Proceedings of 17^{th} Gökova
Geometry-Topology Conference
pp. 1 – 20

Lectures on Symplectic and Contact Homology

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Abstract. We shall give the geometric definition of several holomorphic curve invariants for symplectic and contact manifolds, such as contact homology, symplectic homology and some of their variants. We shall then explain the relationship between these theories, leading to a common algebraic framework. These relations can also be used to translate structural results from one of these invariants to the other ones. This can be illustrated with the effect of some geometric operations on these invariants, such as Legendrian surgery.

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1. Symplectic and contact homology

1.1. Symplectic and contact manifolds

Let X be a smooth manifold of dimension $2n$. A 2-form ω on X is called **symplectic** if it is closed (i.e., $d\omega = 0$) and nondegenerate (i.e., $\omega^{\wedge n} \neq 0$ everywhere). The pair (X, ω) is called a **symplectic manifold**.

Let Y be a smooth manifold of dimension $2n-1$. A hyperplane distribution ξ on Y is called a **contact structure** if, for any 1-form α such that $\xi = \ker\alpha$ on some open subset $U \subset Y$, we have $\alpha \wedge d\alpha^{\wedge(n-1)} \neq 0$ on U. The pair (Y, ξ) is called a **contact manifold**. A 1-form α such that $\xi = \ker\alpha$ on Y is called a **contact form**.

To any contact form α on Y, one can associate a vector field R_α called the **Reeb vector field** and characterized by

$$\begin{cases} \imath(R_\alpha)d\alpha &= 0, \\ \alpha(R_\alpha) &= 1. \end{cases}$$

This vector field strongly depends on the choice of a contact form for a given contact structure.

Even though the above defintions are apparently very different, contact manifolds can be thought of as the odd dimensional counterpart of symplectic manifolds. Observe indeed that, if α is a contact form on (Y, ξ), then $d\alpha$ restricts to a symplectic form on ξ. This symplectic form depends on the choice of α only up to a nonvanishing factor. We now describe two constructions illustrating the interplay between contact and symplectic geometry.

Given a contact manifold (Y, ξ), we can construct a symplectic manifold called the **symplectization** of (Y, ξ). It is the manifold $\mathbb{R} \times Y$, equipped with the symplectic form $d(e^t\alpha)$, where t is the coordinate on the $\mathbb{R}$ factor and α is a contact form for ξ. This symplectic manifold does not depend on the choice of α.

On the other hand, let Y be a hypersurface in a symplectic manifold (X, ω). A vector field v on X is called **Liouville** if $\mathcal{L}_v\omega = \omega$. The hypersurface Y is said to be **of contact type** if there exists a Liouville vector field v on X defined near Y and transverse to Y. In that case, the 1-form $\alpha = \imath(v)\omega$ restricts to a contact form on Y. In particular, if $Y = \partial X$ is of contact type with a transverse Liouville vector field v pointing outside X, we say that (X, ω) has a **convex** boundary. If v is pointing inwards, we say that (X, ω) has a **concave** boundary. If a symplectic manifold (X, ω) has a boundary $\partial X = \partial_+ X \cup \partial_- X$ such that $Y_+ = \partial_+ X$ is convex and $Y_- = \partial_- X$ is concave, we say that (X, ω) is a **symplectic cobordism** from Y_+ to Y_-.

If (X, ω) has a convex boundary, we can complete X to a manifold $\widehat{X}$ given by $X \cup_{\partial X \simeq \{0\} \times Y} \mathbb{R}^+ \times Y$ equipped with the symplectic form $\widehat{\omega}$ given by ω on X and by $d(e^t\alpha)$ on $\mathbb{R}^+ \times Y$.

1.2. Complex structures

Many symplectic and contact invariants can be constructed using a compatible (almost) complex structure. In particular, these allow to define pseudo-holomorphic curves, which were introduced in symplectic geometry by Gromov [14].

On a smooth manifold X of dimension $2n$, an **almost complex structure** is an endomorphism $J : TX \to TX$ such that $J^2 = -I$. An almost complex structure J on a symplectic manifold (X, ω) is said to be **compatible** with ω if

$$\omega(Jv, Jw) = \omega(v, w) \qquad \text{and} \qquad \omega(v, Jv) > 0$$

for any $v, w \in TX$ with $v \neq 0$.

It is a well-known fact that the set of compatible almost complex structures on a symplectic manifold (X, ω) is nonempty and contractible. Therefore, the tangent bundle

TX can be considered as a complex vector bundle up to homotopy. In particular, we can define its **first Chern class** $c_1(X,\omega) \in H^2(X,\mathbb{Z})$.

We say that an almost complex structure J on X is **integrable** if (X,J) is a complex manifold, or more explicitly if X admits an atlas with complex coordinates in which J is represented by i and with biholomorphic coordinate changes.

On a contact manifold (Y,ξ), a **complex structure** J is an endomorphism $J : \xi \to \xi$ such that $J^2 = -I$. Such a complex structure J is said to be **compatible** with $d\alpha$ if

$$d\alpha(Jv, Jw) = d\alpha(v,w) \qquad \text{and} \qquad d\alpha(v, Jv) > 0$$

for any $v, w \in \xi$ with $v \neq 0$.

Similarly, the set of compatible almost complex structures on a contact manifold (Y,ξ) is nonempty and contractible, so that the first Chern class $c_1(\xi) \in H^2(Y,\mathbb{Z})$ can be defined.

A compatible complex structure J on (Y,ξ) can be extended to a compatible almost complex structure on the symplectization $(\mathbb{R} \times Y, d(e^t\alpha))$ by $J\frac{\partial}{\partial t} = R_\alpha$. Like the Reeb vector field, this extension strongly depends on the choice of a contact form α for ξ.

Example 1.1. We now introduce our main class of examples for these lectures. A **Stein manifold** is a complex manifold (X,J) admitting a proper, complex embedding in $\mathbb{C}^N$ for some $N > 0$.

A smooth function $\phi : X \to \mathbb{R}$ is called **exhausting** if it is proper and bounded from below. It is called **strictly plurisubharmonic** if the 2-form $-dd^{\mathbb{C}}\phi$ is symplectic, where $d^{\mathbb{C}}\phi = d\phi \circ J$. These notions allow to formulate an intrinsic characterization of Stein manifolds due to Grauert [13]: a Stein manifold is a complex manifold (X,J) which admits an exhausting, strictly plurisubharmonic function ϕ. In particular, Stein manifolds are symplectic and admit a Liouville vector field which is transverse to the regular level sets of ϕ.

Note that since the above conditions on ϕ are open, we can assume that it is Morse. It then follows from the above definition that the critical points of ϕ have index at most $n = \frac{1}{2}\dim X$. Eliashberg [10] used this to give a topological characterization of Stein manifolds: an open smooth manifold X of dimension $2n > 4$ with an almost complex structure J and an exhausting Morse function ϕ with critical points of index at most n admits a Stein structure. More precisely, J is homotopic through almost complex structures to an integrable complex structure J' such that ϕ is strictly plurisubharmonic.

A Stein manifold (X,J,ϕ) is called **subcritical** if the critical points of ϕ have index strictly less than $n = \frac{1}{2}\dim X$.

1.3. Floer homology

Let (X,ω) be a closed symplectic manifold of dimension $2n$. In this context, a smooth function $H : S^1 \times X \to \mathbb{R}$ will be called a **(time-dependent) Hamiltonian function**. For any $\theta \in S^1$, we write $H_\theta = H|_{\{\theta\}\times X}$.

To any (time-dependent) Hamiltonian function H, one can associate a (time-dependent) vector field X_H^θ, called **Hamiltonian vector field** and characterized by

$$\imath(X_H^\theta)\omega = dH_\theta.$$

A closed curve $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to X$ is called a **1-periodic orbit** of X_H^θ if

$$\dot\gamma(\theta) = X_H^\theta(\gamma(\theta)),$$

for all $\theta \in S^1$. In local coordinates, the above differential equation coincides with Hamilton's equations for classical mechanics.

We say that a 1-period orbit γ of X_H^θ is **nondegenerate** if the graph of the time 1 flow $\varphi_H^1 : X \to X$ of X_H^θ in $X \times X$ is transverse to the diagonal $\Delta \subset X \times X$ at point $(\gamma(0), \gamma(0))$. In that case, after choosing a symplectic trivialization of TX along γ, one associates to γ an integer $\mu_{CZ}(\gamma)$ called the **Conley-Zehnder index** of γ. See [19] for a definition of this index.

The **Novikov ring** Λ_ω of (X, ω) over $\mathbb{Z}$ is a completion of the group ring $\mathbb{Z}[H_2(X, \mathbb{Z})]$; it consists of elements

$$\sum_{A \in H_2(X,\mathbb{Z})} \lambda_A e^A$$

such that the number of nonzero coefficients λ_A with $\omega(A) < C$ is finite for any $C \in \mathbb{R}$. The ring Λ_ω is graded by $|e^A| = -2\langle c_1(X, \omega), A\rangle$.

Let $H : S^1 \times X \to \mathbb{R}$ be a generic Hamiltonian, so that all 1-periodic orbits of X_H^θ are nondegenerate. For each such contractible orbit γ, we choose a spanning disk D_γ and use it to induce a symplectic trivialization of TX along γ. We define a module $FH(X, \omega)$ freely generated over Λ_ω by the contractible 1-periodic orbits of X_H^θ. This module is graded by the Conley-Zehnder index : $|\gamma| = \mu_{CZ}(\gamma)$.

This module will be equipped with a differential in order to imitate the Morse complex for the **action functional**

$$\mathcal{A}_H : C^\infty_{\mathrm{contr}}(S^1, X) \to \mathbb{R} : \gamma \mapsto -\int_{D^2} \sigma^*\omega - \int_{S^1} H(\theta, \gamma(\theta))\, d\theta,$$

where $\sigma : D^2 \to X$ is a smooth homotopy between $\sigma_{\partial D^2} = \gamma$ and a constant loop in X. This action functional is well-defined if (X, ω) is symplectically aspherical, i.e., $\langle [\omega], \pi_2(X)\rangle = 0$. The critical points of $\mathcal{A}_H$ are exactly the contractible 1-periodic orbits of X_H^θ.

To define gradient trajectories in Morse theory, we need to choose a Riemannian metric on the domain of the Morse function. In this case, such a metric will be induced by a (time dependent) compatible almost complex structure J_θ, $\theta \in S^1$, on (X, ω). The equation for gradient trajectories $u : \mathbb{R} \times S^1 \to X$ for $\mathcal{A}_H$ can then be written formally as:

$$\frac{\partial u}{\partial s} + J_\theta(u)\left(\frac{\partial u}{\partial \theta} - X_H^\theta(u)\right) = 0, \tag{1}$$

with asymptotic conditions

$$\lim_{s\to-\infty} u(s,\theta) = \gamma_-(\theta), \qquad \lim_{s\to\infty} u(s,\theta) = \gamma_+(\theta), \tag{2}$$

for some contractible 1-period orbits γ_-, γ_+ of X_H^θ.

These equations form a well-defined elliptic problem, but unlike in Morse theory these do not define a flow. Note that it is not necessary to have a flow in order to define a differential complex. Indeed, the construction of Floer homology is based on the **moduli spaces of Floer trajectories** $\mathcal{M}^A(\gamma_-, \gamma_+)$ consisting of solutions u of (1) and (2), such that the homology class of the sphere obtained by attaching the image of u with the capping disks D_{γ_-} and D_{γ_+} coincides with $A \in H_2(X, \mathbb{Z})$.

If (X, ω) is aspherical, we can find a generic J_θ so that $\mathcal{M}^A(\gamma_-, \gamma_+)$ is a smooth manifold of dimension $|\gamma_-| - |\gamma_+| - |e^A|$. This manifold has an $\mathbb{R}$-action defined by $(\tau \cdot u)(s,\theta) = u(s+\tau, \theta)$. This collection of moduli spaces can be equipped with coherent orientations, i.e. orientations which are compatible with the operation of gluing Floer trajectories. When the quotient $\mathcal{M}^A(\gamma_-, \gamma_+)/\mathbb{R}$ has dimension zero, it consists of finitely many rigid Floer trajectories, each equipped with a sign. We denote by $\#\mathcal{M}^A(\gamma_-, \gamma_+)/\mathbb{R}$ the algebraic count of these trajectories.

We define a **differential** $d_{\mathrm{FH}} : FH_*(X,\omega) \to FH_{*-1}(X,\omega)$ by

$$d_{\mathrm{FH}}\gamma_- = \sum_{\gamma_+, A} \#\mathcal{M}^A(\gamma_-, \gamma_+)/\mathbb{R} e^A \gamma_+.$$

Note that the above sum can be infinite, because $\cup_{\omega(A)\le C}\mathcal{M}^A(\gamma_-, \gamma_+)/\mathbb{R}$ is compact for all $C < \infty$. However, the union of moduli space $\mathcal{M}^A(\gamma_-, \gamma_+)/\mathbb{R}$ over all homology classes $A \in H_2(X, \mathbb{Z})$ may be noncompact. This is why a partial completion is required in the definition of the ring Λ_ω.

Then $d_{\mathrm{FH}} \circ d_{\mathrm{FH}} = 0$ and the homology $F\mathbb{H}(X,\omega) = H(FH(X,\omega), d_{\mathrm{FH}})$ is independent of H and J. It is called the **Floer homology** of (X, ω). Moroever,

$$F\mathbb{H}_*(X,\omega) \simeq H_{*+n}(X, \Lambda_\omega)$$

by comparing the Floer and the Morse complexes for appropriate Hamiltonian functions H.

This theory was originally developed by Floer [12] in order to prove a homological version of the **Arnold conjecture**, stating that the number of contractible 1-periodic orbits for a time-dependent Hamiltonian on a closed symplectic manifold (X, ω) is bounded below by the sum of the Betti numbers of X.

1.4. Symplectic homology

Let (X, ω) be a symplectic manifold with convex boundary $Y = \partial X$. We shall apply the construction of Floer homology on the completed manifold $(\widehat{X}, \widehat{\omega})$. Since this manifold is not compact, it is necessary to specify the behavior of H and J on $\mathbb{R}^+ \times Y$.

We first choose $H : S^1 \times \widehat{X} \to \mathbb{R}$ so that

(i) $H < 0$ is a C^2-small Morse function on X;

(ii) $H(\theta, t, p) = h(t)$ is strictly increasing on $\mathbb{R}^+ \times Y$, such that $h''(t) - h'(t) > 0$ and, for t large enough, $h(t) = ae^t + b$,

with $a > 0$ is generic, so that X_H^θ has no 1-periodic orbit outside a compact set.

We then require that

$$J_\theta \xi = \xi, \qquad J_\theta \frac{\partial}{\partial t} = R_\alpha, \qquad \text{and } J_\theta \text{ is independent of } t,$$

when $t \in \mathbb{R}^+$ is large enough.

In this context, the homology of the Floer complex is denoted by $S\mathbb{H}(H, J)$. This graded module is in fact independent of J, but it does depend on H through the asymptotic slope $a > 0$.

In order to remove this dependence we use a direct limit and define the **symplectic homology** of (X, ω) by

$$S\mathbb{H}(X, \omega) = \varinjlim_{H} S\mathbb{H}(H),$$

where the direct limit uses monotonicity morphisms $\sigma : S\mathbb{H}(H_1) \to S\mathbb{H}(H_2)$ for $H_1 < H_2$, counting solutions of the Floer equation with an s-dependent Hamiltonian interpolating between H_1 and H_2.

To understand how $S\mathbb{H}(H)$ depends on the slope $a > 0$, consider the 1-periodic orbits of X_H^θ. These fall in two categories:

(i) the orbits in X are critical points of H;
(ii) all orbits contained in $\mathbb{R}^+ \times Y$ correspond to closed Reeb orbits on Y, because $X_H^\theta = -e^{-t}h'(t)R_\alpha$. We therefore obtain all closed Reeb orbits with period in $[0, a]$.

These two types of generators for the complex $SH(X, \omega)$ of symplectic homology are distinguished by the values of the action functional $\mathcal{A}_H$: there exists $\varepsilon > 0$ such that

(i) if $\mathcal{A}_H(\gamma) < \varepsilon$, then γ is a critical point of H in X. These generators therefore form a subcomplex $SH^-(X, \omega) \subset SH(X, \omega)$.
(ii) if $\mathcal{A}_H(\gamma) > \varepsilon$, then γ corresponds to a closed Reeb orbit on Y. These orbits generate the quotient complex $SH^+(X, \omega) = SH(X, \omega)/SH^-(X, \omega)$.

It turns out that the complex $(SH^-(X, \omega), \partial)$ coincides with the Morse complex of H on X, so that

$$S\mathbb{H}^-_*(X, \omega) \simeq H_{*+n}(X, \partial X; \Lambda_\omega).$$

Therefore, the short exact sequences of complexes induces in homology the exact triangle

$$\begin{array}{ccc} S\mathbb{H}_*(X,\omega) & \longrightarrow & S\mathbb{H}^+_*(X,\omega) \\ \nwarrow & & \swarrow_{[-1]} \\ & H_{*+n}(X,\omega) & \end{array}$$

For more details about the definition and the properties of symplectic homology, see [18].

Example 1.2. Let ω_0 be the standard symplectic structure on the ball $X = B^{2n}$. Then

$$S\mathbb{H}(B^{2n}, \omega_0) = 0.$$

To see this, if r denotes the radial coordinate, we have $e^t = r^2$ on $\widehat{X} = \mathbb{R}^{2n}$, so that we can choose the Hamiltonian H of the form $Ar^2 - B$ on $\mathbb{R}^{2n}$, with very large $A, B > 0$. For generic $A > 0$, we obtain a single 1-periodic orbit γ which is constant at the origin $r = 0$. Since this generator satisfies $|\gamma| \to \infty$ as $r \to \infty$, the direct limit over H (i.e., as $A \to \infty$) gives a vanishing symplectic homology in each degree.

In fact, it was proved by Cieliebak [7] that symplectic homology remains zero for any subcritical Stein manifold (X, ω). Inserting this result in the exact triangle, we obtain $S\mathbb{H}^+_*(X, \omega) \simeq H_{*+n-1}(X, \partial X; \Lambda_\omega)$.

1.5. Contact homology

Contact homology is a holomorphic curves invariant for a contact manifold (Y, ξ). It is a small part of a much more general framework for holomorphic curves invariants of contact manifolds and of symplectic manifolds with convex and concave boundaries. This framework is called Symplectic Field Theory and was introduced by Eliashberg, Givental and Hofer [11]. In fact, the specific invariant we are going to consider in these lectures is often called **linearized contact homology**, but since we are not going to consider other variants of contact homology in these notes, we will simply write contact homology.

Let us make a tentative definition of it based on the above constructions, noting that the complex SH^+ is generated by closed Reeb orbits in Y. Ignoring the symplectic filling (X, ω), let us concentrate on the symplectization $(\mathbb{R} \times Y, d(e^t \alpha))$ and set $H = 0$.

Tentative definition. We define a complex generated by closed Reeb orbits (of any period). The differential counts holomorphic cylinders, i.e. maps $F = (a, f)$ from $\mathbb{R} \times S^1$ to $\mathbb{R} \times Y$ satisfying $df \circ j = J \circ dF$, such that

$$\lim_{s \to \pm\infty} a(s, \theta) = \pm\infty, \qquad \lim_{s \to \infty} f(s, \theta) = \gamma_+(T_+\theta), \qquad \lim_{s \to -\infty} f(s, \theta) = \gamma_-(T_-\theta),$$

for some parametrizations γ_+ and γ_- of closed Reeb orbits with periods T_+ and T_-. The collection of such holomorphic cylinders, modulo biholomorphisms of the domain, is called **moduli space of holomorphic cylinders**.

This tentative definition has the following two problems :

(i) We need to exclude "bad orbits" in order to guarantee that the moduli spaces of holomorphic curves are orientable [3]. For any closed Reeb orbit γ, we denote by γ_1 the underlying simple orbit, i.e., the orbit with the same image and the minimal period. We say that the orbit γ is **good** if $\mu_{CZ}(\gamma) - \mu_{CZ}(\gamma_1)$ is even, otherwise we say that γ is **bad**.

(ii) The moduli spaces of holomorphic cylinders are compactified by adding configurations of holomorphic curves that are more general than cylinders. For instance, a 1-parameter family of holomorphic cylinders may degenerate in a rigid pair-of-pants with two negative punctures, attached to a rigid holomorphic plane and a vertical cylinder. In order to take these more general degenerations into account, we have to replace the above moduli spaces of holomorphic cylinders with **moduli spaces of capped holomorphic curves**. These moduli spaces $\mathcal{M}_c^A(\gamma_+, \gamma_-)$ consist of tuples of maps $(F_+, F_1, \dots, F_k)$, modulo biholomorphisms of their domains. The map

$$F_+ = (a_+, f_+) : \mathbb{R} \times S^1 \setminus \{y_1, \dots, y_k\} \to \mathbb{R} \times Y$$

satisfies $dF_+ \circ j = J \circ dF_+$ and

$$\lim_{s \to \pm\infty} a_+(s, \theta) = \pm\infty, \qquad \lim_{z \to y_i} a(z) = -\infty,$$

$$\lim_{s \to \infty} f_+(s, \theta) = \gamma_+(T_+\theta), \qquad \lim_{s \to -\infty} f_+(s, \theta) = \gamma_-(T_-\theta),$$

$$\lim_{\rho \to 0} f_+(y_i + \rho e^{2\pi i \vartheta}) = \gamma_i(T_i \vartheta),$$

for some parametrizations $\gamma_+, \gamma_-, \gamma_1, \dots, \gamma_k$ of closed Reeb orbits with periods $T_+, T_-, T_1, \dots, T_k$. For $i = 1, \dots, k$, the maps

$$F_i : \mathbb{C}P^1 \setminus \{\infty\} \to \widehat{X}$$

satisfy $dF_i \circ j = J \circ dF_i$. Moreover, a neighborhood U_∞ of $\infty \in \mathbb{C}P^1$ is mapped by F_i to $\mathbb{R}^+ \times Y$, so that we can write $F_i|_{U_\infty} = (a_i, f_i)$. These components satisfy

$$\lim_{|z| \to \infty} a_i(z) = \infty, \qquad \lim_{\rho \to \infty} f_i(\rho e^{2\pi i \vartheta}) = \gamma_i(T_i \vartheta).$$

For each $i = 1, \dots, k$ we fix a conformal identification of $\mathbb{C}P^1 \setminus \{\infty\} \simeq \mathbb{C}$ with $T_{y_i}\mathbb{R} \times S^1 \simeq \mathbb{C}$ which is compatible with the limits of f_+ and f_i along γ_i. Note that the number of possible conformal identifications is the multiplicity of the orbit γ_i. We finally require that the homology class of the surface obtained by attaching the images of $F_+, F_1, \dots, F_k$ with the capping disks D_{γ_-} and D_{γ_+} coincides with $A \in H_2(X, \mathbb{Z})$.

The above definition of capped holomorphic curves is a particular case of the more general concept of holomorphic building [2], which is very natural in Symplectic Field Theory.

In favorable circumstances, there exists generic J such that $\mathcal{M}_c^A(\gamma_+, \gamma_-)$ is a smooth manifold of dimension $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) + 2\langle c_1(X, \omega), A\rangle$. Note that unlike in symplectic homology, there are no simple assumptions to guarantee the existence of such a generic J. If such J does not exist, then one has to use other types of perturbations, such as in the theory polyfolds developed by Hofer, Wysocki and Zehnder [15, 16, 17].

Taking the above difficulties into account, we can modify the tentative definition of contact homology.

Actual definition. We define the **contact complex** as the module $CH(Y, \alpha)$ freely generated over Λ_ω by all good closed orbits of the Reeb field R_α. The grading of a generator γ is defined by $|\gamma| = \mu_{CZ}(\gamma) + n - 3$.

We define a **differential** $d_{\mathrm{CH}} : CH_*(Y,\alpha) \to CH_{*-1}(Y,\alpha)$ by

$$d_{\mathrm{CH}}\gamma_+ = \sum_{\gamma_-,A} \#\mathcal{M}_c^A(\gamma_+,\gamma_-)/\mathbb{R}\; e^A\gamma_-. \tag{3}$$

The homology $H(CH(Y,\alpha), d_{\mathrm{CH}})$ of this chain complex is independent of J and of the contact form α for ξ. It is denoted by $C\mathbb{H}(X,\omega)$ and called **linearized contact homology** of (Y,ξ), with respect to the filling (X,ω). In some cases, the resulting invariant depends only on the contact manifold (Y,ξ) and not on the choice of a symplectic filling. It is then usual to denote this invariant by $C\mathbb{H}(Y,\xi)$.

Example 1.3. The symplectic manifold $(X,\omega) = (B^{2n},\omega_0)$ has convex boundary $(Y,\xi) = (S^{2n-1}, \xi_0 = \ker\alpha_0)$ with $\alpha_0 = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)|_{S^{2n-1}}$. The perturbed contact form $\widetilde{\alpha} = \frac{1}{2}\sum_{i=1}^n a_i(x_i dy_i - y_i dx_i)|_{S^{2n-1}}$ with $a_i > 0$ defines a contact structure diffeomorphic to ξ_0. If the coefficients $a_1,\ldots,a_n$ are linearly independent over $\mathbb{Q}$, then the images of the simple closed orbits $\gamma_j, j = 1,\ldots,n$ of the Reeb field $R_{\widetilde{\alpha}}$ are given by

$$\{(x_1,y_1,\ldots,x_n,y_n) \in S^{2n-1} \subset \mathbb{R}^{2n} \mid x_i = y_i = 0, i \neq j\}.$$

The grading of γ_j and their multiples is always even, so that $d_{\mathrm{CH}} = 0$ independently of the symplectic filling. We obtain

$$C\mathbb{H}_k(S^{2n-1},\xi_0) = \begin{cases} \mathbb{Z} & \text{if } k \geq 2n-2 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

More generally, we can compute contact homology for the convex boundary (Y,ξ) of a subcritical Stein manifold (X,ω).

Theorem 1.1 (M.-L. Yau [21])**.** *Let (X,ω) be a subcritical Stein manifold such that $c_1(X,\omega) = 0$. Then*

$$C\mathbb{H}_k(X,\omega) = \bigoplus_{m\geq 0} H_{k-2m+2}(X,\partial X;\Lambda_\omega).$$

2. Relation between symplectic and contact homology

2.1. Exact triangles

Both invariants $S\mathbb{H}^+(X,\omega)$ and $C\mathbb{H}(X,\omega)$ are based on solutions of a Cauchy-Riemann type equation, with asymptotes corresponding to closed Reeb orbits in Y. The examples show that these two invariants are not isomorphic. There are indeed a number of differences in their definitions:

(i) The differential for SH counts objects in $\widehat{X}$, while the differential for CH counts objects living mainly in $\mathbb{R}\times Y$.
However, by "stretching the neck" of $\widehat{X}$ along $\{0\}\times Y$, we split off a copy of $\mathbb{R}\times Y$ in which the objects counted by the differential will mainly live in.

(ii) The symplectic complex uses a Hamiltonian function H, while $H = 0$ for the contact complex.

But we can define maps between these complexes, counting solutions of the Floer equation with a Hamiltonian function interpolating between H and 0 in $\mathbb{R} \times Y$.

(iii) Unlike SH, the complex CH is generated by closed orbits with multiplicities. But in fact, the multiples $\gamma^{(1)}, \gamma^{(2)}, \ldots$ of a closed Reeb orbit are realized as 1-periodic orbits of X_H^θ, for increasing values of t.

(iv) The bad orbits are excluded from CH, but not from SH. But a bad orbit gives rise to a pair of generators in SH; these cancel in homology, if we use Novikov coefficients over $\mathbb{Q}$.

(v) A 1-periodic orbit of X_H^θ has a fixed starting point, corresponding to $\theta = 0$, while there is no favorite basepoint on a closed Reeb orbit. This leads to an important difference between the complexes SH and CH, since any closed Reeb orbit gives rise to a pair of generators in SH. Intuitively, CH looks like a version of SH modulo an S^1-action.

The relation between $S\mathbb{H}^+(X,\omega)$ and $C\mathbb{H}(X,\omega)$ is decribed by the following result.

Theorem 2.1 (Bourgeois, Oancea [4])**.** *Let (X,ω) be a symplectically aspherical manifold with convex boundary (Y,ξ) such that there exists a regular J to define contact homology. Then there is an exact triangle with coefficients in $\Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$:*

$$\begin{array}{ccc} C\mathbb{H}(X,\omega) & \xrightarrow{[-2]} & C\mathbb{H}(X,\omega) \\ {\scriptstyle [n-3]}\nwarrow & & \swarrow{\scriptstyle [4-n]} \\ & S\mathbb{H}^+(X,\omega) & \end{array}$$

Note that the above exact triangle is analogous the the Gysin exact sequence for a circle bundle $S^1 \hookrightarrow V \to B$:

$$\begin{array}{ccc} H(B) & \xrightarrow{[-2]} & H(B) \\ {\scriptstyle [0]}\nwarrow & & \swarrow{\scriptstyle [+1]} \\ & H(V) & \end{array}$$

Since $H(B) \simeq H^{S^1}(V)$, this analogy motivates the construction of an S^1-equivariant version of symplectic homology. To that end, let us first recall the Borel construction of S^1-equivariant homology.

Borel construction. Let M be a smooth manifold with an S^1-action. Let ES^1 be a contractible topological space with a free S^1-action. Such a topological space is unique up to homotopy equivalence. We denote by $M \times_{S^1} ES^1$ the quotient of the product $M \times ES^1$ by the diagonal S^1-action. Note that the latter is free, since the action on the second factor is free. The S^1-equivariant homology of M is defined by $H^{S^1}(M) = H(M \times_{S^1} ES^1)$.

It is related to the ordinary homology of M by the exact triangle

$$\begin{array}{ccc} H^{S^1}(M) & \xrightarrow{[-2]} & H^{S^1}(M) \\ {\scriptstyle [0]}\nwarrow & & \swarrow{\scriptstyle [+1]} \\ & H(M) & \end{array}$$

Let us now adapt this classical construction to symplectic homology, following a sketch by Viterbo [20].

Symplectic construction. The sphere S^{2N+1} with the Hopf S^1-action can be viewed as a finite dimensional approximation of ES^1, because we have the S^1-equivariant inclusions $\ldots \hookrightarrow S^{2N+1} \hookrightarrow S^{2N+3} \hookrightarrow \ldots \hookrightarrow S^\infty = ES^1$.

We need the following geometric data:

(i) A family of compatible almost complex structures J_θ^λ, $\theta \in S^1$, $\lambda \in S^{2N+1}$, such that $J_{\theta+\tau}^{\tau\cdot\lambda} = J_\theta^\lambda$ for all $\tau \in S^1$.

(ii) A smooth function $H : S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ such that $H(\theta, t, p, \lambda) = ae^t + b(\lambda)$ for t sufficiently large on $\mathbb{R}^+ \times Y$, for some function $b : S^{2N+1} \to \mathbb{R}$, and $H(\theta + \tau, x, \tau \cdot \lambda) = H(\theta, x, \lambda)$ for any $\tau \in S^1$.

We then define a parametrized version of the action functional $\mathcal{A}_H$ from $C^\infty_{\mathrm{contr}}(S^1, \widehat{X}) \times S^{2N+1}$ to $\mathbb{R}$ by

$$\mathcal{A}_H(\gamma, \lambda) = -\int_{D^2} \sigma^*\omega - \int_{S^1} H(\theta, \gamma(\theta), \lambda)\, d\theta.$$

It follows from the assumptions of H and J that $\mathcal{A}_H$ is invariant under the S^1-action defined by $\tau \cdot (\gamma, \lambda) = (\gamma(\cdot + \tau), \tau \cdot \lambda)$.

In analogy with symplectic homology, we define the ***S*^{1}-equivariant symplectic complex** $SH^{S^1,N}(X, \omega)$ as the module freely generated over Λ_ω by the circles $\tau \cdot (\gamma, \lambda)$ of critical points of $\mathcal{A}_H$. These consist of a 1-periodic orbit γ of $X^\theta_{H_\lambda}$ and of $\lambda \in S^{2N+1}$ such that $\int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta = 0$.

The **differential** $\partial : SH_*^{S^1,N}(X, \omega) \to SH_{*-1}^{S^1,N}(X, \omega)$ counts rigid solutions (modulo the S^1-action) of

$$\begin{cases} \frac{\partial u}{\partial s} + J^\theta_{\lambda(s)}(u) \left(\frac{\partial u}{\partial \theta} - X^\theta_{H_{\lambda(s)}}(u) \right) = 0, \\ \dot{\lambda} - \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, u(s,\theta), \lambda)\, d\theta = 0, \end{cases}$$

such that

$$\lim_{s \to \pm\infty} (u(s, \theta), \lambda(s)) = (\gamma_\pm(\theta), \lambda_\pm).$$

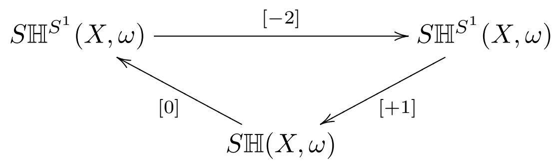
The homology $H(SH^{S^1,N}(X, \omega), \partial)$ of this complex is independent of J and is denoted by $S\mathbb{H}^{S^1,N}(H)$. In order to obtain an S^1-equivariant symplectic invariant, we use a double direct limit to define the ***S*^{1}-equivariant symplectic homology** of (X, ω) by

$$S\mathbb{H}^{S^1}(X, \omega) = \varinjlim_N \varinjlim_H S\mathbb{H}^{S^1,N}(H),$$

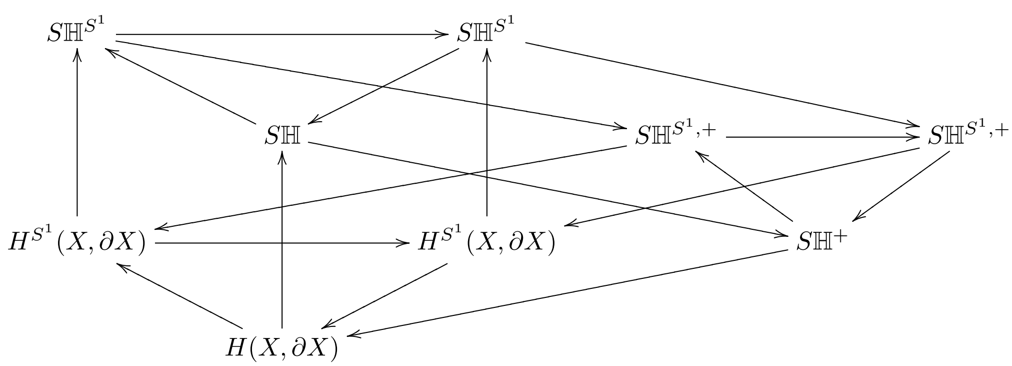
where the first direct limit uses morphisms $S\mathbb{H}^{S^1,N}(X,\omega) \to S\mathbb{H}^{S^1,N+1}(X,\omega)$ induced by the inclusions $S^{2N+1} \hookrightarrow S^{2N+3}$.

The construction of this invariant and its relation to the ordinary symplectic homology are provided by the following result.

Theorem 2.2 (Bourgeois, Oancea [5])**.** *Let (X,ω) be a symplectically aspherical manifold with convex boundary. Then its S^1-equivariant symplectic homology $S\mathbb{H}^{S^1}(X,\omega)$ is well-defined as a symplectic invariant and it fits into the exact triangle with coefficients in $\Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$:*

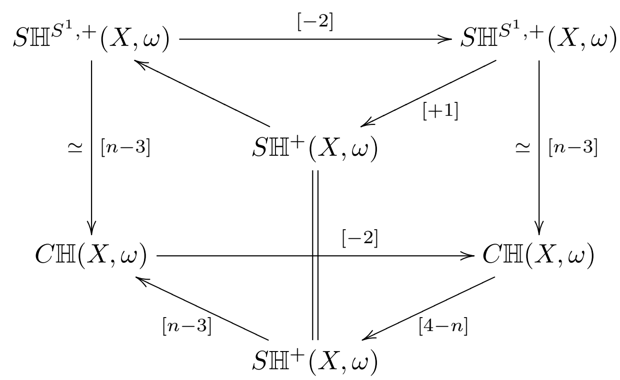


Like in ordinary symplectic homology, one can define a subcomplex $SH^{S^1,N,-}(X,\omega)$ with generators having action less than a small $\epsilon > 0$. This gives rise to an exact triangle involving $S\mathbb{H}^{S^1}(X,\omega)$, $S\mathbb{H}^{S^1,+}(X,\omega)$ and $S\mathbb{H}^{S^1,-}(X,\omega) \simeq H(X,\partial X;\Lambda_\omega) \otimes H(BS^1)$, where $BS^1 = ES^1/BS^1$. Combining with the above theorem, we obtain an exact triangle of exact triangles with coefficients in $\Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$:



The S^1-equivariant version of symplectic homology and its Gysin triangle relating it to the ordinary symplectic homology can lead to a better understanding of contact homology $C\mathbb{H}(X,\omega)$ and its relation to symplectic homology $S\mathbb{H}^+(X,\omega)$. This is the object of the following result.

Theorem 2.3 (Bourgeois, Oancea [6])**.** *Let (X,ω) be a symplectically aspherical manifold with convex boundary (Y,ξ), such that there exists a regular J to define contact homology. Then there is an isomorphism of exact triangles with coefficients in $\Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$:*



This result shows in particular that contact homology $C\mathbb{H}(X,\omega)$ can always be replaced with $S\mathbb{H}^{S^1,+}(X,\omega)$. The disadvantage is that the latter invariant has a more complicated definition, due to the S^{2N+1} factor from the Borel construction. The advantage is that we can always find a regular J for this latter invariant on (X,ω) symplectically aspherical, unlike for contact homology.

Example 2.1. Let (X,ω) be a subcritical Stein manifold with convex boundary (Y,ξ) such that $c_1(X,\omega)=0$. Since $S\mathbb{H}(X,\omega)=0$, we infer that

$$S\mathbb{H}_k^+(X,\omega) \simeq H_{k+n-1}(X,\partial X;\Lambda_\omega).$$

Similarly, we have $S\mathbb{H}^{S^1}(X,\omega)=0$, so that $S\mathbb{H}_k^{S^1,+}(X,\omega) \simeq H_{k+n-1}^{S^1}(X,\partial X;\Lambda_\omega)$.

Since the S^1-action on X is trivial, this implies that

$$C\mathbb{H}_{k+n-3}(X,\omega) \simeq S\mathbb{H}_k^{S^1,+}(X,\omega) \simeq \big(H(X,\partial X;\Lambda_\omega)\otimes H(BS^1)\big)_{k+n-1}.$$

In other words, we get

$$C\mathbb{H}_k(X,\omega) \simeq \bigoplus_{m\geq 0} H_{k-2m+2}(X,\partial X;\Lambda_\omega)$$

as before.

2.2. Common algebraic formalism

The complexes SH and SH^+ for symplectic homology admit an alternative definition which is formulated in terms of generators for the contact complex CH. This common algebraic formalism gives the possibility of a unified treatment of these three invariants. More details about the complexes described in this section can be found in [1].

Let us start with the alternative definition of SH^+. We define $\widetilde{CH}(X,\omega)$ as the module generated by all (good and bad) closed Reeb orbits in Y. We also define $\widehat{CH}(X,\omega)$ to be $\widetilde{CH}(X,\omega)[1]$, i.e., the same as the above module but with grading shifted by 1. Given a generator γ of $CH(Y,\xi)$, we denote by $\check{\gamma}$ and $\widehat{\gamma}$ the corresponding generators in $\widecheck{CH}(X,\omega)$ and in $\widehat{CH}(X,\omega)$ respectively. The **alternative SH^+ complex** is given by

$$SH^+(X,\omega) = \widecheck{CH}(X,\omega) \oplus \widehat{CH}(X,\omega).$$

We denote by $d_{\widecheck{CH}} : \widecheck{CH}_*(X,\omega) \to \widecheck{CH}_{*-1}(X,\omega)$ the differential defined by equation (3) where the summation is now over all (good and bad) orbits, and similarly for $d_{\widehat{CH}} : \widehat{CH}_*(X,\omega) \to \widehat{CH}_{*-1}(X,\omega)$.

We define a linear map $d_M : \widehat{CH}_*(X,\omega) \to \widecheck{CH}_{*-1}(X,\omega)$ by

$$d_M\widehat{\gamma} = \begin{cases} 0 & \text{if } \gamma \text{ is good,} \\ \pm 2\check{\gamma} & \text{if } \gamma \text{ is bad.} \end{cases}$$

On each closed Reeb orbit γ in Y, we fix a point P_γ. We define moduli spaces $\widecheck{\mathcal{M}'}^A_c(\gamma,\gamma')$ and $\widecheck{\mathcal{M}''}^A_c(\gamma,\gamma')$ as follows. The moduli space $\widecheck{\mathcal{M}'}^A_c(\gamma,\gamma')$ consists of elements $(F_+, F_1, \dots, F_k)$ in $\mathcal{M}^A_c(\gamma,\gamma')$ for which there exists $\theta_0 \in S^1$ such that

$$\lim_{s\to\infty} f_+(s,\theta_0) = P_\gamma \qquad \text{and} \qquad \lim_{s\to-\infty} f_+(s,\theta_0) = P_{\gamma'}.$$

The moduli space $\widecheck{\mathcal{M}''}^A_c(\gamma,\gamma')$ consists of pairs of elements

$$((F^1_+, F^1_1, \dots, F^1_{k_1}), (F^2_+, F^2_1, \dots, F^2_{k_2})) \in \mathcal{M}^{A_1}_c(\gamma,\gamma'')/\mathbb{R} \times \mathcal{M}^{A_2}_c(\gamma'',\gamma')$$

with $A_1 + A_2 = A$ and for some closed Reeb orbit γ'' such that the points

$$\lim_{s\to-\infty} f^1_+(s,\theta^1_0), \qquad \lim_{s\to\infty} f^2_+(s,\theta^2_0) \qquad \text{and} \qquad P_{\gamma''}$$

lie on γ'' in the cyclic order induced by the direction of the Reeb field, where $\theta^1_0, \theta^2_0 \in S^1$ are defined by

$$\lim_{s\to\infty} f^1_+(s,\theta^1_0) = P_\gamma \qquad \text{and} \qquad \lim_{s\to-\infty} f^2_+(s,\theta^2_0) = P_{\gamma'}.$$

Finally, let $\widecheck{\mathcal{M}}^A_c(\gamma,\gamma') = \widecheck{\mathcal{M}'}^A_c(\gamma,\gamma') \cup \widecheck{\mathcal{M}''}^A_c(\gamma,\gamma')$. Under favorable circumstances as for contact homology, and for a generic choice of the points P_γ, this moduli space is a smooth manifold of dimension $\mu_{CZ}(\gamma) - \mu_{CZ}(\gamma') + 2\langle c_1(X,\omega), A\rangle - 1$.

We define a linear map $\delta : \widecheck{CH}_*(X,\omega) \to \widehat{CH}_{*-1}(X,\omega)$ by

$$\delta\check{\gamma} = \sum_{\gamma',A} \#\widecheck{\mathcal{M}}^A_c(\gamma,\gamma')/\mathbb{R}\; e^A \widehat{\gamma'}.$$

The **differential** $d_{SH^+} : SH^+_*(X,\omega) \to SH^+_{*-1}(X,\omega)$ is defined, with respect to the above decomposition of SH^+, by

$$d_{SH^+} = \begin{pmatrix} d_{\widecheck{CH}} & d_M \\ \delta & d_{\widehat{CH}} \end{pmatrix}.$$

With these definitions, the homology $H(SH^+(X,\omega), d_{SH^+})$ is isomorphic to the symplectic homology $S\mathbb{H}^+(X,\omega)$ defined in section 1.4. The generators $\check{\gamma}$ and $\widehat{\gamma}$ of SH^+ correspond to nondegenerate 1-periodic Hamiltonian orbits obtained from the closed Reeb orbit γ by a time-dependent perturbation. The part d_M of the differential is responsible for the elimination of bad orbits over rational coefficients, as explained in section 2.1. The differentials $d_{\widetilde{CH}}$ and $d_{\widehat{CH}}$ restricted to good orbits then coincide with the contact homology differentials. Finally, the δ part of the differential induces the contact homology endomorphism of degree -2 in the exact sequence in Theorem 2.1.

We now give the alternative definition of SH. Let $\tilde{f} : X \to \mathbb{R}$ be a Morse function such that $\tilde{f}|_{\partial X}$ is constant. Let $(\mathrm{Morse}(\tilde{f}), d_{\mathrm{Morse}})$ be the Morse complex of $\tilde{f}$. The **alternative *SH* complex** is given by

$$SH(X,\omega) = SH^+(X,\omega) \oplus \mathrm{Morse}(\tilde{f})[-n].$$

Given a closed Reeb orbit γ and a critical point p of $\tilde{f}$, we define a moduli space $\mathcal{M}(\gamma, p)$ consisting of maps

$$F : \mathbb{C}P^1 \setminus \{\infty\} \to \widehat{X}$$

satisfying $dF \circ j = J \circ dF$ modulo biholomorphisms of the domain preserving 0. Moreover, a neighborhood U_∞ of $\infty \in \mathbb{C}P^1$ is mapped by F to $\mathbb{R}^+ \times Y$, so that we can write $F|_{U_\infty} = (a, f)$. These components satisfy

$$\lim_{|z|\to\infty} a(z) = \infty, \qquad \lim_{\rho\to\infty} f(\rho e^{2\pi i\vartheta}) = \gamma(T\vartheta),$$

for some parametrization of the closed Reeb orbit γ with period T. We also require that $F(0)$ belongs to the unstable manifold of p for the Morse function $\tilde{f}$ with respect to a fixed Riemannian metric on X. The gradient trajectories of $\tilde{f}$ are extended to $\widehat{X}$ using integral curves of $\frac{\partial}{\partial t}$ on $\mathbb{R}^+ \times Y$.

Let $\theta : SH^+(X,\omega) = \widetilde{CH}(X,\omega) \oplus \widehat{CH}(X,\omega) \to \mathrm{Morse}(\tilde{f})$ be the linear map defined by

$$\theta\widehat{\gamma} = 0, \qquad \theta\check{\gamma} = \begin{cases} \sum_{p\in\mathrm{Crit}(\tilde{f})} \#\mathcal{M}(\gamma,p)\ p & \text{if } \gamma \text{ is good,} \\ 0 & \text{if } \gamma \text{ is bad.} \end{cases}$$

The **differential** $d_{SH} : SH_*(X,\omega) \to SH_{*-1}(X,\omega)$ is defined, with respect to the above decomposition of SH, by

$$d_{SH} = \begin{pmatrix} d_{SH^+} & 0 \\ \theta & d_{\mathrm{Morse}} \end{pmatrix}.$$

With these definitions, the homology $H(SH(X,\omega), d_{SH})$ is isomorphic to the symplectic homology $S\mathbb{H}(X,\omega)$ defined in section 1.4.

3. Effect of Legendrian surgery

The common formalism described in the previous section can be used to describe simultaneously the effect of a Legendrian surgery on Y (i.e., attaching a critical handle to X) for the three invariants $C\mathbb{H}$, $S\mathbb{H}$ and $S\mathbb{H}^+$.

The Legendrian surgery operation is a particular case of the more general contact surgery. The latter consists of attaching a handle to X along an isotropic sphere of (Y, ξ). For an explicit model of the handles and the attaching procedure, see [22]. Legendrian surgery corresponds to the critical case, i.e., an attachment along a Legendrian sphere Λ in (Y, ξ). Subcritical Stein manifolds are the result of subcritical handle attachments to the standard symplectic ball (B^{2n}, ω_0). As hinted at the end of section 1.4, symplectic homology $S\mathbb{H}(X, \omega)$ is not affected by subcritical handle attachments. On the other hand, general Stein manifolds can be obtained from subcritical Stein manifolds by Legendrian surgery along a collection of Legendrian spheres. As we will see in this section, the three holomorphic curve invariants that we consider in these notes are deeply affected by such an operation.

For simplicity, we will consider only the case of Legendrian surgery along a single Legendrian sphere. In order to understand why the effect of Legendrian surgery differs so much from the effect of subcritical surgeries, let us compare the Reeb dynamics on the boundary of the corresponding handles:

(i) On the boundary $D^k \times S^{2n-k-1}$, $k < n$ of a subcritical handle, we can find:
  - Closed Reeb orbits foliating the sphere $\{0\} \times S^{2n-2k-1}$. These do not contribute to symplectic homology, for the same reason as on S^{2n-1}, like in example 1.2.
  - Reeb trajectories leaving the handle. After a perturbation of the isotropic sphere to which the handle is attached, the return time of these trajectories to the handle can be made arbitrarily long, by reducing the size of attaching locus of the handle around the isotropic sphere. For this reason, closed Reeb orbits that are only partially contained in the handle do not contribute to $S\mathbb{H}$, $S\mathbb{H}^+$ or $C\mathbb{H}$.

(ii) On the boundary ST^*D^n of a critical handle, the Reeb dynamics coincide with the geodesic flow on the disk D^n. Therefore, we can find:
  - No closed Reeb orbits in this critical handle, since all Reeb trajectories project to a piece of line moving across D^n.
  - Reeb trajectories leaving the handle. These can return to the handle after a bounded time, since generically a Legendrian sphere Λ will admit Reeb chords, i.e., integral curves of the Reeb field starting and ending on Λ. Since geodesic segments can join any pair of points on $\partial D^n \simeq \Lambda$, one can easily get convinced that the closed Reeb orbits that are created by a Legendrian surgery along Λ are in bijective correspondence with cyclic tuples of Reeb chords of Λ.

This last category of closed Reeb orbits should contribute to our holomorphic curve invariants. Moreover, these orbits depend on the interactions of the Reeb dynamics with the Legendrian sphere Λ. It is therefore to be expected that the effect of Legendrian surgery will be described in terms of a holomorphic curve invariant of Λ, defined in the same spirit as symplectic and contact homology.

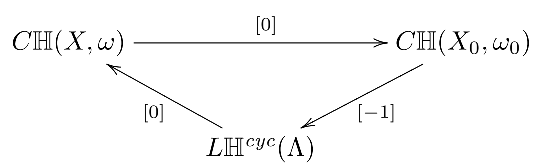
We define $LH(\Lambda)$ as the noncommutative unital algebra freely generated over $\Lambda_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$ by the Reeb chords of Λ. This algebra can be equipped with a differential d_{LH} mapping $LH_*(\Lambda)$ to $LH_{*-1}(\Lambda)$ satisfying the Leibniz rule and counting some holomorphic disks in $\mathbb{R} \times Y$ with boundary on $\mathbb{R} \times \Lambda$. The corresponding homology $L\mathbb{H}(\Lambda) = H(LH(\Lambda), d_{LH})$ is an invariant of the Legendrian isotopy class of Λ. For a definition of this invariant for Legendrian submanifolds, see [8, 9].

Cyclic version. We now define a variant of the complex $(LH(\Lambda), d_{LH})$, which plays an important role to describe the effect of Legendrian surgery on $C\mathbb{H}(X, \omega)$. We denote by $LH^+(\Lambda) = LH(\Lambda)/\langle 1 \rangle$ the module generated by words of Reeb chords for Λ. Let $P : LH^+(\Lambda) \to LH^+(\Lambda)$ be the linear map defined by

$$P(c_1 \dots c_k) = (-1)^{|c_k|(|c_1|+\dots+|c_{k-1}|)} c_k c_1 \dots c_{k-1}.$$

We define $LH^{\mathrm{cyc}}(\Lambda) = LH^+(\Lambda)/\mathrm{im}(I - P)$. It is the module generated by cyclic words of Reeb chords for Λ. Note that, since $\langle 1 \rangle$ and $\mathrm{im}(I - P)$ are subcomplexes, the differential d_{LH} induces a differential d_{cyc} on $LH^{\mathrm{cyc}}(\Lambda)$. The corresponding homology $L\mathbb{H}^{\mathrm{cyc}}(\Lambda) = H(LH^{\mathrm{cyc}}(\Lambda), d_{\mathrm{cyc}})$ is again an invariant of the Legendrian isotopy class of Λ.

Theorem 3.1 (Bourgeois, Ekholm, Eliashberg [1]). *Let (X_0, ω_0) be a symplectic manifold with convex boundary (Y_0, ξ_0) such that $C\mathbb{H}(Y_0, \xi_0)$ is defined. Let Λ be a Legendrian sphere in (Y_0, ξ_0) and let (X, ω) be the result of a critical handle attachment on (X_0, ω_0) along Λ. Then we have an exact triangle*



In the above Theorem, the map $C\mathbb{H}(X, \omega) \to C\mathbb{H}(X_0, \omega_0)$ counts rigid capped holomorphic curves in the symplectic cobordism $X \setminus X_0$. The map $C\mathbb{H}(X_0, \omega_0) \to L\mathbb{H}^{\mathrm{cyc}}(\Lambda)$ counts rigid capped holomorphic curves in the symplectization $\mathbb{R} \times X_0$ with boundary on $\mathbb{R} \times \Lambda$, that are asymptotic to a closed Reeb orbit at the convex end and to Reeb chords of Λ at the concave end.

Noncyclic version. We now define another variant of the complex $(LH(\Lambda), d_{LH})$, which plays an important role to describe the effect of Legendrian surgery on $S\mathbb{H}^+(X, \omega)$. We define a module $LH^{\mathrm{Ho},+}(\Lambda) = \widetilde{LH}^+(\Lambda) \oplus \widehat{LH}^+(\Lambda)$, where $\widetilde{LH}^+(\Lambda) = LH^+(\Lambda)$ and $\widehat{LH}^+(\Lambda) = LH^+(\Lambda)[1]$. If $w = c_1 \dots c_k$ is a word of Reeb chords for Λ in $LH^+(\Lambda)$ we denote by $\breve{w} = \breve{c}_1 c_2 \dots c_k$ and by $\widehat{w} = \widehat{c}_1 c_2 \dots c_k$ the corresponding generators in $\widetilde{LH}^+(\Lambda)$ and in $\widehat{LH}^+(\Lambda)$ respectively. In what follows, we will sometimes write a word with a decoration (check or hat) which is not in the first position. As a convention, we identify such a word with its cyclic permutation starting with the decorated chord, multiplied with a sign as in the definition of P, where $|c| = |\breve{c}| = |\widehat{c}| - 1$.

We denote by $\check{d} : \widecheck{LH}^+_*(X,\omega) \to \widecheck{LH}^+_{*-1}(X,\omega)$ the differential defined by

$$\check{d}\check{w} = \widecheck{(d_{LH} w)}.$$

Let $S : LH(\Lambda) \to \widehat{LH}^+(\Lambda)$ be the linear map defined by $S(1) = 0$ and

$$S(c_1 \dots c_k) = \widehat{c_1} c_2 \dots c_k + \dots + (-1)^{|c_1|+|\dots+|c_{k-1}|} c_1 \dots c_{k-1} \widehat{c_k}.$$

We denote by $\widehat{d} : \widehat{LH}^+_*(X,\omega) \to \widehat{LH}^+_{*-1}(X,\omega)$ the differential defined by

$$\widehat{d}(\widehat{c} w') = S(d_{LH} c) w' + (-1)^{|c|+1} \widehat{c}(d_{LH} w').$$

We define a linear map $\alpha : \widehat{LH}^+_*(\Lambda) \to \widecheck{LH}^+_{*-1}(\Lambda)$ by

$$\alpha(\widehat{c_1} c_2 \dots c_k) = \widecheck{c_1} c_2 \dots c_k - c_1 \dots c_{k-1} \widecheck{c_k}.$$

The **differential** $d_{\mathrm{Ho},+} : LH^{\mathrm{Ho},+}_*(\Lambda) \to LH^{\mathrm{Ho},+}_{*-1}(\Lambda)$ is defined, with respect to the above decomposition of $LH^{\mathrm{Ho},+}$, by

$$d_{\mathrm{Ho},+} = \begin{pmatrix} \check{d} & \alpha \\ 0 & \widehat{d} \end{pmatrix}.$$

Then $d_{\mathrm{Ho},+} \circ d_{\mathrm{Ho},+} = 0$ and the homology $L\mathbb{H}^{\mathrm{Ho},+}(\Lambda) = H(LH^{\mathrm{Ho},+}(\Lambda), d_{\mathrm{Ho},+})$ of the corresponding complex is again an invariant of the Legendrian isotopy class of Λ.

Note that, with the above algebraic definitions, we have an exact triangle

$$\begin{array}{ccc} L\mathbb{H}^{\mathrm{cyc}}(\Lambda) & \longrightarrow & L\mathbb{H}^{\mathrm{cyc}}(\Lambda) \\ \nwarrow & & \swarrow \\ & L\mathbb{H}^{\mathrm{Ho},+}(\Lambda) & \end{array}$$

which is similar to the Connes exact triangle for cyclic and Hochschild homologies. The analogy between the above exact triangle and the exact triangle in Theorem 2.1 motivates the following result.

Theorem 3.2 (Bourgeois, Ekholm, Eliashberg [1])**.** *Under the same assumptions as in Theorem 3.1, we have an exact triangle*

$$\begin{array}{ccc} S\mathbb{H}^+(X,\omega) & \xrightarrow{[0]} & S\mathbb{H}^+(X_0,\omega_0) \\ {\scriptstyle [0]}\nwarrow & & \swarrow{\scriptstyle [-1]} \\ & L\mathbb{H}^{Ho,+}(\Lambda) & \end{array}$$

Expanded version. We finally define another variant of the complex $(LH(\Lambda), d_{LH})$, which plays an important role to describe the effect of Legendrian surgery on $S\mathbb{H}(X,\omega)$. We define an expanded module $LH^{\mathrm{Ho}}(\Lambda) = LH^{\mathrm{Ho},+}(\Lambda) \oplus \langle \tau \rangle$. The unique generator for the second summand formally represents the unique connected component of Λ.

We define a linear map $T : LH^{\mathrm{Ho},+}(\Lambda) \to \langle \tau \rangle$ by

$$\begin{aligned} T(\breve{c}) &= \#\mathcal{M}(c)/\mathbb{R}\ \tau, \\ T(\breve{w}) &= 0 \text{ if } w \text{ contains more than one letter}, \\ T(\widehat{w}) &= 0. \end{aligned}$$

In this definition, the moduli space $\mathcal{M}(c)$ consists of holomorphic disks in $\mathbb{R} \times Y$ with boundary on $\mathbb{R} \times \Lambda$, having one boundary puncture converging at the convex end of $\mathbb{R} \times Y$ to the Reeb chord c.

The **differential** $d_{\mathrm{Ho}} : LH_*^{\mathrm{Ho}}(\Lambda) \to LH_{*-1}^{\mathrm{Ho}}(\Lambda)$ is defined, with respect to the above decomposition of LH^{Ho}, by

$$d_{\mathrm{Ho}} = \begin{pmatrix} d_{\mathrm{Ho},+} & 0 \\ T & 0 \end{pmatrix}.$$

Then $d_{\mathrm{Ho}} \circ d_{\mathrm{Ho}} = 0$ and the corresponding homology $L\mathbb{H}^{\mathrm{Ho}}(\Lambda) = H(LH^{\mathrm{Ho}}(\Lambda), d_{\mathrm{Ho}})$ is again an invariant of the Legendrian isotopy class of Λ.

Theorem 3.3 (Bourgeois, Ekholm, Eliashberg [1]). *Under the same assumptions as in Theorem 3.1, we have an exact triangle*

$$\begin{array}{ccc} S\mathbb{H}(X,\omega) & \xrightarrow{[0]} & S\mathbb{H}(X_0,\omega_0) \\ {\scriptstyle [0]}\nwarrow & & \swarrow{\scriptstyle [-1]} \\ & L\mathbb{H}^{Ho}(\Lambda) & \end{array}$$

In this last result, note that for a subcritical Stein manifold (X_0, ω_0), since we have $S\mathbb{H}(X_0, \omega_0) = 0$, it follows that

$$S\mathbb{H}(X, \omega) \simeq L\mathbb{H}^{\mathrm{Ho}}(\Lambda).$$

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