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Degree one cohomology with twisted coefficients of the mapping class group

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Abstract. Let Γ be the mapping class group of an oriented surface Σ of genus g with r boundary components. We prove that the first cohomology group of the mapping class group $H^1(\Gamma, \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})})^*)$ is non-trivial, where the coefficient module is the dual of the space of algebraic functions on the $\mathrm{SL}_2(\mathbb{C})$ moduli space over Σ.

1. Introduction

Let $\Gamma = \Gamma_{g,r}$ denote the mapping class group of a compact surface $\Sigma = \Sigma_{g,r}$ with genus g and r boundary components. There is an action of Γ on the moduli space $\mathcal{M}_G$ of flat G-connections over Σ. The vector space $\mathcal{O}(\mathcal{M}_G) \subseteq \mathrm{Fun}(\mathcal{M}_G, \mathbb{C})$ of regular functions on the moduli space is naturally a Γ-module.

Presently, we consider the special case of $G = \mathrm{SL}_2(\mathbb{C})$, and we simply write $\mathcal{M}$ for $\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$. In this case, there is an isomorphism of Γ-modules

$$\nu\colon \mathcal{B}(\Sigma) \to \mathcal{O}(\mathcal{M}), \tag{1}$$

where the source denotes the complex vector space freely spanned by the set of multicurves on Σ, i.e., isotopy classes of a closed 1-submanifold. We usually think of this as a finite collection of pairwise non-intersecting, non-trivial, unoriented simple loops on Σ. Note that we explicitly allow a multicurve to contain isotopic connected components, and that it may also contain components parallel to a boundary component of Σ. Letting $B = B(\Sigma)$ denote the set of multicurves on Σ, $\mathcal{B} = \mathcal{B}(\Sigma)$ is simply the complex vector space spanned by B. There is a natural algebra structure on this space; for details on this see [1] and [2]. The isomorphism ν is given on a single simple loop γ by $\nu(\gamma) = -f_{\vec{\gamma}}$, where $\vec{\gamma}$ is any of the two oriented versions of γ, and $f_{\vec{\gamma}}$ is the function which to a gauge equivalence class $[A]$ of flat connections associates the trace of the holonomy of A along $\vec{\gamma}$.

We may think of $\mathcal{B}$ as the set of maps $B \to \mathbb{C}$ which vanish except for a finite number of multicurves. This is naturally embedded in the larger module of *all* maps $\hat{\mathcal{B}} = \mathrm{Map}(B, \mathbb{C})$; this is clearly the same as the algebraic dual $\mathcal{O}(\mathcal{M})^*$ of $\mathcal{O}(\mathcal{M})$. The action of Γ splits B into orbits. Let S denote a set of representatives of these orbits, and for $D \in S$,

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let $\hat{M}_D$ (respectively M_D) denote the space of all maps from the orbit through D to $\mathbb{C}$ (respectively, the maps $\Gamma D \to \mathbb{C}$ which vanish for all but a finite number of multicurves in the orbit). With this notation, we obtain splittings of $\mathcal{B}$ and $\hat{\mathcal{B}}$ as Γ-modules

$$\mathcal{O}(\mathcal{M}) \cong \mathcal{B} \cong \bigoplus_{D \in S} M_D$$
$$\mathcal{O}(\mathcal{M})^* \cong \hat{\mathcal{B}} \cong \prod_{D \in S} \hat{M}_D \tag{2}$$

which induce decompositions in cohomology

$$H^*(\Gamma, \mathcal{B}) \cong \bigoplus_{D \in S} H^*(\Gamma, M_D) \tag{3}$$
$$H^*(\Gamma, \hat{\mathcal{B}}) \cong \prod_{D \in S} H^*(\Gamma, \hat{M}_D). \tag{4}$$

The isomorphism (3) depends on the well-known fact that Γ is finitely generated (see e.g. [3]). A cocycle $u\colon \Gamma \to \mathcal{O}(\mathcal{M})^* = \hat{\mathcal{B}} = \mathrm{Map}(B, \mathbb{C})$ may also be thought as a map $u\colon \Gamma \times B \to \mathbb{C}$ by simply putting $u(\gamma)(E) = u(\gamma, E)$.

Theorem 1.1. *A cocycle $u\colon \Gamma \to \mathcal{O}(\mathcal{M})^* = \hat{\mathcal{B}} = \mathrm{Map}(B, \mathbb{C})$ is a coboundary if and only if for each $D \in S$, the restriction of u to $\Gamma_D \times \{D\}$ is identically 0, where Γ_D denotes the stabilizer of the multicurve D in Γ.*

We will use this theorem to arrive at the main result:

Theorem 1.2. *For every $g, r \geq 0$, the cohomology group $H^1(\Gamma_{g,r}, \mathcal{O}(\mathcal{M})^*)$ is a direct product of summands $H^1(\Gamma, \hat{M}_D)$, each of which is finite-dimensional. Here D runs over a set of representatives of multicurves on Σ.*

In particular, we obtain by explicit examples

Corollary 1.3. *Whenever $\Gamma_{g,r}$ is non-trivial (i.e., when (g, r) is not $(0,0)$ or $(0,1)$), $H^1(\Gamma_{g,r}, \mathcal{O}(\mathcal{M})^*)$ is non-trivial.*

The motivation to study the cohomology of the mapping class group with these coefficients came from [4], particularly Proposition 6, where integrability of certain cocycles turns out to be an obstruction to finding a Γ-invariant equivalence between two equivalent star products on the moduli space. The motivation for studying that problem comes from the expectation that the star products discussed in [4] are equivalent to the star product which is constructed in [1] and which is the same as the ones induced on the $\mathrm{SL}_2(\mathbb{C})$-moduli space from the constructions given in [5] and [6]. For the first results in this direction please see [7].

The main result of this paper, Theorem 1.2, should be compared to the following vanishing results also due to the authors of this paper. The first of these concerns the same moduli space as considered in this paper, with coefficients in the space of algebraic functions.

Theorem 1.4. *For $g \geq 2$ and $r \geq 0$, the cohomology group $H^1(\Gamma, \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}))$ vanishes.*

This result is proved in [8].

In another paper [9], we considered the abelian moduli space of a surface with at most one boundary component, i.e., the space

$$\mathcal{M}_{\mathrm{U}(1)} = \mathrm{Hom}(\pi_1\Sigma, \mathrm{U}(1)) \cong \mathrm{U}(1)^{2g}$$

of flat U(1)-connections on Σ. The mapping class group acts by measure-preserving diffeomorphisms of $\mathcal{M}_{\mathrm{U}(1)}$, so both $C^\infty(\mathcal{M}_{\mathrm{U}(1)})$ and $L^2(\mathcal{M}_{\mathrm{U}(1)})$ are Γ-modules. When the genus is at least 3, we obtained these results:

Theorem 1.5. *The cohomology group $H^1(\Gamma, L^2(\mathcal{M}_{\mathrm{U}(1)}))$ vanishes.*

Theorem 1.6. *The cohomology group $H^1(\Gamma, C^\infty(\mathcal{M}_{\mathrm{U}(1)}))$ vanishes.*

Also in [9] we derived a general result which applies to any unitary representation of the mapping class group.

Theorem 1.7. *Assume $g \geq 3$. Let $\Gamma \to \mathrm{U}(V)$ be a unitary representation of Γ on a real or complex Hilbert space V. For a simple closed curve γ on Σ, let $V_\gamma = V^{\tau_\gamma}$ denote the closed subspace fixed under the twist τ_γ and let $p_\gamma \colon V \to V_\gamma$ be the orthogonal projection. Then, for any cocycle $u\colon \Gamma \to V$, we have $p_\gamma u(\tau_\gamma) = 0$.*

This result should be understood in light of the observation that if u is the coboundary of some element $v \in V$, then $u(\tau_\gamma) = (1 - \tau_\gamma)v$ is killed by p_γ. Hence this necessary condition for the vanishing of $H^1(\Gamma, V)$ is always satisfied whenever $g \geq 3$.

This paper is organized as follows. In Section 2 we develop some of the basic properties of group cohomology which are needed in the calculations, ending with a proof of Theorem 1.1. In Section 3, we develop an algorithm to compute $H^1(\Gamma, \hat{M}_D)$ for any multicurve, which enables us to prove Theorem 1.2. This is used in Section 4 to give a generic example of a multicurve for which the cohomology is non-zero, hence proving Corollary 1.3.

2. Group cohomological background

If Γ is a group and A is a Γ-module, a *cocycle* is a map $u\colon \Gamma \to A$ satisfying the *cocycle condition*

$$u(gh) = u(g) + gu(h) \tag{5}$$

for all $g, h \in \Gamma$. A *coboundary* is a cocycle of the form $g \mapsto a - ga = \delta a(g)$ for some $a \in A$. The cohomology group $H^1(\Gamma, A)$ is the space of cocycles modulo the space of coboundaries.

Theorem 2.1 (Shapiro's Lemma)**.** *Let H be a subgroup of Γ and A a left H-module. Then there are isomorphisms*

$$H_*(H, A) \cong H_*(\Gamma, \mathrm{Ind}_H^\Gamma A) \tag{6}$$

$$H^*(H, A) \cong H^*(\Gamma, \mathrm{Coind}_H^\Gamma A). \tag{7}$$

This is Proposition III.6.2 in [10]. Here Ind_H^Γ is the so-called *induced* module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} A$, where $\mathbb{Z}\Gamma$ is considered as a right H-module via the right action of H on Γ, and the left Γ-module structure is given by $g \cdot (g' \otimes a) = gg' \otimes a$ for $g, g' \in \Gamma$, $a \in A$. Similarly, $\mathrm{Coind}_H^\Gamma A$ is the co-induced module $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$ of H-equivariant maps from the left H-module $\mathbb{Z}\Gamma$ to A. The left action of Γ is defined by

$$(g \cdot f)(g') = f(g'g)$$

for $g, g' \in \Gamma$, $f \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$.

Remark 2.1. If the action of H on A is trivial, there is a canonical bijection between $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$ and $\mathrm{Map}(H\backslash\Gamma, A)$ given by $f \mapsto (Hg \mapsto f(g))$; equipping the latter with the Γ-action $(g \cdot f)(Hg') = f(Hg'g)$ this becomes an isomorphism of Γ-modules. The usual bijection between the sets of left and right cosets given by $Hg \mapsto g^{-1}H$ induces a bijection $\mathrm{Map}(H\backslash\Gamma, A) \to \mathrm{Map}(\Gamma/H, A)$, and the latter also carries a natural left Γ-action making this a Γ-isomorphism, namely $(g \cdot f)(g'H) = f(g^{-1}g'H)$.

We summarize the special case of Shapiro's Lemma we will need in a corollary:

Corollary 2.2. *Let A be an abelian group, and Γ a group which acts transitively on a set R. Consider the Γ-module $\mathrm{Map}(R, A)$ of all maps $R \to A$ with action given by $(g \cdot f)(r) = f(g^{-1}r)$. Let $D \in R$ be any element, and $\Gamma_D \subseteq \Gamma$ the stabilizer subgroup of D. Then there is an isomorphism*

$$H^*(\Gamma, \mathrm{Map}(R, A)) \cong H^*(\Gamma_D, A) \tag{8}$$

where A is considered as a trivial Γ_D-module.

Proof. The bijection $\Gamma/\Gamma_D \to R$ given by $g\Gamma_D \mapsto gD$ clearly induces an isomorphism of Γ-modules $\mathrm{Map}(\Gamma/\Gamma_D, A) \to \mathrm{Map}(R, A)$. Then from Shapiro's Lemma and the isomorphisms mentioned in the above remark we have a sequence of isomorphisms

$$\begin{aligned} H^*(\Gamma_D, A) &\cong H^*(\Gamma, \mathrm{Hom}_{\mathbb{Z}\Gamma_D}(\mathbb{Z}\Gamma, A)) \\ &\cong H^*(\Gamma, \mathrm{Map}(\Gamma_D\backslash\Gamma, A)) \\ &\cong H^*(\Gamma, \mathrm{Map}(\Gamma/\Gamma_D, A)) \\ &\cong H^*(\Gamma, \mathrm{Map}(R, A)). \end{aligned}$$

□

Note that the Γ-module $\mathrm{Map}(R, A)$ can also be considered as the set of all formal A-linear combinations of elements from R (that is, the sum $\sum_{r \in R} m_r r$ corresponds to the map $r \mapsto m_r$).

Specializing to the case $* = 1$, we have

Corollary 2.3. *With assumptions as in Corollary 2.2 above, we have an isomorphism*

$$H^1(\Gamma, \mathrm{Map}(R, A)) \cong H^1(\Gamma_D, A) = \mathrm{Hom}(\Gamma_D, A). \tag{9}$$

We now describe this isomorphism explicitly. First note that a map $u\colon \Gamma \to \mathrm{Map}(R, A)$ can be considered as a map $u\colon \Gamma \times R \to A$ by the adjoint formula $u(g)(r) = u(g, r)$. In this context, the cocycle condition (5) reads

$$u(g_1 g_2, r) = u(g_1, r) + u(g_2, g_1^{-1} r). \tag{10}$$

We wish to derive necessary and sufficient conditions for a cocycle u to be a coboundary δf. For the rest of this section, fix an element $D \in R$ and let $\Gamma_D \subseteq \Gamma$ denote the stabilizer subgroup of D.

Lemma 2.4. *A cocycle $u\colon \Gamma \times R \to A$ is a coboundary if and only if, for every pair $g_1, g_2 \in \Gamma$ with $g_1 g_2^{-1} \in \Gamma_D$, u satisfies the condition*

$$u(g_1, D) = u(g_2, D). \tag{11}$$

Proof. First we prove the necessity of the condition. Suppose that $u = \delta f$ for some $f\colon R \to A$. Since the action is transitive, it is easy to see that the kernel of the map $\delta\colon C^0(\Gamma, \mathrm{Map}(R, A)) \to C^1(\Gamma, \mathrm{Map}(R, A))$ is the set of constant maps $R \to A$. Thus we may without loss of generality assume that $f(D) = 0$. Recall that $u = \delta f$ means that for every $g \in \Gamma$, $r \in R$ we have $u(g, r) = f(r) - f(g^{-1} r)$. In particular,

$$f(g^{-1} D) = -u(g, D). \tag{12}$$

Now if $g_1 g_2^{-1} \in \Gamma_D$, we have $g_1^{-1} D = g_2^{-1} D$, and thus $-u(g_1, D) = f(g_1^{-1} D) = f(g_2^{-1} D) = -u(g_2, D)$ as desired.

Now suppose that u satisfies (11) whenever $g_1 g_2^{-1} D = D$. We need to construct a map $f\colon R \to A$. For $r \in R$, choose $g \in \Gamma$ so that $g^{-1} D = r$, and define f using (12), i.e., $f(r) = f(g^{-1} D) = -u(g, D)$. By assumption, this is a well-defined map (independent of the chosen g), and we only need to check that $u = \delta f$. Let $h \in \Gamma$ and $r \in R$ be arbitrary. To calculate $(\delta f)(h, r)$, we may choose any $g \in \Gamma$ with $g^{-1} D = r$, and we obtain

$$\begin{aligned} (\delta f)(h, r) = f(r) - f(h^{-1} r) &= f(g^{-1} D) - f((gh)^{-1} D) \\ &= -u(g, D) + u(gh, D) = u(h, g^{-1} D) = u(h, r) \end{aligned}$$

by the cocycle condition (10). □

Lemma 2.5. *The restriction of u to $\Gamma_D \times \{D\}$ is a group homomorphism $\tilde{u}\colon \Gamma_D \to A$.*

Proof. Let $g, h \in \Gamma_D$. Then

$$\begin{aligned} \tilde{u}(gh) = u(gh, D) &= u(g, D) + u(h, g^{-1} D) \\ &= u(g, D) + u(h, D) = \tilde{u}(g) + \tilde{u}(h) \end{aligned} \tag{13}$$

as claimed. □

Since A is abelian, $\tilde{u}$ factors through the abelianization $(\Gamma_D)_{\mathrm{ab}}$ of Γ_D, and we have thus established a map

$$\varphi\colon Z^1(\Gamma, \mathrm{Map}(R, A)) \to \mathrm{Hom}(\Gamma_D, A) = \mathrm{Hom}((\Gamma_D)_{\mathrm{ab}}, A).$$

The latter group may be thought of as the cohomology group $H^1((\Gamma_D)_{\mathrm{ab}}, A)$ with trivial action of $(\Gamma_D)_{\mathrm{ab}}$ on A.

Theorem 2.6. *The map φ factors to an isomorphism*

$$H^1(\Gamma, \mathrm{Map}(R, A)) \to H^1((\Gamma_D)_{\mathrm{ab}}, A),$$

which is also denoted φ.

Before we begin the proof, we need an observation: For any cocycle u and any $g \in \Gamma$, $h \in \Gamma_D$ we have

$$\begin{aligned} u(ghg^{-1}, gD) &= u(g, gD) + u(hg^{-1}, D) \\ &= u(g, gD) + u(h, D) + u(g^{-1}, D) \\ &= u(h, D) \end{aligned}$$

using $h^{-1}D = D$ and the fact that $0 = u(1) = u(g^{-1} \cdot g) = u(g^{-1}) + g^{-1}.u(g)$.

Proof of Theorem 2.6. To prove that φ does induce a map on cohomology, we need to show that the restriction of a cobundary δf to $\Gamma_D \times \{D\}$ is identically 0. But this is trivial since

$$\widetilde{\delta f}(h) = (\delta f)(h, D) = f(D) - f(h^{-1}D) = 0$$

for $h \in \Gamma_D$.

Next, assume that the cocycle u restricts to the zero homomorphism $\Gamma_D \to A$. Then for any two elements $g_1, g_2 \in \Gamma$ with $g_1 g_2^{-1} \in \Gamma_D$ we have

$$\begin{aligned} 0 &= u(g_1 g_2^{-1}, D) \\ &= u(g_1, D) + u(g_2^{-1}, g_1^{-1}D) \\ &= u(g_1, D) + u(g_2^{-1})(g_1^{-1}D) \\ &= u(g_1, D) - g_2^{-1}.u(g_2)(g_1^{-1}D) \\ &= u(g_1, D) - u(g_2)(g_2 g_1^{-1}D) \\ &= u(g_1, D) - u(g_2, D) \end{aligned}$$

since $g_2 g_1^{-1} = (g_1 g_2^{-1})^{-1} \in \Gamma_D$, and by Lemma 2.4 we see that u is a coboundary. This shows that φ is injective.

Now, for surjectivity, let $u\colon \Gamma_D \to A$ be any homomorphism. We need to extend u to all of $\Gamma \times R$ in such a way that it becomes a cocycle. To produce this extension, we first assume that an extension exists, and use this to write a formula for a cocycle cohomologous to the given extension. Then we prove that this formula actually defines a cocycle.

Choose a collection $\{h_i\}_{i \in I}$ of representatives for the set $\Gamma_D \backslash \Gamma$ of *right* cosets of Γ_D, and let $1 \in \Gamma$ represent the coset Γ_D. Recall that the map $\Gamma_D \backslash \Gamma \to \Gamma / \Gamma_D$ given by $\Gamma_D x \mapsto x^{-1}\Gamma_D$ is a bijection between the set of right cosets and the set of left cosets of

Γ_D. In particular, $\{h_i^{-1}\}_{i\in I}$ is a collection of representatives of the set of left cosets. We also have a bijection $\Gamma/\Gamma_D \to R$ given by $x\Gamma_D \mapsto xD$.

Now, define $f\colon \Gamma D \to A$ by $f(h_i^{-1}D) = u(h_i, D)$ for $i \in I$. Then since $u(1, D) = 0$, we have $(\delta f)(h_i)(D) = f(D) - f(h_i^{-1}D) = -u(h_i, D)$. Thus, by adding the coboundary of f to the given extension u, we obtain a cohomologous cocycle (again denoted u) satisfying $u(h_i, D) = 0$ for $i \in I$. Furthermore, (the new) u is uniquely determined by its cohomology class and this requirement.

The cocycle condition implies that

$$u(gh_i, D) = u(g, D) + u(h_i, g^{-1}D) = u(g, D) \tag{14}$$

for $i \in I$ and $g \in \Gamma_D$. Since every $x \in \Gamma$ admits a unique factorization as $x = gh_i$ for some $i \in I$ and $g \in \Gamma_D$, this formula extends u to all of $\Gamma \times \{D\}$.

Now consider any $x \in \Gamma$ and $E \in R$. There is a unique $j \in I$ with $h_j^{-1}D = E$, and we have $\Gamma_E = h_j^{-1}\Gamma_D h_j$. Furthermore, the collection $\{h_j^{-1}h_ih_j\}_{i\in I}$ is a collection of representatives for the set $\Gamma_E\backslash\Gamma$ of right cosets of Γ_E. This means that we may factorize x uniquely as $(h_j^{-1}g_0h_j)(h_j^{-1}h_ih_j)$ for some $g_0 \in \Gamma_D$ and $i \in I$. Now we calculate

$$\begin{aligned} u(x, E) &= u(h_j^{-1}g_0h_j \cdot h_j^{-1}h_ih_j, h_j^{-1}D) && (15)\\ &= u(h_j^{-1}g_0h_j, h_j^{-1}D) + u(h_j^{-1}h_ih_j, h_j^{-1}g_0^{-1}h_jh_j^{-1}D) && (16)\end{aligned}$$

By the observation preceding this proof (with $g = h_j^{-1}$ and $h = g_0$), the first term is equal to the known quantity $u(g_0, D)$. For the second term, we apply the cocycle condition a few more times:

$$\begin{aligned} u(h_j^{-1}h_ih_j, h_j^{-1}D) &= u(h_j^{-1}, h_j^{-1}D) + u(h_ih_j, D)\\ &= -u(h_j, D) + u(h_ih_j, D)\\ &= u(h_ih_j, D)\end{aligned}$$

which is also known since u is known on $\Gamma \times \{D\}$. Thus our formula for the extension of u to all of $\Gamma \times R$ reads

$$u(x, E) = u(g_0, D) + u(h_ih_j, D) \tag{17}$$

where $j \in I$ is the unique index such that $h_j^{-1}D = E$, $i \in I$ is the unique index so that x belongs to the right coset of Γ_E represented by $h_j^{-1}h_ih_j$, and $g_0 = h_jgh_j^{-1}$ is the unique element in Γ_D such that $x = g(h_j^{-1}h_ih_j) = (h_j^{-1}g_0h_j)(h_j^{-1}h_ih_j)$. The second term above is defined by (14); thus one must find the $k \in I$ such that h_ih_j is an element of the right coset of Γ_D represented by h_k, say $h_ih_j = g_1h_k$ for $g_1 \in \Gamma_D$, and then $u(h_ih_j, D) = u(g_1, D)$. It remains to check that (17) defines a cocycle.

Let $x, y \in \Gamma$ and $E \in R$ be arbitrary. As above, there is a unique $j \in I$ with $h_j^{-1}D = E$. Let us try to calculate the right-hand side of the cocycle condition given by $u(xy, E) = u(x, E) + u(y, x^{-1}E)$. We must choose $i \in I$ and $g_1 \in \Gamma_D$ such that

$$x = (h_j^{-1}g_1h_j)(h_j^{-1}h_ih_j) \tag{18}$$

and next we choose $k \in I$ and $g_2 \in \Gamma_D$ such that $h_i h_j = g_2 h_k$. Then

$$u(x, E) = u(g_1, D) + u(g_2, D) = u(g_1 g_2, D)$$

Now, the element $x^{-1}E$ of R is the same as

$$x^{-1}E = h_j^{-1} h_i^{-1} g_1^{-1} h_j E = h_j^{-1} h_i^{-1} D = (h_i h_j)^{-1} D = (g_2 h_k)^{-1} D = h_k^{-1} D$$

so in the calculation of $u(y, x^{-1}E)$ it is h_k which plays the role as h_j in the recipe. This recipe then requires us to find $g_3 \in \Gamma_D$ and $\ell \in I$ such that

$$y = (h_k^{-1} g_3 h_k)(h_k^{-1} h_\ell h_k), \tag{19}$$

and $g_4 \in \Gamma_D$ and $m \in I$ such that $h_\ell h_k = g_4 h_m$. Then

$$u(y, x^{-1}E) = u(g_3, D) + u(g_4, D) = u(g_3 g_4, D).$$

Multiplying x and y using the expressions (18) and (19) and the relations defining the various h'es we obtain

$$\begin{aligned} xy &= (h_j^{-1} g_1 h_i h_j)(h_k^{-1} g_3 h_\ell h_k) \\ &= h_j^{-1} g_1 g_2 g_3 g_4 h_m \end{aligned} \tag{20}$$

On the other hand, the recipe requires us to choose $g \in \Gamma_D$ and $n \in I$ such that

$$xy = h_j^{-1} g h_j h_j^{-1} h_n h_j, \tag{21}$$

and $g' \in \Gamma_D$ and $p \in I$ such that $h_n h_j = g' h_p$. Then $u(xy, E) = u(g, D) + u(g', D)$. Comparing (20) and (21) we see that $g_1 g_2 g_3 g_4 h_m = g h_n h_j$, showing that (by uniqueness of g' and p) $h_p = h_m$ and

$$g' = g^{-1} g_1 g_2 g_3 g_4 \tag{22}$$

Finally we conclude that

$$\begin{aligned} u(xy, E) &= u(g, D) + u(g', D) \\ &= u(g_1 g_2 g_3 g_4, D) \\ &= u(g_1 g_2, D) + u(g_3 g_4, D) \\ &= u(x, E) + u(y, x^{-1}E) \end{aligned}$$

showing that the given recipe in fact defines a cocycle $u\colon \Gamma \times R \to A$. The proof is complete. □

Proof of Theorem 1.1. By the splitting (4), a cocycle $u\colon \Gamma \to \hat{\mathcal{B}}$ is the same as a collection of cocycles $u_D\colon \Gamma \to \hat{M}_D$ for $D \in S$. In fact, thinking of u as a map $\Gamma \times B \to \mathbb{C}$, u_D is simply the restriction of u to $\Gamma \times (\Gamma D)$. Specializing Theorem 2.6 to the case $A = \mathbb{C}$ and $R = \Gamma D$, we see that each u_D is a coboundary if and only if u_D restricted to $\Gamma_D \times \{D\}$ is zero. □

In Section 3 below, we are going to need a theorem linking the low-dimensional cohomology groups of the groups appearing in a short exact sequence. Again quoting from [10] (Corollary VII.6.4)

Theorem 2.7. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups, and M a B-module. Then there is an exact sequence of cohomology groups*

$$0 \to H^1(C, M^A) \to H^1(B, M) \to H^1(A, M)^B \to H^2(C, M^A). \tag{23}$$

Here we regard M as an A-module via restriction of scalars, and then clearly $C \cong B/A$ acts on the submodule M^A invariant under A, making sense of $H^*(C, M^A)$. Since A is normal in B, conjugation defines an action on A by automorphisms, so there is an induced action on cohomology $c_b^*\colon H^*(A, M) \to H^*(A, M)$, where c_b is the map $a \mapsto bab^{-1}$. One may show that A acts trivially on $H^1(A, M)$, so there is an induced action of C, and we have $H^1(A, M)^B = H^1(A, M)^C$.

We need a special case of the above.

Corollary 2.8. *If C is a finite group and $M = \mathbb{C}$ as a trivial B-module, both $H^1(C, \mathbb{C})$ and $H^2(C, \mathbb{C})$ are trivial, so we have an isomorphism*

$$\mathrm{Hom}(B, \mathbb{C}) = H^1(B, \mathbb{C}) \cong H^1(A, \mathbb{C})^B = \mathrm{Hom}(A, \mathbb{C})^B. \tag{24}$$

In other words, the space of homomorphisms $B \to \mathbb{C}$ is isomorphic to the space of homomorphisms $f\colon A \to \mathbb{C}$ satisfying

$$(bf)(a) = f(bab^{-1}) = f(a) \tag{25}$$

for all $a \in A$, $b \in B$.

2.1. Abelianization

If M is a trivial Γ-module, a cocycle $\Gamma \to M$ is the same as a group homomorphism, and all coboundaries vanish identically. Hence, in this situation we have

$$H^1(\Gamma, M) = \mathrm{Hom}(\Gamma, M) = \mathrm{Hom}(\Gamma_{\mathrm{ab}}, M), \tag{26}$$

where $\Gamma_{\mathrm{ab}} = H_1(\Gamma, \mathbb{Z})$ is the abelianization of Γ. In the case where M is torsion free, we further have

$$\mathrm{Hom}(\Gamma_{\mathrm{ab}}, M) = \mathrm{Hom}(\Gamma_{\mathrm{ab}} \otimes \mathbb{Q}, M) = \mathrm{Hom}(H_1(\Gamma, \mathbb{Q}), M). \tag{27}$$

The abelianizations of mapping class groups are all known. Abstractly, they are given as follows:

Theorem 2.9. *Let $\Gamma_{g,r}$ denote the mapping class group of a surface of genus g with r boundary components.*

(a) *If $g \geq 3$, both $H_1(\Gamma_{g,r}, \mathbb{Z})$ and $H_1(\Gamma_{g,r}, \mathbb{Q})$ are trivial.*
(b) *For any r, we have $H_1(\Gamma_{2,r}, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$, so $H_1(\Gamma_{2,r}, \mathbb{Q}) = 0$.*
(c) *The group $H_1(\Gamma_{1,0}, \mathbb{Z})$ is cyclic of order 12, whereas for $r \geq 1$, $H_1(\Gamma_{1,r}, \mathbb{Z}) \cong \mathbb{Z}^r$.*
(d) *For $g = 0$, $H_1(\Gamma_{0,r}, \mathbb{Z})$ is a free abelian group of rank $\binom{r}{2} = r(r-1)/2$, so $H_1(\Gamma_{0,r}, \mathbb{Q})$ is a rational vector space of the same dimension.*

See [11] or [12] for easy proofs of (a), (b) and (c). We will need explicit generators in the genus 1 and 0 cases.

Proposition 2.10. *Consider a torus $\Sigma_{1,r}$ with boundary components $\delta_1, \dots, \delta_r$ for some $r \geq 1$. Let t be a Dehn twist in any non-separating curve, and let d_j denote the Dehn twist along δ_j. Then in $H_1(\Gamma_{1,r}, \mathbb{Z})$, we have the relation*

$$12t = d_1 + \cdots + d_r. \tag{28}$$

The homology classes of $d_1, \dots, d_r$ constitute a basis for $H_1(\Gamma_{1,r}, \mathbb{Q})$.

Proof. For $r \leq 2$, this follows from the chain relation (also known as the two-holed torus relation). On Figure 1, we have obtained $\Sigma_{1,r}$ from $\Sigma_{1,r-1}$ by gluing a pair of pants on to the boundary component δ of $\Sigma_{1,r-1}$ (note that the left-hand part of the surface can contain any number of boundary components). The lantern relation applied to the seven

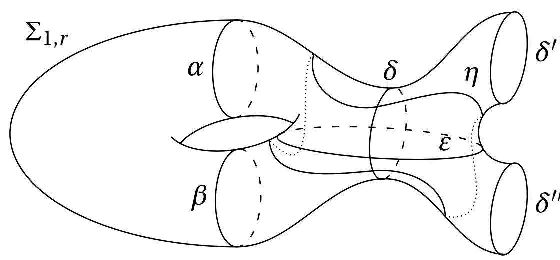

FIGURE 1. Gluing a pair of pants to an $r-1$-holed torus.

curves α, β, δ', δ'', δ, ε, and η yields

$$\tau_\alpha \tau_\beta \tau_{\delta'} \tau_{\delta''} = \tau_\delta \tau_\varepsilon \tau_\eta.$$

Since α, β, ε and η are non-separating in $\Sigma_{1,r}$, the twists in these curves represent the same element in $H_1(\Gamma_{1,r}, \mathbb{Z})$, so we obtain $d' + d'' = d$, where d, d', d'' are the homology classes of τ_δ, $\tau_{\delta'}$ and $\tau_{\delta''}$, respectively.

The last claim follows from the fact that $\Gamma_{1,r}$ is generated by twists in two non-separating curves intersecting in a single point along with the twists in the boundary components. Hence, $H_1(\Gamma_{1,r}, \mathbb{Q})$ is generated by the homology classes of t and $d_1, \dots, d_r$, and (28) shows that t may be omitted. $\square$

It is somewhat more involved to give a symmetric description of $H_1(\Gamma_{0,r}, \mathbb{Z})$. We first recall why this group is free abelian of rank

$$\frac{r(r-1)}{2} = r - 1 + \frac{(r-1)(r-2)}{2}.$$

Let PB_n denote the pure braid group on n strands.

Proposition 2.11. *There is an exact sequence*

$$0 \to \mathbb{Z}^{r-1} \to \Gamma_{0,r} \to \mathrm{PB}_{r-1} \to 1 \tag{29}$$

This sequence splits.

Proof. Think of $\Gamma_{0,r}$ as the mapping class group of a disc with $r-1$ holes. Gluing a disc with a marked point to each of the "inner" boundary components gives a map to the (pure) mapping class group $\Gamma_{0,1}^{r-1}$ of a disc with $r-1$ marked points; the kernel of this map is the free abelian group generated by the twists in the boundary components (see [11], Proposition 3.19), and $\Gamma_{0,1}^{r-1}$ is isomorphic to PB_{r-1} ([11], chapter 9).

Gluing a disc to all of the inner boundary components except the j'th gives a homomorphism from $\Gamma_{0,r}$ to the mapping class group of a cylinder. The latter is infinite cyclic, generated by the Dehn twist in the j'th boundary component. This describes the j'th component of the splitting $\Gamma_{0,r} \to \mathbb{Z}^{r-1}$. □

Since $\Gamma_{0,1}^{r-1}$ is generated by twists τ_{ij} in curves γ_{ij}, $1 \le i < j \le r-1$ encircling the i'th and j'th puncture and all relations are commutation relations (again, see [11], chapter 9), we conclude that $H_1(\Gamma_{0,r}, \mathbb{Z})$ is freely generated by the homology classes of the Dehn twists in $r-1$ of the boundary components, along with Dehn twists along $\binom{r-1}{2} = (r-1)(r-2)/2$ curves separating two of these boundary components from the rest of the surface.

Unfortunately, this way of viewing $H_1(\Gamma_{0,r}, \mathbb{Z})$ singles out one of the boundary components as "special", which makes it inappropriate for our purposes.

Theorem 2.12. *Let A denote the (rank $r + r(r-1)/2$) free abelian group on generators g_i, $1 \le i \le r$ and g_{ij}, $1 \le i < j \le r$. Then $H_1(\Gamma_{0,r}, \mathbb{Z})$ is isomorphic to the quotient of A by the subgroup R generated by the $r+1$ relations*

$$g_k + (r-3)\sum_{i \neq k} g_i = \sum_{\substack{i<j \\ i,j \neq k}} g_{ij} \quad , \quad k = 1, \dots, r \tag{30}$$

$$(r-2)\sum_i g_i = \sum_{i<j} g_{ij}. \tag{31}$$

The isomorphism is induced by the map ψ sending g_i to the homology class of the Dehn twist in the i'th boundary component, and g_{ij} to the homology class of the Dehn twist in a curve which bounds a pair of pants together with the i'th and j'th boundary components.

Proof. By the description of $H_1(\Gamma_{0,r}, \mathbb{Z})$ given above, it is clear that ψ is surjective. To prove that ψ descends to a map on the quotient, we must prove that the relations (30) and (31) hold in $H_1(\Gamma_{0,r}, \mathbb{Z})$. The former immediate follows from the generalized lantern relation [13]. Thus, in $H_1(\Gamma_{0,r+1}, \mathbb{Z})$ one has

$$\tau_{r+1} + (r-2)\sum_{i<r+1} \tau_i = \sum_{i<j<r+1} \tau_{ij},$$

and by the epimorphism $\Gamma_{0,r+1} \to \Gamma_{0,r}$ induced by gluing a disc to the last boundary component we see that also (31) holds in $H_1(\Gamma_{0,r}, \mathbb{Z})$.

Since, by the remarks following Proposition 2.11, $H_1(\Gamma_{0,r}, \mathbb{Z})$ is free abelian on the homology classes of the twists τ_i, $1 \le i < r$ and τ_{ij}, $1 \le i < j < r$, there is a map $H_1(\Gamma_{0,r}, \mathbb{Z}) \to A$ given by $\tau_i \mapsto g_i$ and $\tau_{ij} \mapsto g_{ij}$. We now prove that the composition of this map with the projection $A \to A/R$ is surjective. This amounts to proving that in A/R, the equivalence classes represented by g_r and g_{kr}, $k < r$, can be written entirely in terms of g_i, $i < r$, and g_{ij}, $1 \le i < j \le r-1$.

Using (30) for $k = r$, this is obvious for g_r. Subtracting (30) from (31), we get

$$(r-3)g_k + \sum_{i \neq k} g_i = \sum_{i<k} g_{ik} + \sum_{k<j<r} g_{kj} + g_{kr},$$

from which we see that each g_{kr} can also be expressed using only $g_1, \dots, g_{r-1}$ and g_{ij}, $1 \le i < j < r$ (although g_r occurs on the left-hand side, it can be eliminated by the above). Hence, ψ is invertible. □

3. Computing cohomology

Fix a multicurve D. The purpose of this section is to describe an algorithm to compute $\mathrm{Hom}(\Gamma_D, \mathbb{C})$, the space of homomorphisms from the stabilizer of D to $\mathbb{C}$. We may without loss of generality assume that D does not have components parallel to a boundary component of Σ, since any mapping class preserves each boundary component. We also assume that D is not the empty multicurve (in which case $\Gamma_D = \Gamma$, so $\mathrm{Hom}(\Gamma_D, \mathbb{C})$ is easily computed using Theorem 2.9). We first deal with a simple case.

Lemma 3.1. *If Σ is a closed torus, D consists of parallel copies of a single (non-separating) curve δ. The stabilizer of D is the infinite cyclic group generated by the Dehn twist τ_δ. Hence,* $\mathrm{Hom}(\Gamma_D, \mathbb{C})$ *is one-dimensional, spanned by the map $\tau_\delta \mapsto 1$.* □

From now on, we will assume that Σ is not a closed torus.

Let d denote the number of distinct isotopy classes of loops occuring in D, and let Δ denote the d-component multicurve consisting of one copy of each of these loops. Clearly Γ_D is a subgroup of Γ_Δ of finite index.

Let Q denote the wreath product $C_2 \wr S_d$ of the cyclic group $C_2 = \{\pm 1\}$ by the symmetric group S_d; in other words, the semi-direct product $C_2^d \rtimes S_d$, where S_d acts on C_2^d by permutation of factors. If we fix an enumeration of the components of Δ and an orientation of each component, we obtain a homomorphism $\Theta\colon \Gamma_\Delta \to Q$ as follows: Write $\Delta = \delta_1 \cup \cdots \cup \delta_d$. Then an element $\varphi \in \Gamma_\Delta$ permutes the components of Δ, so there is $\sigma \in S_d$ with $\varphi(\delta_k) = \delta_{\sigma(k)}$ (as unoriented curves) for all k. The k'th sign is $+1$ if the orientation of $\varphi(\delta_k)$ matches the orientation of $\delta_{\sigma(k)}$, and -1 otherwise. Let K denote the kernel of Θ, i.e., the subgroup of Γ which fixes each component and the orientation of each component, and put $P_D = \Theta(\Gamma_D)$, $P_\Delta = \Theta(\Gamma_\Delta)$. These images depend on the topological types of the multicurves D and Δ.

We let Σ' denote the (possibly non-connected) subsurface of Σ obtained by removing a tubular neighbourhood of Δ. The mapping class group Γ' of Σ' is the direct product of the mapping class groups of the connected components of Σ'. There is a homomorphism $\eta\colon \Gamma' \to \Gamma$ given by extending a diffeomorphism of Σ' by the identity. Theorem 3.18 of [11] yields:

Proposition 3.2. *Assume Σ is not a closed torus. Then the kernel of η is the rank d free abelian group*

$$\ker\eta = \langle \tau_{\alpha_1}\tau_{\beta_1}^{-1}, \ldots, \tau_{\alpha_d}\tau_{\beta_d}^{-1}\rangle, \tag{32}$$

where α_k, β_k are the two boundary components of Σ' isotopic (in Σ) to δ_k.

It is clear that the image of η is precisely the group K. The various groups mentioned above are related by the exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \Gamma_D & \xrightarrow{\Theta} & P_D & \longrightarrow & 1 \\ & & \| & & \cap\!\mid & & \cap\!\mid & & \\ 1 & \longrightarrow & K & \longrightarrow & \Gamma_\Delta & \xrightarrow{\Theta} & P_\Delta & \longrightarrow & 1 \end{array} \tag{33}$$

and

$$1 \longrightarrow \mathbb{Z}^d \longrightarrow \Gamma' \longrightarrow K \longrightarrow 1. \tag{34}$$

Using these, together with Theorem 2.7 and Corollary 2.8, we see that one may compute $\mathrm{Hom}(\Gamma_\Delta, \mathbb{C})$ as follows:

Proposition 3.3. *As above, let Σ' denote the surface obtained by removing a tubular neighbourhood of Δ. For each boundary component γ of Σ', pick a generator g_γ, and for each pair of boundary components α, β belonging to a genus 0 component of Σ', pick a generator $g_{\alpha,\beta}$. Then $\mathrm{Hom}(\Gamma_\Delta, \mathbb{C})$ is isomorphic to the complex vector space spanned by the g_γ and $g_{\alpha,\beta}$'s, modulo the following relations:*

- *(a) If γ is a boundary component of Σ' belonging to a component of genus at least 2, then $g_\gamma = 0$.*
- *(b) If γ_1 and γ_2 are boundary components of Σ' arising from the same component of Δ, we have $g_{\gamma_1} = g_{\gamma_2}$.*
- *(c) For each genus 0 component of Σ', the associated generators must satisfy (30) and (31).*
- *(d) If δ_1 and δ_2 are Γ_Δ-related components of Δ (i.e., there is an element of Γ_Δ sending δ_1 to δ_2), the generators arising from the boundary components induced by δ_1 and δ_2 are identified.*
- *(e) If S_1 and S_2 are genus 0 components of Σ', and if there is $\varphi \in \Gamma_\Delta$ with $\varphi(S_1) = S_2$, then for each pair α, β of boundary components of S_1, we have $g_{\alpha,\beta} = g_{\varphi(\alpha),\varphi(\beta)}$.*

Proof. A homomorphism from Γ' to $\mathbb{C}$ is the same as a collection of homomorphisms from the mapping class groups of the connected components of Σ' to $\mathbb{C}$. By Theorem 2.9, there are no non-trivial homomorphisms from surfaces of genus 2 or more, and the homomorphisms from the genus 0 and 1 components are exactly described by the generators g_γ,

$g_{\alpha,\beta}$ and the condition (c). Homomorphisms from K to $\mathbb{C}$ are, by (34) and 3.2, in bijection with homomorphisms from Γ' to $\mathbb{C}$ which vanish on elements of the form $\tau_{\alpha_k}\tau_{\beta_k}^{-1}$. Hence, $\mathrm{Hom}(K, \mathbb{C})$ is described by the given generators and relations (a), (b) and (c). Finally, (d) and (e) comes from Corollary 2.8. $\square$

The same algorithm describes $\mathrm{Hom}(\Gamma_D, \mathbb{C})$, except that fewer identifications of generators take place in steps (d) and (e) (since, in addition to the topological type, the multiplicities of the involved components of Δ must match up).

Proof of Theorem 1.2. It is immediate from the isomorphism (2) and the splitting (4) that

$$H^1(\Gamma, \mathcal{O}(\mathcal{M})^*) \cong \prod_D H^1(\Gamma, \hat{M}_D). \tag{35}$$

By Corollary 2.3, we have $H^1(\Gamma, \hat{M}_D) = H^1(\Gamma, \mathrm{Map}(\Gamma D, \mathbb{C})) \cong \mathrm{Hom}(\Gamma_D, \mathbb{C})$. Finally, it is clear by the above algorithm that the latter vector space has finite dimension. $\square$

4. An example

Let Σ be a surface of genus $g \geq 2$ with r boundary components $\sigma_1, \ldots, \sigma_r$. Let Δ be a multicurve consisting of $g-1$ simple closed curves $\delta_1, \ldots, \delta_{g-1}$ such that the complement of (a tubular neighborhood of) Δ is a connected surface Σ' of genus 1. By Theorem 2.9, $H_1(\Gamma', \mathbb{Q})$ is generated by the homology classes of the twists in the $r + 2g - 2$ boundary components. Hence we see that a basis for $\mathrm{Hom}(K, \mathbb{C})$ is given by the $r + g - 1$ elements

$$\begin{aligned} [\tau_{\delta_i}] &\mapsto 1 \qquad i = 1, \ldots, g-1 \\ [\tau_{\sigma_j}] &\mapsto 1 \qquad j = 1, \ldots, r. \end{aligned}$$

Clearly Γ_Δ acts transitively on the set of components of Δ, so in Γ_Δ, all the twists τ_{δ_j} are conjugate. This implies that $\mathrm{Hom}(\Gamma_\Delta, \mathbb{C})$ has dimension $r + 1$.

This proves Corollary 1.3 in most cases. In the case of a closed torus, we saw above (Lemma 3.1) that any non-empty multicurve contributes a one-dimensional summand. For a torus with r boundary components, the stabilizer of the empty multicurve (or any multicurve consisting of copies of the boundary components) is the entire mapping class group, which thus contributes an r-dimensional summand by Theorem 2.9.

For a surface of genus 0, the same argument gives a contribution to $H^1(\Gamma, \mathcal{O}(\mathcal{M})^*)$ of dimension $r(r-1)/2$. In fact, for any multicurve D in a surface of genus 0, it is clear that no element of Γ_D may permute the components of D (since every mapping class preserves each of the original boundary components, the connected components of the complement of D cannot be permuted). Hence $K = \Gamma_D$ in the notation above, and we see that the dimension of $\mathrm{Hom}(\Gamma_D, \mathbb{C})$ is given by

$$\sum_S \frac{r_S(r_S - 1)}{2} - \#D$$

where the sum runs over the connected components of the complement of (a tubular neighborhood of) D, r_S is the number of boundary components of S, and $\#D$ is the number of components of D.

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