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Nash homotopy spheres are standard

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Abstract. We prove that the infinite family of homotopy 4-spheres constructed by Daniel Nash are all diffeomorphic to S^4.

0. Introduction

In [D], D. Nash constructed an infinite family of smooth homotopy 4-spheres $\Sigma_{p,q,r,s}$ indexed by $(p,q,r,s) \in \mathbb{Z}^4$, and conjectured that they are possibly not diffeomorphic to S^4. Here we prove that they are all diffeomorphic to S^4. In spirit, Nash's construction is an easier version of the construction of the Akhmedov-Park in [AP], namely one starts with a standard manifold $X^4 = X_1 \cup_\partial X_2$ which is a union of two basic pieces along their boundaries, then does the "log transform" operations to some imbedded tori in both sides with the hope of getting an exotic copy of a known manifold M^4. In Nash's case X is the double of $T_0^2 \times T_0^2$ and M is S^4, in Akhmedov-Park case X is the "Cacime surface" of [CCM] (see [A1] for decomposition of X) and M is $S^2 \times S^2$.

1. Log transform operation

First we need to recall the *log transform* operation. Let X be a smooth 4-manifold which contains a torus T^2 with the trivial normal bundle $\nu(T^2) \approx T^2 \times B^2$. Let φ_p $(p \geq 0)$ be the self-diffeomorphism of T^3 induced by the automorphism

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

of $H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z})$ with the obvious basis (a, b, c). The operation of removing $\nu(T)$ from X and then regluing $T^2 \times B^2$ via $\varphi_p : S^1 \times T^2 \to \partial\nu(T)$ is called the p log-transform of X along T^2. In short we will refer this as $(a \times b, b, p)$ log transform. Figure 1 describes this as a handlebody operation (cf. [AY] and [GS]).

2. Handlebody descriptions of $T^2 \times T^2$ and $T_0^2 \times T_0^2$

We follow the recipe of [A1] for drawing surface bundles over surfaces. Here we have the simpler case where the base and the fibers are tori. Figure 2 describes the handlebody of T^2 and its thickening $T^2 \times [0,1]$. Figure 3 is a handlebody of $T^2 \times S^1$. Note that

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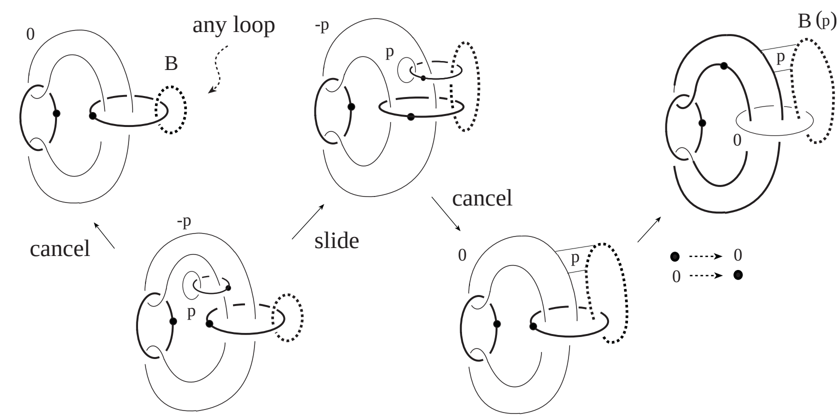

FIGURE 1. p log-transform operation

without the handle v (the green curve) it gives $T_0^2 \times S^1$. So Figure 4 is the handlebody of $T^2 \times T^2$. Note that without 2-handles u and v (the green and blue curves) Figure 4 it is just $T_0^2 \times T_0^2$. By gradually converting the 1-handles of Figure 4 from "pair of balls" notation to the "circle-with-dot" notation of [A2] we get the handlebody pictures of $T^2 \times T^2$ and $T_0^2 \times T_0^2$ in Figure 5 (the first and the second pictures). Then by an isotopy (indicated by the arrow) we obtained the first picture of Figure 6 which is $T_0^2 \times T_0^2$.

3. Nash spheres are standard

Let $X_{p,q}$ be the manifold obtained from $T_0^2 \times T_0^2$ by the log-transformations $(a \times c, a, p)$ and $(b \times c, b, q)$, where a, b, c, d are the circle factors of $T_0^2 \times T_0^2$ indicated of Figure 6. Than Nash homotopy spheres are defined to be:

$$\Sigma_{p,q,r,s} = X_{p,q} \cup_{\phi} -X_{r,s}$$

where ϕ is the involution on $\partial(T_0^2 \times T_0^2)$ flipping $T_0^2 \times S^1$ and $S^1 \times T_0^2$. Notice that if $X^{r,s}$ is the manifold obtained from $T_0^2 \times T_0^2$ by the log transformations $(c \times a, c, r)$ and $(d \times a, d, s)$ then we can identify:

$$\Sigma_{p,q,r,s} = -X_{p,q} \cup_{id} X^{r,s}$$

Theorem 1. $\Sigma_{p,q,r,s} = S^4$

Proof. By using the description of the log-transform in Figure 1, we see that the second picture of Figure 6 is $X_{p,q}$. Now we will turn $X_{p,q}$ upside down and glue it to $X^{r,s}$ along its boundary. For this we take the image of the dual 2-handle curves of $X_{p.q}$ (indicated by the dotted circles in Figure 6) by the diffeomorphism $\partial(X_{p,q}) \xrightarrow{\approx} \partial(X^{r,s})$, and then attach 2-handles to $X^{r,s}$ along the image of these curves. By reversing the log-transform process of Figure 6 we obtain the first picture of Figure 7, which is just $T_0^2 \times T_0^2$, with the dual handle curves indicated! By an isotopy we obtain the second picture of Figure 7, also describing $T^2_0 \times T^2_0$. Now again by using the recipe of Figure 1 we perform the $(c \times a, c, r)$ and $(d \times a, d, s)$ log-transforms and obtain the first picture of Figure 8, which is $X^{r,s}$, with the dual 2-handle curves of $X_{p,q}$ clearly visible in the picture. Now by attaching 2-handles to top of $X^{r,s}$ along these curves we obtain $\Sigma_{p,q,r,s}$, which is also described by the first picture of Figure 8. Here we should mention our convention: when a framing is not indicated in figures it means the zero framing. Now by the obvious handle slides and cancellations in Figure 8 $\rightsquigarrow$ Figure 9 we obtain $\#^4(S^2 \times B^2)$, and the four 3-handles cancel these to give S^4 $\square$.

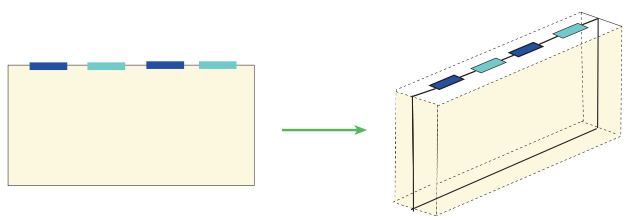

FIGURE 2. $T^2 \to T^2 \times [0,1]$

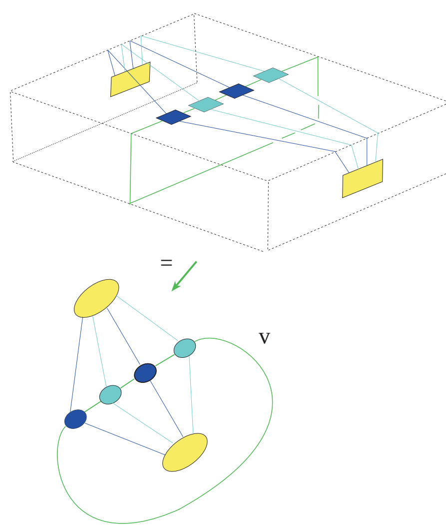

FIGURE 3. $T^2 \times S^1$

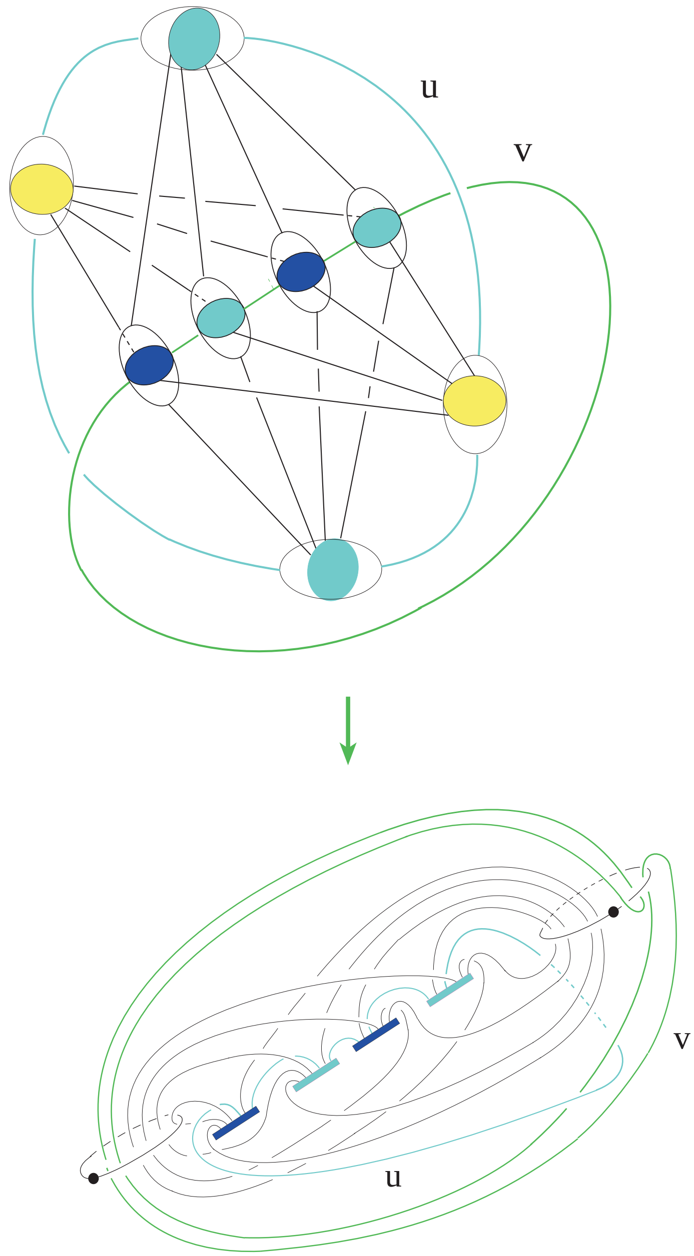

FIGURE 4. $T^2 \times T^2$

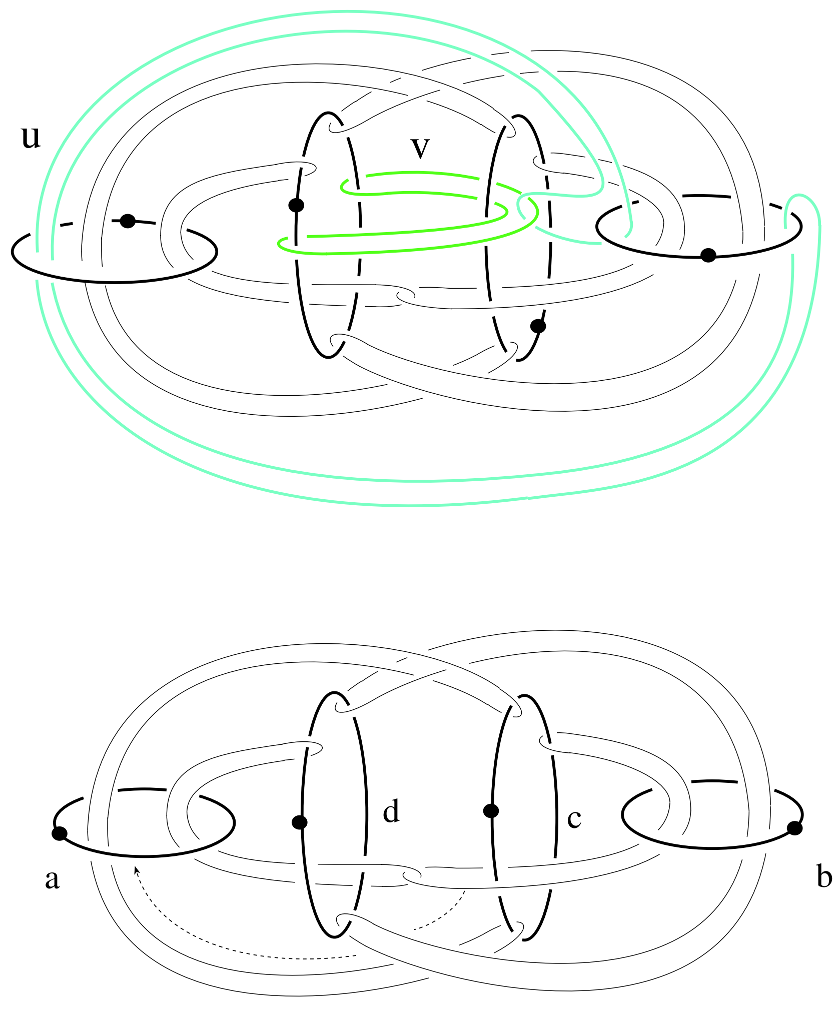

FIGURE 5. $T^2 \times T^2$ and $T_0^2 \times T_0^2$

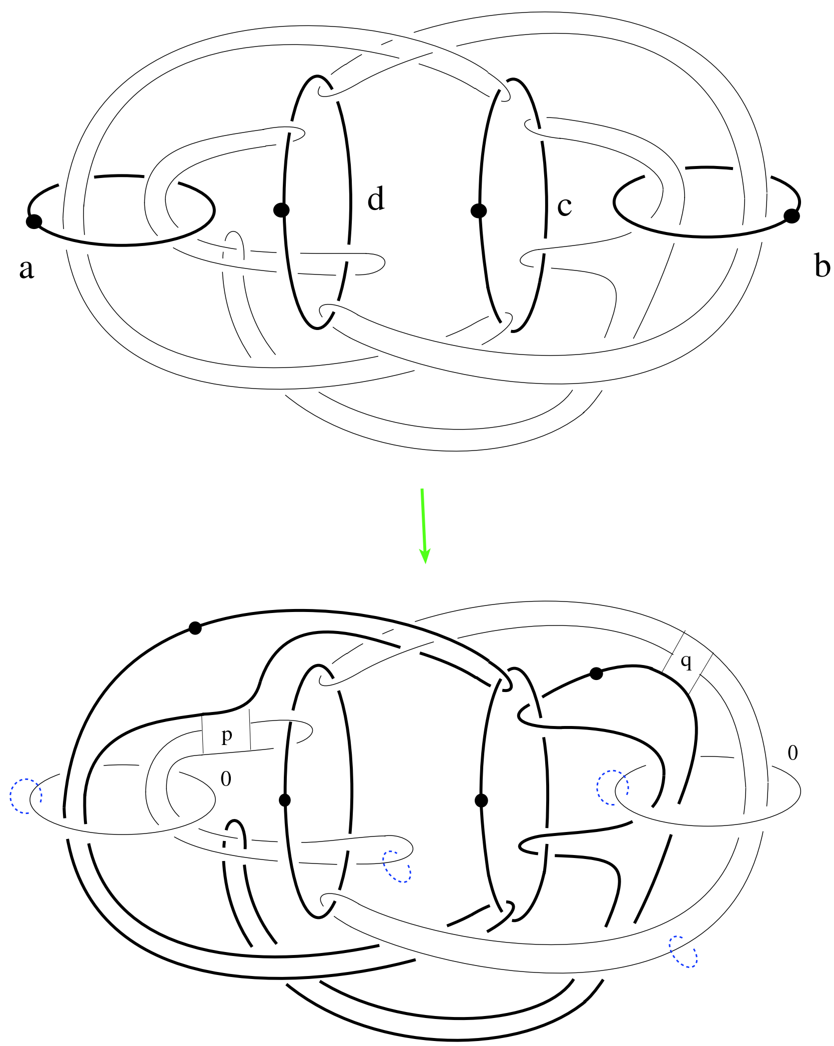

FIGURE 6. log-transformation $T_0^2 \times T_0^2 \to X_{p,q}$

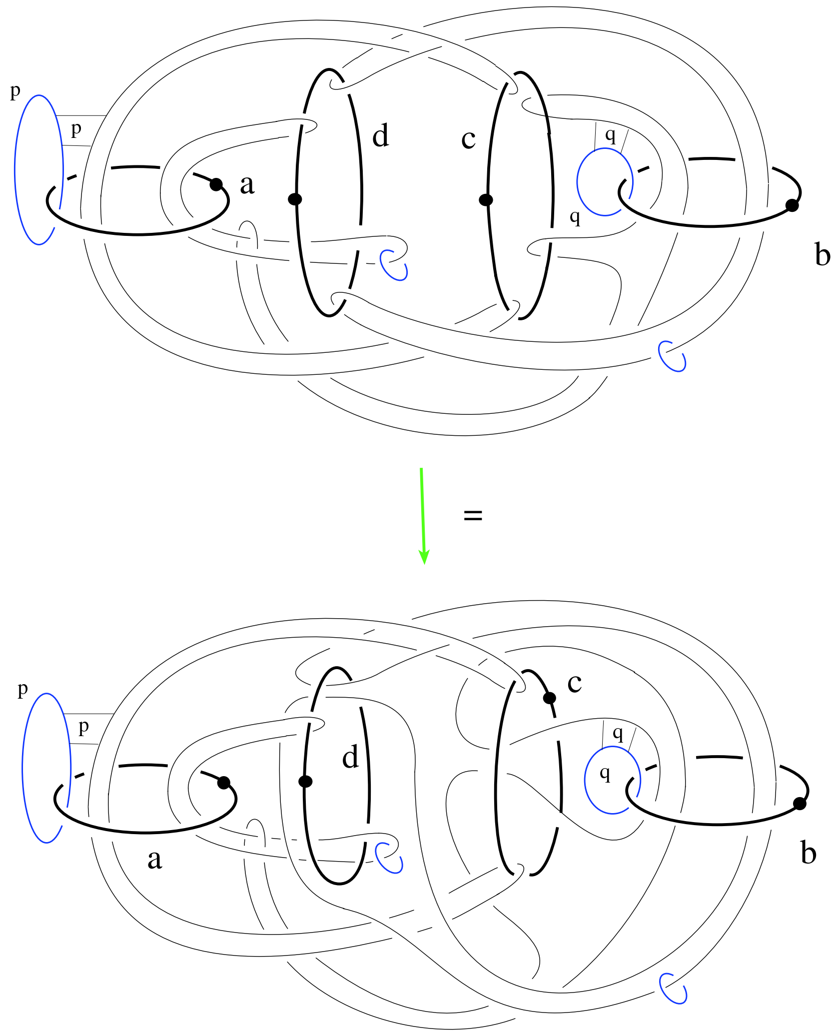

Figure 7. log-transformation undone

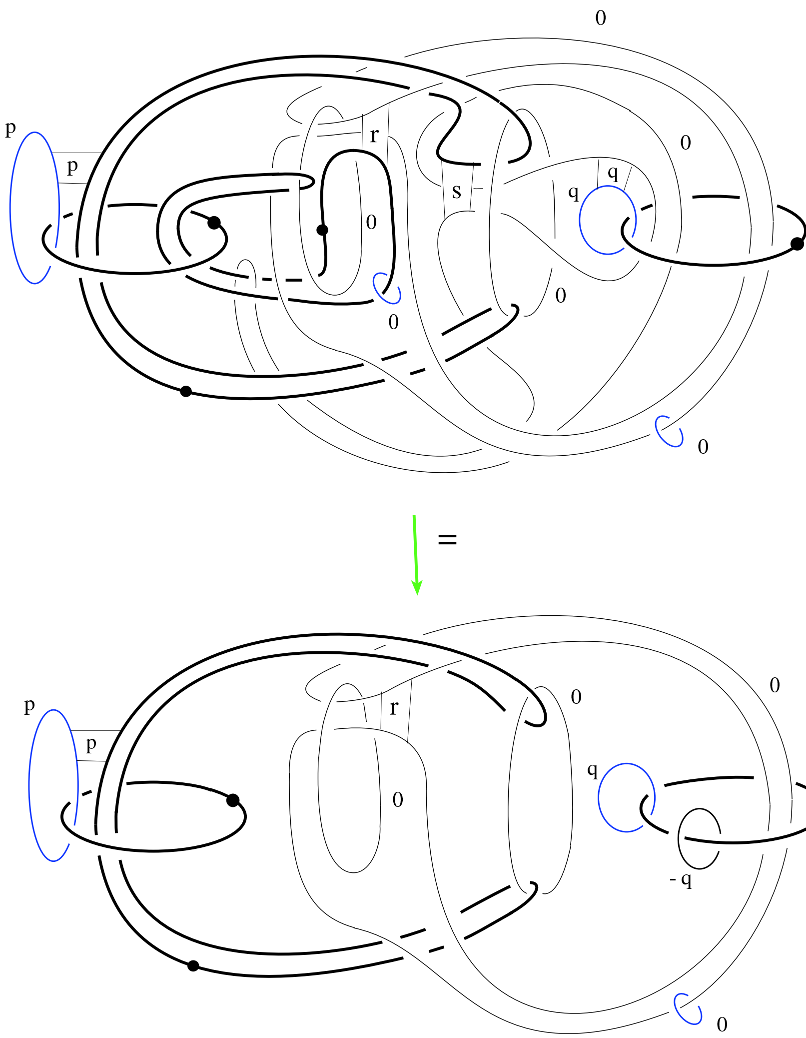

FIGURE 8. $\Sigma_{p,q,r,s}$

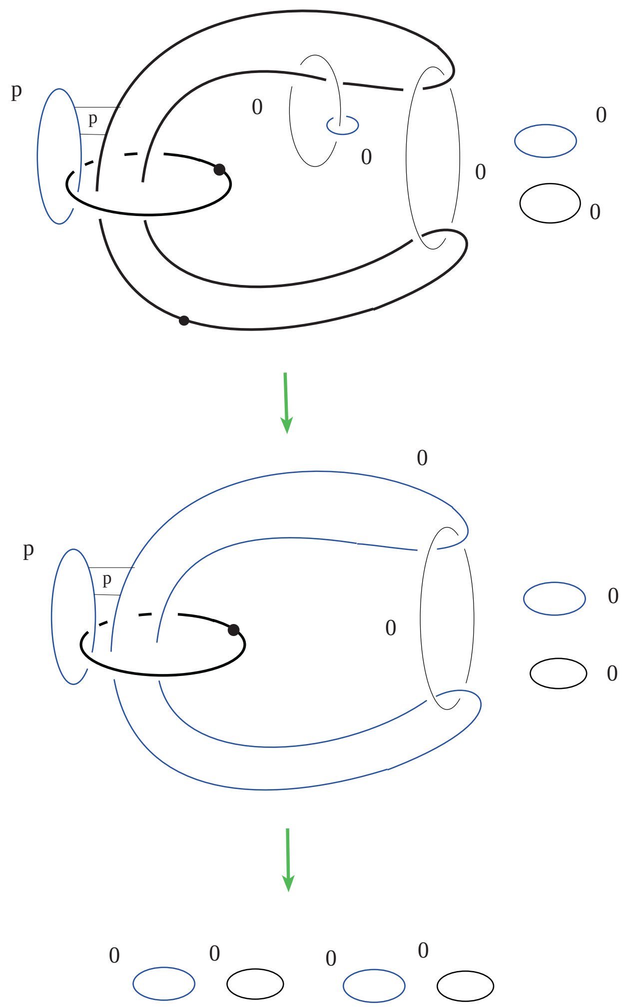

Figure 9

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