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On real determinantal quartics

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Abstract. We describe all possible arrangements of the ten nodes of a generic real determinantal quartic surface in $\mathbb{P}^3$ with nonempty spectrahedral region.

1. Introduction

1.1. Motivation

It is a common understanding that, thanks to the global Torelli theorem for $K3$-surfaces [16] and surjectivity of the period map [13], any reasonable question concerning the topology of singular or real $K3$-surfaces can be reduced to a certain arithmetic problem; many examples, treating the two subjects separately, are found in the literature. However, there are but a few papers where objects that are both real and singular are considered; one can mention [11] and [14], which deal, respectively, with real sextics with a single node in $\mathbb{P}^2$ and real quartic surfaces with a single node in $\mathbb{P}^3$.

In the present paper, we make an attempt to advance this line of research, considering real quartic surfaces with several nodes. Special attention is paid to degenerations of nonsingular quartics, which are used to control the topology of the resulting singular surfaces. Since the classical problem of enumerating all equivariant equisingular deformation types seems rather hopeless (one would expect thousands of classes), we confine ourselves to a very special example arising from convex algebraic geometry. Namely, we describe arrangements of the ten nodes of a generic determinantal quartic with nonempty spectrahedral region, see next subsection for details.

1.2. Principal results

Consider a generic dimension 3 real linear system V of quadrics in $\mathbb{P}^3$. Singular quadrics form a surface $X \subset V \cong \mathbb{P}^3$ of degree 4, which is called a *transversal determinantal quartic* (see Section 4 for details and precise definitions). In other words, we consider a quartic surface $X \subset \mathbb{P}^3$ given by an equation of the form $\det \sum_{i=0}^{3} x_i \bar{q}_i = 0$, where $[x_0 : x_1 : x_2 : x_3]$ are homogeneous coordinates in $\mathbb{P}^3$ and $\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3$ are certain fixed

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nonzero symmetric (4×4)-matrices. Generically, such a surface is known to have ten nodes.

Whenever present, quadrics given by definite quadratic forms constitute a single connected component of the complement $V_{\mathbb{R}} \smallsetminus X_{\mathbb{R}}$; this component is called the *spectrahedral region* of V. This construction is a special case of a more general framework, see [20] and [17] for details. The study of the shapes of various spectrahedra is a major problem of convex algebraic geometry.

A transversal determinantal quartic has ten nodes, and the original question posed to us by B. Sturmfels was whether all ten can be located in the boundary of the spectrahedral region. (The best known example, constructed explicitly, had eight nodes in the boundary.) We answer this question in the affirmative; moreover, we describe all possible arrangements of the nodes with respect to the components of the complement $V_{\mathbb{R}} \smallsetminus X_{\mathbb{R}}$.

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a transversal real determinantal quartic with nonempty spectrahedral region R. Then X has an even number $m \geqslant 0$ of real nodes in the boundary of R and an even number $n \geqslant 0$ of real nodes disjoint from R, so that $2 \leqslant m + n \leqslant 10$. Any pair of even numbers $m, n \geqslant 0$, $2 \leqslant m + n \leqslant 10$, is realized by a quartic as above.*

This theorem is proved in Subsection 5.3.

Remark 1.1. It is worth emphasizing that any transversal real determinantal quartic with nonempty spectrahedral region has at least two real nodes. Note that a similar, and even stronger, statement holds for transversal real determinantal cubics in $\mathbb{P}^3$, which are discriminants of linear systems of plane conics: such a cubic (which is necessarily a Cayley cubic) has nonempty spectrahedral region if and only if at least one of its four nodes is real.

It would be interesting to find explicit matrix representations of transversal real determinantal quartics for all values of m and n allowed by Theorem 1.1.

1.3. Contents of the paper

To prove Theorem 1.1, we analyze the equisingular stratification of the space of complex quartics in $\mathbb{P}^3$, Section 2, and identify the stratum that is formed, up to codimension one subset, by the transversal determinantal quartics, Section 4. Then we describe the sets of cycles that can vanish under certain special nodal degenerations of a real quartic surface, see Section 3. Finally, in Section 5, we show that each transversal real determinantal quartic is obtained by a degeneration of a nonsingular quartic with two nested spheres (the so called *hyperbolic quartic*), and use previously known arithmetical computations in order to construct/prohibit various degenerations of the latter.

2. Singular quartics in $\mathbb{P}^3$

The principal result of this section is Theorem 2.1, which enumerates the equisingular strata of the space of quartics in $\mathbb{P}^3$.

2.1. Integral lattices

A *lattice* is a finitely generated free abelian group L supplied with a symmetric bilinear form $b : L \otimes L \to \mathbb{Z}$. We abbreviate $b(x, y) = x \cdot y$ and $b(x, x) = x^2$. A lattice L is *even* if $x^2 = 0 \bmod 2$ for all $x \in L$. As the transition matrix between two integral bases has determinant ± 1, the determinant $\det L \in \mathbb{Z}$ (*i.e.*, the determinant of the Gram matrix of b in any basis of L) is well defined. A lattice L is called *nondegenerate* if the determinant $\det L \neq 0$; it is called *unimodular* if $\det L = \pm 1$.

Given a lattice L, the bilinear form can be extended to $L \otimes \mathbb{Q}$ by linearity. If L is nondegenerate, the dual group $L^\vee = \operatorname{Hom}(L, \mathbb{Z})$ can be identified with the subgroup

$$\bigl\{x \in L \otimes \mathbb{Q} \bigm| x \cdot y \in \mathbb{Z} \text{ for all } x \in L\bigr\}.$$

In particular, L itself can be identified with a subgroup of $L^\vee$.

The group of isometries of a lattice L is denoted by $\operatorname{Aut} L$. Given a vector $a \in L$, $a^2 \neq 0$, the *reflection* against (the hyperplane orthogonal to) a is the automorphism $r_a \colon L \to L$, $x \mapsto x - 2(x \cdot a)a/a^2$, provided that it is well defined, *i.e.*, takes integral vectors to integral vectors. The reflection is always well defined if $a^2 = \pm 1$ or ± 2; if $a^2 = \pm 4$, the reflection r_a is well defined if and only if $a = 0 \bmod 2L^\vee$.

A nondegenerate lattice L is called *elliptic* or *hyperbolic* if its positive inertia index equals 0 or 1, respectively. To any hyperbolic lattice H one can associate a hyperbolic space $\mathbb{P}(C) := C/\mathbb{R}^*$, where $C = C_H := \{x \in H \otimes \mathbb{R} \mid x^2 > 0\}$ is the *positive cone* of H. In particular, given a lattice L and an isometric involution $c \colon L \to L$ with hyperbolic invariant sublattice $L^c_+ = \{x \in L \mid c(x) = x\}$, one can define the space $\mathbb{P}(C^c_+)$. Any subgroup $G \subset \operatorname{Aut} H$ generated by (some) reflections $r_a \colon H \to H$ defined by vectors $a \in H$ with $a^2 < 0$ admits a polyhedral fundamental domain $\mathfrak{P}_G \subset \mathbb{P}(C)$: it is the closure of (any) connected component of the space $\mathbb{P}(C)$ with all mirrors of G removed.

All lattices considered in the paper are even. A *root* in an even lattice is a vector of square (-2). A *root system* is an elliptic lattice generated by roots. We use the standard notation $\mathbf{A}_p$, $p \geqslant 1$, $\mathbf{D}_q$, $q \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ for the irreducible root systems of the same name. Let $\mathbf{U} = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$, $u_1^2 = u_2^2 = 0$, $u_1 \cdot u_2 = 1$; this lattice is called the *hyperbolic plane*, and any basis (u_1, u_2) as above is called a *standard basis* for $\mathbf{U}$. Given a lattice L and an integer d, the notation $L(d)$ stands for the lattice obtained from L by multiplying the values of the bilinear form by d.

2.2. Singular homological types

Definition 2.1. A *set of* (*simple*) *singularities* is a pair (Σ, σ), where Σ is a root system and σ is a collection of roots of Σ constituting a Weyl chamber of Σ. An isometry $\Sigma_1 \to \Sigma_2$ of two sets of singularities (Σ_i, σ_i), $i = 1, 2$, is *admissible* if it takes σ_1 to σ_2.

Remark 2.1. Any Weyl chamber of a root system Σ can be taken to any other Weyl chamber by an element of the Weyl group of Σ, which extends to any larger lattice containing Σ. For this reason, when speaking about the isomorphism classification of sets of singularities, configurations, and singular homological types (see below), the subset σ in Definition 2.1 can be, and often is disregarded.

Definition 2.2. A *configuration* (extending a given set of singularities (Σ, σ)) is a finite index extension $\tilde{S} \supset S := \Sigma \oplus \mathbb{Z}h$, $h^2 = 4$, satisfying the following conditions:

(1) each root $r \in \tilde{S} \cap (\Sigma \otimes \mathbb{Q})$ belongs to Σ;
(2) $\tilde{S}$ does not contain an element u with $u^2 = 0$ and $u \cdot h = 2$.

An *admissible isometry* of two configurations $\tilde{S}_i \supset S_i = \Sigma_i \oplus \mathbb{Z}h_i$, $i = 1, 2$, is an isometry $\tilde{S}_1 \to \tilde{S}_2$ taking h_1 to h_2 and inducing an admissible isometry $\Sigma_1 \to \Sigma_2$.

Definition 2.3. A *singular homological type* (extending a set of singularities (Σ, σ)) is an extension of the orthogonal direct sum $S := \Sigma \oplus \mathbb{Z}h$, $h^2 = 4$, to a lattice L isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$, such that the primitive hull $\tilde{S}$ of S in L is a configuration. (The singular homological type is also said to *extend* the configuration $\tilde{S} \supset S$.) An *isomorphism* between two singular homological types $L_i \supset S_i \supset \sigma_i \cup \{h_i\}$, $i = 1, 2$, is an isometry $L_1 \to L_2$ taking h_1 to h_2 and σ_1 to σ_2 (as a set).

A singular homological type is uniquely determined by the collection $\mathcal{H} = (L, h, \sigma)$; then $\Sigma = \Sigma_{\mathcal{H}}$ is the sublattice spanned by σ, and $S = S_{\mathcal{H}} = \Sigma \oplus \mathbb{Z}h$.

Given a singular homological type $\mathcal{H}$, the orthogonal complement $S_{\mathcal{H}}^{\perp}$ is a nondegenerate lattice of positive inertia index 2. Hence, the orthogonal projection of any positive definite 2-subspace $\omega_1 \subset S_{\mathcal{H}}^{\perp} \otimes \mathbb{R}$ to any other such subspace ω_2 is an isomorphism of vector spaces; it can be used to compare orientations of ω_1 and ω_2. Thus, a choice of an orientation of a positive definite 2-subspace in $S_{\mathcal{H}}^{\perp} \otimes \mathbb{R}$ defines a coherent orientation of any other.

Definition 2.4. An *orientation* of a singular homological type $\mathcal{H} = (L, h, \sigma)$ is a choice of coherent orientations of positive definite 2-subspaces of $S_{\mathcal{H}}^{\perp} \otimes \mathbb{R}$. Oriented singular homological types $(\mathcal{H}_i, \mathfrak{o}_i)$, $i = 1, 2$, are *isomorphic* if there is an isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$ taking $\mathfrak{o}_1$ to $\mathfrak{o}_2$. A singular homological type $\mathcal{H}$ is called *symmetric* if $(\mathcal{H}, \mathfrak{o}) \cong (\mathcal{H}, -\mathfrak{o})$, *i.e.*, it $\mathcal{H}$ admits an automorphism reversing orientation.

2.3. Classification of singular quartics

Let $X \subset \mathbb{P}^3$ be a quartic surface with simple singularities only. Denote by $\tilde{X} \to X$ the minimal resolution of singularities of X; it is a minimal $K3$-surface. Introduce the following objects:

- $L_X = H_2(\tilde{X}) = H^2(\tilde{X})$, regarded as a lattice *via* the intersection form (we always identify homology and cohomology *via* the Poincaré duality);
- $\sigma_X \subset L_X$, the set of the classes of the exceptional divisors contracted by the blow-up map $\tilde{X} \to X$;
- $h_X \in L_X$, the class of the pull-back of a generic plane section of X;
- $\omega_X \subset L_X \otimes \mathbb{R}$, the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\tilde{X}$ (the *period* of $\tilde{X}$).

Note that ω_X is positive definite. According to [18, 21], a triple $\mathcal{H} = (L, h, \sigma)$ has the form (L_X, h_X, σ_X) for a quartic $X \in \mathbb{P}^3$ as above if and only if it is a singular homological type in the sense of Definition 2.3. If this is the case, the above orientation of ω_X defines an orientation of $\mathcal{H}$.

The following theorem is quite expectable; however, we could not find an explicit statement in the literature. The surjectivity part is contained in [21].

Theorem 2.1. *The map sending a quartic surface $X \subset \mathbb{P}^3$ with simple singularities to the pair consisting of its singular homological type $\mathcal{H}_X = (L_X, h_X, \sigma_X)$ and the orientation of the space ω_X establishes a one-to-one correspondence between the set of equisingular deformation classes of quartics with a given set of simple singularities (Σ, σ) and the set of isomorphism classes of oriented abstract singular homological types extending (Σ, σ).*

Proof. Proof of this theorem repeats, almost literally, the proof of a similar theorem for plane sextic curves, see [5]. It is based on Beauville's construction [1] of a fine period space of marked polarized $K3$-surfaces. We omit the details. □

The equisingular stratum of the space of quartic surfaces in $\mathbb{P}^3$ corresponding to an oriented singular homological type $(\mathcal{H}, \mathfrak{o})$ will be denoted by $\mathcal{M}(\mathcal{H}, \mathfrak{o})$. If $\mathcal{H}$ is symmetric, we abbreviate this notation to $\mathcal{M}(\mathcal{H})$. As part of the proof of Theorem 2.1, one obtains an explicit description of the moduli space of quartics, which results in the following formula for its dimension

$$\dim \mathcal{M}(\mathcal{H}, \mathfrak{o})/PGL(4, \mathbb{C}) = 19 - \operatorname{rk} \Sigma_{\mathcal{H}} \tag{1}$$

(similar to the corresponding formula for plane sextics). Note that $\operatorname{rk} \Sigma_{\mathcal{H}} = \#\sigma$ equals the total Milnor number $\mu(X)$ of X.

Remark 2.2. The equisingular deformation classification of quartic surfaces with isolated singularities and at least one non-simple singular point is found in [3, 4]. With a few exceptions, the deformation class of such a quartic is also determined by its (appropriately defined) singular homological type.

3. Real quartics

In this section, we analyze the position of the vanishing cycles of a degeneration of a nonsingular quartic with respect to its period domain. The principal results are Theorems 3.2 and 3.3.

3.1. Real homological types

Given an isometric involution $c\colon L \to L$ on a lattice L, the (±1)-eigenlattices of c will be denoted by $L^c_\pm = \{x \in L \mid c(x) = \pm x\} \subset L$. If L is nondegenerate, $L^c_\pm$ are the orthogonal complements of each other.

Definition 3.1. A *real homological type* is a triple (L, h, c), where L is a lattice isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$, $h \in L$ is a vector of square 4, and $c\colon L \to L$ is an isometric involution such that

- the sublattice L^c_+ is hyperbolic, and
- one has $h \in L^c_-$.

An *isomorphism* between two real homological types (L_i, h_i, c_i), $i = 1, 2$, is an isometry $\varphi\colon L_1 \to L_2$ such that $\varphi(h_1) = h_2$ and $\varphi \circ c_1 = c_2 \circ \varphi$.

Definition 3.2. Let (L, h, c) be a real homological type. Consider vectors $e \in L^c_+$ of the following three kinds:

(1) $e^2 = -2$, *i.e.*, e is a root;
(2) $e^2 = -4$ and $e = h \bmod 2L$;
(3) $e^2 = -4$, $e \neq h \bmod 2L$, and e is *decomposable*, *i.e.*, $e = r' - r''$ for a pair of roots $r', r'' \in L$ such that $r' \cdot r'' = r' \cdot h = r'' \cdot h = 0$ and $r'' = -c(r')$.

A *fundamental tower* of (L, h, c) is a triple $\mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}} \subset \mathbb{P}(C^c_+)$ of fundamental domains of the subgroups of $\operatorname{Aut} L^c_+$ generated by the reflections defined by all vectors of L^c_+ of type (1)–(3), (1)–(2), and (1), respectively. (Recall that $C^c_+ \subset L^c_+ \otimes \mathbb{R}$ is the positive cone, see Subsection 2.1.)

Note that, in cases (2) and (3), the conditions imposed imply $e = 0 \bmod 2(L^c_+)^\vee$, *i.e.*, e does define a reflection $r_e\colon L^c_+ \to L^c_+$. Moreover, one can easily see that this reflection extends to an automorphism of the homological type. As a consequence, any two fundamental towers are related by an automorphism of the homological type (in fact, by a sequence of reflections).

Definition 3.3. A real homological type (L, h, c) equipped with a distinguished fundamental tower $\mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}}$ is called a *period lattice*, and the polyhedra $\mathfrak{P}$ and $\bar{\mathfrak{P}}$ are called the *period domains* (more precisely, the period domain of real quartics and that of abstract real $K3$-surfaces, respectively).

As explained in Subsection 2.1, the facets of the polyhedra in Definition 3.2 are (parts of) some of the mirrors (walls) of the respective groups, *i.e.*, hyperplanes orthogonal to vectors of corresponding types. We refer to the type of the vector as the *type* of the corresponding wall.

Remark 3.1. The polyhedra $\mathfrak{P}$ and $\bar{\mathfrak{P}}$ have a certain geometric meaning (see Subsection 3.2 below), whereas $\mathfrak{S}$ does not. However, in many examples, $\mathfrak{S}$ is much easier to compute and, on the other hand, a choice of $\mathfrak{S}$ determines the other two polyhedra: $\mathfrak{P}$ is paved by the copies of $\mathfrak{S}$ obtained from $\mathfrak{S}$ by iterated reflections against (the consecutive images of) the walls of type 3.2(3), and, similarly, $\bar{\mathfrak{P}}$ is paved by the copies of $\mathfrak{P}$ obtained by iterated reflections against the walls of type 3.2(2).

3.2. Invariant periods

A quartic $X \subset \mathbb{P}^3$ is called *real* if it is invariant under the complex conjugation involution $\operatorname{conj}\colon \mathbb{P}^3 \to \mathbb{P}^3$. The involution conj restricts to X and, if X is singular, lifts to the minimal resolution $\tilde{X}$ of singularities of X, turning both into *real $K3$-surfaces*. A nonsingular real quartic $X \subset \mathbb{P}^3$ gives rise to a real homological type (L_X, h_X, c_X), where $c_X : L_X \to L_X$ is the involution induced by conj.

Two nonsingular real quartics $X, Y \subset \mathbb{P}^3$ are said to be *coarse deformation equivalent* if X is equivariantly deformation equivalent to either Y or the quartic Y' obtained from Y by an orientation reversing automorphism of $\mathbb{P}^3$. A coarse deformation class consists of one or two components of the space of nonsingular real quartics; in the former case, the quartics are called *amphichiral*, in the latter case, *chiral*. The following statement is found in [15].

Theorem 3.1. *Two nonsingular real quartics $X, Y \subset \mathbb{P}^3$ are coarse deformation equivalent if and only if the corresponding real homological types (L_X, h_X, c_X) and (L_Y, h_Y, c_Y) are isomorphic.* □

A complete classification of nonsingular real quartics in $\mathbb{P}^3$ up to equivariant deformation, addressing in particular the chirality problem, and an interpretation of the result in topological terms are found in [12].

Given a real quartic $X \subset \mathbb{P}^3$ or, more generally, a real $K3$-surface (X, conj), a holomorphic 2-form Ω_X on X can be normalized (uniquely up to a nonzero real factor) so that $\mathrm{conj}^* \Omega_X = \bar{\Omega}_X$; such a form is called *real*. The real part $(\omega_X)_+$ of the class of a real form Ω_X belongs to $(L_X)_+^{c_X} \otimes \mathbb{R}$ and defines a point $[(\omega_X)_+]$ in the associated hyperbolic space; this point is called the *invariant period* of X.

Fix a period lattice $(L, h, c; \mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}})$ with the real homological type (L, h, c) isomorphic to that of X. A *marking* of X is a particular choice of an isomorphism $\varphi\colon (L_X, h_X, c_X) \to (L, h, c)$. A marking φ is called *proper* if $\varphi[(\omega_X)_+] \in \mathfrak{P}$. In fact, if X is nonsingular, the image $\varphi[(\omega_X)_+]$ under a proper marking belongs to the interior $\operatorname{Int}\mathfrak{P}$, see, *e.g.*, [15]. It follows that any two proper markings differ by a symmetry of $\mathfrak{P}$.

3.3. Degenerations

A *degeneration* is a smooth family $X_t \subset \mathbb{P}^3$, $t \in [0,1]$, of real quartics such that all quartics X_t, $t \in (0,1]$ are nonsingular. For simplicity, we confine ourselves to the case when X_0 has simple nodes only as singularities. Recall that the homology groups $H_2(X_t)$ of the nonsingular members of the family are canonically identified *via* the Gauss-Manin connection, and this common group contains a set of *vanishing cycles* (defined up to sign), one for each node of X_0.

The Gauss-Manin connection can be extended to identify the homology of X_1 with the homology of the minimal resolution $\tilde{X}_0$ of X_0, taking (up to sign) the vanishing cycles to the classes of the exceptional divisors contracted in X_0.

At each real node of X_0, the difference of the local Euler characteristics of the real parts of X_0 and X_1 is ± 1; according to this difference, the node is called *positive* or *negative*, respectively. Negative are the nodes whose vanishing cycles are c_{X_1}-invariant. Below, we are interested in the *non-positive* nodal degenerations, *i.e.*, such that each node of X_0 is either not real or real and negative.

Definition 3.4. Let $(L, h, c; \mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}})$ be a period lattice. A collection of roots $r_i, s'_j, s''_j \in L$, $i = 1, \ldots, k$, $j = 1, \ldots l$, is called an *admissible system of cycles* if it satisfies the following conditions:

(1) all roots are orthogonal to each other and to h;
(2) the primitive hull in L of the sublattice spanned by r_i, s'_j, s''_j contains no roots other than $\pm r_i$, $\pm s'_j$, $\pm s''_j$, *cf.* 2.2(1);
(3) each root r_i, $i = 1, \ldots, k$, belongs to L^c_+ and defines a facet of $\mathfrak{P}$, which is necessarily of type 3.2(1);
(4) for each $j = 1, \ldots, l$, one has $c(s'_j) = -s''_j$ and the decomposable invariant vector $s'_j - s''_j$ defines a type 3.2(3) facet of $\mathfrak{S}$.

Theorem 3.2. *Let $(L, h, c; \mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}})$ be a period lattice, and let X_t be a non-positive nodal degeneration with the real homological type of the nonsingular surface $X := X_1$ isomorphic to (L, h, c). Then X admits a proper marking that takes the set of vanishing cycles of X_t to an admissible system of cycles.*

Theorem 3.3. *Given a period lattice $(L, h, c; \mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}})$ and an admissible system of cycles $\sigma = \{r_i, s'_j, s''_j\}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$, there exists a nonsingular real quartic X and a proper marking $\varphi\colon (L_X, h_X, c_X) \to (L, h, c)$ which identifies σ with the set of vanishing cycles of a certain non-positive nodal degeneration of X.*

Proof of Theorem 3.2. Clearly, the vanishing cycles are orthogonal to each other and to h, *i.e.*, satisfy 3.4(1), as they are geometrically disjoint and can be chosen disjoint from a hyperplane section.

Denote by $r_i \in L_X$, $i = 1, \ldots, k$, the vanishing cycles corresponding to the real nodes of X_0, and by $s'_j, s''_j \in L_X$, $j = 1, \ldots, l$, those corresponding to the pairs of complex conjugate nodes; the latter are oriented so that $c_X(s'_j) = -s''_j$.

Let $\tilde{X}_0$ be the minimal resolution of singularities of X_0. Recall that, using the Gauss-Manin connection, we identify the homology of X and $\tilde{X}_0$. Let $\tilde{\omega} \in L_X \otimes \mathbb{C}$ be the class realized by a real holomorphic 2-form on $\tilde{X}_0$, and let $[\tilde{\omega}_+] \in \mathfrak{P}_X$ be its invariant part (the invariant period of $\tilde{X}_0$). Note that $[\tilde{\omega}_+]$ does belong to $\mathfrak{P}_X$ as it is the limit of invariant periods of X_t, which are all in $\operatorname{Int} \mathfrak{P}_X$. One has $\operatorname{Pic} \tilde{X}_0 = \tilde{\omega}^\perp \cap L_X$. Denote $\operatorname{Pic}_h \tilde{X}_0 = (h_X)^\perp$, the orthogonal complement of h_X in $\operatorname{Pic} \tilde{X}_0$. Up to sign, any root in $\operatorname{Pic}_h \tilde{X}_0$ is represented by a unique (-2)-curve contracted in X_0, and these are all (-2)-curves contracted. It follows that the roots of $\operatorname{Pic}_h \tilde{X}_0$ are precisely the vanishing cycles; in particular, this implies condition 3.4(2). Thus, the maximal root system in $\operatorname{Pic}_h \tilde{X}_0$ is $(k + 2l)\mathbf{A}_1$, and all its roots define a common face of all its Weyl chambers. Passing to the c_X-invariant part, one easily concludes that the invariant vanishing cycles r_i, $i = 1, \ldots, k$, define a common face of all $\mathfrak{P}$-like fundamental polyhedra containing $[\tilde{\omega}_+]$, in particular, of $\mathfrak{P}_X$, whereas the decomposable vectors $s'_j - s''_j$, $j = 1, \ldots, l$, define a common face of all $\mathfrak{S}$-like polyhedra containing $[\tilde{\omega}_+]$; for the latter, one can take any polyhedron $\mathfrak{S}'$ containing $[\tilde{\omega}_+]$ and contained in $\mathfrak{P}_X$. Due to Theorem 2.1 and condition 2.2(2) in the definition, one has $s'_j - s''_j \neq h \bmod 2L$; hence the wall defined by this vector is of type 3.2(3).

It remains to consider any proper marking of X_1 and, if necessary, adjust it by a symmetry of $\mathfrak{P}$ to make sure that the polyhedron $\mathfrak{S}'$ constructed above is taken to the preselected polyhedron $\mathfrak{S}$. $\square$

Proof of Theorem 3.3. Let $f_{\mathfrak{P}}$ and $f_{\mathfrak{S}}$ be the intersections of the facets defined in items 3.4(3) and (4), respectively. Notice that $f_{\mathfrak{P}}$ and $f_{\mathfrak{S}}$ are nonempty faces of $\mathfrak{P}$ and $\mathfrak{S}$, respectively (since the facets intersected are mutually orthogonal). One has $f_{\mathfrak{P}} \perp f_{\mathfrak{S}}$ and, since the symmetry about $f_{\mathfrak{S}}$ preserves $\mathfrak{P}$, it also preserves $f_{\mathfrak{P}}$. It follows that the subspace supporting $f_{\mathfrak{S}}$ intersects $f_{\mathfrak{P}}$ at at least one interior point. Let $[\tilde{\omega}_+]$ be such a point, and let $[\tilde{\omega}_-] \subset \mathbb{P}(C^c_{-h})$ be a point in the intersection of the hyperplanes defined by the skew-invariant vectors $s'_j + s''_j$, $j = 1, \ldots, l$, in the hyperbolic space associated with the orthogonal complement L^c_{-h} of h in L^c_-. (Since all hyperplanes are orthogonal to each other, they obviously intersect.) Due to 3.4(2), the pair $([\tilde{\omega}_+], [\tilde{\omega}_-])$ can be chosen generic in the sense that $[\tilde{\omega}_+]$ and $[\tilde{\omega}_-]$ are not simultaneously orthogonal to any root of L which is orthogonal to h and does not belong to $\pm\sigma$.

Let $U \subset \mathbb{P}(C^c_{-h})$ be a sufficiently small neighborhood of $[\tilde{\omega}_-]$. Consider a generic path $([(\omega_+)_t], [(\omega_-)_t]) \in \operatorname{Int}\mathfrak{P} \times U$, $t \in (0, 1]$, converging to the point $([\tilde{\omega}_+], [\tilde{\omega}_-])$. According to [15], it gives rise to a family X_t of properly marked nonsingular real quartics. (Strictly speaking, the path used should avoid a certain codimension 2 subset, see *loc. cit.* for the technical details.) This family can be chosen to converge to a singular quartic X_0 (*cf.*, *e.g.*, [19] for a detailed proof for the similar case of plane sextic, *i.e.*, polarization of square 2), and the limit quartic X_0 is necessarily real. As in the previous proof, considering the Picard group $\operatorname{Pic}_h \tilde{X}_0$ and using the fact that the pair $([\tilde{\omega}_+], [\tilde{\omega}_+])$ is generic, one concludes that the irreducible (-2)-curves contracted in X_0 are precisely those realizing the elements of σ (here, condition 3.4(2) is crucial, which implies that (L, h, σ) is a singular homological type); hence, these elements are the vanishing cycles. $\square$

Remark 3.2. Note that, if negative nodes are present, the real structure does *not* change continuously on the desingularized family $\tilde{X}_t$ of abstract $K3$-surfaces; in fact, the real homological type of the limit surface $\tilde{X}_0$, defined in the obvious way, is not even isomorphic to that of X_t, $t > 0$. However, the real structure does change continuously on the quartics.

4. Complex determinantal quartics

The goal of this section is Theorem 4.8, which identifies the equisingular stratum containing transversal determinantal quartics.

4.1. Notation

Let $\operatorname{Qu}(n) \cong \mathbb{P}^{N(n)}$ be the space of quadrics in $\mathbb{P}^n$; here $N(n) = \frac{1}{2}n(n+3)$. Let, further, $\operatorname{Qu}_r(n) \subset \operatorname{Qu}(n)$, $0 \leqslant r \leqslant n$, be the space of quadrics of corank r. The closure $\Delta(n)$ of $\operatorname{Qu}_1(n)$ is called the *discriminant hypersurface*; it has degree $n + 1$.

The singular locus of a quadric Q of corank $r > 0$ is an $(r-1)$-subspace of $\mathbb{P}^n$. Sending Q to its singular locus, one obtains a locally trivial fibration

$$\mathrm{Qu}_r(n) \to \mathrm{Gr}(n+1, r); \tag{2}$$

its fiber is $\mathrm{Qu}_0(n-r)$. (We let $\mathrm{Qu}(0) = \mathrm{Qu}_0(0) = \mathrm{pt.}$) Thus, $\mathrm{Qu}_r(n)$ is a smooth quasi-projective variety and $\dim \mathrm{Qu}_r(n) = N(n) - \frac{1}{2}r(r-1)$.

Definition 4.1. A *geometric hyperplane* is a hyperplane

$$H_p := \bigl\{Q \in \mathrm{Qu}(n) \bigm| Q \ni p\bigr\}$$

consisting of all quadrics passing through a fixed point $p \in \mathbb{P}^n$.

For each point $p \in \mathbb{P}^n$, fibration (2) restricts to a locally trivial fibration

$$\mathrm{Qu}_r(n) \smallsetminus H_p \to \mathrm{Gr}(n+1, r) \smallsetminus \mathrm{Gr}(n, r-1) \tag{3}$$

with fiber $\mathrm{Qu}_0(n-r) \smallsetminus H_p$. Here, the difference in the right hand side is the space of all $(r-1)$-planes in $\mathbb{P}^n$ not passing through p.

Since, in this paper, we are mainly concerned with quadrics in $\mathbb{P}^3$, we abbreviate the notation as follows: let $\mathrm{Qu} = \mathrm{Qu}(3)$, $\Delta = \Delta(3)$, and let Δ' and Δ'' be the closures of $\mathrm{Qu}_2(3)$ and $\mathrm{Qu}_3(3)$, respectively. Let also $\Delta^\circ = \mathrm{Qu}_1(3) = \Delta \smallsetminus \Delta'$. One has

$$\dim \mathrm{Qu} = 9, \quad \dim \Delta = \dim \Delta^\circ = 8, \quad \dim \Delta' = 6, \quad \dim \Delta'' = 3.$$

Recall also that $\deg \Delta = 4$ and $\deg \Delta' = 10$ (see [10]).

Let V be a subspace of Qu of dimension 3. Unless $V \in \Delta$, the intersection $\Delta_V := V \cap \Delta$ is a quartic in V. Any quartic $X \in \mathbb{P}^3$ such that the pair $(\mathbb{P}^3, X)$ is isomorphic to (V, Δ_V) above is called a *determinantal quartic*.

A 3-space V is called *transversal* if it is transversal to the strata Δ°, $\Delta' \smallsetminus \Delta''$, and Δ''. Any determinantal quartic $X \subset \mathbb{P}^3$ isomorphic to $\Delta_V \subset V$ is also called *transversal*. If V is transversal, the singular locus $\mathrm{Sing}\, \Delta_V$ coincides with $V \cap \Delta'$ and consists of ten type $\mathbf{A}_1$ points. Conversely, if $\mathrm{Sing}\, \Delta_V$ consists of ten type $\mathbf{A}_1$ points, V is transversal.

For a 3-space $V \subset \mathrm{Qu}$ as above, we denote $\Delta_V^\circ = \Delta_V \smallsetminus \Delta'$.

4.2. Some fundamental groups

Observe that the set $\mathrm{Sing}\, \Delta(n) = \mathrm{Qu}_{\geqslant 2}(n)$ of singular points of $\Delta(n)$ has codimension 3 in $\mathrm{Qu}(n)$. Hence a generic plane section of $\Delta(n)$ is a nonsingular plane curve of degree $n+1$ and, due to Zariski's hyperplane section theorem [22], one has $\pi_1(\mathrm{Qu}_0(n)) = \mathbb{Z}_{n+1}$. A generic plane section of the union $\Delta(n) \cup H_p$ is a transversal union of a nonsingular curve and a line; hence, $\pi_1(\mathrm{Qu}_0(n) \smallsetminus H_p) = \mathbb{Z}$.

Proposition 4.1. *One has* $\pi_1(\Delta^\circ) = 0$.

Proof. The Serre exact sequence of a fibration (2) takes the form

$$\begin{array}{ccccccc} \pi_2(\mathbb{P}^3) & \longrightarrow & \pi_1(\mathrm{Qu}_0(2)) & \longrightarrow & \pi_1(\Delta^\circ) & \longrightarrow & \pi_1(\mathbb{P}^3). \\ \Big\| & & \Big\| & & & & \Big\| \\ \mathbb{Z} & & \mathbb{Z}_3 & & & & 0 \end{array}$$

From this sequence, one concludes that $\pi_1(\Delta^\circ) = H_1(\Delta^\circ)$ is a quotient of $\mathbb{Z}_3$ and, moreover, the inclusion homomorphism $H_1(U \smallsetminus \Delta') \to H_1(\Delta^\circ)$ is onto, where U is a regular neighborhood in Δ of a point $Q \in \Delta'$. A normal 3-plane section of Δ' in Δ is a type $\mathbf{A}_1$ singularity and $U \smallsetminus \Delta'$ is homotopy equivalent to its link. Hence, one has $H_1(U \smallsetminus \Delta') = \mathbb{Z}_2$ and the statement follows. $\square$

Proposition 4.2. *One has* $\pi_1(\Delta^\circ \smallsetminus H_p) = \mathbb{Z}_2$ *(for any point* $p \in \mathbb{P}^3$*).*

Proof. Similar to the previous proof, using fibration (3) instead of (2), one concludes that the abelian group $\pi_1(\Delta^\circ \smallsetminus H_p) = H_1(\Delta^\circ \smallsetminus H_p)$ is a quotient of the group $H_1(U \smallsetminus \Delta') = \mathbb{Z}_2$, where U is a regular neighborhood in Δ of a point in Δ'. Thus, it remains to show that $\Delta^\circ \smallsetminus H_p$ admits a nontrivial double covering.

Let $Q \in \Delta^\circ$, $Q \not\ni p$. Denote by q the (only) singular point of Q. Then, there are exactly two planes passing through the line (pq) and tangent to Q along a whole generatrix: the original quadric Q and the two planes are the cones, with the vertex at q, over the section of Q by a generic plane $\alpha \ni p$ and the two tangents to this section passing through p. Clearly, the space of all pairs

$$(Q, \{\text{tangent plane as above}\})$$

is a double covering of $\Delta^\circ \smallsetminus H_p$. This covering is nontrivial: for example, in the family

$$Q_t = \big\{(x_1 - x_3)^2 + x_1^2 - e^{2\pi i t} x_2^2 = 0\big\}, \qquad t \in [0, 1],$$

the two tangents $x_1 = \pm e^{\pi i t} x_2$ are interchanged. (In particular, it follows that the path Q_t, $t \in [0, 1]$, is a non-contractible loop in $\Delta^\circ \smallsetminus H_{(0:0:0:1)}$.) $\square$

Remark 4.1. Similar to Propositions 4.1 and 4.2, one can easily show that all fundamental groups $\pi_1(\mathrm{Qu}_r(n))$ and $\pi_1(\mathrm{Qu}_r(n) \smallsetminus H_p)$ are cyclic.

Corollary 4.3. *Let* $p \in \mathbb{P}^3$*. Then, for a generic transversal* 3*-plane* $V \subset \mathrm{Qu}$*, one has* $\pi_1(\Delta^\circ_V) = 0$ *and* $\pi_1(\Delta^\circ_V \smallsetminus H_p) = \mathbb{Z}_2$*.*

Proof. The statement follows from Propositions 4.1 and 4.2 and Zariski type hyperplane section theorem for quasi-projective varieties (see [9, 7, 8] or recent survey [2], Theorem 5.1). $\square$

4.3. The determinantal stratum

In this subsection, we identify the stratum in the space of quartics formed by the transversal determinantal ones.

Lemma 4.4. *Any transversal determinantal quartic has a quadruple of non-coplanar singular points.*

We postpone the proof of this technical statement till Subsection 4.4.

Lemma 4.5. *The space of quintuples* $(V; Q_1, Q_2, Q_3, Q_4)$*, where* $V \subset \mathrm{Qu}$ *is a transversal* 3*-space and* Q_1*,* Q_2*,* Q_3*,* Q_4 *are four non-coplanar singular points of* Δ_V*, is an irreducible quasi-projective variety of dimension* 24*.*

Proof. The statement is a tautology, as the 3-subspace $V \subset \mathrm{Qu}$ is uniquely determined by a quadruple of its non-coplanar points Q_1, Q_2, Q_3, Q_4. Thus, the space in question is a Zariski open subset of the irreducible variety $(\Delta' \smallsetminus \Delta'')^4$. □

Since $\dim PGL(4, \mathbb{C}) = 15$, dimension of the moduli space of determinantal quartics is $24 - 15 = 9$, and, comparing the dimensions, see (1), one arrives at the following corollary.

Corollary 4.6. *Transversal determinantal quartics* $X \subset \mathbb{P}^3$ *form a Zariski open subset of a single equisingular stratum of the space of quartics.* □

We denote by $\mathcal{M}_{\mathrm{det}}$ the equisingular stratum containing transversal determinantal quartics. The corresponding configuration and singular homological type are denoted by $\tilde{S}_{\mathrm{det}}$ and $\mathcal{H}_{\mathrm{det}}$, respectively; they extend the set of singularities $\Sigma_{\mathrm{det}} := 10\mathbf{A}_1$. Let $a_1, \ldots, a_{10} \in \Sigma_{\mathrm{det}}$ be generators of the $\mathbf{A}_1$ summands.

Lemma 4.7. *The extension* $\tilde{S}_{\mathrm{det}} \supset S_{\mathrm{det}} := \Sigma_{\mathrm{det}} \oplus \mathbb{Z}h$ *is obtained from* S_{det} *by adjoining the element* $\frac{1}{2}(a_1 + \ldots + a_{10} + h)$*. One has* $\tilde{S}_{\mathrm{det}} \cong \mathbf{U} \oplus \mathbf{E}_8(2) \oplus [-4]$*.*

Proof. The requirement that $\tilde{S}$ should be an even integral lattice implies that $\tilde{S}_{\mathrm{det}}$ is generated in $S_{\mathrm{det}} \otimes \mathbb{Q}$ by S_{det} and several elements of the form

(1) $\frac{1}{2}(a_1 + \ldots + a_4)$,
(2) $\frac{1}{2}(a_1 + \ldots + a_8)$,
(3) $\frac{1}{2}(a_1 + a_2 + h)$,
(4) $\frac{1}{2}(a_1 + \ldots + a_6 + h)$,
(5) $\frac{1}{2}(a_1 + \ldots + a_{10} + h)$.

(up to reordering of the basis elements a_i).

Case (1) is impossible as the only nontrivial finite index extension of $4\mathbf{A}_1$ is $\mathbf{D}_4$, which contradicts to 2.2(1).

Consider case (2), *i.e.*, assume that $\tilde{S}_{\mathrm{det}}$ contains $a := \frac{1}{2}(a_1 + \ldots + a_8)$. If $\tilde{S}_{\mathrm{det}}$ contained another element a' of the same form, then, up to a further reordering, one would have $a' = \frac{1}{2}(a_1 + \ldots + a_6 + a_9 + a_{10})$, and the difference $a' - a$ would be as in case (1). Hence, a is the only element of $\tilde{S}_{\mathrm{det}} \bmod S_{\mathrm{det}}$ of this form, and each surface in the stratum has eight distinguished singular points. This contradicts Lemma 4.5.

Case (3) contradicts to 2.2(2).

Since cases (1) and (2) have been eliminated, $\tilde{S}_{\mathrm{det}} \bmod S_{\mathrm{det}}$ may contain at most one element as in (4) or (5). In case (4), each surface in the stratum would have six distinguished singular points, which would contradict Lemma 4.5. Thus, either $\tilde{S}_{\mathrm{det}} = S_{\mathrm{det}}$ or

$\tilde{S}_{\mathrm{det}} \supset S_{\mathrm{det}}$ is the index 2 extension generated by the (only) element (5). Pick a point $p \in \mathbb{P}^3$ and a sufficiently generic transversal quartic Δ_V, so that $H_p \cap \Delta_V$ is nonsingular. Since $\tilde{S}_{\mathrm{det}}$ is the primitive hull of S_{det} in $L \cong H_2(\tilde{\Delta}_V)$, from the Poincaré–Lefschetz duality it follows that $H_1(\Delta_V^\circ \smallsetminus H_p) = \mathrm{Ext}(\tilde{S}_{\mathrm{det}}/S_{\mathrm{det}}, \mathbb{Z})$. Due to Corollary 4.3, one has $[\tilde{S}_{\mathrm{det}} : S_{\mathrm{det}}] = 2$, and the first statement follows. The isomorphism class of $\tilde{S}_{\mathrm{det}}$ is given by a simple computation of the discriminant group and Nikulin's uniqueness theorem [15], Theorem 1.14.2. $\square$

Remark 4.2. Alternatively, case (2) in the proof of Lemma 4.7 can also be eliminated using Corollary 4.3, and case (4) can be eliminated using a refinement of this corollary stating that the group $\pi_1(\Delta_V^\circ \smallsetminus H_p)$ is generated by the group of the link of any of the singular points.

Theorem 4.8. *The configuration $\tilde{S}_{\mathrm{det}}$ given by Lemma 4.7 extends to a unique, up to isomorphism, singular homological type $\mathcal{H}_{\mathrm{det}}$, which is symmetric. Thus, one has $\mathcal{M}_{\mathrm{det}} = \mathcal{M}(\mathcal{H}_{\mathrm{det}})$.*

Proof. The uniqueness of a primitive embedding $\tilde{S}_{\mathrm{det}} \hookrightarrow L$ of the lattice $\tilde{S}_{\mathrm{det}}$ given by Lemma 4.7 follows from [15], Theorem 1.14.4. One has $\tilde{S}_{\mathrm{det}}^\perp \cong \mathbf{U} \oplus \mathbf{E}_8(2) \oplus [4]$. Clearly, $\tilde{S}_{\mathrm{det}}^\perp$ has a vector of square 2, and the reflection against the hyperplane orthogonal to such a vector is an orientation reversing automorphism. $\square$

Remark 4.3. Alternatively, the fact that $\mathcal{H}_{\mathrm{det}}$ is symmetric follows from the obvious existence of real determinantal quartics.

4.4. Proof of Lemma 4.4

We prove a stronger statement: a plane $W \subset V$ cannot contain more than six singular points of a transversal determinantal quartic $\Delta_V \subset V$.

Assume that seven singular points $Q_1, \ldots, Q_7$ of Δ_V belong to a single plane $W \subset V$. Since Δ_V is irreducible, the intersection $\Delta_W := \Delta_V \cap W$ is a curve, which is of degree 4. Furthermore, each point $Q_1, \ldots, Q_7$ is singular for Δ_W.

Since a reduced plane quartic has at most six singular points, Δ_W must have multiple components.

Lemma 4.9. *A pencil $U \subset \mathrm{Qu}$ not contained entirely in Δ' intersects Δ' at at most three points.*

Proof. First, assume that U has a base point singular for all quadrics. Projecting from this point, one obtains a pencil of plane conics, singular conics corresponding to the elements of the intersection $U \cap \Delta'$, and the statement follows from the fact that $\deg \Delta(2) = 3$.

Now, assume that U does not have a singular base point. Let $P_1, P_2 \in U \cap \Delta'$ be two distinct members of U of corank at least 2; they generate U. Since P_1 and P_2 have no common singular points, one has $P_1 = \{x_0x_1 = 0\}$ and $P_2 = \{x_2x_3 = 0\}$ in appropriate homogeneous coordinates, and it is immediate that P_1 and P_2 are the only singular members of U. $\square$

Lemma 4.9 rules out the possibility that Δ_W contains a double line: at most two double lines in Δ_W would contain at most six quadrics in Δ'.

The remaining possibility is that Δ_W is a double conic. In this case Δ_W has no linear components; in particular, no two quadrics $P_1, P_2 \in \Delta_W$ have a common singular point. The quadric Q_1 splits into two distinct planes α_1, α_2. The linear system W restricts to a pencil of conics in α_1 containing at least six distinct singular members, namely, the restrictions of $Q_2, \ldots, Q_7$. (If the restrictions of two distinct quadrics Q_i, Q_j coincided, Q_i and Q_j would have a common singular point.) Since $\deg \Delta(2) = 3 < 6$, all members of the restricted pencil are singular. Hence, the vertex of each quadric $P \in \Delta_W \smallsetminus \Delta'$ belongs to α_1. The same argument shows that the vertex of P also belongs to α_2, *i.e.*, P and Q_1 have a common singular point. This contradiction concludes the proof of Lemma 4.4. $\square$

Remark 4.4. Using the surjectivity of the period map and the Riemann–Roch theorem for $K3$-surfaces, one can easily show that the stratum $\mathcal{M}_{\mathrm{det}}$ does contain a quartic with all ten singular points coplanar (lying in a curve of degree two). In particular, determinantal quartics form a proper subset of $\mathcal{M}_{\mathrm{det}}$.

5. Real determinantal quartics

In this section, we discuss the topology of a determinantal quartic with nonempty spectrahedral domain and prove Theorem 1.1.

5.1. Geometric real structures

We always consider the space $\mathrm{Qu}(n)$ with its *geometric real structure*, *i.e.*, the one induced by the complex conjugation $\mathrm{conj}\colon \mathbb{P}^n \to \mathbb{P}^n$. All real quadratic forms constitute a linear space $\mathbb{R}^{N(n)+1}$, and one has a double covering $S^{N(n)} \to \mathrm{Qu}(n)_{\mathbb{R}}$, where $S^{N(n)}$ is the unit sphere in $\mathbb{R}^{N(n)+1}$.

We reserve the notation $\bar{\ }$ for the lift from $\mathrm{Qu}(n)_{\mathbb{R}}$ to $S^{N(n)}$; in particular, one has real discriminant hypersurfaces $\bar{\Delta}(n) \subset S^{N(n)}$ and $\bar{\Delta} \subset S^9$.

Recall that a real quadratic form $\bar{q}$ has a well defined *index* $\operatorname{ind} \bar{q}$ (the negative inertia index of $\bar{q}$); one has $0 \leqslant \operatorname{ind} \bar{q} \leqslant n+1$.

A *real determinantal quartic* is a real quartic $X \subset \mathbb{P}^3$ equivariantly isomorphic to a quartic $\Delta_V \subset V$, where $V \subset \mathrm{Qu}$ is a 3-subspace real with respect to the geometric real structure. Given such a quartic X, the *spectrahedral region* of X is the (only) connected component of the complement $\mathbb{P}^3_{\mathbb{R}} \smallsetminus X_{\mathbb{R}}$ constituted by the quadrics represented by quadratic forms of index 0 (equivalently, those of maximal index 4).

Lemma 5.1. *Let $X \subset \mathbb{P}^3$ be a real determinantal quartic. Then any real line meeting the spectrahedral region of X intersects X at four real points* (*counted with multiplicities*)*. In other words, all intersection points are real.*

Proof. Identify $(\mathbb{P}^3, X)$ with a real pair (V, Δ_V) and let W be the image of the line in question. Consider the lift $\bar{W} \subset S^9$. The index function $\mathrm{ind}\colon \bar{W} \to \mathbb{Z}$ is locally constant on $\bar{W} \smallsetminus \bar{\Delta}$, and its increment δ_p at an intersection point $p \in \bar{W} \cap \bar{\Delta}$ of multiplicity m_p is subject to the conditions $|\delta_p| \leqslant m_p$, $\delta_p = m_p \bmod 2$. Any point Q in the spectrahedral region of Δ_V lifts to a pair $\bar{q}$, $-\bar{q}$ of quadratic forms, so that one has $\operatorname{ind}\bar{q} = 4$, $\operatorname{ind}(-\bar{q}) = 0$. Since the two indices differ by 4, the intersection $\bar{W} \cap \bar{\Delta}$ must consist of at least eight points (counted with multiplicities). □

Corollary 5.2. *Let $X \subset \mathbb{P}^3$ be a real determinantal quartic with nonempty spectrahedral region. Then X is a non-positive degeneration of a nonsingular real quartic $X' \subset \mathbb{P}^3$ with the real part $X'_{\mathbb{R}}$ constituted by a nested pair of spheres.* □

5.2. Preliminary computation

Consider the real homological type (L, h, c) corresponding to nonsingular real quartics with two nested spheres; such quartics are amphichiral. According to [15], one has

$$L^c_+ \cong \mathbf{E}_8(2) \oplus 2\mathbf{A}_1 \oplus \mathbf{U}, \qquad L^c_- \cong \mathbf{E}_8(2) \oplus 2\mathbf{A}_1(-1). \tag{4}$$

Fix standard bases $e_1, \ldots, e_8$, v_1, v_2, and u_1, u_2 for $\mathbf{E}_8(2)$, $2\mathbf{A}_1$, and $\mathbf{U}$, respectively, and let $e'_1, \ldots, e'_8$, v'_1, v'_2 be the 'matching' standard basis for L^c_-, so that the sum $r + r'$ of two basis vectors $r \in L^c_+$, $r' \in L^c_-$ of the same name is divisible by 2 in L. In view of [15],

$$h = v'_1 + v'_2, \qquad h = v_1 + v_2 \bmod 2L. \tag{5}$$

Let $\mathfrak{S} \subset \mathfrak{P} \subset \bar{\mathfrak{P}}$ be a fundamental tower of (L, h, c). The polyhedron $\mathfrak{S}$ is finite; its Coxeter scheme, computed in [14], is shown in Figure 1, and $\mathfrak{S}$ can be chosen to be bounded by the walls orthogonal to the vectors indicated in the figure. (In the figure, walls of type 3.2(1) and (3) are represented by $\circ$ and $\bullet$, respectively, and the only wall of type 3.2(2) is represented by a circled bullet. Whenever the hyperplanes supporting two walls intersect at an angle π/n, $n \geqslant 2$, the corresponding vertices are connected by $(n-2)$ edges. Note that, in the case under consideration, any two walls do intersect.)

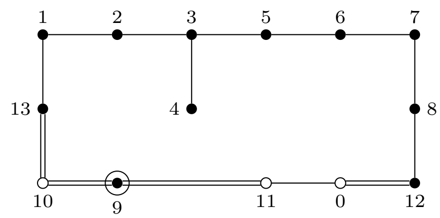
Clearly, $\mathfrak{P}$ is paved by the (infinitely many) copies of $\mathfrak{S}$ obtained by iterated reflections against the walls of type 3.2(3) (vertices $e_1, \ldots, e_8$, e_{12}, and e_{13} in the figure), and $\bar{\mathfrak{P}}$ is the union of $\mathfrak{P}$ and its image under the reflection against the only wall e_9 of type 3.2(2); see Definition 3.2 and Remark 3.1.

Let $X \subset \mathbb{P}^3$ be a properly marked nonsingular quartic of type (L, h, c).

Lemma 5.3. *The classes realized in L^c_+ by the inner and outer spheres of $X_{\mathbb{R}}$ are $\mathrm{sp}_{in} = e_{11}$ and $\mathrm{sp}_{out} = e_{11} + e_9$, respectively.*

Proof. Let G be the graph obtained from the Coxeter scheme of $\bar{\mathfrak{P}}$ by removing all but simple edges. According to [6], Theorem 16.1.1, any vertex of G of valency > 2 is a class realized by a spherical component of $X_{\mathbb{R}}$. Clearly, e_{11} and $e_{11} + e_9$ (obtained from e_{11} by reflection) are two such vertices. The outer sphere is definitely not contractible. Hence, the vector e_{11}, which defines a wall of $\mathfrak{P}$ and thus can serve as a vanishing cycle, see Theorem 3.3, represents the inner sphere. □

Denote by $L^0_+ \subset L^c_+$ the sublattice spanned by $e_1, \ldots, e_8$, e_{12}, and e_{13}.



$$\begin{aligned} e_0 &= u_1 - u_2, & e_{11} &= u_2 - v_1, \\ e_9 &= v_1 - v_2, & e_{12} &= 2u_2 + e_8^*, \\ e_{10} &= v_2, & e_{13} &= 2(u_1 + u_2) - v_1 - v_2 + e_1^*, \end{aligned}$$

$$\begin{aligned} e_1^* &= -4e_1 - 7e_2 - 10e_3 - 5e_4 - 8e_5 - 6e_6 - 4e_7 - 2e_8, \\ e_8^* &= -2e_1 - 4e_2 - 6e_3 - 3e_4 - 5e_5 - 4e_6 - 3e_7 - 2e_8. \end{aligned}$$

FIGURE 1. The fundamental polyhedron $\mathfrak{S}$

Lemma 5.4. *Each real vanishing cycle $r \in L_+^c$ of a non-positive nodal degeneration of X is of one of the following three forms*:

(1) $e_{11} = \mathrm{sp}_{in}$ (*the inner sphere shrinks to a point*);
(2) $e_0 + d$, $d \in L_+^0$ (*a common point of the two spheres*);
(3) $e_{10} + d$, $d \in L_+^0$ (*a node in the outer sphere*).

For each pair s', s'' of conjugate vanishing cycles, the invariant decomposable vector $s' - s''$ belongs to L_+^0.

Proof. Each real vanishing cycle is a wall of $\mathfrak{P}$ of type 3.2(1), see Theorem 3.2. From the description of $\mathfrak{P}$ in terms of $\mathfrak{S}$ and Figure 1 it follows that any such wall either is e_{11} or is obtained from e_0 or e_{10} by iterated reflections against the walls $e_1, \ldots, e_8$, e_{12}, and e_{13} (and their consecutive images). The geometry of the corresponding degeneration is easily seen from comparing the vanishing cycle r against the classes of the spheres: one has $r = \mathrm{sp}_{\mathrm{in}}$ in case (1), $r \cdot \mathrm{sp}_{\mathrm{in}} = r \cdot \mathrm{sp}_{\mathrm{out}} = 1$ in case (2), and $r \cdot \mathrm{sp}_{\mathrm{in}} = 0$, $r \cdot \mathrm{sp}_{\mathrm{out}} = 2$ in case (3).

For a pair s', s'' of conjugate vanishing cycles, the vector $s' - s''$ is either one of the type 3.2(3) walls of $\mathfrak{S}$ or one of their consecutive images under reflections, see Theorem 3.2. $\square$

Lemma 5.5. *The lattice L_-^c does not contain a quintuple t_i, $i = 1, \ldots, 5$, of pairwise orthogonal vectors of square (-4) such that $t_1 + \ldots + t_5 = h \bmod 2L_-^c$.*

Proof. In view of (4) and (5), the vector h is characteristic, *i.e.*, $a^2 + a \cdot h = 0 \bmod 4$ for any $a \in L_-^c$. Hence, $(h + 2a)^2 = 4 \bmod 16$ for any $a \in L_-^c$. On the other hand, $(t_1 + \ldots + t_5)^2 = -20 = -4 \bmod 16$. $\square$

5.3. Proof of Theorem 1.1

We keep the notation of Subsection 5.2. The assumption that the spectrahedral region R of X is nonempty rules out real vanishing cycles of type 5.4(1). Assume that X has m vanishing cycles of type 5.4(2) and n vanishing cycles of type (3) (and $10-m-n$ imaginary vanishing cycles split into conjugate pairs). Using Lemma 5.4 and the description of the vectors involved given in Figure 1, one can easily see that the parities of the coefficients of v_1 and v_2 in the sum of all ten vanishing cycles differ by $n \bmod 2$. Due to (5) and Lemma 4.7, n is even, and so is m.

If $m = n = 0$, then X has five pairs of complex conjugate vanishing cycles s'_j, $s''_j = -c(s'_j)$, $j = 1, \ldots, 5$, and the skew-invariant vectors $s'_j + s''_j \in L^c_-$ form a quintuple contradicting Lemma 5.5.

For the construction, relabel the nine vertices of type 3.2(3) in the edges of the Coxeter scheme consecutively, *i.e.*, let $e_{13} = w_1$, $e_i = w_{i+1}$ for $i = 1, 2, 3$, $e_i = w_i$ for $i = 5, \ldots, 8$, and $e_{12} = w_9$. Pick a pair m, n of even integers as in the statement, let p be $5 - \frac{1}{2}(m+n)$, and consider the following vectors:

$$\begin{aligned} r'_i &= e_0 + w_9 + \ldots + w_{11-i}, && i = 1, \ldots, m \text{ (if } m > 0), \\ r''_j &= e_{10} + w_1 + \ldots + w_{j-1}, && j = 1, \ldots, n \text{ (if } n > 0), \\ t_k &= w_{n+2k-1}, && k = 1, \ldots, p \text{ (if } p > 0). \end{aligned}$$

It is straightforward to check that:

(1) each r'_i is obtained by a sequence of reflections from the vertex e_0, *i.e.*, is as in Lemma 5.4(2);
(2) each r''_j is obtained by a sequence of reflections from the vertex e_{10}, *i.e.*, is as in Lemma 5.4(3);
(3) each t_k is a wall of $\mathfrak{S}$ of type 3.2(3);
(4) all vectors are pairwise orthogonal;
(5) all vectors are linearly independent in $L^c_+/2L^c_+$;
(6) the sum of all ten vectors equals $h \bmod 2L$.

Furthermore, assuming that $p \leqslant 4$, one can easily find pairwise orthogonal vectors $t'_1, \ldots, t'_p \in L^c_-$ such that $t'^2_k = -4$, $t'_k \cdot h = 0$, and $t'_k = t_k \bmod 2L$, $k = 1, \ldots, p$. Indeed, consider the vectors $w'_1 = v'_1 - v'_2 + e'^*_1$, $w'_3 = e'_2$, $w'_5 = e'_5$, $w'_7 = e'_7$, $w'_9 = e'^*_8$. They are orthogonal to h and have square (-4), and any sequence of up to four consecutive vectors is orthogonal. (In fact, all five vectors are pairwise orthogonal except that $w'_1 \cdot w'_9 \neq 0$.) Now, one can take for t'_k the 'matching' vectors w'_*.

Finally, the set σ constituted by the ten vectors

$$r'_1, \ldots, r'_m,\ r''_1, \ldots, r''_n,\ \tfrac{1}{2}(t'_1 \pm t_1), \ldots, \tfrac{1}{2}(t'_p \pm t_p)$$

is an admissible system of cycles, see Definition 3.4; due to Theorem 3.3, it can serve as the set of vanishing cycles of a non-positive nodal degeneration of X. On the other hand, the set σ satisfies Lemma 4.7; hence, according to Corollary 4.6 and Theorem 4.8, a generic degeneration of X contracting these vanishing cycles is a transversal determinantal quartic. □

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