrm{d}^*\eta_2, \mathrm{d}\eta_2)$ from which it immediately follows that η_2 is closed. Now, $\mathrm{d}\eta_0 = -\mathrm{d}^*\eta_2$ implies that η_0, η_2 lie in the kernel of $(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})^q_{2+m,-\gamma}$. Again by [7, Proposition 3.8], $\ker((\mathrm{d} + \mathrm{d}^*)^q_{2+m,-\gamma}) = \ker((\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})^q_{2+m,-\gamma})$, and both kernels consist of smooth closed and coclosed forms. This shows both that η_2 is coclosed, and that η_0 is closed; η_0 is coclosed by definition, and smoothness, in both cases, follows from elliptic regularity (Theorem 3.5). Hence, $\ker((\mathrm{d}^* + \mathrm{d})^q_{2+m,-\gamma})$ consists of smooth, closed, coclosed 0- and 2-forms on L. □

Theorem 4.2. *Let $p > 1$, $l \geq 0$ and $\gamma \in \mathbb{R}$. Then the operator (3) is* not *Fredholm if and only if either of the following conditions hold:*

(1) $\gamma = 0$,
(2) *γ^2 is a positive eigenvalue of $\Delta = \mathrm{d}^*\mathrm{d}$ on functions on N.*

Proof. Throughout this proof let $\mathrm{d}^*_L, *_L$ operate on L and $\mathrm{d}^*, *$ operate on N. A smooth section of $(T^*(N \times \{p\} \times \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C})$ which is invariant under translations in $\mathbb{R}$ can be written uniquely as $\eta + f_1\mathrm{d}t$ where $\eta \in C^\infty(T^*N \otimes_{\mathbb{R}} \mathbb{C})$, the smooth sections of $T^*N \otimes_{\mathbb{R}} \mathbb{C}$, and $f_1 : N \to \mathbb{C}$ is smooth. A smooth section of $\Lambda^3 T^*(N \times \{p\} \times \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ invariant under translations in $\mathbb{R}$ can be written uniquely as $f_2 dvol_N \wedge \mathrm{d}t$ with $f_2 : N \to \mathbb{C}$ smooth. Then by Theorem 3.7, $(\mathrm{d} + \mathrm{d}^*_L)^p_{2+l,\gamma}$ is not Fredholm if and only if there exist f_1, f_2, η, with f_1, f_2, η not all zero, and $\delta \in \mathbb{R}$ such that

$$(\mathrm{d} + \mathrm{d}^*_L)(e^{(\gamma+i\delta)t}(\eta + f_1\mathrm{d}t), e^{(\gamma+i\delta)t}(f_2 dvol_N \wedge \mathrm{d}t)) \equiv (0,0).$$

Thus:

$$\begin{aligned} 0 &\equiv \mathrm{d}(e^{(\gamma+i\delta)t}(\eta + f_1\mathrm{d}t)) + \mathrm{d}^*_L(e^{(\gamma+i\delta)t}(f_2 dvol_N \wedge \mathrm{d}t)) \\ &= e^{(\gamma+i\delta)t}[-(\gamma+i\delta)\eta \wedge \mathrm{d}t + \mathrm{d}\eta + \mathrm{d}f_1 \wedge \mathrm{d}t] - *_L\mathrm{d}(e^{(\gamma+i\delta)t}f_2) \\ &= e^{(\gamma+i\delta)t}[-(\gamma+i\delta)\eta \wedge \mathrm{d}t + \mathrm{d}\eta + \mathrm{d}f_1 \wedge \mathrm{d}t] - e^{(\gamma+i\delta)t} *_L [(\gamma+i\delta)f_2\mathrm{d}t + \mathrm{d}f_2] \\ &= e^{(\gamma+i\delta)t}[-(\gamma+i\delta)\eta \wedge \mathrm{d}t + \mathrm{d}\eta + \mathrm{d}f_1 \wedge \mathrm{d}t - (\gamma+i\delta)f_2 dvol_N - (*\mathrm{d}f_2) \wedge \mathrm{d}t] \end{aligned}$$

and

$$\begin{aligned} 0 &\equiv \mathrm{d}^*_L(e^{(\gamma+i\delta)t}(\eta + f_1\mathrm{d}t)) = - *_L \mathrm{d} *_L (e^{(\gamma+i\delta)t}(\eta + f_1\mathrm{d}t)) \\ &= - *_L \mathrm{d}[e^{(\gamma+i\delta)t}(*\eta \wedge \mathrm{d}t + f_1 dvol_N)] \\ &= - *_L [(\gamma+i\delta)e^{(\gamma+i\delta)t}\mathrm{d}t \wedge f_1 dvol_N + e^{(\gamma+i\delta)t}\mathrm{d}(*\eta \wedge \mathrm{d}t)] \\ &= -e^{(\gamma+i\delta)t}[(\gamma+i\delta)f_1 + *\mathrm{d} * \eta] \\ &= -e^{(\gamma+i\delta)t}[(\gamma+i\delta)f_1 - \mathrm{d}^*\eta]. \end{aligned}$$

This yields the following system of equations in 0-, 1- and 2-forms on N respectively:

$$(\gamma+i\delta)f_1 - \mathrm{d}^*\eta \equiv 0,$$

$$\mathrm{d}f_1 - (\gamma+i\delta)\eta - *\mathrm{d}f_2 \equiv 0,$$

$$-(\gamma+i\delta)f_2 dvol_N + \mathrm{d}\eta \equiv 0.$$

From here, we immediately see that if $\gamma = 0$ then $\delta = 0$, $\eta = 0$ and $f_1 \equiv 1$, $f_2 \equiv 1$ yields a solution to the above system of equations, in which case $(\mathrm{d}+*_L\mathrm{d}_L^*)^p_{2+l,0}$ is not Fredholm. Assume now that $\gamma + i\delta \neq 0$. Since N is compact, Hodge theory yields $\eta = (\eta_0, \eta_1, \eta_2)$ where η_0 is a harmonic 1-form on N, η_1 is an exact 1-form on N and η_2 is a coexact 1-form on N. Splitting the above system up by harmonic, exact and coexact 1-forms on N yields the following system of equations:

$$\begin{aligned}(\gamma + i\delta)\eta_0 &= 0,\\ \mathrm{d}f_1 - (\gamma + i\delta)\eta_1 &= 0,\\ - * \mathrm{d}f_2 - (\gamma + i\delta)\eta_2 &= 0,\\ (\gamma + i\delta)f_1 - \mathrm{d}^*\eta_1 &= 0,\\ (\gamma + i\delta)f_2 - *\mathrm{d}\eta_2 &= 0.\end{aligned}$$

Because we are assuming that $\gamma + i\delta \neq 0$, $\eta_0 = 0$ we first obtain:

$$\mathrm{d}f_1 = (\gamma + i\delta)\eta_1, \tag{4}$$

$$(\gamma + i\delta)f_1 = \mathrm{d}^*\eta_1. \tag{5}$$

Notice that $f_1 = 0$ if and only if $\eta_1 = 0$. Since f_1 and η cannot simultaneously be zero, neither can be zero.

Taking d^* of (4), then substituting (5) yields:

$$\mathrm{d}^*\mathrm{d}f_1 = (\gamma + i\delta)\mathrm{d}^*\eta_1 = (\gamma + i\delta)^2 f_1,$$

so that $(\gamma + i\delta)^2$ is an eigenvalue of $\mathrm{d}^*\mathrm{d}$ on functions on N. Similarly, the remaining two equations

$$\begin{aligned}*\mathrm{d}f_2 - (\gamma + i\delta)\eta_2 &= 0,\\ (\gamma + i\delta)f_2 - *\mathrm{d}\eta_2 &= 0.\end{aligned}$$

also imply that $(\gamma + i\delta)^2$ is an eigenvalue of $\mathrm{d}^*\mathrm{d}$ on functions on N.

Since such eigenvalues must be positive, this shows that $\delta = 0$, and so γ^2 is an eigenvalue of $\mathrm{d}^*\mathrm{d}$ on functions on N. Conversely, assume $\Delta f = \mathrm{d}^*\mathrm{d}f = \gamma^2 f$ for some nonzero $\gamma \in \mathbb{R}$ and some smooth nonzero $f : N \to \mathbb{C}$; then $\eta = \gamma^{-1}\mathrm{d}f$ is a smooth nonzero 1-form on N which satisfies the above equations.

Now, take d of (5) and substitute (4) to get:

$$\mathrm{d}\mathrm{d}^*\eta_1 = (\gamma + i\delta)\mathrm{d}f = (\gamma + i\delta)^2\eta_1.$$

Because η_1 is an exact 1-form on N, $(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\eta_1 = (\gamma + i\delta)^2\eta_1$ which shows that $(\gamma + i\delta)^2$ is an eigenvalue of $\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = \mathrm{d}\mathrm{d}^*$ on exact 1-forms on N, and so $\delta = 0$ by the same reasoning as above. Hence γ^2 is an eigenvalue of $\mathrm{d}\mathrm{d}^*$ on exact 1-forms on N. Conversely, assume that $\mathrm{d}\mathrm{d}^*\eta = \gamma^2\eta$ for some nonzero $\gamma \in \mathbb{R}$ and some smooth nonzero exact 1-form η on N; then $f = \gamma^{-1}\mathrm{d}^*\eta$ is a smooth nonzero function on N which satisfies (4) and (5). Now, note that any eigenvalue of the Laplacian on exact 1-forms is an eigenvalue of the Laplacian on functions, since if $\Delta(df) = \lambda(df)$ with $\int_N f = 0$ then $\Delta f = \lambda f$ and hence we obtained the same set of eigenvalues. One could also see this from the set of equations (4)(5) which gives a bijection between the function f_1 and the 1-form η_1. $\square$

5. Proof of Theorem 1.1

Let (X, ω, g_X, Ω) be an asymptotically cylindrical Calabi-Yau 3-fold with decay rate $\alpha < 0$, asymptotic to the cylindrical Calabi-Yau 3-fold $(X_0, \omega_0, g_{X_0}, \Omega_0)$, $X = M \times S^1 \times \mathbb{R}$ where M is a compact connected Calabi-Yau 2-fold as in Definition 2.3. Let $K \subset X$ be a compact subset, $R > 0$ and $\Psi : M \times S^1 \times (R, \infty) \to X \setminus K$ a diffeomorphism with the following properties:

(1) $\Psi^*(\omega_{X_0}) = \omega + \mathrm{d}\xi_1$, for some ξ_1, a 1-form on X with $|\nabla^k \xi_1| = O(e^{\alpha t})$ for all $k \geq 0$;

(2) $\Psi^*(\Omega_{X_0}) = \Omega + \mathrm{d}\xi_2$, for some ξ_2, a complex 2-form on X with $|\nabla^k \xi_2| = O(e^{\alpha t})$ for all $k \geq 0$.

Let L be an asymptotically cylindrical special Lagrangian 3-submanifold of X with decay rate β ($\alpha \leq \beta < 0$), asymptotic to the cylindrical special Lagrangian 3-submanifold $L_0 = N \times \{p\} \times \mathbb{R}$ of X_0, where N is a compact special Lagrangian 2-submanifold of M and $p \in S^1$ as in Definition 2.6. Let $K' \subset L$ be a compact subset, $R' > R$, v a normal vector field on $N \times \{p\} \times (R', \infty)$ with $|\nabla^k v| = O(e^{\beta t})$ for all $k \geq 0$ and $\Phi : N \times \{p\} \times (R', \infty) \to L \setminus K'$ a diffeomorphism making Diagram (1) commute.

Let $\gamma < 0$ be strictly greater than β and be such that $(0, \gamma^2]$ does not contain eigenvalues of the Laplacian $\Delta_N = \mathrm{d}^*\mathrm{d}$ on complex-valued functions on N. Let $p > 3$, $l \geq 1$ and the map

$$(\mathrm{d} + \mathrm{d}^*)^p_{2+l,\gamma} : L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3 T^*L) \to L^p_{1+l,\gamma}(\Lambda^0 T^*L \oplus \Lambda^2 T^*L)$$

be as in Equation (3). By Theorem 4.2, the conditions on γ imply this operator is Fredholm, so the results of Theorem 4.1 are applicable.

We begin by constructing an identification of small sections of the normal bundle of L with X near L that is compatible with the data on these manifolds. Let ν_N be the normal bundle of N in M with exponential map $\exp_N : \nu_N \to M$; let $\epsilon > 0$ such that $\exp_N : B_{2\epsilon}(\nu_N) \to T_N$ is a diffeomorphism of the subbundle $B_{2\epsilon}(\nu_N)$ whose fiber above each point is the ball of radius 2ϵ about 0, with a tubular neighborhood T_N of N in M. Then $B_{2\epsilon}(\nu_N) \times \{p\} \times \mathbb{R}$ is a subbundle of the normal bundle $\nu_N \times \{p\} \times \mathbb{R}$ of $N \times \{p\} \times \mathbb{R}$ in $M \times S^1 \times \mathbb{R}$, $T_N \times \{p\} \times \mathbb{R}$ is a tubular neighborhood (also of $N \times \{p\} \times \mathbb{R}$ in $M \times S^1 \times \mathbb{R}$) and $\exp_N \times \iota \times id : B_{2\epsilon}(\nu_N) \times \{p\} \times \mathbb{R} \to T_N \times \{p\} \times \mathbb{R}$ is a diffeomorphism, where ι is the natural inclusion map and id the identity map on $\mathbb{R}$. Notice that v from above is a section of $\nu_N \times \{p\} \times (R', \infty)$, so since v is decaying, we can assume that the graph of v lies in $B_{2\epsilon}(\nu_N) \times \{p\} \times (R', \infty)$ (making K', R' larger if necessary).

Let $\pi : B_\epsilon(\nu_N) \times \{p\} \times (R', \infty) \to N \times \{p\} \times (R', \infty)$ be the natural projection map. Let $\Xi : B_\epsilon(\nu_N) \times \{p\} \times (R', \infty) \to X \setminus K$ be given by $\Xi(w) = \Psi[(\exp_N \times \iota \times id)(v|_{\pi(w)} + w)]$. First, notice that since Ξ is a composition of diffeomorphisms (onto their images), Ξ is also a diffeomorphism (onto its image). Second, viewing $N \times \{p\} \times (R', \infty)$ as the zero section of $B_\epsilon(\nu_N) \times \{p\} \times (R', \infty)$, $\Xi|_{N \times \{p\} \times (R', \infty)} = \Psi \circ (\exp_N \times \iota \times id)|_{N \times \{p\} \times (R', \infty)} = \Psi \circ \exp_v = \Phi$ where the last two equalities follow from Definition 2.6 and the commutativity of Diagram (1). Third, $\mathrm{d}\Xi : T(B_\epsilon(\nu_N) \times \{p\} \times (R', \infty)) \to \Xi^*(T(X \setminus K)) = \Phi^*(T(X \setminus K))$ is an isomorphism.

Since $\Phi : N \times \{p\} \times (R', \infty) \to L \setminus K'$ is a diffeomorphism, the induced map $\mathrm{d}\Phi : T(N \times \{p\} \times (R', \infty))) \to T(L \setminus K')$ is an isomorphism. In fact, by the above discussion, $\mathrm{d}\Phi = \mathrm{d}\Xi|_{N\times\{p\}\times(R',\infty)} : T(B_\epsilon(\nu_N) \times \{p\} \times (R', \infty))|_{N\times\{p\}\times(R',\infty)} \to \Phi^*(T(L \setminus K'))$. Thus, define $\zeta = \mathrm{d}\Xi|_{N\times\{p\}\times(R',\infty)}$, so that

$$\begin{aligned} \zeta : \nu_N \times \{p\} \times (R', \infty) &\cong \frac{T(B_\epsilon(\nu_N) \times \{p\} \times (R', \infty))|_{N\times\{p\}\times(R',\infty)}}{T(N \times \{p\} \times (R', \infty))} \\ &\to \Phi^*(T(X \setminus K))/\Phi^*(T(L \setminus K')) \cong \Phi^*(T(X \setminus K)/T(L \setminus K')) = \Phi^*(\nu_L), \end{aligned} \tag{6}$$

where ν_L is the normal bundle of L; ζ is a vector bundle isomorphism of $\nu_N \times \{p\} \times (R', \infty)$ and $\Phi^*(\nu_L)$ by construction.

Finally, let $\Theta : B_{\epsilon'}(\nu_L) \to T_L$ denote an identification of $B_{\epsilon'}(\nu_L)$ (for some small $\epsilon' > 0$) with T_L, a tubular neighborhood of L in X, satisfying the following properties: first, thinking of L as the zero section of $B_{\epsilon'}(\nu_L)$, $\Theta|_L = id_L$; also, ϵ' should be small enough so that $\zeta^*(\Phi^*(B_{\epsilon'}(\nu_L))) \subset B_\epsilon(\nu_N) \times \{p\} \times (R', \infty) \subset \nu_N \times \{p\} \times (R', \infty)$ and $\Theta \circ \zeta = \Xi$ on $\xi^*(\Phi^*(B_{\epsilon'}(\nu_L)))$. Notice that the first condition implies that we have $\mathrm{d}\Theta|_L = id_{TL} : TL \subset T(B_{\epsilon'}(\nu_L))|_L \to TL \subset T(T_L)|_L$; the last condition uniquely defines Θ and T_L on $B_{\epsilon'}(\nu_L)|_{L\setminus K'}$, so one need only smoothly extend Θ and T_L to the compact subset K' of L.

To summarize this construction, we have used the maps Ξ and ζ to define Θ, giving an identification of small sections of the normal bundle ν_L of L with the ambient manifold X near L, in such a way that is compatible with the diffeomorphisms Φ and Ψ. This allows us to identify small sections of ν_L with 3-submanifolds of X near L and detect the asymptotic convergence of such a submanifold to $N \times \{p\} \times (R', \infty)$ by the asymptotic convergence of small sections of ν_L to zero. Further, since $\nu_L \cong T^*L$ on the special Lagrangian submanifold L, we will regard $\Theta : B_{\epsilon'}(T^*L) \to T_L$.

This has the advantage that now smooth sections η of the space $L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))$ (note that this is an open Banach subspace of the Banach manifold $L^p_{2+l,\gamma}(T^*L)$, so it is itself a Banach manifold) correspond to smooth 3-submanifolds of X near L, and since η is by definition a map $\eta : L \to L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))$, $\Theta \circ \eta : L \to T_L$. Hence we define our deformation map as $F : L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to \Lambda^2T^*L \oplus \Lambda^3T^*L$, $F(\eta) = [(\Theta \circ \eta)^*(-\omega), (\Theta \circ \eta)^*(\operatorname{Im}\Omega)]$. Letting Γ_η denote the graph of η in $L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))$ and $\tilde{L} = \Theta(\Gamma_\eta)$ its image in X, $\tilde{L}$ is special Lagrangian precisely when $\omega|_{\tilde{L}} \equiv 0$ and $\operatorname{Im}\Omega|_{\tilde{L}} \equiv 0$, but this is equivalent to $F(\eta) = (0, 0)$. Thus, $F^{-1}(0, 0)$ parameterizes the special Lagrangian 3-submanifolds $\tilde{L}$ near L.

This completes our setup. Our first step now will be to prove that F extends to a smooth map of Banach manifolds, and then extend F over the space of three forms on L $\tilde{F} = \tilde{F}^p_{2+l,\gamma} : L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3T^*L) \to L^p_{1+l,\gamma}(\Lambda^2T^*L \oplus \Lambda^0T^*L)$ in order to get an elliptic operator. The linearization at 0 of this operator will be given by the operator $\mathrm{d}\tilde{F}^p_{2+l,\gamma}(0) = (\mathrm{d} + \mathrm{d}^*)^p_{2+l,\gamma}$ from above; moreover, we will show that $\tilde{F}$ actually maps into the image of $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$. The point is to ultimately use these results to invoke the Implicit Mapping Theorem for Banach Manifolds (see, e. g., [6, Theorem 1.2.5]).

Proposition 5.1. *$F : L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to L^p_{1+l,\gamma}(\Lambda^2T^*L \oplus \Lambda^3T^*L)$ is a smooth map of Banach manifolds with linearization at 0 given by $\eta \mapsto (\mathrm{d}\eta, *\mathrm{d}^*\eta)$.*

Proof. Begin by noting that our assumptions $p > 3$ and $l \geq 1$ yield, by the Weighted Sobolev Embedding Theorem 3.3, the continuous inclusion $L^p_{2+l,\gamma} \hookrightarrow C^1_\gamma$, so that locally $F(\eta) \in L^p_{1+l}$. Now $F(\eta)$ is simply the restriction of $\Theta^*(-\omega)$, a 2-form on $B_{\epsilon'}(T^*L)$, and of $\Theta^*(\operatorname{Im}\Omega)$, a 3-form on $B_{\epsilon'}(T^*L)$, to Γ_η. From the properties of Θ, this means that $F(\eta)$ is equal to $\Xi^*(-\omega)$ and $\Xi^*(\operatorname{Im}\Omega)$ on $B_{\epsilon'}(\nu_N) \times \{p\} \times (R', \infty)$; further, the asymptotic properties of Ψ, Φ and v we built into Ξ imply that $\Xi^*(-\omega)$ is the sum of the pullback of the negative of the cylindrical Kähler 2-form ω_0 on $M \times S^1 \times \mathbb{R}$ to $B_{\epsilon'}(\nu_N) \times \{p\} \times (R', \infty)$ and an error term which decays at a rate of $O(e^{\beta t})$; similarly, $\Xi^*(\operatorname{Im}\Omega)$ is the sum of the pullback of the 3-form $\operatorname{Im}\Omega$ on $M \times S^1 \times \mathbb{R}$ to $B_{\epsilon'}(\nu_N) \times \{p\} \times (R', \infty)$ and an error term which decays at a rate of $O(e^{\beta t})$.

Hence because $\beta < \gamma$, $F(\eta) \in L^p_{1+l,\gamma}(\Lambda^2T^*L \oplus \Lambda^3T^*L)$. Smoothness of F follows by construction, so the first claim follows. That the linearization of F at 0 is $(\mathrm{d} + *\mathrm{d}^*)^p_{2+l,\gamma}$ follows exactly as in Theorem 2.7 since the calculation is local. $\square$

Lemma 5.2. *The image of F lies in exact 2-forms and exact 3-forms; specifically,*

$$\begin{aligned} F(L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))) &\subset \mathrm{d}(L^p_{1+l,\gamma}(T^*L \oplus \Lambda^2T^*L)) \\ &\subset L^p_{l,\gamma}(\Lambda^2T^*L \oplus \Lambda^3T^*L). \end{aligned}$$

Proof. Recall that ω and $\operatorname{Im}\Omega$ are closed forms, so they determine the de Rham cohomology classes $[\omega]$ and $[\operatorname{Im}\Omega]$. In particular, $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$ since L is special Lagrangian, so $[\omega|_L] = 0$ and $[\operatorname{Im}\Omega|_L] = 0$; moreover, since T_L, the tubular neighborhood of L from above, retracts onto L, $[\omega|_{T_L}] = [\omega|_L]$ and $[\operatorname{Im}\Omega|_{T_L}] = [\operatorname{Im}\Omega|_L]$. Thus, there exist $\tau_1 \in C^\infty(T^*T_L)$ such that $\omega|_{T_L} = \mathrm{d}\tau_1$ and $\tau_2 \in C^\infty(\Lambda^2T^*T_L)$ such that $\operatorname{Im}\Omega|_{T_L} = \mathrm{d}\tau_2$. Now, since $\omega|_L \equiv 0$, we can assume that $\tau_1|_L \equiv 0$; second, because ω and all its derivatives decay at a rate $O(e^{\beta t})$ to the translation-invariant 2-form ω_0 on $M \times S^1 \times \mathbb{R}$, we can assume that τ_1 and all its derivatives decay at a rate $O(e^{\beta t})$ to a translation invariant 1-form on $T_N \times \{p\} \times \mathbb{R}$. We can make similar assumptions regarding τ_2 based on the properties of $\operatorname{Im}\Omega$.

From this, we calculate the following for $\eta \in L^p_{2+l,\gamma}(T^*L)$:

$$\begin{aligned} F(\eta) &= ((\Theta \circ \eta)^*(-\omega), (\Theta \circ \eta)^*(\operatorname{Im}\Omega)) \\ &= ((\Theta \circ \eta)^*(-\mathrm{d}\tau_1), (\Theta \circ \eta)^*(\mathrm{d}\tau_2)) \\ &= (\mathrm{d}(\Theta \circ \eta)^*(-\tau_1)), \mathrm{d}((\Theta \circ \eta)^*(\tau_2))). \end{aligned}$$

The result now follows by Proposition 5.1. $\square$

F as defined is not elliptic; we need to extend F over a space of 3-forms in order to make it so. First, using the Hodge star isomorphism, we can interpret F as a map $L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to L^p_{1+l,\gamma}(\Lambda^2 T^*L \oplus \Lambda^0 T^*L)$. We can then make it elliptic by defining

$$\tilde{F} : L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L) \to L^p_{1+l,\gamma}(\Lambda^2 T^*L \oplus \Lambda^0 T^*L)$$

where $\tilde{F}(\eta_1, \eta_3) = F(\eta_1) + \mathrm{d}^*\eta_3$. Note that the linearization of F at $\eta = 0$ is now given by $\mathrm{d}F(0)(\tau) = (\mathrm{d}\tau, \mathrm{d}^*\tau)$, so the linearization of $\tilde{F}$ at $(\eta_1, \eta_3) = (0,0)$ is given by $\mathrm{d}\tilde{F}(0,0)(\tau_1, \tau_3) = (\mathrm{d}\tau_1 + \mathrm{d}^*\tau_3, \mathrm{d}^*\tau_1)$.

Proposition 5.3. *Let $\mathcal{C}$ denote the image of the operator $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$. Then*

$$\tilde{F} : L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L) \to \mathcal{C}.$$

Proof. By Theorem 4.1, $\operatorname{coker}((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \cong (\ker((\mathrm{d}+\mathrm{d}^*)^q_{2+m,-\gamma}))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m \geq 1$, so $\tilde{F}(\eta_1, \eta_3) = F(\eta_1) + \mathrm{d}^*\eta_3$ is in $\mathcal{C}$ if and only if

$$\langle F(\eta_1) + \mathrm{d}^*\eta_3, (\chi_0, \chi_2)\rangle_{L^2} \equiv 0 \text{ for all } (\chi_0, \chi_2) \in \ker((\mathrm{d}+\mathrm{d}^*)^q_{2+m,-\gamma}).$$

By the previous lemma $F(\eta_1) = (\mathrm{d}\tau_1, \mathrm{d}\tau_2)$ for some $\tau_1 \in L^p_{1+l,\gamma}(T^*L)$ and $\tau_2 \in L^p_{1+l,\gamma}(\Lambda^2 T^*L)$; then Theorem 4.1 implies $*\chi_0$ and χ_2 are closed and coclosed 3- and 2-forms respectively which yields the following:

$$\begin{aligned}\langle F(\eta_1) + \mathrm{d}^*\eta_3, (*\chi_0, \chi_2)\rangle_{L^2} &= \langle \mathrm{d}\tau_2, *\chi_0\rangle_{L^2} + \langle \mathrm{d}\tau_1 + \mathrm{d}^*\eta_3, \chi_2\rangle_{L^2} \\ &= \langle \tau_2, \mathrm{d}^*(*\chi_0)\rangle_{L^2} + \langle \tau_1, \mathrm{d}^*\chi_2\rangle_{L^2} + \langle \eta_3, \mathrm{d}\chi_2\rangle_{L^2} = 0.\end{aligned}$$

□

The next step is to use the Implicit Mapping Theorem for Banach Spaces. Let $\mathcal{A} = \ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$ and $\mathcal{B}$ denote the subspace of $L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3 T^*L)$ that is L^2-orthogonal to $\mathcal{A}$. Because $\mathcal{A}$ is finite-dimensional and the L^2-inner product is continuous on $L^p_{2+l,\gamma}(T^*L)$, $\mathcal{A}$ and $\mathcal{B}$ are Banach spaces such that $\mathcal{A} \oplus \mathcal{B} = L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3 T^*L)$. Choose open neighborhoods $\mathcal{U}$, $\mathcal{V}$ of 0 in $\mathcal{A}$, $\mathcal{B}$, respectively, such that $\mathcal{U} \times \mathcal{V} \subset L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L)$. Then, by Proposition 5.1 and Proposition 5.3, $\tilde{F} : \mathcal{U} \times \mathcal{V} \to \mathcal{C}$ is a smooth map of Banach manifolds, $\tilde{F}(0,0) = (0,0)$ and $\mathrm{d}\tilde{F}(0,0) = (\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma} : \mathcal{A} \oplus \mathcal{B} \to \mathcal{C}$; moreover, $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}|_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ is an isomorphism of vector spaces by construction, and it is a homeomorphism of topological spaces by the Open Mapping Theorem. The Implicit Mapping Theorem for Banach Spaces now guarantees the existence of a connected open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of 0 and a smooth function $G : \mathcal{U}' \to \mathcal{V}$ such that $G(0) = 0$ and $\tilde{F}(x) = (x, G(x)) \equiv (0,0)$ for all $x \in \mathcal{U}'$. Hence we conclude that near $(0,0)$, $\tilde{F}^{-1}(0,0) = \{(x, G(x)) : x \in \mathcal{U}'\}$, so that $\tilde{F}^{-1}(0,0)$ is smooth, finite-dimensional and locally isomorphic to $\mathcal{A} = \ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$.

Lemma 5.4. $\tilde{F}^{-1}(0,0) = F^{-1}(0,0) \times \{0\}$

Proof. We clearly have $F^{-1}(0,0)\times\{0\} \subseteq \tilde{F}^{-1}(0,0)$. Suppose then that $(\eta_1,\eta_3) \in \tilde{F}^{-1}(0,0)$, so $F(\eta_1) + \mathrm{d}^*\eta_3 = 0$. By Lemma 5.2, we know that $F(\eta_1) = (\mathrm{d}\tau_1, *\mathrm{d}\tau_2)$ for some $\tau_1 \in L^p_{1+l,\gamma}(T^*L)$ and $\tau_2 \in L^p_{1+l,\gamma}(\Lambda^2 T^*L)$;in particular, we must have $\mathrm{d}\tau_1 = -\mathrm{d}^*\eta_3$ and $*\mathrm{d}\tau_2 = 0$. Then

$$||\mathrm{d}^*\eta_3||^2_{L^2} = \langle \mathrm{d}^*\eta_3, \mathrm{d}^*\eta_3\rangle_{L^2} = -\langle \mathrm{d}^*\eta_3, \mathrm{d}\tau_1\rangle_{L^2} = 0.$$

Thus, $\mathrm{d}^*\eta_3 = 0$, and η_3 is constant since η_3 is a 3-form. Finally, since $\eta_3 \to 0$ near infinity in L, $\eta_3 = 0$ from which the claim now follows. ☐

The last part of the proof consists of defining a map from $F^{-1}(0,0)$ to the moduli space M^γ_L of asymptotically cylindrical special Lagrangian deformations of L near L; however, one technical step involved in showing the map is well defined is to show that the sections η in $F^{-1}(0,0)$ are smooth. Theorem 1.1 will then follow from these results, Theorem 4.1 and the fact that, by the previous lemma, $F^{-1}(0,0)$ is smooth, finite-dimensional and locally isomorphic to $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$.

Lemma 5.5. *If $\eta \in F^{-1}(0,0)$, then $\eta \in L^p_{2+m,\gamma}(T^*L)$ for all $m \geq 1$.*

Proof. We begin by noting that the functional form of $F(\eta)$ is given by $H(x, \eta|_x, \nabla\eta|_x)$ where $x \in L$ and H is a smooth function. Fix $m \geq 1$, and let ∇ denote the Levi-Cività connection of g_L on L. We are going to apply the Laplacian $\Delta_L = g^{ij}_L \nabla_i\nabla_j$ to F which will allow us to split F up in such a way that we can use a regularity result (which we will prove in the course of this argument) to increase the regularity of η.

Let ∇^x denote the derivative in the x-direction; let ∂^y and ∂^z denote the derivatives in the y- and z-directions respectively, where $y = \eta$ and $z = \nabla\eta$. Then

$$\begin{aligned}
\Delta_L(F(\eta)) &= g^{ij}_L \nabla_i\nabla_j(H(x,y,z)) \\
&= g^{ij}_L \nabla_i[(\nabla^x_j H)(x,y,z) + (\partial^y H)(x,y,z)\cdot\nabla_j\eta + (\partial^z H)(x,y,z)\cdot\nabla_j\nabla\eta] \\
&= g^{ij}_L [(\nabla^x_i\nabla^x_j H)(x,y,z) + (\nabla^x_i\partial^y H)(x,y,z)\cdot\nabla_j\eta + (\partial^y H)(x,y,z)\cdot\nabla_i\nabla_j\eta \\
&\quad +(\nabla^x_i\partial^z H)(x,y,z)\cdot\nabla_j\nabla\eta + (\partial^z H)(x,y,z)\cdot\nabla_i\nabla_j\nabla\eta \\
&\quad +(\partial^y\nabla^x_j H)(x,y,z)\cdot\nabla_i\eta + (\partial^y\partial^y H)(x,y,z)\cdot(\nabla_i\eta\otimes\nabla_j\eta) \\
&\quad +(\partial^y\partial^z H)(x,y,z)\cdot(\nabla_i\eta\otimes\nabla_j\nabla\eta) + (\partial^z\nabla^x_j H)(x,y,z)\cdot\nabla_i\nabla\eta \\
&\quad +(\partial^z\partial^y H)(x,y,z)\cdot(\nabla_i\nabla\eta\otimes\nabla_j\eta) + (\partial^z\partial^z H)(x,y,z)\cdot(\nabla_i\nabla\eta\otimes\nabla_j\nabla\eta)] \\
&= (\partial^z H)(x,y,z)\cdot\Delta_L\nabla\eta + (\partial^y H)(x,y,z)\cdot\Delta_L\eta + g^{ij}_L[(\nabla^x_i\nabla^x_j H)(x,y,z) \\
&\quad +(\nabla^x_i\partial^y H)(x,y,z)\cdot\nabla_j\eta + (\nabla^x_i\partial^z H)(x,y,z)\cdot\nabla_j\nabla\eta \\
&\quad +(\partial^y\nabla^x_j H)(x,y,z)\cdot\nabla_i\eta + (\partial^y\partial^y H)(x,y,z)\cdot(\nabla_i\eta\otimes\nabla_j\eta) \\
&\quad +(\partial^y\partial^z H)(x,y,z)\cdot(\nabla_i\eta\otimes\nabla_j\nabla\eta) + (\partial^z\nabla^x_j H)(x,y,z)\cdot\nabla_i\nabla\eta \\
&\quad +(\partial^z\partial^y H)(x,y,z)\cdot(\nabla_i\nabla\eta\otimes\nabla_j\eta) + (\partial^z\partial^z H)(x,y,z)\cdot(\nabla_i\nabla\eta\otimes\nabla_j\nabla\eta)].
\end{aligned}$$

Notice that Δ_L splits F into two pieces: the only term $(\partial^z H)(x,y,z)\cdot\Delta_L\nabla\eta$ involving the third derivatives of η, and everything else which depends only on η up to its second derivatives, which we will denote from now on by $E(x,\eta,\nabla\eta,\nabla^2\eta)$. Now, for η fixed, let $\tilde{H}_\eta : L^p_{3+s}(T^*L) \to L^p_s(\Lambda^2T^*L) \oplus L^p_s(\Lambda^3T^*L)$ denote the map defined by $\sigma \mapsto \tilde{H}_\eta(\sigma) = (\partial^z H)(x,\eta|_x,\nabla\eta|_x) \cdot \Delta_L\nabla\sigma|_x$, so that $\tilde{H}_\eta$ is a third-order linear elliptic operator ($\tilde{H}_\eta$ is basically $\Delta_L \mathrm{d}$ with d the exterior derivative operator). Further, since the coefficients of $\tilde{H}_\eta$ only depend on η and $\nabla\eta$, they are L^p_{1+m} locally, and so the maximum regularity we can get from $\tilde{H}_\eta(\sigma)$ is L^p_{1+m}; this forces us to take s such that $0 \le s \le m+1$. Of course, since the coefficients of $\tilde{H}_\eta$ are only L^p_{1+m} locally (rather than smooth) we cannot use the elliptic regularity result, Theorem 3.5, so we need the following result:

Lemma 5.6. *Assume $\sigma \in L^p_3(T^*L)$ and $\tilde{H}_\eta(\sigma) \in L^p_m(\Lambda^2T^*L) \oplus L^p_m(\Lambda^3T^*L)$. Then $\sigma \in L^p_{3+m}(T^*L)$, and there exists a constant $\tilde{C} > 0$ such that*

$$\|\sigma\|_{L^p_{3+m,\gamma}} \le \tilde{C}(\|\tilde{H}_\eta(\sigma)\|_{L^p_{m,\gamma}} + \|\sigma\|_{L^p_{3,\gamma}}).$$

Proof. We will use results from Morrey [15, Section 6.2] to prove this lemma. If the coefficients of $\tilde{H}_\eta$ are C^m, then [15, Theorem 6.2.5] guarantees that σ is locally L^p_{3+m}. Further, [15, Theorem 6.2.6] provides a local interior estimate of the form above where $\tilde{C} > 0$ depends on m, p, the domains involved, C^m-bounds on the coefficients of $\tilde{H}_\eta$ and a modulus of continuity for their m-th derivatives. Notice that if we have Hölder $C^{0,\alpha}$, $\alpha \in (0,1)$, bounds for the m-th derivatives, we get our modulus of continuity; we can further simplify the problem to finding $C^{1+m,\alpha}$ bounds for η since this will give $C^{m,\alpha}$ bounds for $\tilde{H}_\eta$ giving us the desired bounds. Recall that we are assuming $p > 3$, so that with $\alpha = 1 - p/3$, the Sobolev Embedding Theorem embeds L^p_{2+m} into $C^{1+m,\alpha}$. Since $\eta \in L^p_{2+m,\gamma}$ and $\gamma < 0$, we get the desired control on η which yields the modulus of continuity and the result. □

To finish the proof, let $\eta \in F^{-1}(0,0)$, so that $\eta \in L^p_{2+m,\gamma}(T^*L)$. From the above computation and the fact that $F(\eta) = 0$, we have $\tilde{H}_\eta(\eta) = -E(x,\eta,\nabla\eta,\nabla^2\eta)$, so that $\tilde{H}_\eta(\eta) \in L^p_{m,\gamma}(\Lambda^2T^*L) \oplus L^p_{m,\gamma}(\Lambda^3T^*L)$. By the regularity result above, we now have that $\eta \in L^p_{3+m,\gamma}(T^*L)$. The result now follows simply from induction on m. □

Proposition 5.7. *Let M^γ_L be the moduli space of nearby asymptotically cylindrical special Lagrangian deformations of L with decay rate γ and asymptotic to $N \times \{p\} \times (R',\infty)$. Define $S : F^{-1}(0,0) \to \{3\text{-submanifolds of } X\}$ by $\eta \mapsto \Theta(\Gamma_\eta)$ where Γ_η is the graph of η in $B_{\epsilon'}(T^*L)$. Then S is a homeomorphism of $F^{-1}(0,0)$ with a neighborhood of L in M^γ_L.*

Proof. Let $\eta \in F^{-1}(0,0)$, let $\tilde{L} = S(\eta) \subset T_L \subset X$. By the Lemma 5.5, $\tilde{L}$ is smooth; further,

$$(0,0) = F(\eta) = ((\Theta\circ\eta)^*(-\omega), (\Theta\circ\eta)^*(\operatorname{Im}\Omega)) = (-\omega|_{\tilde{L}}, \operatorname{Im}\Omega|_{\tilde{L}}),$$

which proves $\tilde{L}$ is a special Lagrangian 3-submanifold of X.

We now appeal to Definition 2.6 to prove that $\tilde{L}$ is asymptotically cylindrical with decay rate γ. $\Theta\circ\eta : L \to \tilde{L}$ is a diffeomorphism, so let $\tilde{K}' = (\Theta\circ\eta)(K')$; then $\tilde{K}'$

is a compact subset of $\tilde{L}$. Let $\tilde{\Phi} = \Theta \circ \eta \circ \Phi : N \times \{p\} \times (R', \infty) \to \tilde{L} \setminus \tilde{K}'$; then $\tilde{\Phi}$ is a diffeomorphism. Finally, notice $\eta \in T^*L \cong \nu_L$, so $\zeta^* \circ \Phi^*(\eta)$ is a section of $\nu_N \times \{p\} \times (R', \infty)$, ζ defined by (6); v is also a section of $\nu_N \times \{p\} \times (R', \infty)$, so we can define $\tilde{v} = v + \zeta^* \circ \Phi^*(\eta) \in \nu_N \times \{p\} \times (R', \infty)$. With this data, $\tilde{L}, \tilde{K}', \tilde{\Phi}, \tilde{v}$, Diagram (1) commutes, that is, $\Psi \circ \exp_{\tilde{v}} = \tilde{\Phi}$ on $N \times \{p\} \times (R', \infty)$.

We need finally to show that $\tilde{L}$ has the correct decay rate. First, $|\nabla^k v| = O(e^{\gamma t})$ for all $k \geq 0$ on $N \times \{p\} \times (R', \infty)$ since L is asymptotically cylindrical with decay rate $\beta < \gamma$. Next, by Lemma 5.5, $\eta \in L^p_{l+2,\gamma}(T^*L)$ for all $l \geq 1$; by the Sobolev Embedding Theorem 3.3, $|\nabla^k \eta| = O(e^{\gamma\rho})$ (see Definition 3.2) for all $k \geq 0$ on L, so $|\nabla^k(\zeta^* \circ \Phi^*)(\eta)| = O(e^{\gamma t})$ for all $k \geq 0$ on $N \times \{p\} \times (R', \infty)$ since ζ and Φ are asymptotically cylindrical. Thus, $|\nabla^k \tilde{v}| = O(e^{\gamma t})$ on $N \times \{p\} \times (R', \infty)$ for all $k \geq 0$ on $N \times \{p\} \times (R', \infty)$, and so $\tilde{L}$ is asymptotically cylindrical special Lagrangian with decay rate γ; hence, the map $S : F^{-1}(0,0) \to M^\gamma_L$ is well defined.

Conversely, assume that $\tilde{L}$ is close to L in M^γ_L, and let $\tilde{K}, \tilde{\Phi}, \tilde{v}$ be as in Definition 2.6 for $\tilde{L}$. Then there exists a unique smooth section η of the bundle $B_{\epsilon'}(T^*L)$ with $\Theta \circ \eta : L \to \tilde{L}$ a diffeomorphism since $\tilde{L}$ and L are C^1 close; moreover, because $\omega|_{\tilde{L}} \equiv 0$ and $\operatorname{Im} \Omega_{\tilde{L}} \equiv 0$, $F(\eta) = (0,0)$.

Now here again, as in the above construction, we have $\tilde{v} = v + \zeta^* \circ \Phi^*(\eta)$, and since $|\nabla^k \tilde{v}| = O(e^{\gamma t})$ and $|\nabla^k v| = O(e^{\gamma t})$ for all $k \geq 0$, we have $|\nabla^k \eta| = O(e^{\gamma\rho})$ for all $k \geq 0$ on L. Unfortunately, this estimate is only good enough to show $\eta \in L^p_{2+l,\gamma'}(B_{\epsilon'}(T^*L))$ for any $\gamma' > \gamma$; however, if F' denotes F with the new decay rate γ', we now have $\eta \in F'^{-1}(0,0)$. If we further take $\gamma' > \gamma$ with $[\gamma, \gamma'] \cap \mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0} = \emptyset$, then all of the above arguments apply also to F', so that both F, F' are smooth, finite-dimensional and locally isomorphic to $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$ and $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma'})$ respectively. Recall that these kernels depend only on the connected components of $\mathcal{D}_{(\mathrm{d}+\mathrm{d}^*)_0}$ in which the decay rates lie, so in fact these kernels are equal. Since $F^{-1}(0,0) \subseteq F'^{-1}(0,0)$, we have that $F^{-1}(0,0) = F'^{-1}(0,0)$ near 0 and, hence, $\eta \in F^{-1}(0,0) \subset L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))$.

It is left to consider the topology. Since we have identified submanifolds of X with sections of the cotangent bundle T^*L of L, we have an induced topology on M^γ_L coming from some Banach norm on the sections η of T^*L. Recall that $F^{-1}(0,0)$ with the topology defined by the $L^p_{2+l,\gamma}$ Banach norm is locally homeomorphic to the finite-dimensional vector space $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma})$. This shows that all Banach norms on the sections η of T^*L will induce the same topology on M^γ_L, so that S is indeed a local homeomorphism. □

6. Proof of Theorem 1.2

Assume that (X, ω, g, Ω), α, $(X_0 = M \times S^1 \times \mathbb{R}, \omega_0, g_0, \Omega_0)$, $K \subset X$, R, Ψ, L, β, $L_0 = N \times \{p\} \times \mathbb{R}$, $K' \subset L$, R', v, Φ, ν_N and Θ are as in the proof of Theorem 1.1.

We begin with the following lemmas.

Lemma 6.1. *N is special Lagrangian in M if and only if $N \times \{p\} \times \mathbb{R}$ is special Lagrangian in $M \times S^1 \times \mathbb{R}$.*

Proof. Assume that N is special Lagrangian in M. First note that for any $p \in S^1$, $\{p\} \times \mathbb{R}$ is Lagrangian in $S^1 \times \mathbb{R}$ since $(\mathrm{d}\theta \wedge \mathrm{d}t)|_{\{p\}\times\mathbb{R}} \equiv 0$. Then $\omega_0|_{N\times\{p\}\times\mathbb{R}} = (\kappa_I + \mathrm{d}\theta\wedge\mathrm{d}t)|_{N\times\{p\}\times\mathbb{R}} = \kappa_I|_N + (\mathrm{d}\theta\wedge\mathrm{d}t)|_{\{p\}\times\mathbb{R}} \equiv 0$ since N is Lagrangian in M and $\{p\}\times\mathbb{R}$ is Lagrangian in $S^1 \times \mathbb{R}$. Next, $\operatorname{Im}\Omega_0 = \operatorname{Im}[(\kappa_J + i\kappa_K)\wedge(\mathrm{d}t + i\mathrm{d}\theta)] = \kappa_J \wedge \mathrm{d}\theta + \kappa_K \wedge \mathrm{d}t$. Then $(\operatorname{Im}\Omega_0)|_{N\times\{p\}\times\mathbb{R}} = (\kappa_J \wedge \mathrm{d}\theta)|_{N\times\{p\}} + (\kappa_K \wedge \mathrm{d}t)|_{N\times\mathbb{R}} \equiv 0$. Thus we have that $N \times \{p\} \times \mathbb{R}$ is special Lagrangian in $M \times S^1 \times \mathbb{R}$.

Conversely, assume that $N \times \{p\} \times \mathbb{R}$ is special Lagrangian in $M \times S^1 \times \mathbb{R}$. Then $0 \equiv (\kappa_I + \mathrm{d}\theta\wedge\mathrm{d}t)|_{N\times\{p\}\times\mathbb{R}} = \kappa_I|_N$. Further, $0 \equiv (\kappa_J\wedge\mathrm{d}\theta + \kappa_K\wedge\mathrm{d}t)|_{N\times\{p\}\times\mathbb{R}} = \kappa_K|_N \wedge \mathrm{d}t|_{\mathbb{R}}$. Since $\mathrm{d}t$ is the volume form on $\mathbb{R}$, we must have $\kappa_K|_N \equiv 0$, thus proving that N is special Lagrangian in M. □

Lemma 6.2. *If $L_0 = N\times\{p\}\times[a,b]$ is special Lagrangian in $X_0 = M\times S^1\times[a,b]$ for some $a, b \in \mathbb{R}$, then any special Lagrangian $\tilde{L_0}$ homologous to L_0 is of the form $\tilde{N} \times \{\tilde{p}\} \times [a,b]$ for some special Lagrangian submanifold $\tilde{N}$ of M, for some $\tilde{p} \in S^1$.*

Proof. Begin by normalizing the volume of $[a,b]$ to 1. By the previous result N is special Lagrangian. Let $\tilde{L_0}$ be any special Lagrangian of X which is homologous to L_0. We begin with the following calculation.

$$\begin{aligned}\operatorname{vol}(\tilde{L_0}) &= \int_{\tilde{L_0}} \mathrm{d}vol_{\tilde{L_0}} = \int_{\tilde{L_0}} \operatorname{Re}(\Omega_0)\\ &= \int_{N\times\{p\}\times[a,b]} \operatorname{Re}(\Omega_0) = \int_{[a,b]} \operatorname{vol}(N)\mathrm{d}t = \operatorname{vol}(N)\end{aligned}$$

Now, let $\tilde{L_{0,t}} = \tilde{L_0} \cap (M \times \{\tilde{p}\} \times \{t\})$ for each $t \in [a,b]$, so that $\operatorname{vol}(\tilde{L_{0,t}}) \geq \operatorname{vol}(N)$ since N is a calibrated submanifold of M where equality happens if $\tilde{L_{0,t}}$ is also calibrated. Further, we always have $\mathrm{d}vol_{\tilde{L_0}} \geq \mathrm{d}vol_{\tilde{L_{0,t}}} \wedge \mathrm{d}t$ locally since $\tilde{L_{0,t}} \subset \tilde{L_0}$ which, by Fubini's Theorem, yields that $\operatorname{vol}(\tilde{L_0}) \geq \int_{[a,b]} \operatorname{vol}(\tilde{L_{0,t}})\mathrm{d}t$ where equality occurs precisely when $\tilde{L_0}$ is a product with $[a,b]$. Finally, we see

$$\begin{aligned}\operatorname{vol}(N) = \operatorname{vol}(\tilde{L_0}) &\geq \int_{[a,b]} \operatorname{vol}(\tilde{L_{0,t}})\mathrm{d}t\\ &\geq \int_{[a,b]} \operatorname{vol}(N)\mathrm{d}t = \operatorname{vol}(N).\end{aligned}$$

□

The next step will be to parameterize the special Lagrangian submanifolds near N in M. Recall that N is compact, so the moduli space of special Lagrangian deformations of N near N in M can be identified with the harmonic 1-forms on N which, again since N is compact, can by identified with $H^1(N,\mathbb{R})$, the first de Rham cohomology group. This, however, is just a direct sum of $b^1(N)$ copies of $\mathbb{R}$, $b^1(N)$ the first betti number of N. For the moment then let $\mathcal{U}$ be an open, connected, simply-connected subset of $\mathbb{R}^d$ for some $d \leq b^1(N)$ containing 0. Now $N = N_0$ can be identified with $0 \in \mathcal{U}$, and we can talk about the special Lagrangian submanifolds N_s close to N_0 for s close to 0 in $\mathcal{U}$. We want to restrict $\mathcal{U}$ in such a way that the elements of $\mathcal{U}$ correspond to the special Lagrangian submanifolds N_s in M which are a boundary at infinity of an asymptotically cylindrical special Lagrangian deformation of L.

Recall that since N is special Lagrangian there is an identification of a tubular neighborhood T_N of N with 1-forms on N. Identifying N with the zero section of T^*N, we know that N_s is identified with the graph of some harmonic 1-form ξ_s on N; in fact, we may realize ξ_s as a translation-invariant, harmonic 1-form on $N \times \{p\} \times (R', \infty)$, so the graph of ξ_s corresponds to $N_s \times \{p\} \times (R', \infty)$. We then get a 1-form σ_s on L via the diffeomorphism $\Phi : N \times \{p\} \times (R', \infty) \to L \setminus K'$ (Φ gives σ_s on $L \setminus K'$, so smoothly extend σ_s over the compact set K'). Finally, the map Θ identifies the graph of σ_s with a 3-submanifold L_s of X asymptotic to $N_s \times \{p\} \times \mathbb{R}$.

Notice that for each s we have chosen a section σ_s, so that we need to show the independence of this choice on the problem. What we are interested in are the values of ω and $\operatorname{Im}\Omega$ when restricted to the submanifolds L_s, so we need only show independence of the choice of L_s on these values. Note that when L_s is asymptotic to $N_s \times \{p\} \times (R', \infty)$, then near infinity $\omega|_{L_s} = O(e^{\gamma t})$ and $(\operatorname{Im}\Omega)|_{L_s} = O(e^{\gamma t})$. (We will prove this fact in Proposition 6.3.) On the other hand, since N_s is a special Lagrangian, we can take $\omega = 0$, $\operatorname{Im}\Omega = 0$ in a small neighborhood of the boundary N_s in $N_s \times \{p\} \times (R', \infty)$, so that $\omega|_{L_s}$, $(\operatorname{Im}\Omega)|_{L_s}$ are compactly supported; i.e., we can take $[\omega|_{L_s}]$ in $H^2_{cs}(L_s, \mathbb{R})$, and $[(\operatorname{Im}\Omega)|_{L_s}]$ in $H^3_{cs}(L_s, \mathbb{R})$. Moreover, since N_0 and N_s are isotopic we can take $[\omega|_{L_s}] \in H^2_{cs}(L_0, \mathbb{R})$ and $[(\operatorname{Im}\Omega)|_{L_s}] \in H^3_{cs}(L_0, \mathbb{R})$ which are independent of the choice of L_s; by Lemma 6.2, we in fact have that each L_s is asymptotically cylindrical special Lagrangian, asymptotic to $N_s \times \{p\} \times \mathbb{R}$.

From topology, we have a short exact sequence of differential forms:

$$0 \to \Lambda^*_{cs}(T^*L) \to \Lambda^*(T^*L) \to \Lambda^*(T^*N) \to 0.$$

Let $i : \Lambda^*_{cs}(T^*L) \to \Lambda^*(T^*L)$ and $\varrho : \Lambda^*(T^*L) \to \Lambda^*(T^*N)$ be the corresponding chain maps. This yields a long exact sequence in de Rham cohomology given by the zig-zag lemma:

$$\begin{array}{c} 0 \to H^0(L) \to H^0(N) \to H^1_{cs}(L) \to H^1(L) \to H^1(N) \\ \downarrow \\ 0 \leftarrow H^3_{cs}(L) \leftarrow H^2(N) \leftarrow H^2(L) \leftarrow H^2_{cs}(L) \end{array} \tag{7}$$

In particular, consider the boundary map $\Upsilon : H^1(N) \to H^2_{cs}(L)$. This map is defined as follows: for a closed 1-form η on N, there is a 1-form $\tilde{\eta}$ on L by the short exact sequence, such that $\varrho(\tilde{\eta}) = \eta$. Notice that $\varrho(\mathrm{d}\tilde{\eta}) = \mathrm{d}\varrho(\tilde{\eta}) = \mathrm{d}\eta = 0$. Again, by the short exact sequence above, there then exists $\widehat{\eta}$, a compactly-supported 2-form on L, such that $i(\widehat{\eta}) = \mathrm{d}\tilde{\eta}$. Noting that $i(\mathrm{d}\widehat{\eta}) = \mathrm{d}i(\widehat{\eta}) = \mathrm{d}\mathrm{d}\tilde{\eta} = 0$, we have $\mathrm{d}\widehat{\eta} = 0$ since i is injective, that is, $\widehat{\eta}$ is a closed, compactly-supported 2-form on L. Thus, we define $\Upsilon([\eta]) = [\widehat{\eta}]$. From here, we have that $[\widehat{\eta}] = 0$ if and only if $\widehat{\eta} = \mathrm{d}\overline{\eta}$ for a compactly-supported 1-form $\overline{\eta}$ on L. By Theorem 1.1, $\overline{\eta}$ corresponds to an asymptotically cylindrical special Lagrangian

3-fold near L in X. Thus, we define $\mathcal{U}$ to be an open, connected, simply-connected subset of $\ker \Upsilon$.

Recall from the proof of Theorem 1.1 we had the map $\Theta : B_{\epsilon'}(T^*L) \to T_L \subset X$, simply the explicit identification of small sections of T^*L with the tubular neighborhood T_L which is compatible with our asymptotic identifications. Now, we can assume without loss of generality that for $\eta \in L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))$ and σ_s defined above we have a smooth section of the bundle $B_{\epsilon'}(T^*L)$ given by $\eta + \sigma_s : L \to B_{\epsilon'}(T^*L)$, so that $\Theta \circ (\eta + \sigma_s) : L \to T_L$ makes sense. Thus we can define our deformation map as

$$G : \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to \{\text{2-forms on } L\} \oplus \{\text{3-forms on } L\}$$

by

$$G(s,\eta) = ((\Theta \circ (\eta + \sigma_s))^*(-\omega), (\Theta \circ (\eta + \sigma_s))^*(\operatorname{Im}\Omega)).$$

Note that this map is equivalent to the one defined by

$$G(s,\eta) = (\pi_*(-\omega|_{\Gamma(\eta+\sigma_s)}), \pi_*((\operatorname{Im}\Omega)|_{\Gamma(\eta+\sigma_s)}))$$

where π is the projection onto L and $\Gamma(\eta + \sigma_s)$ is the graph of $\Gamma(\eta + \sigma_s)$ in X, that is, a 3-submanifold $\tilde{L}$ of X. Finally, notice that $\tilde{L}$ is special Lagrangian if and only if $G(s,\eta) = (0,0)$ so that $G^{-1}(0,0)$ parameterizes the special Lagrangian submanifolds near L in X.

We now need to show that $G^{-1}(0,0)$ is smooth, finite-dimensional and locally isomorphic to $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \subset \mathcal{U} \times L^p_{2+l,\gamma}(T^*L)$. This is accomplished in much that same way as in the fixed boundary case. We show that G is a smooth map when considered as a map between weighted Sobolev spaces, extend G to an elliptic operator $\tilde{G}$, prove that the image of $\tilde{G}$ lies in the image of the asymptotically cylindrical linear elliptic operator $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$ which will then allow us to use the Implicit Mapping Theorem for Banach Manifolds.

Proposition 6.3. *$G : \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to L^p_{1+l,\gamma}(\Lambda^2T^*L) \oplus L^p_{1+l,\gamma}(\Lambda^3T^*L)$ is a smooth map of Banach manifolds with linearization at* 0 *given by* $\mathrm{d}G_{(0,0)}(s,\eta) = (\mathrm{d}(\eta+\sigma_s), *\mathrm{d}^*(\eta+\sigma_s))$.

Proof. The functional form of G is

$$G(s,\eta)|_x = H(s,x,\eta|_x,\nabla\eta|_x),\ x \in L$$

where H is a smooth function of its arguments. Since $p > 3$ and $l \geq 1$ by Sobolev embedding theorem we have $L^p_{2+l,\gamma}(T^*L) \hookrightarrow C^1_\gamma(T^*L)$. General arguments then show that locally $G(s,\eta)$ is L^p_{1+l}.

From Proposition 5.1, we know that $G(0,\eta)$ is in $L^p_{1+l,\gamma}(\Lambda^2T^*L) \oplus L^p_{1+l,\gamma}(\Lambda^3T^*L)$. When $s \neq 0$, then $G(s,0) = (\pi_*(-\omega|_{\Gamma(\sigma_s)}), \pi_*((\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)}))$. By construction $\Gamma(\sigma_s)$ is asymptotic to $N_s \times \{p\} \times \mathbb{R}$ which is special Lagrangian in $M \times S^1 \times \mathbb{R}$ as N_s is a special Lagrangian 2-fold. Then ω and $\operatorname{Im}\Omega$ on $X \setminus K$ can be written as $\omega = \kappa_I + \mathrm{d}\theta \wedge \mathrm{d}t + O(e^{\alpha t})$ and $\operatorname{Im}\Omega = \operatorname{Im}[(\kappa_J + i\kappa_K) \wedge (\mathrm{d}\theta + i\mathrm{d}t)] + O(e^{\alpha t})$ where α is the rate for X converging to $M \times S^1 \times \mathbb{R}$. $\Gamma(\sigma_s)$ is the graph of σ_s which is equal to $N_s \times \{p\} \times (R+1,\infty)$

in $M \times S^1 \times (R+1, \infty)$. Since N_s is special Lagrangian, for $t > R+1$, $\omega|_{\Gamma(\sigma_s)} = (\kappa_I + \mathrm{d}\theta \wedge \mathrm{d}t + O(e^{\alpha t}))|_{N_s \times \{p\} \times (R+1,\infty)} = 0 + O(e^{\alpha t})|_{N_s \times \{p\} \times (R+1,\infty)}$ and $(\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)} = (\operatorname{Im}[(\kappa_J + i\kappa_K) \wedge (\mathrm{d}\theta + i\mathrm{d}t)] + O(e^{\alpha t}))|_{N_s \times \{p\} \times (R+1,\infty)} = 0 + O(e^{\alpha t})|_{N_s \times \{p\} \times (R+1,\infty)}$.

Therefore for $\Gamma(\sigma_s)$ the error terms $\omega|_{\Gamma(\sigma_s)}$ and $(\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)}$ come from the degree of the asymptotic decay. In particular, as $\alpha < \gamma$ we can assume $\omega|_{\Gamma(\sigma_s)} \equiv O(e^{\gamma t})$ and $(\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)} \equiv O(e^{\gamma t})$. We can easily arrange to choose σ_s such that $\pi_*(\omega|_{\Gamma(\sigma_s)})$ lies in $L^p_{1+l,\gamma}(\Lambda^2 T^*L)$ and $\pi_*((\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)}) \in L^p_{1+l,\gamma}(\Lambda^3 T^*L)$ and that we have inequalities $\|\pi_*(\omega|_{\Gamma(\sigma_s)})\|_{L^p_{1+l,\gamma}} \leq c_1|s|$ and $\|\pi_*((\operatorname{Im}\Omega)|_{\Gamma(\sigma_s)})\|_{L^p_{1+l,\gamma}} \leq c_2|s|$ for some constants c_1 and c_2. This implies that $G(s,\eta) \in L^p_{1+l,\gamma}(\Lambda^2 T^*L) \oplus L^p_{1+l,\gamma}(\Lambda^3 T^*L)$.

The linearization of G is $\mathrm{d} + *\mathrm{d}^*$ by the same calculation as before since we are still only working locally. □

Proposition 6.4. *The image of G lies in exact 2-forms and exact 3-forms; specifically,*

$$\begin{aligned} G(\mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L))) &\subset \mathrm{d}(L^p_{1+l,\gamma}(T^*L)) \oplus \mathrm{d}(L^p_{1+l,\gamma}(\Lambda^2 T^*L)) \\ &\subset L^p_{l,\gamma}(\Lambda^2 T^*L) \oplus L^p_{l,\gamma}(\Lambda^3 T^*L). \end{aligned}$$

Proof. We need only modify slightly the proof from Lemma 5.2. Recall that ω and $\operatorname{Im}\Omega$ are closed forms, so they determine the de Rham cohomology classes $[\omega]$ and $[\operatorname{Im}\Omega]$. In particular, $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$ since L is special Lagrangian, so $[\omega|_L] = 0$ and $[\operatorname{Im}\Omega|_L] = 0$; moreover, since T_L, the tubular neighborhood of L from above, retracts onto L, $[\omega|_{T_L}] = [\omega|_L]$ and $[\operatorname{Im}\Omega|_{T_L}] = [\operatorname{Im}\Omega|_L]$ in $H^2(T_L)$ and $H^3(T_L)$. Thus, there exists $\tau_1 \in C^\infty(T^*T_L)$ such that $\omega|_{T_L} = \mathrm{d}\tau_1$ and $\tau_2 \in C^\infty(\Lambda^2 T^*T_L)$ such that $\operatorname{Im}\Omega|_{T_L} = \mathrm{d}\tau_2$. Now, since $\omega|_L \equiv 0$, we can assume that $\tau_1|_L \equiv 0$; second, because ω and all its derivatives decay at a rate $O(e^{\beta t})$ to the translation-invariant 2-form ω_0 on $M \times S^1 \times \mathbb{R}$, we can assume that τ_1 and all its derivatives decay at a rate $O(e^{\beta t})$ to a translation invariant 1-form on $T_N \times \{p\} \times \mathbb{R}$. We can make similar assumptions regarding τ_2 based on the properties of $\operatorname{Im}\Omega$. In particular, since $\beta < \gamma$, we have that these forms and all of their derivatives decay at $O(e^{\gamma t})$. Thus, we have $\tau_1 \in L^p_{1+l,\gamma}(T^*L)$ and $\tau_2 \in L^p_{1+l,\gamma}(\Lambda^2 T^*L)$.

By Proposition 6.3, the map G maps $\mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to L^p_{1+l,\gamma}(\Lambda^2 T^*L) \oplus L^p_{1+l,\gamma}(\Lambda^3 T^*L)$. Thus for $(s,\eta) \in \mathcal{U} \times L^p_{2+l,\gamma}(T^*L)$ we have:

$$\begin{aligned} G(s,\eta) &= ((\Theta \circ (\eta + \sigma_s))^*(-\omega), (\Theta \circ (\eta + \sigma_s))^*(\operatorname{Im}\Omega)) \\ &= ((\Theta \circ (\eta + \sigma_s))^*(-\mathrm{d}\tau_1), (\Theta \circ (\eta + \sigma_s))^*(\mathrm{d}\tau_2)) \\ &= (\mathrm{d}(\Theta \circ (\eta + \sigma_s))^*(-\tau_1), \mathrm{d}(\Theta \circ (\eta + \sigma_s))^*(\tau_2)). \end{aligned}$$

□

We now extend the map G over the space of 3-forms on L to get an elliptic operator $\tilde{G}$ as we did in the proof of Theorem 1.1. Using the Hodge star isomorphism, we can view $G : \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L)) \to L^p_{1+l,\gamma}(\Lambda^2 T^*L \oplus \Lambda^0 T^*L)$, then define $\tilde{G} : \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L) \to L^p_{1+l,\gamma}(\Lambda^2 T^*L \oplus \Lambda^0 T^*L)$ by $\tilde{G}(s,\eta_1,\eta_3) =$

$G(s,\eta_1)+\mathrm{d}^*\eta_3$. In this way, $\mathrm{d}G(0,0)(s,\eta_1) = (\mathrm{d}(\eta_1+\sigma_s), \mathrm{d}^*(\eta_1+\sigma_s))$ by Proposition 6.3, so $\mathrm{d}\tilde{G}(0,0)(s,\eta_1,\eta_3) = (\mathrm{d}(\eta_1+\sigma_s)+\mathrm{d}^*\eta_3, \mathrm{d}^*(\eta_1+\sigma_s))$

Proposition 6.5. *Let $\mathcal{C}$ denote the image of the operator $(\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}$. Then*

$$\tilde{G} : \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L) \to \mathcal{C}.$$

Proof. Let $(s,\eta_1,\eta_3) \in \mathcal{U} \times L^p_{2+l,\gamma}(B_{\epsilon'}(T^*L) \oplus \Lambda^3 T^*L)$. From Theorem 4.1, we have $\operatorname{coker}((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \cong (\ker((\mathrm{d}+\mathrm{d}*)^q_{2+m,-\gamma}))^*$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m \geq 1$, so $\tilde{G}(s,\eta_1,\eta_3) = G(s,\eta_1)+\mathrm{d}^*\eta_3 \in \mathcal{C}$ if and only if

$$\langle G(s,\eta_1)+\mathrm{d}^*\eta_3, (\chi_0,\chi_2)\rangle_{L^2} \equiv 0 \text{ for all } (\chi_0,\chi_2) \in \ker((\mathrm{d}+\mathrm{d}*)^q_{2+m,-\gamma}).$$

By the previous proposition $G(s,\eta_1) = (\mathrm{d}\tau_1, \mathrm{d}\tau_2)$ for some $\tau_1 \in L^p_{1+l,\gamma}(T^*L)$ and $\tau_2 \in L^p_{1+l,\gamma}(\Lambda^2 T^*L)$; then Theorem 4.1 shows $*\chi_0$ and χ_2 are harmonic 3- and 2-forms respectively which yields the following:

$$\begin{aligned}\langle G(s,\eta_1)+\mathrm{d}^*\eta_3, (*\chi_0,\chi_2)\rangle_{L^2} &= \langle \mathrm{d}\tau_1+\mathrm{d}^*\eta_3, \chi_2\rangle_{L^2} + \langle \mathrm{d}\tau_2, *\chi_0\rangle_{L^2} \\ &= \langle \tau_1, \mathrm{d}^*\chi_2\rangle_{L^2} + \langle \eta_3, \mathrm{d}\chi_2\rangle_{L^2} + \langle \tau_2, \mathrm{d}^*(*\chi_0)\rangle_{L^2} = 0.\end{aligned}$$

□

We are now able to invoke to the Implicit Mapping Theorem for Banach Spaces to conclude that $\tilde{G}^{-1}(0,0)$ is smooth, finite-dimensional and locally isomorphic to $\ker((\mathrm{d}+\mathrm{d}^*)^p_{2+l,\gamma}) \subset \mathcal{U} \times L^p_{2+l,\gamma}(T^*L)$ as in the proof of Theorem 1.1. By a completely analogous proof to the that of Lemma 5.4, we have $\tilde{G}^{-1}(0,0) = G^{-1}(0,0) \times \{0\}$.

The final step is to calculate the dimension of the moduli space which we do by using Fredholm index arguments using the elliptic operator $\tilde{G}$. The rest of the proof of Theorem 1.2 then follows exactly as in the proof of Theorem 1.1, so we will not repeat it here.

Proposition 6.6. *Let $\mathcal{U} \subset H^1(N,\mathbb{R})$ be a subspace of special Lagrangian deformations of the boundary N. Also, let $\mathrm{d}\tilde{G}_{(0,0)}(s,\eta)$ represent the linearization of the deformation map G at 0 with moving boundary and $\mathrm{d}\tilde{G}^f_{(0,0)}(\eta)$ represent the linearization of the deformation map $\tilde{G}^f$ at 0 with fixed boundary. Then*

$$\operatorname{Ind}(\mathrm{d}\tilde{G}_{(0,0)}) = \dim \mathcal{U} + \operatorname{Ind}(\mathrm{d}\tilde{G}^f_{(0,0)}),$$

where Ind *denotes the index of a map.*

Proof. At $s=0$

$$\tilde{G}(0,\eta) = \tilde{G}^f(\eta) = (\pi_*(-\omega|_{\Gamma_\eta}), *\pi_*((\operatorname{Im}\Omega)|_{\Gamma_\eta}))$$

with linearization at $(0,0)$

$$\mathrm{d}\tilde{G}_{(0,0)}(0,\eta) = \mathrm{d}\tilde{G}^f_{(0,0)}(\eta).$$

Since $\mathrm{d}\tilde{G}_{(0,0)}$ is linear, we have

$$\mathrm{d}\tilde{G}_{(0,0)}(s,\eta) = \mathrm{d}\tilde{G}_{(0,0)}(s,0) + \mathrm{d}\tilde{G}^f_{(0,0)}(0,\eta)$$

where $\mathrm{d}\tilde{G}_{(0,0)}(s,0)$ is finite-dimensional, $s \in T_0\mathcal{U} \cong \mathbb{R}^d$, $d = \dim\ker\Upsilon$. Then

$$\begin{aligned}
&\mathrm{Ind}(\mathrm{d}\tilde{G}_{(0,0)} : \mathcal{U} \times L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3T^*L) \to L^p_{1+l,\gamma}(\Lambda^2T^*L \oplus \Lambda^0T^*L))\\
&= \mathrm{Ind}(\mathrm{d}\tilde{G}^f_{(0,0)} : \mathcal{U} \times L^p_{2+l,\gamma}(T^*L \oplus \Lambda^3T^*L) \to L^p_{1+l,\gamma}(\Lambda^2T^*L \oplus \Lambda^0T^*L))\\
&= \dim\mathcal{U} + \mathrm{Ind}(\mathrm{d}\tilde{G}^f_{(0,0)}).
\end{aligned}$$

□

Proposition 6.7. *The dimension of $\mathcal{M}^\gamma_L$, the moduli space of special Lagrangian deformations of an asymptotically cylindrical special Lagrangian submanifold L asymptotic to $N_s \times \{p\} \times (R,\infty)$, $s \in \mathcal{U}$, with decay rate γ is*

$$\dim\mathcal{M}^\gamma_L = \dim V + b^2(L) - b^1(L) + b^0(L) - b^2(N) + b^1(N).$$

If we further allow $p \in S^1$ to vary, we get a total dimension equal to $\dim\mathcal{M}^\gamma_L + b^0(N)$.

Proof. Let $b^k(N)$, $b^k_{cs}(L)$ and $b^k(L)$ be the corresponding Betti numbers. We know from the previous proposition that

$$\mathrm{Ind}(\mathrm{d}\tilde{G}_{(0,0)}) = \dim\mathcal{U} + \mathrm{Ind}(\mathrm{d}\tilde{G}^f_{(0,0)}).$$

In particular,

$$\dim(\ker\mathrm{d}\tilde{G}_{(0,0)}) = \dim(\ker\Upsilon) + \dim(\ker\mathrm{d}\tilde{G}^f_{(0,0)}),$$

that is, the dimension of the moduli space for the moving boundary case is the sum of the dimension of the moduli space for the fixed boundary case and the dimension of the kernel of Υ. Taking alternating sums of dimensions in the long exact sequence (7) shows that the dimension of the kernel of Υ is

$$\begin{aligned}
\dim(\ker\Upsilon) &= b^1(N) - b^2_{cs}(L) + b^2(L) - b^2(N) + b^3_{cs}(L)\\
&= b^2(L) - b^1(L) + b^0(L) - b^2(N) + b^1(N).
\end{aligned}$$

Finally, we show that the moduli space $\mathfrak{M}$ of asymptotically cylindrical special Lagrangian submanifolds $\tilde{L}$, asymptotic to $N_s \times \{\tilde{p}\} \times \mathbb{R}$, which are isotopic to L is equivalent to the product of S^1 with the moduli space $\mathcal{M}^\gamma_L$ of asymptotically cylindrical special Lagrangian submanifolds $\overline{L}$, asymptotic to $N_s \times \{p\} \times \mathbb{R}$, which are isotopic to L, i.e., the total moduli space of asymptotically cylindrical special Lagrangian submanifolds of X isotopic to L is a smooth manifold of dimension $\dim(\mathfrak{M}) = \dim(\mathcal{M}^\gamma_L) + b^0(N)$.

Note that for any $\tilde{p} \in S^1$, the moduli space of asymptotically cylindrical special Lagrangians, asymptotic to $N_s \times \{\tilde{p}\} \times \mathbb{R}$, isotopic to L, is equivalent to the moduli space of asymptotically cylindrical special Lagrangians, asymptotic to $N_s \times \{p\} \times \mathbb{R}$. To see this, let $\tilde{L}$ be asymptotic to $N_s \times \{\tilde{p}\} \times \mathbb{R}$; then there exists an isotopy from $N_s \times \{\tilde{p}\} \times \mathbb{R}$ to $N_s \times \{p\} \times \mathbb{R}$. By the data of an asymptotically cylindrical special Lagrangian, this gives an isotopy on $\tilde{L}$ via the pullback map. Since $\tilde{L}$ is isotopic to L, we have an asymptotically cylindrical special Lagrangian submanifold, asymptotic to $N_s \times \{p\} \times \mathbb{R}$ and isotopic to L. Conversely, if $\tilde{L}$ is asymptotic to $N_s \times \{p\} \times \mathbb{R}$, there is an isotopy from $N_s \times \{p\} \times \mathbb{R}$ to $N_s \times \{\tilde{p}\} \times \mathbb{R}$, giving an isotopy on $\tilde{L}$, and hence, an asymptotically cylindrical special Lagrangian submanifold, asymptotic to $N_s \times \{\tilde{p}\} \times \mathbb{R}$ and isotopic to L. Finally, if N_1,...,N_k are the connected components of N, $k = b^0(N)$, then the dimension of the moduli space of asymptotically cylindrical special Lagrangians limiting to the union of $N_i \times p_i$ for $p_1, ..., p_k$ in S^1 will have an additional $b^0(N)$ as we allow p_i to vary. □

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