

A solution to the Arf-Kervaire invariant problem

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ABSTRACT. This paper gives the history and background of one of the oldest problems in algebraic topology, along with an outline of our solution to it. A rigorous account can be found in our preprint [HHR]. The third author has a website with numerous links to related papers and talks we have given on the subject since announcing our result in April, 2009.



Drawing by Carolyn Snaith, 1981

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CONTENTS

| | |
|--|----|
| 1. Background and history | 24 |
| 1.1. Pontryagin's early work on homotopy groups of spheres | 24 |
| 1.2. Our main result | 27 |
| 1.3. The manifold formulation | 28 |
| 1.4. The unstable formulation | 30 |
| 1.5. Questions raised by our theorem | 32 |
| 2. Our strategy | 33 |
| 2.1. Ingredients of the proof | 33 |
| 2.2. The spectrum Ω | 33 |
| 2.3. How we construct Ω | 34 |
| 2.4. The norm | 34 |
| 2.5. Fixed points and homotopy fixed points | 35 |
| 2.6. $RO(G)$ -graded homotopy | 36 |
| 3. The slice filtration | 37 |
| 3.1. Postnikov towers | 37 |
| 3.2. Slice cells | 38 |
| 3.3. The slice tower | 38 |
| 3.4. The Slice Theorem | 39 |
| 3.5. Refining $\pi_*^u MU^{(G)}$ | 40 |
| 3.6. The definition of $\Omega_{\mathbb{O}}$ | 40 |
| 4. The Gap Theorem | 41 |
| 4.1. Derivation from the Slice Theorem | 41 |
| 4.2. An easy calculation | 41 |
| 5. The Periodicity Theorem | 44 |
| 5.1. Geometric fixed points | 44 |
| 5.2. Some slice differentials | 46 |
| 5.3. Some $RO(G)$ -graded calculations | 47 |
| 5.4. An even trickier $RO(G)$ -graded calculation | 47 |
| 5.5. The proof of the Periodicity Theorem | 48 |
| 6. The Homotopy Fixed Point Theorem | 49 |
| 7. The Detection Theorem | 50 |
| 7.1. θ_j in the Adams-Novikov spectral sequence | 50 |
| 7.2. Formal A -modules | 52 |
| 7.3. The relation between $MU^{(C_8)}$ and formal A -modules | 53 |
| 7.4. Deriving the Detection Theorem from Lemma 7.1 | 54 |
| 7.5. The proof of Lemma 7.1 | 54 |
| 8. The Slice and Reduction theorems | 55 |
| 8.1. Deriving the Slice Theorem from the Reduction Theorem | 56 |
| 8.2. The proof of the Reduction Theorem | 57 |
| References | 61 |

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}^S$ do not exist for $j \geq 7$.*

Here π_k^S denotes the k th stable homotopy group of spheres, which will be defined shortly.

The k th (for a positive integer k) homotopy group of the topological space X , denoted by $\pi_k(X)$, is the set of continuous maps to X from the k -sphere S^k , up to continuous deformation. For technical reasons we require that each map send a specified point in S^k (called a *base point*) to a specified point $x_0 \in X$. When X is path connected the choice of these two points is irrelevant, so it is usually omitted from the notation. When X is not path connected, we get different collections of maps depending on the path connected component of the base point.

This set has a natural group structure, which is abelian for $k > 1$. The word *natural* here means that a continuous base point preserving map $f : X \rightarrow Y$ induces a homomorphism $f_* : \pi_k X \rightarrow \pi_k Y$, sometimes denoted by $\pi_k f$.

It is known that the group $\pi_{n+k} S^n$ is independent of n for $n > k$. There is a homomorphism $E : \pi_{n+k} S^n \rightarrow \pi_{n+k+1} S^{n+1}$ defined as follows. S^{n+1} [respectively S^{n+k+1}] can be obtained from S^n [S^{n+k}] by a double cone construction known as *suspension*. The cone over S^n is an $(n+1)$ -dimensional ball, and gluing two such balls together along their common boundary gives an $(n+1)$ -dimensional sphere. A map $f : S^{n+k} \rightarrow S^n$ can be canonically extended (by suspending both its source and target) to a map $Ef : S^{n+k+1} \rightarrow S^{n+1}$, and this leads to the suspension homomorphism E . The Freudenthal Suspension Theorem [Fre38], proved in 1938, says that it is onto for $k = n$ and an isomorphism for $n > k$. For this reason the group $\pi_{n+k} S^n$ is said to be *stable* when $n > k$, and it is denoted by π_k^S and called the *stable k -stem*.

The Main Theorem above concerns the case $k = 2^{j+1} - 2$. The θ_j in the theorem is a hypothetical element related to a geometric invariant of certain manifolds studied originally by Pontryagin starting in the 1930s, [Pon38], [Pon50] and [Pon55]. The problem came into its present form with a theorem of Browder [Bro69] published in 1969. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved, namely that all of the θ_j exist.

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. Browder's theorem says that such things can exist only in dimensions that are 2 less than a power of 2.

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j . They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the DOOMSDAY HYPOTHESIS.

After 1980, the problem faded into the background because it was thought to be too hard. In 2009, just a few weeks before we announced our theorem, Snaithe published a book [Sna09] on the problem "to stem the tide of oblivion." On the difficulty of the problem, he wrote

In the light of . . . the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll.

Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.

1. Background and history

1.1. Pontryagin’s early work on homotopy groups of spheres

The Arf-Kervaire invariant problem has its origins in Pontryagin’s early work on a geometric approach to the homotopy groups of spheres, [Pon38], [Pon50] and [Pon55].

Pontryagin’s approach to maps $f : S^{n+k} \rightarrow S^n$ is to assume that f is smooth and that the base point y_0 of the target is a regular value. (Any continuous f can be continuously deformed to a map with this property.) This means that $f^{-1}(y_0)$ is a closed smooth k -manifold M in S^{n+k} . Let D^n be the closure of an open ball around y_0 . If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n)$ in S^{n+k} is an $(n+k)$ -manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. It is also a tubular neighborhood of M^k and comes equipped with a map $p : V^{n+k} \rightarrow M^k$ sending each point to the nearest point in M . For each $x \in M$, $p^{-1}(x)$ is homeomorphic to a closed n -ball B^n . The pair $(p, f|_{V^{n+k}})$ defines an explicit homeomorphism

$$V^{n+k} \xrightarrow[\approx]{(p, f|_{V^{n+k}})} M^k \times D^n.$$

This structure on M^k is called a *framing*, and M is said to be *framed in \mathbf{R}^{n+k}* . A choice of basis of the tangent space at $y_0 \in S^n$ pulls back to a set of linearly independent normal vector fields on $M \subset \mathbf{R}^{n+k}$.

Conversely, suppose we have a closed sub- k -manifold $M \subset \mathbf{R}^{n+k}$ with a closed tubular neighborhood V and a homeomorphism h to $M \times D^n$ as above. This is called a *framed sub- k -manifold* of \mathbf{R}^{n+k} . Some remarks are in order here.

- The existence of a framing puts some restrictions on the topology of M . All of its characteristic classes must vanish. In particular it must be orientable.
- A framing can be twisted by a map $g : M \rightarrow SO(n)$, where $SO(n)$ denotes the group of orthogonal $n \times n$ matrices with determinant 1. Such matrices act on D^n in an obvious way. The twisted framing is the composite

$$\begin{aligned} V &\xrightarrow{h} M^k \times D^n \longrightarrow M^k \times D^n \\ & \quad (m, x) \longmapsto (m, g(m)(x)). \end{aligned}$$

We will say more about this later.

- If we drop the assumption that M is framed, then the tubular neighborhood V is a (possibly nontrivial) disk bundle over M . The map $M \rightarrow y_0$ needs to be replaced by a map to the classifying space for such bundles, $BO(n)$. This leads to unoriented bordism theory, which was analyzed by Thom in [Tho54]. Two helpful references for this material are the books by Milnor-Stasheff [MS74] and Stong [Sto68].

Pontryagin constructs a map $P(M, h) : S^{n+k} \rightarrow S^n$ as follows. We regard S^{n+k} as the one point compactification of \mathbf{R}^{n+k} and S^n as the quotient $D^n/\partial D^n$. This leads to a diagram

$$\begin{array}{ccccc}
 (V, \partial V) & \xrightarrow{h} & M \times (D^n, \partial D^n) & \xrightarrow{p_2} & (D^n, \partial D^n) \\
 \downarrow & & & & \downarrow \\
 (\mathbf{R}^{n+k}, \mathbf{R}^{n+k} - \text{int}V) & \longrightarrow & (S^{n+k}, S^{n+k} - \text{int}V) & \xrightarrow{P(M, h)} & (S^n, \{\infty\}) \\
 & & S^{n+k} - \text{int}V & \xrightarrow{P(M, h)} & \{\infty\}
 \end{array}$$

The map $P(M, h)$ is the extension of $p_2 h$ obtained by sending the complement of V in S^{n+k} to the point at infinity in S^n . For $n > k$, the choice of the embedding (but not the choice of framing) of M into the Euclidean space is irrelevant. Any two embeddings (with suitably chosen framings) lead to the same map $P(M, h)$ up to continuous deformation.

To proceed further, we need to be more precise about what we mean by continuous deformation. Two maps $f_1, f_2 : X \rightarrow Y$ are *homotopic* if there is a continuous map $h : X \times [0, 1] \rightarrow Y$ (called a *homotopy between f_1 and f_2*) such that

$$h(x, 0) = f_1(x) \quad \text{and} \quad h(x, 1) = f_2(x).$$

Now suppose $X = S^{n+k}$, $Y = S^n$, and the map h (and hence f_1 and f_2) is smooth with y_0 as a regular value. Then $h^{-1}(y_0)$ is a framed $(k+1)$ -manifold N whose boundary is the disjoint union of $M_1 = f_1^{-1}(y_0)$ and $M_2 = f_2^{-1}(y_0)$. This N is called a *framed cobordism* between M_1 and M_2 , and when it exists the two closed manifolds are said to be *framed cobordant*.

Let $\Omega_{k,n}^{\text{fr}}$ denote the cobordism group of framed k -manifolds in \mathbf{R}^{n+k} . The above construction leads to Pontryagin's isomorphism

$$\Omega_{k,n}^{\text{fr}} \xrightarrow{\cong} \pi_{n+k} S^n.$$

First consider the case $k = 0$. Here the 0-dimensional manifold M is a finite set of points in \mathbf{R}^n . Each comes with a framing which can be obtained from a standard one by an element in the orthogonal group $O(n)$. We attach a sign to each point corresponding to the sign of the associated determinant. With these signs we can count the points algebraically and get an integer called the *degree of f* . Two framed 0-manifolds are cobordant iff they have the same degree.

Sidebar 1 The Hopf-Whitehead J -homomorphism

Suppose our framed manifold is S^k with a framing that extends to a D^{k+1} . This will lead to the trivial element in $\pi_{n+k}(S^n)$, but twisting the framing can lead to nontrivial elements. The twist is determined up to homotopy by an element in $\pi_k(SO(n))$. Pontryagin's construction thus leads to the homomorphism

$$\pi_k SO(n) \xrightarrow{J} \pi_{n+k} S^n$$

introduced by Hopf [Hop35] and Whitehead [Whi42]. Both source and target are known to be independent of n for $n > k + 1$. In this case the source group for each k (denoted simply by $\pi_k SO$ since n is irrelevant) was determined by Bott [Bot59] in his remarkable periodicity theorem. He showed

$$\pi_k SO = \begin{cases} \mathbf{Z} & \text{for } k \equiv 3 \text{ or } 7 \pmod{8} \\ \mathbf{Z}/2 & \text{for } k \equiv 0 \text{ or } 1 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

Here is a table showing these groups for $k \leq 10$.

| | | | | | | | | | | |
|-------------|----------------|---|--------------|---|---|---|--------------|----------------|----------------|----|
| k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\pi_k(SO)$ | $\mathbf{Z}/2$ | 0 | \mathbf{Z} | 0 | 0 | 0 | \mathbf{Z} | $\mathbf{Z}/2$ | $\mathbf{Z}/2$ | 0 |

In each case where the group is nontrivial, the image under J of its generator is known to generate a direct summand. In the j th case we denote this image by β_j and its dimension by $\phi(j)$, which is roughly $2j$. The first three of these are the Hopf maps $\eta \in \pi_1^S$, $\nu \in \pi_3^S$ and $\sigma \in \pi_7^S$. After that we have $\beta_4 \in \pi_8^S$, $\beta_5 \in \pi_9^S$, $\beta_6 \in \pi_{11}^S$ and so on. For the case $\pi_{4m-1} SO = \mathbf{Z}$, the image under J is known to be a cyclic group whose order a_m is the denominator of $B_m/4m$, where B_m is the m th Bernoulli number. Details can be found in [Ada66] and [MS74]. Here is a table showing these values for $m \leq 10$.

| | | | | | | | | | | |
|-------|----|-----|-----|-----|-----|--------|----|--------|--------|--------|
| m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| a_m | 24 | 240 | 504 | 480 | 264 | 65,520 | 24 | 16,320 | 28,728 | 13,200 |

Now consider the case $k = 1$. M is a closed 1-manifold, i.e., a disjoint union of circles. Two framings on a single circle differ by a map from S^1 to the group $SO(n)$, and it is known that

$$\pi_1 SO(n) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

It turns out that any disjoint union of framed circles is cobordant to a single framed circle. This can be used to show that

$$\pi_{n+1}S^n = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

The case $k = 2$ is more subtle. As in the 1-dimensional case we have a complete classification of closed 2-manifolds, and it is only necessary to consider ones which are path connected and oriented. The existence of a framing implies that the surface is orientable, so it is characterized by its genus.

If the genus is zero, namely if $M = S^2$, then there is a framing which extends to a 3-dimensional ball. This makes M cobordant to the empty set, which means that the map is *null homotopic* (or, more briefly, *null*), meaning that it is homotopic to a constant map. Any two framings on S^2 differ by an element in $\pi_2 SO(n)$. This group is known to vanish (as is π_2 of any Lie group), so any two framings on S^2 are equivalent, and the map $f : S^{n+2} \rightarrow S^n$ is null.

Now suppose the genus is one and suppose we can find an embedded arc on which the framing extends to a disk in S^{n+2} . Then there is a cobordism which effectively cuts along the arc, leaving a surface bounded by two circles, and attaches two framed disks to them. This process is called *framed surgery*. If we can do this, then we have converted the torus to a 2-sphere and shown that the map $f : S^{n+2} \rightarrow S^n$ is null.

When can we find such a closed curve in M ? It must represent a generator of $H_1(M)$ and carry a trivial framing. This leads to a map

$$\varphi : H_1(M; \mathbf{Z}/2) \rightarrow \mathbf{Z}/2 \tag{1}$$

defined as follows. Each class in H_1 can be represented by a closed curve which is framed either trivially or nontrivially. It can be shown that homologous curves have the same framing invariant, so φ is well defined. At this point Pontryagin made a famous mistake which went undetected for over a decade: he assumed that φ was a homomorphism. We now know this is not the case, and we will say more about it below in §1.3.

On that basis he argued that φ must have a nontrivial kernel, since the source group is $(\mathbf{Z}/2)^2$. Therefore there is a closed curve along which we can do the surgery. It follows that M can be surgered into a 2-sphere, leading to the erroneous conclusion that $\pi_{n+2}(S^n) = 0$ for all n . Freudenthal [Fre38] and later George Whitehead [Whi50] both proved that it is $\mathbf{Z}/2$ for $n \geq 2$. Pontryagin corrected his mistake in [Pon50], and in [Pon55] he gave a complete account of the relation between framed cobordism and homotopy groups of spheres.

1.2. Our main result

Our main theorem can be stated in three different but equivalent ways:

A solution to the Arf-Kervaire invariant problem

| x | 0 | a | b | $a + b$ | Arf invariant |
|----------|---|-----|-----|---------|---------------|
| $q(x)$ | 0 | 0 | 0 | 1 | 0 |
| $q'(x)$ | 0 | 1 | 1 | 1 | 1 |
| $q''(x)$ | 0 | 1 | 0 | 0 | 0 |

The value each refinement on $a+b$ is determined by those on a and b , and q'' is isomorphic to q . Thus the vote is three to one in each case. When \overline{H} has rank 4, it is 10 to 6.

Let M be a $2m$ -connected smooth closed manifold of dimension $4m + 2$ with a framed embedding in \mathbf{R}^{4m+2+n} . We saw above that this leads to a map $f : S^{n+4m+2} \rightarrow S^n$ and hence an element in $\pi_{n+4m+2}S^n$.

Let $H = H_{2m+1}(M; \mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2m+1} \looparrowright M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

In 1960 Kervaire [Ker60] defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The *Kervaire invariant* $\Phi(M)$ is defined to be the Arf invariant of q . In the case $m = 0$, when the dimension of the manifold is 2, Kervaire's q is Pontryagin's map φ of (1).

What can we say about $\Phi(M)$?

- Kervaire [Ker60] showed it must vanish when $k = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure. Let N be a smooth 10-manifold with boundary given as the union of two copies of the tangent disk bundle of S^5 , glued together along a common copy of $D^5 \times D^5$ where the fibers in one copy are parallel to the base in the other. The boundary is homeomorphic to S^9 . Thus we can get a closed topological manifold X by gluing on a 10-ball along its common boundary with n , or equivalently collapsing ∂N to a point. X then has nontrivial Kervaire invariant. On the other hand, Kervaire proved that any smooth framed manifold must have trivial Kervaire invariant. Therefore the topological framed manifold X cannot have a smooth structure. Equivalently, the boundary ∂N cannot be diffeomorphic to S^9 . It must be an exotic 9-sphere.
- For $k = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant. Pontryagin used it in [Pon50] (after some false starts in the 30s) to show $\pi_{n+2}S^n = \mathbf{Z}/2$ for all $n \geq 2$.
- There are similar constructions for $k = 1$ and $k = 3$, where the framed manifolds are $S^3 \times S^3$ and $S^7 \times S^7$ respectively. Like S^1 , S^3 and S^7 are both parallelizable, meaning that their tangent bundles are trivial. The framings can be twisted in such a way as to yield a nontrivial Kervaire invariant.
- Brown-Peterson [BP66b] showed that it vanishes for all positive even k . This means that apart from the 2-dimensional case, any smooth framed manifold with nontrivial Kervaire invariant must have a dimension congruent to 6 modulo 8.
- Browder [Bro69] showed that it can be nontrivial only if $k = 2^{j-1} - 1$ for some positive integer j . This happens iff the element h_j^2 is a permanent cycle in the

Adams spectral sequence, which was originally introduced in [Ada58]. (More information about it can be found in [Rav86] and [Rav04].) The corresponding element in $\pi_{n+2^j+1-2}^S$ is θ_j , the subject of our theorem. *This is the stable homotopy theoretic formulation of the problem.*

- θ_j is known to exist for $1 \leq j \leq 3$, i.e., in dimensions 2, 6, and 14. In these cases the relevant framed manifold is $S^{2^j-1} \times S^{2^j-1}$ with a twisted framing as discussed above. The framings on S^{2^j-1} represent the elements h_j in the Adams spectral sequence. The Hopf invariant one theorem of Adams [Ada60] says that for $j > 3$, h_j is not a permanent cycle in the Adams spectral sequence because it supports a nontrivial differential. (His original proof was not written in this language, but had to do with secondary cohomology operations.) This means that for $j > 3$, a smooth framed manifold representing θ_j (i.e., having a nontrivial Kervaire invariant) cannot have the form $S^{2^j-1} \times S^{2^j-1}$.
- θ_j is also known to exist for $j = 4$ and $j = 5$, i.e., in dimensions 30 and 62. In both cases the existence was first established by purely homotopy theoretic means, without constructing a suitable framed manifold. For $j = 4$ this was done by Barratt, Mahowald and Tangora in [MT67] and [BMT70]. A framed 30-manifold with nontrivial Kervaire invariant was later constructed by Jones [Jon78]. For $j = 5$ the homotopy theory was done in 1985 by Barratt-Jones-Mahowald in [BJM84].
- Our theorem says θ_j does *not* exist for $j \geq 7$. The case $j = 6$ is still open.

Kervaire’s 10-manifold with boundary that is described above can be generalized to a $(4k+2)$ -manifold with boundary constructed in a similar way. In all cases except $k = 0, 1$ or 3 , any framing of this manifold will do because the tangent bundle of S^{2k+1} is nontrivial and leads to a nontrivial invariant. The boundary is homeomorphic to S^{4k+1} , but may or may not be diffeomorphic to the standard sphere. If it is, then attaching a $(4k+2)$ -disk to it will produce a smooth framed manifold with nontrivial Kervaire invariant. If it is not, then we have an exotic $(4k+1)$ -sphere bounding a framed manifold and hence not detected by framed cobordism.

1.4. The unstable formulation

Assume all spaces in sight are localized at the prime 2. For each $n > 0$ there is a fiber sequence due to James, [Jam55], [Jam56a], [Jam56b] and [Jam57]

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

Here $\Omega X = \Omega^1 X$ where $\Omega^k X$ denotes the space of continuous base point preserving maps to X from the k -sphere S^k , known as the k th loop space of X . This leads to a long exact sequence of homotopy groups

$$\dots \longrightarrow \pi_{m+n} S^n \xrightarrow{E} \pi_{m+n+1} S^{n+1} \xrightarrow{H} \pi_{m+n+1} S^{2n+1} \xrightarrow{P} \pi_{m+n-1} S^n \longrightarrow \dots$$

A solution to the Arf-Kervaire invariant problem

Here

- E stands for *Einhängung*, the German word for suspension.
- H stands for *Hopf* invariant.
- P stands for Whitehead *p*roduct.

Assembling these for fixed m and various n leads to a diagram

$$\begin{array}{ccccccc}
 & & \pi_{m+n+1}S^{2n-1} & & \pi_{m+n+2}S^{2n+1} & & \pi_{m+n+3}S^{2n+3} \\
 & & \downarrow P & & \downarrow P & & \downarrow P \\
 \dots & \xrightarrow{E} & \pi_{m+n-1}S^{n-1} & \xrightarrow{E} & \pi_{m+n}S^n & \xrightarrow{E} & \pi_{m+n+1}S^{n+1} \xrightarrow{E} \dots \\
 & & \downarrow H & & \downarrow H & & \downarrow H \\
 & & \pi_{m+n-1}(S^{2n-3}) & & \pi_{m+n}S^{2n-1} & & \pi_{m+n+1}S^{2n+1}
 \end{array}$$

where

- Sequences of arrows labeled H, P, E, H, P (or any subset thereof) in that order are exact.
- The groups in the top and bottom rows are inductively known, and we can compute those in the middle row by induction on n .
- The groups in the top and bottom rows vanish for large n , making E an isomorphism.
- An element in the middle row has trivial suspension (is killed by E) iff it is in the image of P .
- It desuspends (is in the image of E) iff its Hopf invariant (image under H) is trivial.

When $m = n - 1$ this diagram is

$$\begin{array}{ccccccc}
 & & & & \pi_{2n+1}S^{n+1} & & \\
 & & & & \downarrow H & & \\
 & & \pi_{2n}S^{2n-1} & & \pi_{2n+1}S^{2n+1} = \mathbf{Z} & & \pi_{2n+2}S^{2n+3} = 0 \\
 & & \downarrow P & & \downarrow P & & \downarrow P \\
 \dots & \xrightarrow{E} & \pi_{2n-2}S^{n-1} & \xrightarrow{E} & \pi_{2n-1}S^n & \xrightarrow{E} & \pi_{2n}S^{n+1} \xrightarrow{E} \dots \\
 & & \downarrow H & & \downarrow H & & \downarrow H \\
 & & \pi_{2n-2}S^{2n-3} & & \pi_{2n-1}S^{2n-1} = \mathbf{Z} & & \pi_{2n}S^{2n+1} = 0
 \end{array}$$

The image under P of the generator of $\pi_{2n+1}S^{2n+1}$ is denoted by $w_n \in \pi_{2n-1}S^n$ and is called the *Whitehead square*.

- When n is even, $H(w_n) = 2$ and w_n has infinite order.
- w_n is trivial for $n = 1, 3$ and 7 . In those cases the generator of $\pi_{2n+1}S^{2n+1}$ is the Hopf invariant (image under H) of one of the three Hopf maps in $\pi_{2n+1}S^{n+1}$,

$$S^3 \xrightarrow{\eta} S^2, \quad S^7 \xrightarrow{\nu} S^4 \quad \text{and} \quad S^{15} \xrightarrow{\sigma} S^8.$$

- For other odd values of n , twice the generator of the upper \mathbf{Z} is $H(w_{n+1})$, so w_n has order 2.
- It turns out that w_n is divisible by 2 iff $n = 2^{j+1} - 1$ and θ_j exists, in which case $w_n = 2\theta_j$.
- Each Whitehead square $w_{2n+1} \in \pi_{4n+1}S^{2n+1}$ (except the cases $n = 0, 1$ and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_j (see Sidebar 1, page 26). More precisely we have

$$H(w_{(2s+1)2^j-1}) = \beta_j$$

for each $j > 0$ and $s \geq 0$. This result is essentially Adams' 1962 solution to the vector field problem [Ada62].

Recall the EHP sequence

$$\dots \longrightarrow \pi_{m+n}S^n \xrightarrow{E} \pi_{m+n+1}S^{n+1} \xrightarrow{H} \pi_{m+n+1}S^{2n+1} \xrightarrow{P} \pi_{m+n-1}S^n \longrightarrow \dots$$

Given some $\beta_j \in \pi_{\phi(j)+2n+1}S^{2n+1}$ for $\phi(j) < 2n$, one can ask about the Hopf invariant of its image under P , which vanishes when β_j is in the image of H . In most cases the answer is known and is due to Mahowald, [Mah67] and [Mah82]. The remaining cases have to do with θ_j . The answer that he had hoped for is the following, which can be found in [Mah67]. (To our knowledge, Mahowald never referred to this as the World Without End Hypothesis. We chose that term to emphasize its contrast with the Doomsday Hypothesis.)

World Without End Hypothesis (Mahowald 1967).

- *The Arf-Kervaire element $\theta_j \in \pi_{2^j+1-2}^S$ exists for all $j > 0$.*
- *It desuspends to $S^{2^{j+1}-1-\phi(j)}$ and its Hopf invariant is β_j .*
- *Let $j, s > 0$ and suppose that $m = 2^{j+2}(s+1) - 4 - \phi(j)$ and $n = 2^{j+1}(s+1) - 2 - \phi(j)$. Then $P(\beta_j)$ has Hopf invariant θ_j .*

This describes the systematic behavior in the EHP sequence of elements related to the image of J , and the θ_j are an essential part of the picture. Because of our theorem, we now know that this hypothesis is incorrect.

1.5. Questions raised by our theorem

EHP sequence formulation. The World Without End Hypothesis was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since θ_j does not exist for $j \geq 7$. *This means the behavior of the indicated elements $P(\beta_j)$ for $j \geq 7$ is a mystery.*

Adams spectral sequence formulation. We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. *We have no idea what their targets are.*

Manifold formulation. Here our result does *not* lead to any obvious new questions. It appears rather to be the final page in the story.

Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future.

2. Our strategy

2.1. Ingredients of the proof

Our proof has several ingredients.

- It uses methods of stable homotopy theory, which means it uses spectra instead of topological spaces. Recall that a space X has a homotopy group $\pi_k X$ for each positive integer k . A spectrum X has an abelian homotopy group $\pi_k X$ defined for every integer k . For the sphere spectrum S^0 , $\pi_k S^0$ is the stable k -stem homotopy group π_k^S . The hypothetical θ_j is an element of this group for $k = 2^{j+1} - 2$.
- It uses complex cobordism theory, the subject of Sidebar 2. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor [Mil60], Novikov ([Nov60], [Nov62] and [Nov67]) and Quillen [Qui69] in the 60s. A pivotal tool in the subject is the theory of formal group laws, the subject of Sidebar 4.
- It also makes use of newer less familiar methods from equivariant stable homotopy theory. A helpful introduction to this subject is the paper of Greenlees-May [GM95]. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers \mathbf{Z} , but by $RO(G)$, the orthogonal representation ring of G . Our calculations make use of this richer structure.

2.2. The spectrum Ω

We will produce a map $S^0 \rightarrow \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties:

- (i) *Detection Theorem.* It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. *This means that if θ_j exists, we will see its image in $\pi_* \Omega$.*
- (ii) *Periodicity Theorem.* It is 256-periodic, meaning that $\pi_k \Omega$ depends only on the reduction of k modulo 256.

(iii) *Gap Theorem.* $\pi_k\Omega = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool that we call the slice spectral sequence.

If $\theta_7 \in \pi_{254}S^0$ exists, (i) implies it has a nontrivial image in $\pi_{254}\Omega$. On the other hand, (ii) and (iii) imply that $\pi_{254}\Omega = 0$, so θ_7 cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \pmod{256}$ for $j \geq 7$.

2.3. How we construct Ω

The construction of Ω requires the use of *equivariant stable homotopy theory*. It is discussed extensively in [HHR, §2 and Appendices A and B]. Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\Omega_{\mathbb{O}}$.

To construct it we start with the complex cobordism spectrum MU . It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The resulting C_2 -equivariant spectrum is denoted by $MU_{\mathbf{R}}$ and is called *real cobordism theory*. This terminology follows Atiyah's definition of real K -theory [Ati66], by which he meant complex K -theory equipped with complex conjugation.

2.4. The norm

Next we use a formal tool we call the *norm* N_H^G for inducing up from an H -spectrum to a G -spectrum when H is a subgroup of G . It is treated in [HHR, §2.3]. Since the only groups we consider here are cyclic, we will denote N_H^G by N_h^g where $h = |H|$ and $g = |G|$.

The analogous construction on the space level is easy to describe. Let X be a space that is acted on by a subgroup H of G . Let

$$Y = \text{Map}_H(G, X),$$

the space of H -equivariant maps from G to X . Here the action of H on G is by left multiplication, and the resulting object has an action of G by right multiplication. As a set, $Y = X^{|G/H|}$, the $|G/H|$ -fold Cartesian power of X . A general element of G permutes these factors, each of which is left invariant by the subgroup H . For an H -spectrum X , the analogous G -spectrum is denoted by $N_H^G X$.

In particular for a finite cyclic 2-group $G = C_{2^{n+1}}$ we get a G -spectrum

$$MU^{(G)} = N_2^g MU_{\mathbf{R}}$$

underlain by $MU^{(2^n)}$. We are most interested in the case $n = 2$. This spectrum is not periodic, but it has a close relative $\Omega_{\mathbb{O}}$ (to be defined below in §5) which is periodic.

Sidebar 2 Complex cobordism

The complex cobordism spectrum MU is a critical tool in modern stable homotopy theory. Its use in computing the stable homotopy groups of spheres is the subject of [Rav86]. Much of the needed background material on vector bundles and Thom spaces can be found in [MS74].

Let $U(n)$ denote the n -dimensional unitary group and $BU(n)$ its classifying space. The latter is the space of complex n -planes in an infinite dimensional complex vector space. It is the base space of an n -dimensional complex vector bundle where the fiber at each point is the set of vectors in the given n -plane. We denote the one point compactification of its total space (also known as the Thom space of the vector bundle) by $MU(n)$. The inclusion map $U(n) \rightarrow U(n+1)$ leads to a map from $\Sigma^2 MU(n)$ to $MU(n+1)$, which is adjoint to the map $MU(n) \rightarrow \Omega^2 MU(n+1)$.

Such maps can be looped and iterated, allowing us to define

$$MU_k = \lim_{\rightarrow} \Omega^{2n-k} MU(n)$$

with MU_k homeomorphic to ΩMU_{k+1} . The collection $\{MU_k\}$ is the spectrum MU . Its homotopy groups are defined by

$$\pi_i MU = \lim_{\rightarrow} \pi_{2n+i} MU(n) = \pi_{i+k} MU_k.$$

Its homology and cohomology can be similarly defined. The inclusions of $U(m) \times U(n)$ into $U(m+n)$ lead to maps $MU(m) \wedge MU(n) \rightarrow MU(m+n)$ and $MU \wedge MU \rightarrow MU$. This makes MU into an E_∞ -ring spectrum. There are generalized homology and cohomology theories MU_* and MU^* defined by

$$MU_i(X) = \pi_i X \wedge MU \quad \text{and} \quad MU^i(X) = [X, MU_i],$$

with the latter being endowed with cup products.

We know that

$$\pi_* MU = \mathbf{Z}[x_i : i > 0] \quad \text{and} \quad H_* MU = \mathbf{Z}[m_i : i > 0]$$

where $|x_i| = |m_i| = 2i$. After localization at a prime p , MU splits into a wedge of copies of a spectrum BP (for Brown-Peterson [BP66a]) with

$$\pi_* BP = \mathbf{Z}_{(p)}[v_i : i > 0] \quad \text{and} \quad H_* BP = \mathbf{Z}_{(p)}[\ell_i : i > 0]$$

where $|v_i| = |\ell_i| = 2(p^i - 1)$. This is related to a corresponding fact about formal groups laws described in Sidebar 4 (page 51). *In this paper we will assume that MU has been localized at the prime 2.*

2.5. Fixed points and homotopy fixed points

For a G -space X , X^G is the subspace fixed by all of G , which is the same as the space of equivariant maps from a point to X , $\text{Map}_G(*, X)$. To get X^{hG} , we replace the point

here by a free contractible G -space EG . The homotopy type of $X^{hG} = \text{Map}_G(EG, X)$ is known to be independent of the choice of EG . The unique map $EG \rightarrow *$ leads to a map $\varphi : X^G \rightarrow X^{hG}$.

We construct a C_8 -spectrum $\Omega_{\mathbb{O}}$ and show that

- $\Omega_{\mathbb{O}}^{hC_8}$ satisfies the detection and Periodicity Theorems.
- $\Omega_{\mathbb{O}}^{C_8}$ satisfies the Gap Theorem.

Hence our proof depends on a fourth property:

- (iv) *Fixed Point Theorem.* The map $\Omega_{\mathbb{O}}^{C_8} \rightarrow \Omega_{\mathbb{O}}^{hC_8}$ is an equivalence. Its proof is given in [HHR, §10]. This fixed point set is our spectrum Ω .

We will come back to the definition of $\Omega_{\mathbb{O}}$ below.

2.6. $RO(G)$ -graded homotopy

Let $RO(G)$ denote the orthogonal representation ring of G and let S^V denote the one point compactification of the orthogonal representation V . For a G -space or spectrum X define

$$\pi_V^G X = [S^V, X]_G.$$

Note that when the action of G on V is trivial and V has dimension $|V|$, we denote $\pi_V^G X$ by $\pi_{|V|}^G X$, and an equivariant map $S^V = S^{\dim V} \rightarrow X$ must land in X^G , so

$$\pi_n^G X = \pi_n X^G.$$

In the stable category we can make sense of this for virtual as well as actual representations, so we get homotopy groups indexed by $RO(G)$, which we denote collectively by $\pi_*^G X$. We denote the ordinary homotopy of the underlying spectrum of X by $\pi_*^u X$.

We use the norm in two ways:

- Externally as a functor from H -spectra to G -spectra. In particular, for a representation V of H , $N_h^g S^V = S^{\text{ind} V}$, the one point compactification of the induced representation of V .
- Internally as a homomorphism

$$N_h^g : \pi_*^H X \rightarrow \pi_*^G X$$

for a G -ring spectrum X . Here we are using the forgetful functor to regard X as an H -spectrum with $RO(H)$ -graded homotopy. The external norm takes us to $\pi_*^G N_h^g X$. Then $N_h^g X$ is $X^{(g/h)}$ as a G -spectrum, and we use the multiplication on X to get a map $N_h^g X \rightarrow X$.

We will use the same notation for both.

Recall that

$$\pi_* MU = \mathbf{Z}_{(2)}[x_1, x_2, \dots] \quad \text{with } |x_i| = 2i.$$

(We are assuming that MU has been localized at 2.) It turns out that any choice of generator $x_i : S^{2i} \rightarrow MU$ is the image of the forgetful functor of a map

$$S^{i\rho} \xrightarrow{\bar{x}_i} MU_{\mathbf{R}}.$$

Here ρ denotes the regular real representation of C_2 , which is the same thing as the complex numbers \mathbf{C} acted on by conjugation.

For $G = C_{2^{n+1}}$, $\pi_*^u MU^{(G)}$ is a graded polynomial algebra over $\mathbf{Z}_{(2)}$ where

- there are 2^n generators in each positive even dimension $2i$.
- they are acted on transitively by G .

For a group generator $\gamma \in G$ and polynomial generator $r_i \in \pi_{2i}$, the set

$$\{\gamma^j r_i : 0 \leq j < 2^n\}$$

is algebraically independent, and $\gamma^{2^n} r_i = (-1)^i r_i$.

3. The slice filtration

Now we introduce our main technical tool, which is treated in [HHR, §6]. It is an equivariant analog of the Postnikov tower.

3.1. Postnikov towers

First we need to recall some things about the classical Postnikov towers. The m th Postnikov section $P^m X$ of a space or spectrum X is obtained by killing all homotopy groups of X above dimension m by attaching cells. The fiber of the map $X \rightarrow P^m X$ is $P_{m+1} X$, the m -connected cover of X .

These two functors have some universal properties. Let \mathcal{S} and $\mathcal{S}_{>m}$ denote the categories of spectra and m -connected spectra.

The functor P^m is *Farjoun nullification* [Far96] with respect to the subcategory $\mathcal{S}_{>m}$. This means the map $X \rightarrow P^m X$ is universal among maps from X to spectra which are $\mathcal{S}_{>m}$ -null in the sense that all maps to them from m -connected spectra are null. In other words,

- The spectrum $P^m X$ is $\mathcal{S}_{>m}$ -null.
- For any $\mathcal{S}_{>m}$ -null spectrum Z , the map $\mathcal{S}(P^m X, Z) \rightarrow \mathcal{S}(X, Z)$ is an equivalence.

Since $\mathcal{S}_{>m} \subset \mathcal{S}_{>m-1}$, there is a natural transformation $P^m \rightarrow P^{m-1}$, whose fiber is denoted by $P_m^m X$. It is an Eilenberg-Mac Lane spectrum with homotopy concentrated in dimension m .

In what follows G will be a finite cyclic 2-group, and $g = |G|$. Let \mathcal{S}^G denote the category of G -equivariant spectra. We need an equivariant analog of $\mathcal{S}_{>m}$. Our choice for this is somewhat novel.

3.2. Slice cells

Recall that $\mathcal{S}_{>m}$, the category of m -connected spectra, is the category of spectra built up out of spheres of dimension $> m$ using arbitrary wedges and mapping cones.

For a subgroup H of G with $|H| = h$ and an integer k , let

$$\widehat{S}(k\rho_H) = G_+ \wedge_H S^{k\rho_H}$$

where ρ_H denotes the regular real representation of H . Its underlying spectrum is a wedge of g/h spheres of dimension kh which are permuted by elements of G and are invariant under H .

We will replace the set of sphere spectra by the collection

$$\mathcal{A} = \left\{ \widehat{S}(k\rho_H), \Sigma^{-1}\widehat{S}(k\rho_H) : H \subset G, k \in \mathbf{Z} \right\}.$$

Definition 3.1. *We will refer to the elements in the set \mathcal{A} above as slice cells or simply as cells. (Note that $\Sigma^{-2}\widehat{S}(k\rho_H)$ and larger desuspensions are not cells.) A free cell is one of the form $\widehat{S}(k\rho_{\{e\}})$, a wedge of g k -spheres permuted by G . Note that*

$$\Sigma^{-1}\widehat{S}(k\rho_{\{e\}}) = \widehat{S}((k-1)\rho_{\{e\}}).$$

Nonfree cells are said to be isotropic. Cells of the form $\widehat{S}(k\rho_H)$ (rather than $\Sigma^{-1}\widehat{S}(k\rho_H)$) are said to be pure.

In order to define $\mathcal{S}_{>m}^G$, we need to assign a dimension to each element in \mathcal{A} . We do this in terms of the underlying wedge summands, namely

$$\dim \widehat{S}(k\rho_H) = kh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(k\rho_H) = kh - 1.$$

Then $\mathcal{S}_{>m}^G$ is the category built up out of elements in \mathcal{A} of dimension $> m$ using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

3.3. The slice tower

With this definition it is possible to construct functors P_{m+1}^G and P_m^G with the same formal properties as in the classical case. Thus we get a tower

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_G^{m+1}X & \longrightarrow & P_G^m X & \longrightarrow & P_G^{m-1}X & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & {}^G P_{m+1}^{m+1}X & & {}^G P_m^m X & & {}^G P_{m-1}^{m-1}X & & \end{array} \quad (2)$$

in which the homotopy colimit is X and the homotopy limit is contractible.

We call this the *slice tower*. ${}^G P_m^m X$ is the n th slice and the decreasing sequence of subgroups of $\pi_* X$ is the *slice filtration*. We also get slice filtrations of the $RO(G)$ -graded homotopy $\pi_*^G X$ and the homotopy groups of fixed point sets $\pi_* X^H$. *From now on we will drop the symbol G from the functors P^m , P_m and P_m^m .*

There is an important difference between this tower and the classical one. In the classical case the map $X \rightarrow P^m X$ does not change homotopy groups in dimensions $\leq m$. *This is not true in this equivariant case.*

In the classical case, $P_m^m X$ is an Eilenberg-Mac Lane spectrum whose n th homotopy group is that of X . In our case, $\pi_* P_m^m X$ need not be concentrated in dimension m .

This means the slice filtration leads to a (possibly noncollapsing) *slice spectral sequence* converging to $\pi_* X$ and its variants.

One variant has the form

$$E_2^{s,t} = \widehat{\pi}_{t-s}^G P_t^t X \implies \pi_{t-s}^G X.$$

Recall that $\pi_*^G X$ is by definition $\pi_* X^G$, the homotopy of the fixed point set.

This is the spectral sequence we will use to study $MU^{(G)}$ and its relatives.

3.4. The Slice Theorem

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties.

Slice Theorem. *In the slice tower for $MU^{(G)}$, every odd slice is contractible and P_{2m}^{2m} decomposes as $\widehat{W}_{2m} \wedge H\mathbf{Z}_{(2)}$, where \widehat{W}_{2m} is a certain wedge of $2m$ -dimensional slice cells (to be named later) and $H\mathbf{Z}_{(2)}$ is the 2-local integer Eilenberg-Mac Lane spectrum with trivial G action. \widehat{W}_{2m} never has any free summands.*

We will encounter other spectra with similar properties, so we give them a name.

Definition 3.2. *A 2-local spectrum X is pure (isotropic) if each of its odd slices is contractible and each even slice has the form $\widehat{W} \wedge H\mathbf{Z}_{(2)}$ where \widehat{W} is a wedge of pure (isotropic) cells (see Definition 3.1).*

In order to specify \widehat{W}_{2m} , we need the following definition.

Definition 3.3. *Suppose X is a (2-local) G -spectrum such that its underlying homotopy group $\pi_k^u X$ is a free abelian group (a free $\mathbf{Z}_{(2)}$ -module). A refinement of $\pi_k^u X$ is an equivariant map*

$$c : \widehat{W} \rightarrow X$$

in which \widehat{W} is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u X$.

Recall that in $\pi_*^u(MU_{\mathbf{R}})$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{m\rho_2}$.

$\pi_*^u MU^{(C_8)}$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $\gamma^j r_i$ for $0 \leq j \leq 3$. The action of $\gamma^4 \in G = C_8$ is given by

$$\gamma^4 r_i = (-1)^i r_i.$$

We will explain how $\pi_*^u MU^{(C_8)}$ can be refined.

$\pi_2^u MU^{(C_8)}$ has 4 generators $\gamma^j r_1$ that are permuted up to sign by G . It is refined by an equivariant map

$$\widehat{W}_2 = \widehat{S}(\rho_2) \rightarrow MU^{(C_8)}.$$

Recall that the underlying spectrum of $\widehat{S}(\rho_2)$ is a wedge of 4 copies of S^2 .

In $\pi_4^u MU^{(C_8)}$ there are 14 monomials that fall into 4 orbits under the action of G , each corresponding to a map from a slice cell.

$$\begin{aligned} \widehat{S}(2\rho_2) &\longleftrightarrow \{r_1^2, \gamma r_1^2, \gamma^2 r_1^2, \gamma^3 r_1^2\} \\ \widehat{S}(2\rho_2) &\longleftrightarrow \{r_1 \cdot \gamma r_1, \gamma r_1 \cdot \gamma^2 r_1, \gamma^2 r_1 \cdot \gamma^3 r_1, \gamma^3 r_1 \cdot r_1\} \\ \widehat{S}(2\rho_2) &\longleftrightarrow \{r_2, \gamma r_2, \gamma^2 r_2, \gamma^3 r_2\} \\ \widehat{S}(\rho_4) &\longleftrightarrow \{r_1 \cdot \gamma^2 r_1, \gamma r_1 \cdot \gamma^3 r_1\} \end{aligned}$$

Recall that $\widehat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$. It follows that $\pi_4^u MU^{(C_8)}$ is refined by an equivariant map from

$$\widehat{W}_4 = \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(\rho_4) \vee \widehat{S}(2\rho_2).$$

3.5. Refining $\pi_*^u MU^{(G)}$

More generally, we can proceed as follows. If we regard $MU^{(G)}$ as a C_2 -spectrum, then $\pi_*^u MU^{(G)}$ is refined by a map from a wedge of $S^{k\rho_2}$ s. More specifically, for each $i > 0$ we have a map

$$S^0[\bar{r}_i] = \bigvee_{m \geq 0} S^{mi\rho_2} \rightarrow MU^{(G)}$$

representing the powers of a certain generator $\bar{r}_i \in \pi_{2i}^{C_2} MU^{(G)}$ defined in [HHR, §5.4.2]. Using the multiplication on $MU^{(G)}$, we get a map

$$S^0[\bar{r}_1, \bar{r}_2, \dots] = \bigwedge_{i > 0} S^0[\bar{r}_i] \rightarrow MU^{(G)}.$$

This spectrum is a wedge of $S^{m\rho_2}$ s representing each monomial in the r_i s.

Now we apply the internal norm functor N_2^g to this map and get a map

$$A = N_2^g S^0[\bar{r}_1, \bar{r}_2, \dots] \rightarrow MU^{(G)} \tag{3}$$

which refines the underlying homotopy of the target as a G -spectrum. A is a wedge of even dimensional slice cells, and the \widehat{W}_{2m} in the Slice Theorem is the wedge of $2m$ -dimensional slice cells in A .

3.6. The definition of $\Omega_{\mathbb{O}}$

Our G -spectrum $\Omega_{\mathbb{O}}$ (where $G = C_8$) is obtained from the E_{∞} -ring spectrum $MU^{(G)}$ by inverting a certain element $D \in \pi_{19\rho_G}^G$ specified in (6) below. The choices of G and D are the simplest ones leading to a homotopy fixed point set with the detection property, as shown in [HHR, §11]. The slice tower for $\Omega_{\mathbb{O}}$ has similar properties to that of $MU^{(G)}$.

4. The Gap Theorem

4.1. Derivation from the Slice Theorem

Assuming the Slice Theorem, the Gap Theorem follows from the fact that π_{-2}^G vanishes for each isotropic slice, i.e., for each one of the form

$$\widehat{S}(k\rho_H) \wedge H\mathbf{Z}_{(2)}$$

for nontrivial H . Proving this amounts to computing the homology of a certain chain complex. Details can be found in [HHR, §3.2]. *These calculations do not require localization at 2, so we use $H\mathbf{Z}$ instead of $H\mathbf{Z}_{(2)}$ throughout this section.*

In order to give a feel for these calculations, a picture of $\pi_*^G S^{k\rho_G} \wedge H\mathbf{Z}$ for $G = C_8$ and various integers k is shown in Figure 1.

In order to derive the Gap Theorem from the Slice Theorem we need to find the groups

$$\pi_*^G W(m\rho_h) \wedge H\mathbf{Z} = \pi_*^H S^{m\rho_h} \wedge H\mathbf{Z}.$$

We need this for *all* integers m because eventually we will obtain the spectrum $\Omega_{\mathbb{O}}$ by inverting a certain element in $\pi_{19\rho_8}^G MU^{(C_8)}$. Here is what we will learn.

Vanishing Theorem.

- For $m \geq 0$, $\pi_k^H S^{m\rho_h} \wedge H\mathbf{Z} = 0$ unless $m \leq k \leq mh$.
- For $m < 0$ and $h > 1$, $\pi_k^H S^{m\rho_h} \wedge H\mathbf{Z} = 0$ unless $mh \leq k \leq m - 2$. The upper bound can be improved to $m - 3$ except in the case $(h, m) = (2, -2)$ in which case we have $\pi_{-4}^H S^{-2\rho_2} \wedge H\mathbf{Z} = \mathbf{Z}_{(2)}$.

Gap Corollary. For $h > 1$ and all integers m , $\pi_k^H S^{m\rho_h} \wedge H\mathbf{Z} = 0$ for $-4 < k < 0$.

This means a similar statement must hold for $\pi_*^{C_8} \Omega_{\mathbb{O}} = \pi_* \Omega$, which gives the Gap Theorem.

Assuming the Slice Theorem, the Gap Theorem (the statement that $\pi_{-2}\Omega = 0$) follows immediately from the Gap Corollary above, as does the following.

Vanishing Line Corollary. The slice spectral sequence for $\pi_*^G MU^{(G)}$ is confined to the first quadrant and

$$E_2^{s,t} = 0 \quad \text{for } s > (g-1)(t-s),$$

where $g = |G|$. If we invert any element in $\pi_{k\pi_G}^G MU^{(G)}$, the spectral sequence is confined to the first and third quadrants, and in the latter

$$E_2^{s,t} = 0 \quad \text{for } s < \min(0, (g-1)(t-s+4) + 4 - g).$$

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

4.2. An easy calculation

We begin by constructing an equivariant cellular chain complex $C(m\rho_g)_*$ for $S^{m\rho_g}$, where $m \geq 0$. In it the cells are permuted by the action of G , and the action of any

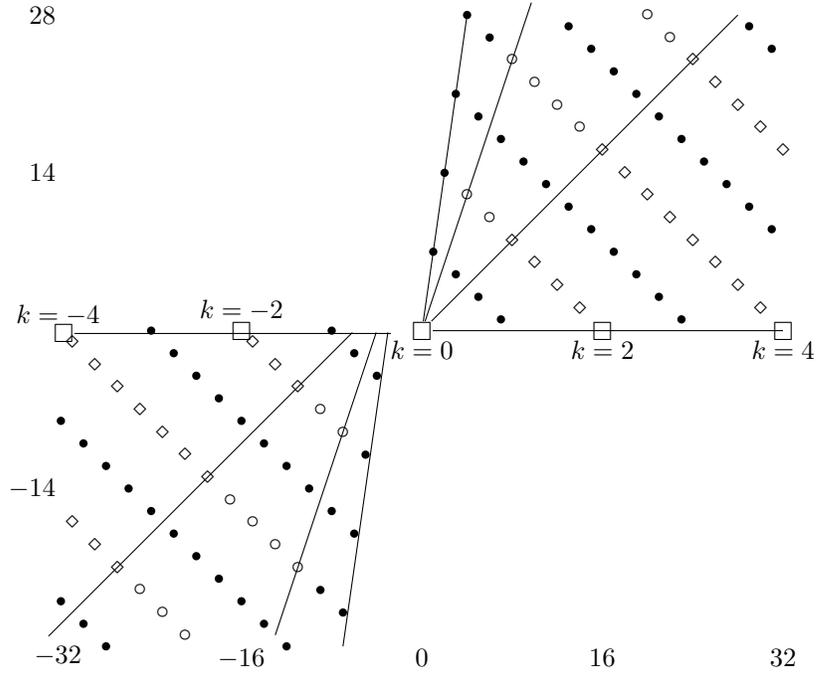


FIGURE 1. The homotopy of the slices $S^{k\rho_s} \wedge H\mathbf{Z}$ for small k . The horizontal and vertical axes are $t - s$ and s in the slice spectral sequence. The numbers along the $(t - s)$ -axis are even values of k . The groups $\pi_*^G S^{k\rho_s} \wedge H\mathbf{Z}$ lie along diagonals where $t = 8k$. Squares, bullets, circles and diamonds stand for \mathbf{Z} , $\mathbf{Z}/2$, $\mathbf{Z}/4$ and $\mathbf{Z}/8$ respectively. Lines of slopes 1, 3 and 7 separate regions with various types of torsion for even k . The gap can be seen in the center of this chart. The vanishing regions seen here also occur in the slice spectral sequence for $\pi_*^G MU^{(G)}$.

subgroup H on the H -invariant cells is trivial. It is a complex of $\mathbf{Z}[G]$ -modules and is uniquely determined by fixed point data of $S^{m\rho_g}$.

For $H \subset G$ we have

$$(S^{m\rho_g})^H = S^{mg/h}.$$

This means there is a G -CW-complex with one cell in dimension m , two cells in each dimension from $m + 1$ to $2m$, four cells in each dimension from $2m + 1$ to $4m$, and so on.

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{unless } m \leq k \leq gm \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \text{ and } h < g. \end{cases}$$

Each of these is a cyclic $\mathbf{Z}[G]$ -module. The boundary operator is uniquely determined by the fact that $H_*(C(m\rho_g)) = H_*(S^{gm})$.

Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

These groups are nontrivial only for $m \leq k \leq gm$, which gives the Vanishing Theorem for $m \geq 0$.

We will look at the bottom three groups in the complex $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$. Since $C(m\rho_g)_k$ is a cyclic $\mathbf{Z}[G]$ -module, the Hom group is always \mathbf{Z} .

For $m > 1$ our chain complex $C(m\rho_g)$ has the form

$$\begin{array}{ccccccc} & C(m\rho_g)_m & C(m\rho_g)_{m+1} & C(m\rho_g)_{m+2} & & & \\ & \parallel & \parallel & \parallel & & & \\ 0 & \longleftarrow \mathbf{Z} & \xleftarrow{\epsilon} \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} \dots & & \end{array}$$

Applying $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ to this gives (in dimensions $\leq 2m$ for $m > 4$)

$$\begin{array}{ccccccc} \mathbf{Z} & \xleftarrow{2} \mathbf{Z} & \xleftarrow{0} \mathbf{Z} & \xleftarrow{2} \mathbf{Z} & \xleftarrow{0} \mathbf{Z} & \xleftarrow{\dots} & \dots \\ m & m+1 & m+2 & m+3 & m+4 & & \end{array}$$

It follows that for $m \leq k < 2m$,

$$\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \pmod{2} \\ 0 & \text{otherwise.} \end{cases}$$

For negative multiples of ρ_g , $S^{-m\rho_g}$ (with $m > 0$) is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$. This means that

$$\pi_*^G S^{-m\rho_g} \wedge H\mathbf{Z} = H^*(\text{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

Applying the functor $\text{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex $C(m\rho_g)$

$$\begin{array}{ccccccc} \mathbf{Z} & \xleftarrow{\epsilon} \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1-\gamma} \dots & & \\ m & m+1 & m+2 & m+3 & & & \end{array}$$

gives a cochain complex beginning with

$$\begin{array}{ccccccc} \mathbf{Z} & \xrightarrow{1} \mathbf{Z} & \xrightarrow{0} \mathbf{Z} & \xrightarrow{2} \mathbf{Z} & \xrightarrow{0} \mathbf{Z} & \longrightarrow & \dots \\ -m & -m-1 & -m-2 & -m-3 & -m-4 & & \end{array}$$

Here is a diagram showing both functors in the case $m \geq 4$.

$$\begin{array}{ccccccccc}
 m & & m+1 & & m+2 & & m+3 & & m+4 & & \\
 \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{\dots} & \dots \\
 & & & & \uparrow & & & & & & \\
 & & & & \text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot) & & & & & & \\
 \mathbf{Z} & \xleftarrow{\epsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \dots \\
 & & & & \downarrow & & & & & & \\
 & & & & \text{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z}) & & & & & & \\
 \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{\dots} & \dots \\
 -m & & -m-1 & & -m-2 & & -m-3 & & -m-4 & &
 \end{array}$$

Note the difference in the behavior of the map $\epsilon : \mathbf{Z}[C_2] \rightarrow \mathbf{Z}$ under the two functors $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\text{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the gap.

5. The Periodicity Theorem

We now outline the proof of the Periodicity Theorem (assuming the Slice Theorem), which is treated in [HHR, §9].

We establish some differentials in the slice spectral sequence and show that certain elements become permanent cycles after inverting a certain $D \in \pi_{19\rho_G}^G MU^{(G)}$ (defined below in (6)), where $G = C_8$. This leads to an equivariant self map

$$\Sigma^{256}\Omega_{\mathbb{O}} \rightarrow \Omega_{\mathbb{O}}.$$

It is an ordinary homotopy equivalence, and we will see that this implies formally that it induces an equivalence on homotopy fixed point sets.

5.1. Geometric fixed points

The key tool for studying differentials in the slice spectral sequence is the *geometric fixed point spectrum*, denoted by $\Phi^G X$ for a G -spectrum X . A detailed account and references can be found in [HHR, §2.5]. It has much nicer properties than the usual fixed point spectrum X^G , which is awkward for two reasons:

- it fails to commute with smash products and
- it fails to commute with infinite suspensions.

The *geometric fixed set* $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the *isotropy separation sequence*, which is the cofiber sequence

$$E\mathcal{P}_+ \rightarrow S^0 \rightarrow \tilde{E}\mathcal{P}. \quad (4)$$

where $E\mathcal{P}$ is a G -space such that for a subgroup H of G ,

$$E\mathcal{P}^H = \begin{cases} \text{empty} & \text{for } H = G \\ \text{contractible} & \text{otherwise.} \end{cases}$$

Such a space can be constructed for any finite group G and it is unique up to equivariant homotopy equivalence. For cyclic 2-group G , the space EC_2 with G acting via its projection onto C_2 has the required properties. The symbol \mathcal{P} here stands for the family of proper subgroups of G . A similar space $E\mathcal{F}$ can be constructed for any family of subgroups \mathcal{F} closed under inclusion and conjugation.

Definition 5.1. For a G -spectrum X the geometric fixed point set is

$$\Phi^G X = (\tilde{E}\mathcal{P} \wedge X)^G.$$

This functor has the following properties:

Theorem 5.2.

- (i) For G -spectra X and Y , $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.
- (ii) For a G -space X , $\Phi^G \Sigma^\infty X = \Sigma^\infty (X^G)$.
- (iii) A map $f : X \rightarrow Y$ is a G -equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the suspension property we can deduce that for any finite cyclic 2-group G ,

$$\Phi^G MU^{(G)} = MO,$$

the unoriented cobordism spectrum. Its homotopy type has been well understood since Thom's work [Tho54] in the 50s.

Geometric Fixed Point Theorem. Let G be a finite cyclic 2-group and let σ denote its sign representation. Then for any G -spectrum X , $\pi_* \tilde{E}C_2 \wedge X = a_\sigma^{-1} \pi_* X$, where $a_\sigma \in \pi_{-\sigma} X$ is induced by the inclusion of the fixed point set $S^0 \rightarrow S^\sigma$.

In [HHR, §5.4.2] we define specific polynomial generators

$$\bar{r}_i^G \in \pi_{i\rho_2}^{C_2} MU^{(G)} \tag{5}$$

that are convenient for our purposes. We denote the underlying homotopy classes by r_i that lie in $\pi_{2i}^u MU^{(G)}$.

Recall that $\pi_* MO = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. It is not hard to show that

$$\pi_*^u MU^{(C_8)} = \mathbf{Z}_{(2)}[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]$$

where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$. In $\pi_{i\rho_8} MU^{(4)}$ we have the element

$$Nr_i = r_i \gamma(r_i) \gamma^2(r_i) \gamma^3(r_i).$$

Applying the functor Φ^G to the map $N_2^8 r_i : S^{i\rho_8} \rightarrow MU^{(C_8)}$ gives a map $S^i \rightarrow MO$.

Lemma 5.3. *The generators r_i and y_i satisfy*

$$\Phi^G N_2^g r_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

5.2. Some slice differentials

We know that the slice spectral sequence for $MU^{(G)}$ has a vanishing line of slope $g - 1$. We will describe the subring of elements lying on it.

Let $f_i \in \pi_i(MU^{(G)})$ be the composite

$$S^i \xrightarrow{a_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU^{(G)},$$

where $a_{i\rho_g}$ is the inclusion of the fixed point set. The following facts about f_i are easy to prove.

- It appears in the slice spectral sequence in $E_2^{(g-1)i, gi}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the f_i .
- Under the map

$$\pi_* MU^{(G)} \rightarrow \pi_* \Phi^G MU^{(G)} = \pi_* MO$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

- Any differential landing on the vanishing line must have a target in the ideal given by (f_1, f_3, f_7, \dots) . A similar statement can be made after smashing with $S^{2^k\sigma}$.

For an oriented representation V there is a map $u_V : S^{|V|} \rightarrow \Sigma^V H\mathbf{Z}_{(2)}$, which lies in $\pi_{V-|V|}(H\mathbf{Z}_{(2)})$. It satisfies $u_{V+W} = u_V u_W$, so $u_{2^k\sigma} = u_{2^k}^{2^k-1}$.

Slice Differentials Theorem. *In the slice spectral sequence for $\Sigma^{2^k\sigma} MU^{(G)}$ for $k > 0$, we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + g(2^k - 1)$, and*

$$d_{1+g(2^k-1)}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

Sketch of proof: Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each power of $u_{2^k\sigma}$ has to support a nontrivial differential. The only way this can happen is as indicated in the theorem.

Typically one proves theorems about differentials in such spectral sequences by means of some sort of extended power construction. In our case, all of the necessary geometry is encoded in the relation between $\pi_^u MU^{(G)}$ and $\pi_* MO$!*

5.3. Some $RO(G)$ -graded calculations

For a cyclic 2-group G let

$$\begin{aligned}\bar{\mathfrak{d}}_k^G = N_2^g \bar{r}_{2^k-1} &= \bar{r}_{2^k-1} \gamma(\bar{r}_{2^k-1}) \cdots \gamma^{g/2-1}(\bar{r}_{2^k-1}) \\ &\in \pi_{(2^k-1)\rho_g} MU^{(G)}\end{aligned}$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, for $G = C_8$ it is confined to the first and third quadrants with vanishing lines of slopes 0 and 7.

The differential d_r on $u_{2\sigma}^{2^k}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\bar{\mathfrak{d}}_k^G$, then $u_{2\sigma}^{2^k}$ will be a permanent cycle.

We have

$$\begin{aligned}f_{2^{k+1}-1} \bar{\mathfrak{d}}_k^G &= (a_{\rho_g}^{2^{k+1}-1} N_2^g \bar{r}_{2^{k+1}-1})(N_2^g \bar{r}_{2^k-1}) \\ &= a_{\rho_g}^{2^k} N_2^g \bar{r}_{2^{k+1}-1} (a_{\rho_g}^{2^k-1} N_2^g \bar{r}_{2^k-1}) \\ &= a_{\rho_g}^{2^k} \bar{\mathfrak{d}}_{k+1}^G f_{2^k-1} \\ &= a_V^{2^k} \bar{\mathfrak{d}}_{k+1}^G a_\sigma^{2^k} f_{2^k-1} \quad \text{where } V = \rho_g - \sigma \\ &= a_V^{2^k} p \bar{\mathfrak{d}}_{k+1}^G d_{1+8(2^k-1)}(u_{2^k\sigma}).\end{aligned}$$

Corollary. *In the $RO(G)$ -graded slice spectral sequence for $(\bar{\mathfrak{d}}_k^G)^{-1} MU^{(G)}$, the class given by $u_{2^{k+1}\sigma} = u_{2\sigma}^{2^k}$ is a permanent cycle.*

5.4. An even trickier $RO(G)$ -graded calculation

The Corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle. *We need to invert something to make a power of $u_{2\rho_8}$ a permanent cycle.*

We will get this by using the norm property of u . It says that if V is an oriented representation of a subgroup $H \subset G$ with $V^H = 0$ and V' is the induced representation of V , then the norm functor N_h^g from H -spectra to G -spectra satisfies $N_h^g(u_V)u_{V''} = u_{V'}$, where V'' is the induced representation of the trivial representation of degree $|V|$.

From this we can deduce that

$$u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2}),$$

where σ_g denotes the sign representation on C_g .

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\bar{\mathfrak{d}}_{k_m}^{C_{2^m}}$ for $1 \leq m \leq 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j . It turns out that k_m must be chosen so that $8|2^m k_m$.

- Inverting $\bar{\mathfrak{d}}_4^{C_2}$ makes $u_{32\sigma_2}$ a permanent cycle.

- Inverting $\bar{d}_2^{C_4}$ makes $u_{8\sigma_4}$ a permanent cycle.
- Inverting $\bar{d}_1^{C_8}$ makes $u_{4\sigma_8}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8} = u_{2\rho_8}^{16}$ a permanent cycle.

Let

$$D = \bar{d}_1^{C_8} N_4^8(\bar{d}_2^{C_4}) N_2^8(\bar{d}_4^{C_2}) = N_2^8(\bar{r}_1^{C_8} \bar{r}_3^{C_4} \bar{r}_{15}^{C_2}) \in \pi_{19\rho_8} MU^{(C_8)}. \quad (6)$$

Then we define

$$\Omega_{\mathbb{O}} = D^{-1} MU^{(C_8)} \quad \text{and} \quad \Omega = \Omega_{\mathbb{O}}^{hC_8}. \quad (7)$$

Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*\Omega = \pi_*^{C_8}\Omega_{\mathbb{O}}$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0 .

5.5. The proof of the Periodicity Theorem

Preperiodicity Theorem. *Let $\Delta_1^{(8)} = u_{2\rho_8}(\bar{d}_1^{C_8})^2 \in E_2^{16,0}(\Omega_{\mathbb{O}})$. Then $(\Delta_1^{(8)})^{16}$ is a permanent cycle.*

To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}(\bar{d}_1^{C_8})^{32}$. Both $u_{32\rho_8}$ and $\bar{d}_1^{C_8}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

Hence we have an equivariant map $\Pi : \Sigma^{256}\Omega_{\mathbb{O}} \rightarrow \Omega_{\mathbb{O}}$ where

- $u_{32\rho_8} : S^{256-32\rho_8} \rightarrow \Omega_{\mathbb{O}}$ induces to the unit map from S^0 on the underlying ring spectrum and
- $\Delta_1^{(8)}$ is invertible because it is a factor of D .

The above imply that the underlying map $i_0\Pi$ of ordinary spectra is a homotopy equivalence. It is known that any such map induces an equivalence of homotopy fixed point sets, so

$$\Sigma^{256}\Omega_{\mathbb{O}}^{hC_8} \xrightarrow[\simeq]{\Pi^{hC_8}} \Omega_{\mathbb{O}}^{hC_8}.$$

Unfortunately the slice spectral sequence tells us nothing about this homotopy fixed point set. We know it detects all of the θ_j , but there is no direct way of showing that it has the gap property.

Fortunately we have a theorem (the subject of the next section) stating that in this case the homotopy fixed set is equivalent to the actual fixed point set Ω . The slice spectral sequence tells us that the latter has the gap property. Thus we have proved

Periodicity Theorem. *For Ω as in (7), $\Sigma^{256}\Omega$ is equivalent to Ω .*

6. The Homotopy Fixed Point Theorem

In order to finish the proof of the main theorem, we need to show that the actual fixed point set of $\Omega_{\mathbb{O}} = D^{-1}MU^{(C_8)}$ is equivalent to its homotopy fixed point set. A detailed account can be found in [HHR, §10]. The slice spectral sequence computes the homotopy of the former while the Hopkins-Miller spectral sequence, also known as the homotopy fixed point spectral sequence (which is known to detect θ_j), computes that of the latter.

Here is a general approach to showing that actual and homotopy fixed points are equivalent for a G -spectrum X .

We have an equivariant map $EG_+ \rightarrow S^0$. Mapping both into X gives a map of G -spectra $\varphi : X_+ \rightarrow F(EG_+, X_+)$. Passing to fixed points would give a map from X^G to X^{hG} , which we would like to be an equivalence. We will prove the stronger statement that φ is a G -equivalence.

The case of interest is $X = \Omega_{\mathbb{O}}$ and $G = C_8$. We will argue by induction on the order of the subgroups H of G , the statement being obvious for the trivial group. We will smash φ with the cofiber sequence

$$EG_+ \rightarrow S^0 \rightarrow \tilde{E}G.$$

This gives us the following diagram in which both rows are cofiber sequences.

$$\begin{array}{ccccc} EG_+ \wedge \Omega_{\mathbb{O}} & \longrightarrow & \Omega_{\mathbb{O}} & \longrightarrow & \tilde{E}G \wedge \Omega_{\mathbb{O}} \\ \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\ EG_+ \wedge F(EG_+, \Omega_{\mathbb{O}}) & \longrightarrow & F(EG_+, \Omega_{\mathbb{O}}) & \longrightarrow & \tilde{E}G \wedge F(EG_+, \Omega_{\mathbb{O}}) \end{array}$$

The map φ' is an equivalence because $\Omega_{\mathbb{O}}$ is nonequivariantly equivalent to $F(EG_+, \Omega_{\mathbb{O}})$, and EG_+ is built up entirely of free G -cells.

Thus it suffices to show that φ'' is an equivalence, which we will do by showing that both its source and target are contractible. Both have the form $\tilde{E}G \wedge X$ where X is a module spectrum over $\Omega_{\mathbb{O}}$, so it suffices to show that $\tilde{E}G \wedge \Omega_{\mathbb{O}}$ is contractible.

We will show that it is H -equivariantly contractible by induction on the order of the subgroups H of G . Over the trivial group $\tilde{E}G$ itself is contractible. Let H be a subgroup, $H' \subset H$ the subgroup of index 2 and $H_2 = H/H'$.

We will smash our spectrum with the isotropy separation sequence of (4), which in our case is

$$EH_{2+} \rightarrow S^0 \rightarrow \tilde{E}H_2.$$

Then $\tilde{E}H_2 \wedge \tilde{E}G \wedge \Omega_{\mathbb{O}}$ is contractible over H' , so it suffices to show that its H -fixed point set is contractible. It is

$$\Phi^H(\tilde{E}G \wedge \Omega_{\mathbb{O}}) = \Phi^H(\tilde{E}G) \wedge \Phi^H(\Omega_{\mathbb{O}}),$$

and $\Phi^H(\Omega_{\mathbb{O}})$ is contractible because $\Phi^H(D) = 0$. Thus it remains to show the H -contractibility of $EH_{2+} \wedge \tilde{E}G \wedge \Omega_{\mathbb{O}}$. But this is equivalent to the H' -contractibility of $\tilde{E}G \wedge \Omega_{\mathbb{O}}$, which we have by induction.

Sidebar 3 Formal group laws I

Formal groups laws are a key tool in complex cobordism theory, which is part of the infrastructure of our proof. The definitive reference is Hazewinkel's book [Haz78]. A much briefer account can be found in [Rav86, Appendix 2]. A *formal group law* F over a ring R is a power series $F(x, y) \in R[[x, y]]$ satisfying three conditions, namely

- (i) COMMUTATIVITY $F(y, x) = F(x, y)$
- (ii) IDENTITY $F(x, 0) = F(0, x) = x$
- (iii) ASSOCIATIVITY $F(F(x, y), z) = F(x, F(y, z))$.

Elementary examples include the additive and multiplicative formal group laws, $x + y$ and $1 + x + y + xy$ respectively. When R is torsion free, F is determined by a power series over $R \otimes \mathbf{Q}$ in one variable called the *logarithm of F* satisfying

$$\log_F(F(x, y)) = \log_F(x) + \log_F(y).$$

For our two elementary examples it is x and $\sum_{n \geq 0} (-1)^n x^{n+1} / (n+1)$.

There is a universal example due to Lazard [Laz55] of a formal group law G over a ring L with the property that for any formal group law F over any ring R , there is a unique ring homomorphism $\theta : L \rightarrow R$ sending G to F . It is given by

$$G(x, y) = \sum_{i,j} a_{i,j} x^i y^j \quad \text{and} \quad L = \mathbf{Z}[a_{i,j}] / I$$

where I is the ideal generated by the relations among the coefficients $a_{i,j}$ forced by conditions (i)-(iii) above. To describe L explicitly, it is useful to give it a grading such that $G(x, y)$ is homogeneous of degree 2 if $|x| = |y| = 2$. This means that $|a_{i,j}| = 2(1 - i - j)$. Lazard then shows that

$$L = \mathbf{Z}[x_i : i > 0] \quad \text{where } |x_i| = -2i.$$

7. The Detection Theorem

The Detection Theorem is the subject of [HHR, §11].

7.1. θ_j in the Adams-Novikov spectral sequence

Browder's Theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \text{Ext}_{\mathcal{A}}^{2, 2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2),$$

where \mathcal{A} denotes the mod 2 Steenrod algebra. This element is known to be the only one in its bidegree.

Sidebar 4 Formal group laws II

There is a formal group law F defined over the complex cobordism ring MU^* defined as follows. We know that

$$MU^*(\mathbf{C}P^\infty) = MU^*[[x]]$$

where $x \in MU^2\mathbf{C}P^\infty$ is the first Chern class of the canonical complex line bundle. Similarly,

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = MU^*[[x \otimes 1, 1 \otimes x]].$$

There is a map $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \rightarrow \mathbf{C}P^\infty$ related to the tensor product of line bundles. It sends x to a power series $F(x \otimes 1, 1 \otimes x)$, which can easily be shown to be a formal group law. Quillen [Qui69] showed that the Lazard classifying map $L \rightarrow MU^*$ is an isomorphism.

Cartier [Car67] showed that when R is a torsion free $\mathbf{Z}_{(p)}$ -algebra, F is canonically isomorphic to a formal group law F' which is p -typical, meaning that its logarithm has the form

$$\log_{F'}(x) = x + \sum_{n>0} \ell_n x^{p^n} \in R \otimes \mathbf{Q}[[x]].$$

The analog of the Lazard ring for such groups is much smaller than L , having polynomial generators v_n only in each dimension of the form $2(1 - p^n)$.

The topological analog of Cartier's theorem is the splitting of $MU_{(p)}$ into a wedge of suspensions of spectra known as BP , originally constructed by Brown and Peterson [BP66a].

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \text{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of i and j . When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in [Rav86, Chapter 5].

Here are the first few of these in the relevant bidegrees.

$$\begin{aligned} \theta_3 : \beta_{4/4} \text{ and } \beta_3 &\in \text{Ext}_{MU_*(MU)}^{2,16}(MU_*, MU_*), \\ \theta_4 : \beta_{8/8} \text{ and } \beta_{6/2} &\in \text{Ext}_{MU_*(MU)}^{2,32}(MU_*, MU_*), \\ \theta_5 : \beta_{16/16}, \beta_{12/4} \text{ and } \beta_{11} &\in \text{Ext}_{MU_*(MU)}^{2,64}(MU_*, MU_*), \end{aligned}$$

and so on. In the bidegree of θ_j , only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1 \alpha_{2^j-1}$. Thus

$$\text{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$$

is an elementary abelian 2-group of rank $[(j+3)/2]$. An element in it maps to h_j^2 in the corresponding classical Adams Ext group iff the coefficient of $\beta_{2^{j-1}/2^{j-1}}$ (with respect to the evident basis) is nontrivial.

We need to show that any such element has nontrivial image the Adams-Novikov spectral sequence for Ω .

Detection Theorem. *Let $j \geq 6$ and let $x \in \text{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\text{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 . Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*\Omega_{\mathbb{O}})$ is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler Ext group defined in terms of another algebraic tool, *the theory of formal A -modules*, where A is the ring of integers in a suitable field. In that setting the group in the relevant bidegree is $(\mathbf{Z}/8)^4$. We will show that $\beta_{2^{j-1}/2^{j-1}}$ maps to an element of order 2, while the other basis elements map to 0.

7.2. Formal A -modules

Recall the a formal group law (see Sidebar 4) over a ring R is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m , and we get a homomorphism

$$\begin{aligned} \tau : \mathbf{Z} &\rightarrow \text{End}(F) \\ m &\mapsto [m](x) \end{aligned}$$

from the integers to the endomorphism ring of F .

If the ground ring R is an algebra over the p -local integers $\mathbf{Z}_{(p)}$ or the p -adic integers \mathbf{Z}_p , then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or \mathbf{Z}_p .

Now suppose R is an algebra over a larger ring A , such as the ring of localized integers in a number field or a finite extension of the p -adic numbers. We say that the formal group law F is a *formal A -module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \pmod{x^2} \text{ for } a \in A.$$

The theory of formal A -modules is well developed. Lubin-Tate [LT65] used it to do local class field theory.

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. It is the ring of integers in a cyclotomic extension of the 2-adic numbers \mathbf{Q}_2 . The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A -module F over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x, y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}. \quad (8)$$

We need to consider the cohomology of C_8 with coefficients in R_* , where the action of the group is trivial on A and $\gamma w = \zeta w$ for a group generator γ . $H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_* \xrightarrow{\gamma^{-1}} R_* \xrightarrow{\text{Trace}} R_* \xrightarrow{\gamma^{-1}} \dots$$

where Trace is multiplication by $1 + \gamma + \dots + \gamma^7$.

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in s with period 2. The part of it that matters to us is

$$H^2(C_8; R_{16m}) = w^{8m} A / (8) = w^{8m} \mathbf{Z} / (8) \{1, \pi, \pi^2, \pi^3\}.$$

7.3. The relation between $MU^{(C_8)}$ and formal A -modules

What does this have to do with our spectrum $\Omega_{\mathbb{O}} = D^{-1}MU^{(C_8)}$? Recall that D is given by $\bar{\mathfrak{d}}_1^{C_8} N_4^8(\bar{\mathfrak{d}}_2^{C_4}) N_2^8(\bar{\mathfrak{d}}_4^{C_2})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\bar{\mathfrak{d}}$. They are the smallest ones that satisfy the second part of the following.

Lemma 7.1. *The classifying homomorphism $\lambda : \pi_* MU \rightarrow R_*$ for F factors through $\pi_* MU^{(C_8)}$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*^u MU^{(C_8)} \rightarrow R_*$ is equivariant, where C_8 acts on $\pi_*^u MU^{(C_8)}$ as before, it acts trivially on A and $\gamma w = \zeta w$ for a generator γ of C_8 .*
- *The element $D \in \pi_* MU^{(C_8)}$ that we invert to get $\Omega_{\mathbb{O}}$ goes to a unit in R_* .*

We will sketch the proof of this later.

7.4. Deriving the Detection Theorem from Lemma 7.1

Lemma 7.1 implies that there is a diagram

$$\begin{array}{ccc}
 S^0 & \longrightarrow & MU^{(C_8)} \\
 & & \parallel \\
 \text{Ext}_{MU_*(MU)}^{s,t}(MU_*, MU_*) & \longrightarrow & \text{Ext}_{MU_*(MU)}^{s,t}(MU_*, MU_*(MU^{(C_8)})) \\
 & & \parallel \\
 & & H^s(C_8; \pi_t^u MU^{(C_8)}) \\
 & & \downarrow \\
 H^s(C_8; \pi_t \Omega_{\mathbb{D}}) & \xlongequal{\quad\quad\quad} & H^s(C_8; \pi_t D^{-1} MU^{(C_8)}) \xrightarrow{\quad\quad\quad} H^s(C_8; R_t) \\
 & & \swarrow \lambda^{(4)*}
 \end{array}$$

where the unit map at the top induces the homomorphism below it. We prove the Detection Theorem by showing that the composite map

$$\begin{array}{ccc}
 \text{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*) & \longrightarrow & H^2(C_8; R_{2^{j+1}}) \\
 \parallel & & \parallel \\
 \mathbf{Z}/2 \{ \alpha_1 \alpha_{2^j-1}, \beta_{2^{j-1}/2^{j-1}}, \beta_{3 \cdot 2^{j-3}/2^{j-3}}, \dots \} & & w^{2^j} A/(8)
 \end{array}$$

sends $\beta_{2^{j-1}/2^{j-1}}$ to a nontrivial element of order 2, and each of the other basis elements to 0. This is a calculation requiring tools from *BP*-theory which is spelled out in [HHR, §11.4].

7.5. The proof of Lemma 7.1

To prove the first part, consider the following diagram for an arbitrary ring K .

$$\begin{array}{ccccc}
 & & \pi_*^u MU^{(C_4)} & & \\
 & & \parallel & & \\
 & & \pi_* MU^{(2)} & & \\
 & & \parallel & & \\
 & & MU_*(MU) & & \\
 \nearrow \eta_L & & \downarrow \lambda^{(2)} & & \nwarrow \eta_R \\
 \pi_* MU & & & & \pi_* MU \\
 \searrow \lambda_1 & & & & \swarrow \lambda_2 \\
 & & K & &
 \end{array}$$

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K . The Hopf algebra $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a strict isomorphism between F_1 and F_2 .

Similarly consider the following diagram.

$$\begin{array}{ccccc}
 & & \pi_*^u MU^{(C_8)} & & \\
 & & \parallel & & \\
 & & \pi_* MU^{(4)} & & \\
 \eta_1 \nearrow & & \leftarrow \eta_2 & & \eta_4 \searrow \\
 \pi_* MU & \xrightarrow{\quad} & \pi_* MU & \xrightarrow{\quad} & \pi_* MU \\
 \lambda_1 \searrow & & \downarrow \lambda^{(4)} & & \lambda_3 \nearrow \\
 & & K & & \\
 \lambda_2 \searrow & & \leftarrow \lambda_3 & & \lambda_4 \nearrow \\
 \pi_* MU & \xrightarrow{\quad} & \pi_* MU & \xrightarrow{\quad} & \pi_* MU
 \end{array}$$

The four maps η_j are induced by the four unit maps from MU to $MU^{(4)}$, the spectrum underlying $MU^{(C_8)}$. The existence of $\lambda^{(4)}$ is equivalent to that of strict isomorphisms between the formal group laws F_j classified by the λ_j .

Now suppose that $K = R_*$ and each F_j is the formal group law given by (8). Then we have the required isomorphisms, so $\lambda^{(4)}$ exists. The inclusions η_j are related to the C_8 -action on $\pi_* MU^{(4)}$ by

$$\gamma \eta_j = \eta_{j+1} \quad \text{for } 1 \leq j \leq 3,$$

and $\gamma \eta_4$ differs from η_1 by $(-1)^i$ in dimension $2i$. The λ_j can be chosen so that

$$\gamma \lambda_j = \lambda_{j+1} \quad \text{for } 1 \leq j \leq 3.$$

It follows that $\lambda^{(4)}$ is equivariant with respect to the C_8 -actions on its source and target. This proves the first part of Lemma 7.1.

For the second part, recall that

$$D = N_2^8(\bar{r}_{15}^{C_2} \bar{r}_3^{C_4} \bar{r}_1^{C_8}).$$

The norm sends products to products, and $N(x)$ is a product of conjugates of x under the action of C_8 . Hence its image in R_* is a unit multiple of that of a power of x , so it suffices to show that each of the three elements $\bar{r}_{15}^{C_2}$, $\bar{r}_3^{C_4}$ and $\bar{r}_1^{C_8}$ maps to a unit in R_* . We refer the reader to [HHR, §11.5] for the details.

8. The Slice and Reduction theorems

Recall that a pivotal step in our proof is the Slice Theorem, which identifies the layers in the slice tower for $MU_{\mathbf{R}}$ and its relatives. In [HHR, §6] it is derived from the Reduction Theorem, which we now state.

For each cyclic 2-group $G = C_{2^{n+1}}$ there is a certain equivariant (noncommutative) ring spectrum A , defined in (3), which is a certain wedge of slice cells. It maps to $MU^{(G)}$ in such a way that the underlying wedge of spheres hits all of the underlying homotopy of $MU^{(G)}$. We can map it to S^0 (with trivial group action) by sending all of the 0-connected summands to a point. Thus both $MU^{(G)}$ and S^0 are A -modules.

Reduction Theorem. *Let $R(\infty)^{(G)} = MU^{(G)} \wedge_A S^0$. The map*

$$f_G : R(\infty)^{(G)} \rightarrow H\mathbf{Z}_{(2)}$$

given by the 0th slice is an equivariant equivalence.

Recall that we have localized MU at the prime 2. We will denote $R(\infty)^{(G)}$ and f_G simply by $R(\infty)$ and f when the value of G is clear from the context.

The nonequivariant version of this is obvious. It says that if we kill off all of the positive dimensional homotopy of the spectrum underlying $MU^{(G)}$, we get $H\mathbf{Z}_{(2)}$, its zeroth Postnikov section. The case $G = C_2$ was proved by Hu-Kriz in [HK01].

The derivation of the Slice Theorem from the Reduction Theorem is essentially formal.

The proof of the Reduction Theorem is the hardest calculation in our paper and is treated in [HHR, §7].

8.1. Deriving the Slice Theorem from the Reduction Theorem

It is possible to define an A -module structure on M_d , the wedge of all $(d-1)$ -connected summands of A . Then $M_{2d-1} = M_{2d}$ and \widehat{W}_{2d} (defined in 3.5) is the cofiber of the inclusion $M_{2d+2} \rightarrow M_{2d}$, which is also an A -module.

Let

$$K_d = MU^{(G)} \wedge_A M_d \quad \text{and} \quad \tilde{P}^d = MU^{(G)} / K_{d-1}$$

Then it is not difficult to show that the fiber of the inclusion $\tilde{P}^{2d} \rightarrow \tilde{P}^{2d-1}$ is $R(\infty) \wedge \widehat{W}_{2d}$. The Reduction Theorem implies that this is $H\mathbf{Z}_{(2)} \wedge \widehat{W}_{2d}$. It also enables us to identify the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{P}^{2d+1} & \longrightarrow & \tilde{P}^{2d} & \longrightarrow & \tilde{P}^{2d-1} & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & * & & R(\infty) \wedge \widehat{W}_{2d} & & * & & \\ & & & & \parallel & & & & \\ & & & & \wedge \widehat{W}_{2d} & & & & \end{array}$$

with the slice tower (2) for $MU^{(G)}$, i.e., to show that \tilde{P}^m is equivalent to $P^m MU^{(G)}$. The Slice Theorem follows.

There is a converse [HHR, Prop. 6.9] which says roughly that if a ring spectrum R (such as $MU^{(G)} \wedge MU^{(G)}$) with properties similar to those of $MU^{(G)}$ satisfies the Slice Theorem, then the Reduction Theorem follows. Since this converse is used in the proof of the Reduction Theorem itself, we will state it more precisely.

For a finite cyclic 2-group G , let R be a connective 2-local G -ring spectrum with $\pi_*^u R$ a polynomial algebra over $\mathbf{Z}_{(2)}$ on positive dimensional generators. We suppose further that $\pi_*^u R$ is refined by a map from a spectrum A of the form

$$A = N_2^g S^0[\bar{x}_1, \bar{x}_2, \dots]$$

where $\bar{x}_i \in \pi_{k_i \rho_2}^{C_2} X$ with $k_i > 0$ for each i . Then we can define M_d, K_d and \tilde{P}^d for R as we did for $MU^{(G)}$ above. In particular \tilde{P}^0 would be the analog of $R(\infty)$ as above.

Then similar arguments to those above give

Proposition 8.1. *For R as above, if \tilde{P}^0 is equivalent to $H\mathbf{Z}_{(2)}$, then R is pure and isotropic.*

Here is our converse.

Proposition 8.2. *Let R be a connective 2-local G -ring spectrum as above, and suppose that it is pure (Definition 3.2). Then the tower $\{\tilde{P}^d\}$ is the slice tower for R . In particular, \tilde{P}^0 is $P^0 R = H\mathbf{Z}_{(2)}$.*

8.2. The proof of the Reduction Theorem

We will use a double induction argument. Assuming that the map f_G (which is defined only for nontrivial G) is H -equivariant for a subgroup $H \subseteq G$, we will show that

- (i) When H is a proper subgroup of G , it can be replaced by the next larger subgroup of G .
- (ii) G can be replaced by the next larger supergroup G' of G .

We have the statement for trivial H and any G since that is the nonequivariant case. Hence the Reduction Theorem will follow from the two inductive steps.

The second inductive step follows immediately from the fact that the restriction of $R(\infty)^{(G')}$ to G is $R(\infty)^{(G)}$. This means that we need only show the first step in the case where H is the index two subgroup of G .

For A as in (3) and an integer $i \geq 0$ we define an A -module

$$R(i)^{(G)} = MU^{(G)} \wedge_A A'_i \quad \text{where} \quad A'_i = N_2^g S^0[\bar{r}_{i+1}, \bar{r}_{i+2}, \dots]. \quad (9)$$

Roughly speaking, $R(i)^{(G)}$ is obtained from $MU^{(G)}$ by killing the subring of the underlying homotopy generated by elements below dimension $2i$. We will often denote it simply by $R(i)$.

Here are three consequences of the inductive hypothesis, proved in [HHR, Propositions 7.2 and 7.3].

Proposition 8.3. *Let $H \subset G$ be the index 2 subgroup. If the inductive hypothesis above holds in this case, then*

- (i) *The Slice Theorem holds for H when $|H| > 1$.*
- (ii) *$i_H^* MU^{(G)}$ is a pure isotropic spectrum.*
- (iii) *$i_H^* R(i)^{(G)}$ (see (9)) is pure and isotropic.*

Proof. (i) Since $i_H^* R(\infty)^{(G)} = R(\infty)^{(H)}$, the hypothesis of this proposition is the same as the Reduction Theorem for H . It implies the Slice Theorem for H .

(ii) $MU^{(G)}$ and $MU^{(H)} \wedge MU^{(H)}$ are underlain by the same smash power of MU . Thus $i_H^* MU^{(G)}$ is a wedge of suspensions of $MU^{(H)}$, and this splitting can be shown to be equivariant. Hence $i_H^* MU^{(G)}$ is pure isotropic because $MU^{(H)}$ is.

(iii) $i_H^* R(i)^{(G)}$ has an equivariant splitting similar to that of $i_H^* MU^{(G)}$ in which each summand is a suspension of $R(i)^{(H)}$. The reduction theorem for H implies that $R(i)^{(H)}$ is pure and isotropic by Proposition 8.1, so the same is true of $i_H^* R(i)^{(G)}$. \square

Now we smash the map of the Reduction Theorem with the isotropy separation sequence of (4) and get a diagram

$$\begin{array}{ccccc} EP_+ \wedge R(\infty) & \longrightarrow & R(\infty) & \longrightarrow & \tilde{E}\mathcal{P} \wedge R(\infty) \\ \downarrow f' & & \downarrow f & & \downarrow f'' \\ EP_+ \wedge H\mathbf{Z}_{(2)} & \longrightarrow & H\mathbf{Z}_{(2)} & \longrightarrow & \tilde{E}\mathcal{P} \wedge H\mathbf{Z}_{(2)} \end{array}$$

We want to show that f is an equivariant equivalence. The inductive hypothesis implies that f' is one, so it suffices to show that f'' is one. We see from Theorem 5.2 (iii) this boils down to showing that the map

$$\pi_* \Phi^G f = \pi_*^G f'' : \pi_* \Phi^G R(\infty) \rightarrow \pi_* \Phi^G H\mathbf{Z}_{(2)} \quad (10)$$

is an isomorphism.

It is not hard to show that the source and target of (10) are abstractly isomorphic and of finite type [HHR, Prop. 7.5].

Proposition 8.4.

$$\pi_* \Phi^G H\mathbf{Z}_{(2)} = \mathbf{Z}/2[b] \quad \text{where } b = u_{2\sigma} a_\sigma^{-2} \in \pi_2 \Phi^G H\mathbf{Z}_{(2)} \subset a_\sigma^{-1} \pi_*^G H\mathbf{Z}_{(2)}.$$

and

$$\pi_n \Phi^G R(\infty) = \begin{cases} \mathbf{Z}/2 & \text{for } n \geq 0 \text{ and even} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\pi_* \Phi^G R(\infty)$ lacks a multiplication because it is not obvious that $R(\infty)^{(G)}$ is a ring spectrum. We only know it is as a consequence of the Reduction Theorem.

Proof. The first assertion follows from computations similar to those of 4.2.

For the second we will compute $\pi_* \Phi^G R(i)$ by induction on i . Since $R(0) = MU^{(G)}$, $\Phi^G R(0) = MO$, so

$$\pi_* \Phi^G R(0) = \pi_* MO = \mathbf{Z}/2[h_j, j \neq 2^k - 1]$$

where $h_j \in \pi_j MO$. We can identify the fiber of the map $\Phi^G R(i-1) \rightarrow \Phi^G R(i)$ and we have a cofiber sequence

$$\Sigma^k \Phi^G R(i-1) \xrightarrow{\Phi^G \bar{\tau}_i} \Phi^G R(i-1) \longrightarrow \Phi^G R(i)$$

where

$$\Phi^G \bar{\tau}_i = \begin{cases} h_i & \text{for } i \neq 2^k - 1 \\ 0 & \text{for } i = 2^k - 1. \end{cases} \quad (11)$$

The indicated value of $\pi_* \Phi^G R(\infty)$ follows. \square

Thus it suffices to show that the homomorphism of (10) is onto. This job is simplified by the following, which is [HHR, Lemma 7.7].

Let

$$B_i = N_2^g S^0[\bar{r}_i] \quad \text{and} \quad X_i = MU^{(G)} \wedge_{B_i} S^0. \quad (12)$$

X_i is the spectrum obtained from $MU^{(G)}$ by killing the polynomial generators in dimension $2i$ of its underlying homotopy.

Lemma 8.5. *If for every $k \geq 1$, the class $b^{2^{k-1}}$ is in the image of*

$$\pi_{2^k} \Phi^G MU^{(G)} \wedge_{B_{2^{k-1}}} S^0 \rightarrow \pi_{2^k} \Phi^G H\mathbf{Z}_{(2)},$$

then the map of (10) is onto.

Proof. We have

$$R(\infty) = X_1 \wedge_{MU^{(G)}} X_2 \wedge_{MU^{(G)}} \cdots$$

This implies that if $b^{2^{k-1}}$ is in the image for each $k \geq 1$, then every power of b is. \square

Let $c_k = 2^k - 1$. To show that the hypothesis of Lemma 8.5 is met we need to study the element

$$N_2^g \bar{r}_{c_k} \in \pi_{c_k \rho_G}^G MU^{(G)},$$

which we will see to be related to an element mapping to $b^{2^{k-1}}$ under the homomorphism of Lemma 8.5. Its image in $\pi_{c_k \rho_G}^G \tilde{E}\mathcal{P} \wedge X_{c_k}$ is zero for two reasons: we are coning it off to form X_{c_k} , and it is killed by the functor Φ^G . We will verify the hypothesis of Lemma 8.5 by chasing this element around the diagram

$$\begin{array}{ccccc} E\mathcal{P}_+ \wedge MU^{(G)} & \longrightarrow & MU^{(G)} & \longrightarrow & \tilde{E}\mathcal{P} \wedge MU^{(G)} \\ \downarrow & & \downarrow & & \downarrow \\ E\mathcal{P}_+ \wedge X_{c_k} & \longrightarrow & X_{c_k} & \longrightarrow & \tilde{E}\mathcal{P} \wedge X_{c_k} \\ \downarrow & & \downarrow & & \downarrow \\ E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)} & \longrightarrow & H\mathbf{Z}_{(2)} & \longrightarrow & \tilde{E}\mathcal{P} \wedge H\mathbf{Z}_{(2)} \end{array} \quad (13)$$

Here the rows are cofiber sequences, but the columns are not.

Since $S^{i\rho_G}$ is obtained from S^i by attaching induced cells, one can show [HHR, Remark 2.34] that the restriction map

$$\pi_{i\rho_G}^G \tilde{E}\mathcal{P} \wedge X \rightarrow \pi_i^G \tilde{E}\mathcal{P} \wedge X$$

is an isomorphism for any G -spectrum X . The image of $N_2^g \bar{r}_{c_k}$ in

$$\pi_{c_k \rho_G}^G \tilde{E}\mathcal{P} \wedge MU^{(G)} \approx \pi_{c_k}^G \tilde{E}\mathcal{P} \wedge MU^{(G)} \approx \pi_{c_k} MO$$

is zero by (11). Thus in the top row of (13), $N_2^g \bar{r}_{c_k}$ has a preimage

$$y_k \in \pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge MU^{(G)}.$$

Proposition 8.6. *Any choice of y_k as above has nontrivial image in*

$$\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)}.$$

Proof of the Reduction Theorem modulo Proposition 8.6. We have already seen that the Reduction Theorem follows from the hypothesis of Lemma 8.5, that b^{c_k} is in the image of the map induced by f . The image of y_k in $\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge X_{c_k}$ has trivial image in $\pi_{c_k \rho_G}^G X_{c_k}$, so it has a preimage

$$\tilde{y}_k \in \pi_{c_k \rho_G + 1}^G \tilde{E}\mathcal{P}_+ \wedge X_{c_k}.$$

The image of \tilde{y}_k in $\pi_{c_k \rho_G + 1}^G \tilde{E}\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)}$ is nontrivial by Proposition 8.6.

Now consider the diagram

$$\begin{array}{ccc} \pi_{c_k \rho_G + 1}^G \tilde{E}\mathcal{P}_+ \wedge X_{c_k} & \xrightarrow{\approx} & \pi_{2^k}^G \tilde{E}\mathcal{P}_+ \wedge X_{c_k} \\ \downarrow & & \downarrow \\ \pi_{c_k \rho_G + 1}^G \tilde{E}\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)} & \xrightarrow{\approx} & \pi_{2^k}^G \tilde{E}\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)} \end{array}$$

where the horizontal maps are restrictions along the inclusion $S^{2^k} \rightarrow S^{c_k \rho_G + 1}$. The lower right group is $\mathbf{Z}/2$ generated by $b^{2^{k-1}}$. We have just shown that \tilde{y}_k maps nontrivially to the lower left group, so $b^{2^{k-1}}$ is in the image of the map induced by f . \square

Proposition 8.6 concerns the map

$$E\mathcal{P}_+ \wedge MU^{(G)} \rightarrow E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)}.$$

Each cell of both source and target is induced up from the index 2 subgroup H , so we can use the inductive hypothesis. We will use the factorization

$$E\mathcal{P}_+ \wedge MU^{(G)} \rightarrow E\mathcal{P}_+ \wedge R(2^k - 2)^{(G)} \rightarrow E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)},$$

with $R(2^k - 2)^{(G)}$ as in (9), and replace y_k by its image in $\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge R(2^k - 2)^{(G)}$. We will study the smash product of $E\mathcal{P}_+$ with the slice tower for $R(2^k - 2)^{(G)}$. The following is [HHR, Lemma 7.14].

Lemma 8.7. *For $0 < m < c_k g$,*

$$\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge P_m^m R(2^k - 2)^{(G)} = 0$$

and there is an exact sequence

$$\begin{array}{ccc} \pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge P_{c_k g} R(2^k - 2)^{(G)} & \longrightarrow & \pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge R(2^k - 2)^{(G)} \\ & & \downarrow \\ & & \pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)} = \mathbf{Z}/2 \end{array}$$

Proof. The smash product of $E\mathcal{P}_+$ with the slice tower for $R(2^k - 2)^{(G)}$ is determined by the restriction of the latter to H , which has the expected properties by induction. The result follows from an examination of its equivariant homotopy groups. \square

The exact sequence of Lemma 8.7 shows that y_k has nontrivial image in the group $\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge H\mathbf{Z}_{(2)}$ iff it is not in the image of $\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge P_{c_k g} R(2^k - 2)^{(G)}$. For this we will identify a property of this image not shared by y_k . The following is [HHR, Prop. 7.15].

Proposition 8.8. *The image of*

$$\pi_{c_k \rho_G}^G E\mathcal{P}_+ \wedge P_{c_k g} R(2^k - 2)^{(G)} \longrightarrow \pi_{c_k \rho_G}^G R(2^k - 2)^{(G)} \xrightarrow{i_0^*} \pi_{c_k g}^u R(2^k - 2)^{(G)}$$

is contained in the image of $(\gamma - 1)$, where γ is a generator of G and i_0 denotes restriction to the trivial group.

The class y_k does not have this property because its image in $\pi_{c_k g}^u R(2^k - 2)^{(G)}$ is $i_0^* N_2^g \bar{\tau}_{c_k}$, which generates a summand isomorphic to the sign representation of G . Hence this result implies the Reduction Theorem. For its proof we refer the reader to the last two pages of [HHR, §7].

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A solution to the Arf-Kervaire invariant problem

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