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On the curvature of the real amoeba

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Abstract. For a real smooth algebraic curve $A \subset (\mathbb{C}^*)^2$, the amoeba $\mathcal{A} \subset \mathbb{R}^2$ is the image of A under the map Log : $(x, y) \mapsto (\log |x|, \log |y|)$. We describe a universal bound for the total curvature of the real amoeba $\mathcal{A}_{\mathbb{R}A}$ and we prove that this bound is reached if and only if the curve A is a simple Harnack curve in the sense of Mikhalkin.

1. Introduction

Let $A \subset (\mathbb{C}^*)^2$ be a smooth real algebraic curve with a non-degenerated Newton polygon Δ defined by an equation $f = 0$, where $f : \mathbb{C}^2 \to \mathbb{C}$ is a reduced polynomial with real coefficients. Let $\mathbb{R}A \subset (\mathbb{R}^*)^2$ stand for the real part of A and *Log* be the map $Log : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(z_1, z_2) \mapsto (\log |z_1|, \log |z_2|)$, $L = Log_{|A}$ the restriction of Log to the curve A, $\mathcal{A}_A = L(A)$, $\mathcal{A}_{\mathbb{R}A} = L(\mathbb{R}A)$, $F \subset A$ the critical set of L. The set $\mathcal{A}_A$ is called the *amoeba* of A, and $\mathcal{A}_{\mathbb{R}A}$ its *real amoeba* (see [1]). Notice that $\mathcal{A}_{\mathbb{R}A}$ is a real curve since for each quadrant $Q \subset (\mathbb{R}^*)^2$, $Log_{|Q}$ is a diffeomorphism.

G. Mikhalkin ([5]) defined "simple Harnack curves" (see below), and used the notion of amoebas to prove the uniqueness of the topological type of the pair $(\mathbb{R}\bar{A}, T_\Delta)$ when A is a simple Harnack curve and $\bar{A}$ its closure in the toric surface T_Δ. In the paper [8], it is proved that $\text{Area}(\mathcal{A}) \leq \pi^2 \text{Area}(\Delta)$ and in [7] the auhors prove A is a simple Harnack curve if and only if there is equality above, i.e., if and only if the Area of $\mathcal{A}$ is maximal.

In this paper, we prove a similar result, but for the total curvature of the real amoeba (instead of the volume of the amoeba); notice that for a hypersurface (in higher dimension), the total curvature of the real amoeba is always finite, but not the volume of the amoeba which is not bounded in general. We then hope to generalize the results of this paper in higher dimension (at least for surfaces).

2. Amoebas and curvature

Let T_Δ be the toric surface associated to Δ, $\tilde{l}_i$ $(1 \leq i \leq n)$ the sides of Δ, d_i the integer length of $\tilde{l}_i$, $\bar{A}$ (resp. $\mathbb{R}\bar{A}$) the closure of the image of A in T_Δ (resp. in $\mathbb{R}T_\Delta$). We denote by l_i the component of the divisor $T_\Delta^\infty := T_\Delta \setminus ((\mathbb{C}^*)^2)$ corresponding to the side $\tilde{l}_i \in \partial\Delta$. We assume that $\bar{A}$ is smooth and transverse to T_Δ^∞.

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Set γ for the logarithmic Gauss map $\gamma : A \to \mathbb{C}P^1$ (see [5]); the map γ is defined by $\gamma(z_1, z_2) = [z_1 \partial f/\partial z_1, z_2 \partial f/\partial z_2]$. The real Logarithmic Gauss map $\gamma_{\mathbb{R}} : \mathbb{R}A \to \mathbb{R}P^1$ is the restriction of γ to $\mathbb{R}A$. We have $F = \gamma^{-1}(\mathbb{R}P^1)$ ([5], Lemma 3), and the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}A & & \\ {\scriptstyle L}\downarrow & \searrow^{\gamma_{\mathbb{R}}} & \\ \mathcal{A}_{\mathbb{R}A} & \xrightarrow[g]{} & \mathbb{R}P^1 \end{array}$$

where g is the usual Gauss map, defined on the smooth part of $\mathcal{A}_{\mathbb{R}A}$. The logarithmic Gauss map $\gamma : A \to \mathbb{C}P^1$ extends to a map $\bar{\gamma} : \bar{A} \to \mathbb{C}P^1$ of degree $2vol(\Delta)$ ([5]), therefore the map $\gamma_{\mathbb{R}}$ has finite fibers of cardinality $\leq 2vol(\Delta)$ which implies that the fibers of g are also of cardinality $\leq 2vol(\Delta)$.

If k denotes the curvature function on the curve $\mathcal{A}_{\mathbb{R}A}$ (for any orientation), we have

$$\int_{\mathcal{A}_{\mathbb{R}A}} |k| \leq 2\pi vol(\Delta) \tag{1}$$

(see [9] or [4]), since $\mathrm{vol}(\mathbb{R}P^1) = \pi$ and because $\int_{\mathcal{A}_{\mathbb{R}A}} |k|$ is the volume of $\mathrm{im}(g) \subset \mathbb{R}P^1$ (counted with multiplicities, the multiplicity of $x \in \mathbb{R}P^1$ being the cardinality of the fiber $g^{-1}(x)$ which is $\leq 2vol(\Delta)$).

Definition 2.1. We say that the real amoeba $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature if there is equality in (1).

Lemma 2.1. *Let $A \subset (\mathbb{C}^*)^2$ be a real smooth algebraic curve. Then the following conditions are equivalent:*

1. *$\mathcal{A}_{\mathbb{R}A}$ has maximal curvature*
2. *The logarithmic Gauss map $\gamma : A \to \mathbb{C}P^1$ is totally real (i.e., $\gamma^{-1}(x) \subset \mathbb{R}A$ for all $x \in \mathbb{R}P^1$).*

Proof. 1. $\Rightarrow$ 2. If γ is not totally real, there would exist an open set $U \subset \mathbb{R}P^1$ such that $\mathrm{card}(\gamma_{\mathbb{R}}^{-1}(x)) < 2vol(\Delta)$ $\forall x \in U$; then we would have also $\mathrm{card}(g^{-1}(x)) < 2vol(\Delta)$ $\forall x \in U$, which implies that $\mathrm{vol}(\mathrm{im}(g)) < 2\pi vol(\Delta)$.
2. $\Rightarrow$ 1. Since the map γ is totally real, we have that the fibers of g are generically of cardinal $2vol(\Delta)$, which implies that $\int_{\mathcal{A}_{\mathbb{R}A}} |k| = \mathrm{vol}(\mathrm{im}(g)) = 2\pi \mathrm{vol}(\Delta)$, i.e., the real amoeba $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature. □

Remark 2.1. Notice that if $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature, then:

- For $x \in \mathbb{R}P^1$, $\gamma^{-1}(x) \subset \mathbb{R}A$, which implies $F \subset \mathbb{R}A$ since $F = \gamma^{-1}(\mathbb{R}P^1)$ ([5]) and therefore $F = \mathbb{R}A$.

- The only possibilities for $\mathcal{A}_{\mathbb{R}A}$ to have inflection points are at "pinching points" (cf. [5]), which are also the only possible singular points of $\mathcal{A}_{\mathbb{R}A}$ (we will see that in fact if $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature, then $\mathcal{A}_{\mathbb{R}A}$ is smooth).

3. Simple Harnack curves

Definition 3.1. 1) A real algebraic smooth curve $A \subset (\mathbb{C}^*)^2$ with Newton polygon Δ is called maximal (or an M-curve) if the number of connected components of $\mathbb{R}\bar{A}$ is $g+1$, where g is the genus of $\bar{A}$ (or the number of integer points in the interior of Δ).
2) A is said to be "in weak maximal position" if $\bar{A}$ is smooth and transverse to T^∞_Δ and if for $1 \leq i \leq n$, $\bar{A} \cap \mathbb{R}l_i = \bar{A} \cap l_i$ (which means that $\bar{A} \cap T^\infty_\Delta$ is totally real).

Let us look at the following properties for an M-curve A in weak maximal position.

(1) There exists a connected component $C \subset \mathbb{R}\bar{A}$ such that for $1 \leq i \leq n$, we have $C \cap \mathbb{R}l_i = \bar{A} \cap l_i$ (i.e. $C \cap \mathbb{R}l_i$ is made of d_i points).
(2) Condition (1) is verified and for $1 \leq i \leq n$, $C \cap \mathbb{R}l_i$ is contained in an arc C_i of C, such that $C_i \cap C_j = \emptyset$ for $i \neq j$.
(3) Condition (2) is verified and there is an orientation of $\partial\Delta$ such that C is cyclicly in maximal position (see [5]).

Notice that condition (3) is the definition of a "simple Harnack curve", cf. [5].

Remark 3.1. (1) It is proved in [5] that a simple Harnack curve verifies also the following property:
For $1 \leq i \leq n$ the order of the points $C_i \cap \mathbb{R}l_i$ is the same on C_i and $\mathbb{R}l_i$ (for some orientations).
(2) An alternative to conditions (2) and (3) is given by the following lemma:

Lemma 3.1. *With the above notations, assume that A is an M-curve in weak maximal position and that condition* (1) *is fulfilled. Then the following conditions are equivalent:*

(a) *The curve A is a simple Harnack curve (i.e., conditions* (2) *and* (3) *are realised)*
(b) *The amoeba $\mathcal{A}_{\mathbb{R}A}$ is smooth.*

Proof. (a) $\Rightarrow$ (b) is proved in [5] (Corollary 9).
(b) $\Rightarrow$ (a) We have to prove that conditions (2) and (3) above are fulfilled if the amoeba $\mathcal{A}_{\mathbb{R}A}$ is smooth. Assume by absurd that (2) is not fulfilled for a component l_i of T^∞_Δ. Using the moment map, we can see (topologically) the amoeba $\mathcal{A}_{\mathbb{R}A}$ as a subset of $\mathring{\Delta}$. If $C \subset \mathbb{R}A$, set $\tilde{C}$ for its image in $\mathring{\Delta}$.
If condition (2) is not fulfilled , there would exist an arc $\tilde{C}_1$ of $\tilde{C}$ joining the side $\tilde{l}_i$ of $\partial\Delta$ to another one $\tilde{l}_j$ and such that if $\overline{\tilde{C}_1} \cap \tilde{l}_i = \{P\}$, there are points of $\tilde{C} \cap \tilde{l}_i$ on both sides of P. Therefore $\Delta \setminus \tilde{C}_1$ has two connected components, and there must exist an arc $\tilde{C}' \subset \tilde{C}$ joining theses two components hence intersecting $\tilde{C}_1$, which implies that the amoeba $\mathcal{A}_{\mathbb{R}A}$ is not smooth.
The proof of condition (3) is similar. □

Proposition 3.2. *If A is a simple Harnack curve, then $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature (Definition 2.1).*

Proof. The logarithmic Gauss map (restricted to the real part) $\gamma_{\mathbb{R}} : \mathbb{R}\bar{A} \to \mathbb{R}P^1$ is totally real (i.e., the cardinality of each fiber is $2vol(\Delta)$) (see Lemma 2.1). Therefore it is also true for the Gauss map $g : \mathcal{A}_{\mathbb{R}A} \to \mathbb{R}P^1$ ($\gamma_{\mathbb{R}} = g \circ L$) since the hypothesis implies that $L|_{\mathbb{R}A}$ is a bijection onto $\mathcal{A}_{\mathbb{R}A}$; then the total curvature of $\mathcal{A}_{\mathbb{R}A}$ is $\text{vol}(\text{Im}(g)) = 2\pi vol(\Delta)$. □

Theorem 3.3. *Let $A \subset (\mathbb{C}^*)^2$ be a smooth real algebraic curve. Assume that $\mathcal{A}_{\mathbb{R}A}$ has maximal curvature. Then A is a simple Harnack curve.*

Proof. Let us first prove the theorem under the supplementary assumption that the real amoeba $\mathcal{A}_{\mathbb{R}A}$ is smooth. Let p be the number of compact connected components of $\mathcal{A}_{\mathbb{R}A}$. Let t be the cardinality of $\mathbb{R}\bar{A} \cap T^{\infty}_{\Delta}$, $s = \sum d_i = \text{card}(\partial\Delta \cap \mathbb{Z}^2)$. Set $g = \text{card}(\mathring{\Delta} \cap \mathbb{Z}^2)$; g is the genus of the curve $\bar{A}$.

i) By Remarks 2.1 and the smoothness assumption, the real amoeba has no inflection points. Let us prove that since $\mathcal{A}_{\mathbb{R}A}$ is smooth and has no inflection points, one has

$$\int_{\mathcal{A}_{\mathbb{R}A}} |k| \leq 2\pi p + \pi t \tag{2}$$

There are exactly t arcs of $\mathcal{A}_{\mathbb{R}A}$ which are not compact, each one of total curvature $\leq \pi$ (since it has no inflection points), which proves (2) since each compact component have total curvature equal to 2π.

ii) Recall Pick's formula :

$$vol(\Delta) = g + s/2 - 1 \tag{3}$$

The hypothesis of maximal curvature and Pick's formula imply:

$$\int_{\mathcal{A}_{\mathbb{R}A}} |k| = 2\pi vol(\Delta)) = 2\pi g + \pi s - 2\pi$$

which gives, applying (2):

$$2\pi g + \pi s - 2\pi \leq 2\pi p + \pi t. \tag{4}$$

Let us prove that A is an M-curve in weak maximal position. We have necesseraly $t > 0$ because otherwise $s \geq 6$, because the integer length of each side of Δ would then be even, which is not possible by (4) since $p \leq g + 1$. We have therefore $p \leq g$.

(a) Let us first prove that $t = s$ ($t \leq s$ by definition). If we assume by absurd that $t \leq s - 1$, we get from (4) that $2\pi g \leq 2\pi p + \pi$, or $2g \leq 2p + 1$ which implies $g = p$. As above, using the moment map, we can see topologically the amoeba $\mathcal{A}_{\mathbb{R}A}$ as a subset of $\mathring{\Delta}$. Let us denote by c_{ij} ($i \neq j$) an arc of $\mathcal{A}_{\mathbb{R}A}$ (if it exists) joining $\tilde{l}_i$ to $\tilde{l}_j$; if α_{ij} denotes the angle between $\tilde{l}_i$ and $\tilde{l}_j$, then the total curvature of c_{ij} is α_{ij}. Since $g = p$, there exists only one connected component $C \subset \mathbb{R}\bar{A}$ such that $C \cap T^{\infty}_{\Delta} \neq \emptyset$. Therefore if C intersects

k components l_r ($k \leq n$) of the divisor T_Δ^∞, there exist k arcs $c_{rr'}$ of $\mathcal{A}_{\mathbb{R}A}$ such that each side $\tilde{l}_r$ intersects two of them. Since the sum of the angles between two consecutive sides of Δ is $(n-2)\pi$, we have $\sum \alpha_{ij} \leq (n-2)\pi$. Now there is at most $t-n$ other non compact arcs in $\mathcal{A}_{\mathbb{R}A}$, each one of total curvature $\leq \pi$ (since they have no inflection points), which gives

$$2\pi g + \pi s - 2\pi = \int_{\mathcal{A}_{\mathbb{R}A}} |k| \leq 2\pi g + \pi(t-n) + \pi(n-2) = 2\pi g + \pi t - 2\pi$$

or $s = t$, contrary to the hypothesis.

(b) The relation (4) implies now that $2\pi g + \pi s - 2\pi \leq 2\pi p + \pi s$ (since $t = s$) which implies $g - 1 \leq p \leq g$. But since $p \geq g - 1$, there is at least one arc c of $\mathbb{R}\bar{A}$ which intersects two different components l_i and l_j of T_Δ^∞; therefore the total curvature of the corresponding arc of $\mathcal{A}_{\mathbb{R}A}$ is $< \pi$. We have then $2\pi g + \pi s - 2\pi < 2\pi p + \pi s$, which implies that necessarily $p = g$.

We have then that A is an M-curve in weak maximal position satisfying conditon (1). Now A is a simple Harnack curve by Lemma 3.3, since we have assumed $\mathcal{A}_{\mathbb{R}A}$ is smooth. □

4. On the smoothness of the real amoeba

To achieve the proof of Theorem 3.5, We must prove that under the hypothesis of maximal curvature, the real amoeba $\mathcal{A}_{\mathbb{R}A}$ is smooth. Since A is smooth and that $\mathcal{A}_{\mathbb{R}A}$ does not have smooth inflection points (Remark 2.1), the only *a priori* possible inflection points (or singular points) for $\mathcal{A}_{\mathbb{R}A}$ are pinching points (see [5]). At a pinching point P there are two smooth branches of $\mathcal{A}_{\mathbb{R}A}$ crossing at P, each one with an inflection point at P (see [5]). Let α ($0 \leq \alpha \leq \pi/2$) be the angle between these two branches at P.

Lemma 4.1. *Assume that $\mathcal{A}_{\mathbb{R}A}$ has a pinching point P. Then the real amoeba $\mathcal{A}_{\mathbb{R}A}$ does not have maximal curvature, i.e., there exists $\epsilon > 0$ such that*

$$\int_{\mathcal{A}_{\mathbb{R}A}} |k| \leq 2\pi g + \pi(s-2) - \epsilon.$$

Proof. Let $B_P \subset \mathbb{R}^2$ be a ball centered at P small enough such that it contains no other singular point of $\mathcal{A}_{\mathbb{R}A}$ than the point P. Since P is a pinching point and the curve A is smooth, we have that $L^{-1}(P) = \{P_1, P_2\}$ where $P_i \in \mathbb{R}A$ are two points in different quadrants of $(\mathbb{R}^*)^2$. Then $L^{-1}(\mathcal{A}_{\mathbb{R}A} \cap B_P)$ has two connected components C_1 and C_2 (in different quadrants of $(\mathbb{R}^*)^2$) such that $P_1 \in C_1$ and $P_2 \in C_2$. Let γ_1 be the logarithmic Gauss map restricted to C_1. Since C_1 has by hypothesis a logarithmic inflection point at P_1, the map γ_1 has a local extremum at P_1 (in a chart of $\mathbb{R}P^1$). If $\alpha_1 = \gamma_1(P_1)$, where $\alpha_1 \in]-\pi/2, \pi/2[$, and if α_1 is for instance a local maximum at P_1 for γ_1, then there exists $\varepsilon > 0$ such that for $\theta \in]\alpha_1, \alpha_1 + \varepsilon[$, $\gamma_1^{-1}(\theta)$ has two non real (conjugate) points in $L^{-1}(B_P)$.

Then the logarithmic Gauss map $\gamma : A \to \mathbb{C}P^1$ is not totally real, and the amoeba $\mathcal{A}_{\mathbb{R}A}$ does not have maximal curvature (Lemma 2.1). □

Remark 4.1. If the real amoeba $\mathcal{A}_{\mathbb{R}A}$ has not maximal curvature, it may have a pinching point P. However, if α is the angle between the two branches of $\mathcal{A}_{\mathbb{R}A}$ crossing at P, it is possible to prove that the angle α is necessarily > 0.

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