Proceedings of 18^{th} Gökova
Geometry-Topology Conference
pp. 1 – 30

Flat branes on tori and Fourier transforms in the SYZ programme

Kaileung Chan, Conan Naichung Leung and Chit Ma

Abstract. The Strominger-Yau-Zaslow programme says that there should be a theory of fiberwise Fourier transforms along special Lagrangian fibrations which would explain mirror symmetry. This programme has been verified successfully in special cases when quantum corrections do not arise. In this article we will describe a theory of Fourier transforms of flat branes on tori and how it can be applied in the family setting. And in certain special circumstances this explains quantum corrections in SYZ transformations.

1. Mirror symmetry via the SYZ transforms

Roughly speaking, closed string theory on X can be regarded as quantum mechanics on its loop space $\mathfrak{L}X$. A path in $\mathfrak{L}X$ gives a surface Σ in X. This theory can be interpreted as a 2-dimensional σ-model which studies $Map(\Sigma, X)$. Supersymmetric considerations force X to be a Calabi-Yau manifold. Open string theory concerns paths in X instead, families of paths give surfaces with boundaries. The boundary conditions of these surfaces are specified by geometric objects in X, those satisfying supersymmetry are known as (BPS) branes. In [1], Witten suggests two ways of twisting the σ-model, namely the A-twist and the B-twist. The twisted models are known as A-model and B-model, with corresponding branes known as A-branes and B-branes respectively. The A-model is related to the symplectic geometry of X and the B-model is related to the complex geometry of X. An A-brane in X is a Lagrangian submanifold of X with a flat unitary complex vector bundle on it. Energy minimizing A-branes are special Lagrangian submanifolds. A B-brane in X is a complex submanifold of X with a unitary complex vector bundle (or more generally, a complex of coherent sheaves), such that its connection defines a holomorphic structure on the bundle. Energy minimizing B-branes are Hermitian Yang-Mills bundles.

The mirror symmetry conjecture says that the symplectic geometry (A-model) of a Calabi-Yau manifold X is equivalent to the complex geometry (B-model) of another Calabi-Yau manifold $\check{X}$, and vice versa. Kontsevich [2] formulated the conjecture as an equivalence between the category of A-branes and the category of B-branes, which is known as the homological mirror symmetry (HMS) conjecture. It is also generalized to

Key words and phrases. Mirror Symmetry, SYZ conjecture.

manifolds which are not Calabi-Yau.

Strominger-Yau-Zaslow [3] proposed a geometric explanation to mirror symmetry, using physical arguments. Near a large complex structure limit, the quantum effect should be suppressed. To see the relation between a pair of mirror Calabi-Yau manifolds X and $\check{X}$ (of complex dimension n), one can view $\check{X}$ as the moduli space of points in $\check{X}$ and treat it as a family of energy minimizing B-branes in $\check{X}$, it should correspond to a family of energy minimizing A-branes in X. As $\check{X}$ is filled up by points, X is expected to admit a fibration by corresponding A-branes, namely a special Lagrangian torus fibration over a base affine manifold B (of real dimension n). Fix a point $b \in B$, equipping the fiber torus F_b with different flat unitary connections on the trivial line bundle gives a family of A-branes (of real dimension n), which can be parametrized by the dual torus $\check{F}_b$. Therefore, using the fibration structure of X, it is observed that $\check{X}$ is the total space of the dual torus fibration of X over B. Furthermore, one expects that both A- and B-branes should admit tropical limits. Namely they are families of *flat branes* on tori over tropical subvarieties in B. The mirror correspondence should be a fiberwise Fourier-type transform in that case.

The SYZ proposal has been realized in some special cases [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In particular, it can be carried out mathematically in certain semi-flat cases, where the Lagrangian fibration for X does not admit singular fibers and the mirror manifold $\check{X}$ is constructed by taking its dual torus fibration. Fiberwise Fourier transform interchanges the symplectic and complex structures between X and $\check{X}$. For A- and B-branes constructed by taking smooth families of flat branes on fiber tori, Fourier transform can be applied to give an explicit correspondence between them. A brief discussion will be given in section 4.

A typical torus fibration admits singular fibers [16]. If we perform the above naive Fourier transform, we only expect to obtain the duality between A- and B-models in their classical limits. To restore the duality, one needs to include the so called quantum corrections in the A-model side.

Quantum corrections for closed string theory are contributed by holomorphic maps from compact Riemann surfaces into X. There is a well developed theory of Gromov-Witten invariants which allows one to "count" the Riemann surfaces contributing to quantum corrections. Quantum corrections for open string theory are related to open Gromov-Witten invariants defined by Fukaya-Oh-Ohta-Ono in [17, 18, 19], which "count" holomorphic disks mapping into X with Lagrangian boundary conditions. Quantum corrections for open string theory are more complicated, both to define and to compute, than their closed counterpart.

The construction of mirror manifold $\check{X}$ by incorporating quantum corrections of X can be carried out explicitly when X is a toric Fano manifold [9] or a non-compact toric Calabi-Yau manifold [8]. In both cases, X admits a "good" torus fibration $\mu : X \to B$ to an integral affine manifold B. The quantum corrections involved are so called genus zero one-pointed open Gromov-Witten invariants, with boundary loops lying in a fiber torus of the fibration. There is a lattice bundle Λ over B parametrizing fiberwise homotopy classes of loops. Generating functions on Λ are defined by "counting" holomorphic disks bounded by a fiberwise loop in Λ. The Fourier transforms of these functions are used to correct the semi-flat complex structure to give a mirror manifold $\check{X}$. In other words, these generating functions appear as "higher Fourier modes" of the corrected mirror complex structure. We give a brief review for these constructions in section 5.

Despite having an explicit construction in the above cases, the reason for using quantum corrections coming from holomorphic disks to correct the mirror complex structure from the semi-flat complex structure is not completely clear. Furthermore, the general procedure for using quantum corrections in the construction of mirror complex structure is not known yet, due to the lack of computational methods for open Gromov-Witten invariants. As we have seen that the corrections appear as "higher Fourier modes" in the above cases, in order to get a better understanding of quantum corrections, it would be important to develop a finite dimensional path space model and establish the Fourier transform in the semi-flat case. Moreover, the Fourier transform defined for flat branes can be used to give a mirror correspondence between A- and B-branes, as a geometric construction for the mirror correspondence. We also give an example to illustrate Fourier transforms between branes incorporating quantum corrections in section 5.

An ongoing project of the authors aims at laying a foundation for SYZ mirror transforms by defining the category of flat branes on tori and constructing a Fourier-type transform. The purpose of this article is to review the classical Fourier transform and provide an outline of the theory of flat branes and their Fourier transform [20]. Explicit examples will be given to highlight some features of the Fourier transform. We will also describe briefly how to apply this construction along Lagrangian torus fibers and give a geometric explanation to certain mirror symmetric phenomena.

2. Fourier transform on tori

In algebraic geometry, we have the Fourier-Mukai transform between an abelian variety X and its dual abelian variety $\check{X} = \mathrm{Pic}_0(X)$: Let $\pi : X \times \check{X} \to X$ and $\check{\pi} : X \times \check{X} \to \check{X}$ be the natural projection maps, $\mathcal{P}$ be the Poincaré line bundle (also called universal bundle) on $X \times \check{X}$. Then the Fourier-Mukai transform functor $\mathcal{FM} : D(X) \to D(\check{X})$ given by

$$\mathcal{FM}(\mathcal{S}) = R\check{\pi}_*(\pi^*\mathcal{S} \overset{L}{\otimes} \mathcal{P})$$

is an equivalence of categories, where $D(X)$, $D(\check{X})$ are the derived categories of coherent sheaves on X and $\check{X}$ respectively. In fact, $\mathcal{FM}$ transforms the Pontrjagin product into the tensor product and vice versa.

Also, $\mathcal{FM}$ induces maps $\mathcal{F}_K$ on K-theory and $\mathcal{F}_H$ on singular cohomology, with the following commutative diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathcal{F}_K} & K(\check{X}) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^*_{sing}(X) & \xrightarrow{\mathcal{F}_H} & H^*_{sing}(\check{X}), \end{array}$$

where ch is the Chern character map.

We are seeking an analogue transform of $\mathcal{F}_H$ between a real torus T and its dual torus T^*. Using the deRham isomorphism $H^*_{sing}(T) \cong H^*_{dR}(T)$, it becomes a transform between deRham cohomology groups, which will be the classical Fourier-Mukai transform

$$\mathcal{F}_{cl} : H^*_{dR}(T) \to H^*_{dR}(T^*).$$

In this section, we are going to review the definition of $\mathcal{F}_{cl}$ and explore how $\mathcal{F}_{cl}$ can be generalized to an isomorphism in the level of differential forms by incorporating Fourier series. We begin by introducing some notations which will be used throughout the paper.

2.1. Notations

Let V be a real vector space generated by a lattice Λ, with corresponding dual space V^* and dual lattice Λ^*. We use $(\cdot,\cdot)$ as the natural pairing between V and V^*. For a subspace $C_\mathbb{R}$ of V generated by a sublattice $C_\mathbb{Z}$ of Λ, we have the following exact sequence of vector spaces

$$0 \to C_\mathbb{R} \to V \to V/C_\mathbb{R} \to 0,$$

and the following exact sequence of Abelian groups

$$0 \to C_\mathbb{Z} \to \Lambda \to \Lambda/C_\mathbb{Z} \to 0.$$

Then we have a subtorus $C = C_\mathbb{R}/C_\mathbb{Z}$ in $T = V/C_\mathbb{R}$. Given $a \in V/C_\mathbb{R}$ (or $a \in T/C$), we denote by C_a the affine subtorus given by translation of C by a.

On the other hand, we have dual exact sequences

$$0 \to (V/C_\mathbb{R})^* \to V^* \to (C_\mathbb{R})^* \to 0,$$

and

$$0 \to (\Lambda/C_\mathbb{Z})^* \to \Lambda^* \to (C_\mathbb{Z})^* \to 0.$$

We denote $(V/C_\mathbb{R})^*$ and $(V/C_\mathbb{Z})^*$ by $\check{C}_\mathbb{R}$ and $\check{C}_\mathbb{Z}$ respectively. Then $\check{C} := \check{C}_\mathbb{R}/\check{C}_\mathbb{Z}$ defines a subtorus in $T^* = V^*/\Lambda^*$. $\check{C}$ is called the mirror object of C and satisfies $\check{\check{C}} = C$. Similarly, we denote by $\check{C}_{\check{a}} = \check{C} + \check{a}$ the affine subtorus in T^*.

Suppose y_j's are coordinates of T (resp. y^j's are dual coordinates of T^*). Then, given $\check{a} = (a^1, \cdots, a^n) \in V^*$ (resp. $a = (a_1, \cdots, a_n) \in V$), we denote by

$$\nabla_{\check{a}} := d - 2\pi i \sum_{j=1}^{n} a^j dy_j \qquad \text{(resp.} \qquad \check{\nabla}_a := d - 2\pi i \sum_{j=1}^{n} a_j dy^j)$$

the connection 1-form determined by $\check{a}$ (resp. a) on T (resp. T^*).

2.2. Fourier transform

From the fact that

$$T^* = H^1(T, \mathbb{R})/H^1(T, \mathbb{Z}) \overset{\exp}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, U(1)),$$

we can see that points in T^* parametrize flat unitary connections on the trivial line bundle over T, up to gauge equivalence. The roles of T and T^* can be reversed.

To see this concretely, given a point $\check{a} \in V^*$, we equip the trivial bundle $\mathbb{C} \cdot 1$ with the connection form $\nabla_{\check{a}}$. This determines a map from V^* to space of flat $U(1)$ line bundles on T. As $\nabla_{\check{a}}$ is gauge equivalent to $\nabla_{\check{a}+\check{\lambda}}$ for $\check{\lambda} \in \Lambda^*$, the map descends to T^* as an isomorphism.

We can describe the isomorphism using the Poincaré line bundle $\mathcal{P}$. We define the universal connection by the connection 1-form

$$\nabla_{\mathcal{P}} := d + \pi i \sum_{j=1}^{n} (y_j dy^j - y^j dy_j)$$

on the trivial complex line bundle $\tilde{\mathcal{P}}$ over $V \times V^*$, the universal cover of $T \times T^*$. The natural action of $\Lambda \times \Lambda^*$ on $V \times V^*$ can be lifted to an action on $\tilde{\mathcal{P}}$ given by

$$(\lambda, \check{\lambda}) \cdot (y, \check{y}, t) = (y + \lambda, \check{y} + \check{\lambda}, e^{\pi i [(\check{\lambda}, y) - (\check{y}, \lambda)]} t).$$

Taking the quotient of $\tilde{\mathcal{P}}$ by $\Lambda \times \Lambda^*$, we get the Poincaré line bundle $\mathcal{P}$ over $T \times T^*$ which is a nontrivial bundle. The connection $\nabla_{\mathcal{P}}$ descends to $T \times T^*$ as a unitary connection.

We have the universal property that $\nabla_{\mathcal{P}}|_{T \times \{\check{a}\}}$ and $\nabla_{\mathcal{P}}|_{\{a\} \times T^*}$ can be written as $\nabla_{\check{a}}$ and $\check{\nabla}^*_a$ respectively. This provides an isomorphism from T^* to the moduli space of flat unitary line bundles on T, by sending $\check{y}$ to $\mathcal{P}|_{T \times \{\check{y}\}}$.

Also, the curvature of $\nabla_{\mathcal{P}}$ is given by

$$F_{\mathcal{P}} = 2\pi i \sum_{j=1}^{n} dy_j \wedge dy^j.$$

We use this together with the projection maps π and $\breve{\pi}$ from $T \times T^*$ to T and T^* respectively to define the following transform for differential forms.

Definition 2.1. The classical Fourier-Mukai transform $\mathcal{F}_{cl} : \Omega^k(T) \to \Omega^{n-k}(T^*)$ is defined by

$$\begin{aligned}\mathcal{F}_{cl}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} \breve{\pi}_*(\pi^*\alpha \wedge e^{\frac{i}{2\pi}F_{\mathcal{P}}}) \\ &= (-1)^{\frac{n(n-1)}{2}} \int_T \pi^*\alpha \wedge e^{\frac{i}{2\pi}F_{\mathcal{P}}}.\end{aligned}$$

For example $\mathcal{F}_{cl}(1) = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$.

We may also define the inverse Fourier transform $\mathcal{F}_{cl}^{-1} : \Omega^*(T^*) \to \Omega^*(T)$ by

$$\begin{aligned}\mathcal{F}_{cl}^{-1}(\breve{\alpha}) &= (-1)^{\frac{n(n-1)}{2}} \pi_*(\breve{\pi}^*\breve{\alpha} \wedge e^{-\frac{i}{2\pi}F_{\mathcal{P}}}) \\ &= (-1)^{\frac{n(n-1)}{2}} \int_{T^*} \breve{\pi}^*\breve{\alpha} \wedge e^{-\frac{i}{2\pi}F_{\mathcal{P}}}.\end{aligned}$$

Even though $\mathcal{F}_{cl}^{-1}$ is called the inverse transform, $\mathcal{F}_{cl}$ is not an isomorphism and $\mathcal{F}_{cl}^{-1} \circ \mathcal{F}_{cl} : \Omega^*(T) \to \Omega^*(T)$ is just the harmonic projection. Nevertheless, $\mathcal{F}_{cl}$ descends to an isomorphism on cohomologies

$$\mathcal{F}_{cl} : H^*_{dR}(T, \mathbb{R}) \to H^*_{dR}(T^*, \mathbb{R}).$$

Under the deRham isomorphism, $\mathcal{F}_{cl}$ is consistent with the earlier discussion on duality of subtori: We first note that $H_*^{sing}(T)$ is generated by homology classes $[C]$'s of subtori C's in T and $H^*_{sing}(T)$ is generated by their Poincaré duals PD$[C]$'s. Via the deRham isomorphism, $\mathcal{F}_{cl}$ (PD$[C]$) =PD$[\check{C}]$. Furthermore $\mathcal{F}_{cl}$ (PD$[C_1 \cap C_2]$) =PD$[\check{C}_1 + \check{C}_2]$.

The classical Fourier transform $\mathcal{F}_{cl}$ works nicely in the study of mirror symmetry without corrections, see [11] and [13]. However, we have to extend it by incorporating Fourier series in order to obtain an isomorphism on the level of forms which corresponds to quantum corrections in mirror symmetry.

2.3. Fourier series

In the classical Fourier series setting, a function $f : \mathbb{Z} \to \mathbb{C}$ and a function $\check{f} : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ can be interchanged by

$$f(\lambda) = \int_{S^1} \check{f}(y) e^{2\pi i \lambda y} dy$$

and

$$\check{f}(y) = \sum_{\lambda \in \mathbb{Z}} f(\lambda) e^{-2\pi i \lambda y}.$$

This can be easily generalized to higher dimensional cases for tori and lattices. Roughly speaking, instead of considering functions on T and T^*, we consider functions on $T \times \Lambda$

and $T^* \times \Lambda^*$. The idea of defining a transform between them is transforming functions on T (resp. Λ) to functions on Λ^* (resp. T^*). Let π and $\breve{\pi}$ be the projection maps from $T \times \Lambda \times T^* \times \Lambda^*$ to $T \times \Lambda$ and $T^* \times \Lambda^*$ respectively. Then we have the following definition by combining the Fourier series and our earlier Fourier-Mukai transform.

Definition 2.2. Let $(y, \lambda) \in T \times \Lambda$ and $(\breve{y}, \breve{\lambda}) \in T^* \times \Lambda^*$. The Fourier transform $\mathcal{F} : \Omega^k(T \times \Lambda) \to \Omega^{n-k}(T^* \times \Lambda^*)$ is defined by

$$\begin{aligned}\mathcal{F}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} \breve{\pi}_* \Big(\pi^* \alpha \wedge e^{(\frac{i}{2\pi} F_{\mathcal{P}} + 2\pi i (y, \breve{\lambda}) - 2\pi i (\lambda, \breve{y}))} \Big) \\ &= (-1)^{\frac{n(n-1)}{2}} \int_T \Big(\sum_{\lambda \in \Lambda} (\pi^* \alpha) e^{-2\pi i (\lambda, \breve{y})} \Big) e^{2\pi i (y, \breve{\lambda})} \wedge e^{\frac{i}{2\pi} F_{\mathcal{P}}},\end{aligned}$$

and the inverse transform $\mathcal{F}^{-1} : \Omega^k(T^* \times \Lambda^*) \to \Omega^{n-k}(T \times \Lambda)$ is defined by

$$\begin{aligned}\mathcal{F}^{-1}(\breve{\alpha}) &= (-1)^{\frac{n(n-1)}{2}} \pi_* \Big(\breve{\pi}^* \breve{\alpha} \wedge e^{-(\frac{i}{2\pi} F_{\mathcal{P}} + 2\pi i (y, \breve{\lambda}) - 2\pi i (\lambda, \breve{y}))} \Big) \\ &= (-1)^{\frac{n(n-1)}{2}} \int_{T^*} \Big(\sum_{\breve{\lambda} \in \Lambda^*} (\breve{\pi}^* \breve{\alpha}) e^{-2\pi i (\breve{\lambda}, y)} \Big) e^{2\pi i (\lambda, \breve{y})} \wedge e^{-\frac{i}{2\pi} F_{\mathcal{P}}}.\end{aligned}$$

Example 2.3. If $f(y, \lambda) = f(\lambda)$, then

$$\mathcal{F}(f(y, \lambda)) = \mathcal{F}(f(\lambda)) = \begin{cases} \breve{f}(\breve{y}) dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n & \text{if } \breve{\lambda} = 0, \\ 0 & \text{if } \breve{\lambda} \neq 0, \end{cases}$$

where $\breve{f}(\breve{y})$ is the ordinary Fourier transform of $f(\lambda)$. Conversely, if

$$f(y, \lambda) = \begin{cases} f(y) & \text{if } \lambda = 0, \\ 0 & \text{if } \lambda \neq 0, \end{cases},$$

then

$$\mathcal{F}(f(y, \lambda)) = \breve{f}(\breve{\lambda}) dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n,$$

where $\breve{f}(\breve{\lambda})$ is the ordinary Fourier series of $f(y)$. In general, suppose $I = \{i_1, \cdots, i_k\}$ is a subset of $\{1, \cdots, n\}$ with $i_1 < \cdots < i_k$, $\bar{I} = \{i_{k+1}, \cdots, i_n\} = \{1, \cdots, n\} - I$ with $\{i_1, \cdots, i_k, i_{k+1}, \cdots, i_n\}$ an even permutation of $\{1, \cdots, n\}$, we have

$$\begin{aligned}&\mathcal{F}(f(y, \lambda) dy_I) \\ = \; & (-1)^{\frac{n(n-1)}{2}} \int_T \Big(\sum_{\lambda \in \Lambda} (f(y, \lambda) dy_I) e^{-2\pi i (\lambda, \breve{y})} \Big) e^{2\pi i (y, \breve{\lambda})} \wedge e^{\frac{i}{2\pi} F_{\mathcal{P}}} \\ = \; & (-1)^{\frac{n(n-1)}{2} + \frac{|\bar{I}|(|\bar{I}|+1)}{2}} \int_T \Big(\sum_{\lambda \in \Lambda} f(y, \lambda) e^{-2\pi i (\lambda, \breve{y})} \Big) e^{2\pi i (y, \breve{\lambda})} dy_I \wedge dy^{\bar{I}} \wedge dy_{\bar{I}} \\ = \; & (-1)^{\frac{n(n-1)}{2} + \frac{|\bar{I}|(|\bar{I}|+1)}{2} + |I||\bar{I}|} \int_T \Big(\sum_{\lambda \in \Lambda} f(y, \lambda) e^{-2\pi i (\lambda, \breve{y})} \Big) e^{2\pi i (y, \breve{\lambda})} dy^{\bar{I}} \wedge dy_I \wedge dy_{\bar{I}} \\ = \; & (-1)^{\frac{n(n-1)}{2} + \frac{|\bar{I}|(|\bar{I}|-1)}{2} + |I||\bar{I}|} \breve{f}(\breve{y}, \breve{\lambda}) dy^{\bar{I}}\end{aligned}$$

where

$$\check{f}(\check{y},\check{\lambda}) = \int_T \Big(\sum_{\lambda\in\Lambda} f(y,\lambda) e^{-2\pi i(\lambda,\check{y})} \Big) e^{2\pi i (y,\check{\lambda})},$$

and $dy^{\bar{I}} = dy^{i_{k+1}} \wedge \cdots \wedge dy^{i_n}$, $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$.

As in the classical case, given a function $f : \mathbb{Z} \to \mathbb{C}$, $\check{f}(\check{y}) = \sum_{\lambda\in\mathbb{Z}} f(\lambda) e^{-2\pi i \lambda \check{y}}$ may not converge. Therefore, in our case, we have to restrict to those $f(y,\lambda)dy_I \in \Omega^*(T \times \Lambda)$ such that for a fixed $\lambda \in \Lambda$, $f(y,\lambda) = f(y)$ is a L^2-function and for a fixed $y \in T$, $f(y,\lambda) = f(\lambda)$ is a rapid-decay function. We define $\Omega^*_{(2)}(T \times \Lambda)$ to be the subspace of $\Omega^*(T \times \Lambda)$ consisting of all forms with the above property. Then, we have:

Proposition 2.4. *$\mathcal{F} : \Omega^*_{(2)}(T \times \Lambda) \to \Omega^*_{(2)}(T^* \times \Lambda^*)$ is an isomorphism and $\mathcal{F}^{-1} \circ \mathcal{F}$ is the identity map on $\Omega^*_{(2)}(T \times \Lambda)$.*

Remark 2.5. As our motivations come from string theory which is roughly quantum mechanics on loop spaces, it is worth noticing that $T \times \Lambda$ is simply the space of all geodesic loops in T. When we generalize this to the family of tori T in a Lagrangian fibration in a symplectic manifold X, we should consider the space of fiberwise geodesic loops in X. In [9], K.W. Chan and the second author used this method in toric setting to obtain the mirror Landau-Ginzburg model predicted by Kontsevich and Hori-Vafa [21].

3. Flat branes in tori

As mentioned in section 1, an A-brane is a pair (L, E) where L is a special Lagrangian submanifold and E is a flat unitary vector bundle E on L, a B-brane is a pair $(\check{L}, \check{E})$ where $\check{L}$ is a complex submanifold and $\check{E}$ is unitary complex vector bundle such that its connection defines a holomorphic structure on $\check{E}$. In a large structure limit, mirror manifolds are expected to admit dual Lagrangian (possibly singular) tori fibrations over a singular affine manifold B [13]. Furthermore, one expects that both A- and B-branes should admit tropical limits. Namely they are families of flat branes on tori over tropical subvarieties in B. According to SYZ, the mirror correspondence is predicted to be a fiber-wise Fourier-type transform on these flat branes on dual tori, incorporating with quantum corrections. This would give a geometric understanding for the mirror correspondence. However, the mathematical description for this correspondence is not completely clear yet, much more work is needed to clarify how singular the families of flat branes can be and how to include quantum corrections.

In order to understand the correspondence better, we study flat branes in a single flat torus, which should be thought of a smooth fiber of the Lagrangian fibration. We are going to discuss how to obtain a bijection between the set of flat branes in a torus and its dual via Fourier transform. Furthermore, we will define a complex using the space of mininal geodesic paths from a flat brane to another to study the quantum intersection between two flat branes.

We start by giving the definition of flat branes. Recall that, with a subtorus $C = C_{\mathbb{R}}/C_{\mathbb{Z}}$ in T and $a \in V/C_{\mathbb{R}}$ (or $a \in T/C$), we denote by C_a the affine translation of C by a.

Definition 3.1. A flat brane $\mathcal{B}$ in a torus T is a pair (C_a, E), where C_a is an affine subtorus in T and E is a flat unitary complex vector bundle over C_a.

Remark 3.2. E splits orthogonally into a direct sum of flat unitary line bundles.

Suppose $\mathcal{B}_i = (C_{i,a_i}, E_i)$, $i = 1, 2$, are two flat branes in a torus T, we will define a complex to study the quantum intersection of two branes. In general, we are supposed to study the path space from C_{1,a_1} to C_{2,a_2}. When the torus T is a fiber of a Lagrangian fibration at a large complex structure limit, the intersection theory should localize to the subspace consisting of minimal geodesics, which is a finite dimensional manifold.

Definition 3.3. For two affine subtori C_{1,a_1} and C_{2,a_2}, an instanton from C_{1,a_1} to C_{2,a_2} is a connected minimal geodesic segment $\gamma : [0,1] \to T$ such that $\gamma(0) \in C_{1,a_1}$ and $\gamma(1) \in C_{2,a_2}$.

Passing to the universal cover, an affine subtorus C is lifted to infinitely many affine linear subspaces. An instanton from C_{1,a_1} to C_{2,a_2}, upon lifting to the universal cover V, is a line segment joining an affine subspace covering C_{1,a_1} and to an affine subspace covering C_{2,a_2}, which is orthogonal to both subspaces. Two points in distinct flat branes are said to be intersecting each other in the quantum sense if they are joined by an instanton. The space of instantons is denoted by $M(C_{1,a_1}, C_{2,a_2})$, or simply $M_{1,2}$ if there is no confusion. There are two natural evaluation maps

$$\begin{array}{ccccc} & & M(C_{1,a_1}, C_{2,a_2}) & & \\ & \swarrow^{ev_s} & & \searrow^{ev_t} & \\ C_{1,a_1} & & & & C_{2,a_2}. \end{array}$$

Remark 3.4. In the case that both C_{i,a_i}'s are affine translations of the same torus C, we have a map

$$M(C_{a_1}, C_{a_2}) \xrightarrow{\sim} C_{a_1} \times \Lambda/C_{\mathbb{Z}}$$

given by

$$\gamma \mapsto (\gamma(0), \gamma' + \tilde{a}_1 - \tilde{a}_2),$$

where $\tilde{a}_i$'s are some liftings of a_i's. This gives an explicit parametrization for the space of instantons.

The quantum intersection complex can be defined using the space $M_{1,2}$ in the following manner.

First, we can form the homomorphism bundle over $M_{1,2}$ from $ev_s^*E_1$ and $ev_t^*E_2$, denoted by $\mathrm{Hom}_{qu}(E_1,E_2)=\mathrm{Hom}(ev_s^*E_1, ev_t^*E_2)$. There is a natural flat unitary connection given by $\nabla_{1,2}=ev_s^*(\nabla_1^*)\otimes ev_t^*(\nabla_2)$. This gives a differential complex

$$(\Omega^*(M_{1,2},\mathrm{Hom}_{qu}(E_1,E_2)),\nabla_{1,2}).$$

Second, we define a bundle $\mathcal{N}_{1\vee 2}$ over the space $M_{1,2}$. Denoting by $P_\gamma : T_{\gamma(0)}T\to T_{\gamma(1)}T$ the parallel transport along the path $\gamma\in M_{1,2}$, we define

$$\mathcal{N}_{1\vee 2}|_\gamma = T_{\gamma(0)}T/(T_{\gamma(0)}C_{1,a_1}+P_\gamma^{-1}T_{\gamma(1)}C_{2,a_2}).$$

The metric on T induces a connection on $\mathcal{N}_{1\vee 2}$ which is flat.

Indeed, if we identify tangent space at every point of T as V, the tangent space of C_{i,a_i} is naturally identified with $C_{i,\mathbb{R}}$. Then we have $\mathcal{N}_{1\vee 2}\simeq V/(C_{1,\mathbb{R}}+C_{2,\mathbb{R}})$ at every path γ. Thus $\mathcal{N}_{1\vee 2}$ is a trivial bundle with trivial connection under this identification.

$s(\gamma)=2\pi i\gamma'\in\mathcal{N}_{1\vee 2}$ defines a flat section of the trivial bundle $\wedge^1\mathcal{N}_{1\vee 2}$ over $M_{1,2}$. Therefore, we can define

$$\delta:\Gamma(M_{1,2},\wedge^q\mathcal{N}_{1\vee 2})\to\Gamma(M_{1,2},\wedge^{q+1}\mathcal{N}_{1\vee 2})$$

by $\delta(\varphi):=s\wedge\varphi$. We can verify that $\delta^2=0$.

Remark 3.5. There is an equivalent complex defined using the dual bundle $\mathcal{N}_{1\vee 2}^*$, with the differential

$$\breve{\delta}:\Gamma(M_{1,2},\wedge^q\mathcal{N}_{1\vee 2}^*)\to\Gamma(M_{1,2},\wedge^{q-1}\mathcal{N}_{1\vee 2}^*)$$

given by $\breve{\delta}(\varphi):=\iota_s(\varphi)$. The two complexes are identified by an isomorphism $\mathcal{N}_{1\vee 2}^*\simeq\mathcal{N}_{1\vee 2}$ using metric and then the Hodge star operator, up to a factor of $(-1)^{q-1}$ on sections of $\wedge^q\mathcal{N}_{1\vee 2}^*$.

Remark 3.6. The reason for introducing the complex $\wedge^*\mathcal{N}_{1\vee 2}$ will become clear when we will consider families of flat branes. When we consider A-branes, $\mathcal{N}_{1\vee 2}$ is identified with a subbundle of cotangent bundles of two A-branes, using symplectic form. In this case, all non constant paths will contribute and result in a quantum theory. When we consider B-branes, the complex $\wedge^*\mathcal{N}_{1\vee 2}$ is exact off the classical intersection of two B-branes. This gives a classical intersection theory.

We let $\Omega^{p,q}(\mathcal{B}_1,\mathcal{B}_2):=\Omega^p(M_{1,2},\mathrm{Hom}_{qu}(E_1,E_2)\otimes\wedge^q\mathcal{N}_{1\vee 2})$. Then δ and $\nabla_{1,2}$ induce two commuting differentials

$$\delta:\Omega^{p,q}(\mathcal{B}_1,\mathcal{B}_2)\to\Omega^{p,q+1}(\mathcal{B}_1,\mathcal{B}_2)\text{ and }\nabla_{1,2}:\Omega^{p,q}(\mathcal{B}_1,\mathcal{B}_2)\to\Omega^{p+1,q}(\mathcal{B}_1,\mathcal{B}_2).$$

Therefore, $\Omega^{*,*}(\mathcal{B}_1,\mathcal{B}_2)$ is a double complex. Furthermore, we define the total complex $\Omega^n(\mathcal{B}_1,\mathcal{B}_2):=\oplus_{p+q=n}\Omega^{p,q}(\mathcal{B}_1,\mathcal{B}_2)$ and $D:\Omega^n(\mathcal{B}_1,\mathcal{B}_2)\to\Omega^{n+1}(\mathcal{B}_1,\mathcal{B}_2)$ by defining

$D := \nabla_{1,2} + (-1)^{p+q+1}\delta$ on $\Omega^{p,q}(\mathcal{B}_1, \mathcal{B}_2)$. Then we have $D^2 = 0$ and hence $(\Omega^*(\mathcal{B}_1, \mathcal{B}_2), D)$ is a cochain complex.

We can define another complex $\breve{\Omega}^{p,-q} = \Omega^p(M_{1,2}, \mathrm{Hom}_{qu}(E_1, E_2) \otimes \wedge^q \mathcal{N}^*_{1\vee 2})$, with two differentials $(\nabla_{1,2}, (-1)^p \breve{\delta})$. There is an isomorphism between the two double complexes, as well as their total complexes, by a degree shifting.

Remark 3.7. In fact we should consider those $\phi \in \Omega^*(\mathcal{B}_1, \mathcal{B}_2)$ with decay condition $|\gamma^{'}|^k |\phi(\gamma)| \to 0$ as $|\gamma^{'}| \to \infty$ for all $k \in \mathbb{Z}_{\geq 0}$. We continue to denote this subcomplex by $(\Omega^*(\mathcal{B}_1, \mathcal{B}_2), D)$.

Example 3.8. In the case where $C_{a_1} = \{a_1\}$ and $C_{a_2} = \{a_2\}$ are translations of the zero subtorus C, $M_{1,2}$ is parametrized by the lattice Λ. Let $f(\lambda) \in \Gamma(M_{1,2}, \wedge^0 \mathcal{N}_{1\vee 2})$, we have

$$\delta(f(\lambda)) = 2\pi i f(\lambda)(\lambda + \tilde{a}_2 - \tilde{a}_1).$$

If $a_1 \neq a_2$, then $\tilde{a}_2 - \tilde{a}_1$ is not in Λ and $\lambda + \tilde{a}_2 - \tilde{a}_1$ is not zero. In this case, $H^0_\delta(\Gamma(\wedge^* \mathcal{N}_{1\vee 2}))$ is trivial which means C_{a_1} and C_{a_2} do not intersect in the classical sense. However, if $a_1 = a_2$, then $\tilde{a}_2 - \tilde{a}_1 \in \Lambda$ and we can check that $H^0_\delta(\Gamma(\wedge^* \mathcal{N}_{1\vee 2})) = \mathbb{C}$, which means C_{a_1} and C_{a_2} do intersect in the classical sense. Although the spaces of instantons in both cases are the same, the difference in the differential δ helps to distinguish whether the flat branes C_{a_1} and C_{a_2} intersect or not.

When two flat branes are the same, denoted by $\mathcal{B} = (C_a, E)$, we can define a product on $\Omega^*(\mathcal{B}, \mathcal{B})$. We consider a natural map

$$p : M(C_a, C_a) \ {}_{ev_t}\!\times_{ev_s} M(C_a, C_a) \to M(C_a, C_a)$$

defined by joining the paths. More precisely, given a pair of paths (γ_1, γ_2) such that $\gamma_1(1) = \gamma_2(0)$, we let $p(\gamma_1, \gamma_2) = \gamma_1 \circ \gamma_2$ be the unique geodesic path with $\gamma_1 \circ \gamma_2(0) = \gamma_1(0)$ and $(\gamma_1 \circ \gamma_2)' = \gamma_1' + \gamma_2'$. For two $\alpha, \beta \in \Omega^*(\mathcal{B}, \mathcal{B})$, we define

$$\alpha * \beta = p_*(\pi_1^* \alpha \wedge \pi_2^* \beta).$$

In the case where C is just a point in T, we can identify $M(C_a, C_a)$ with the lattice Λ. The product is a convolution product along the lattice. When C is T, it gives the ordinary wedge product. In general, it is a mixture of both. The complex $\big(\Omega^*(\mathcal{B}, \mathcal{B}), D\big)$, together with the product $*$, gives a differential graded algebra.

3.1. Fourier transform

In this section, we define the Fourier transform. A check "$\vee$" will be added to notations for objects in T^* to distinguish them from objects in T.

Transform of flat branes

We have to define a transform between flat branes in T and T^* by using the Poincaré line bundle $\mathcal{P}$ and the universal connection $\nabla_{\mathcal{P}}$.

Given a flat brane $\mathcal{B} = (C_a, E)$ in T, without loss of generality, we consider the case that E is a line bundle over C_a, with a flat unitary connection ∇_E.

For each $\check{y} \in T^*$, we have an embedding $C_a \times \{\check{y}\} \to T \times T^*$. We define

$$\check{E}|_{\check{y}} := H^0(E \otimes \mathcal{P}^*|_{C_a \times \{\check{y}\}}, \nabla_E \otimes \nabla^*_{\mathcal{P}}|_{C_a \times \{\check{y}\}}),$$

and we consider the subset $\{\check{y} \in T^* : \check{E}|_{\check{y}} \neq 0\}$ in T^*. It turns out this subset equals to the affine subtorus $\check{C}_{\check{a}}$ in T^*, for a suitable choice of $\check{a}$ (determined by ∇_E).

We consider the bundle $E \otimes \mathcal{P}^*$ over $C_a \times \check{C}_{\check{a}}$. We define $\check{E}|_{\check{y}} = H^0(E \otimes \mathcal{P}^*|_{\check{y}})$. It has a natural metric constructed from the L^2 metric of the space $H^0(E \otimes \mathcal{P}^*|_{\check{y}})$. Parallel transport along paths in $\check{C}_{\check{a}}$ defines a connection $\check{\nabla}_{\check{E}}$ on $\check{E}$. We define $\mathcal{F}(\mathcal{B})$ to be the flat brane $(\check{C}_{\check{a}}, \check{E})$.

This can be seen in local coordinates. Given C_a, we can fix a local coordinate $y = (y_1, \ldots, y_n)$ of V, such that

$$C_{\mathbb{R}} = \{y \in V \, : \, y_{k+1} = 0, \cdots, y_n = 0\},$$

and let $\check{y} = (y^1, \ldots, y^n)$ be the corresponding dual coordinates. We can write the flat $U(1)$-connection ∇_E on C_a as the connection 1-form

$$\nabla_{\check{a}} = d - 2\pi i \sum_{j=1}^{k} a^j dy_j$$

with respect to some trivialization 1_E, for some $\check{a} = (a^1, \cdots, a^k) \in C^*_{\mathbb{R}}$.

Fixing a point $\check{y} \in V^*$, the function $e^{-\pi i(\check{y}, y)}$ on $V \times V^*$ descends to a trivialization of $\mathcal{P}|_{T \times \{\check{y}\}}$, with corresponding connection 1-form given by $d - 2\pi i(\check{y}, dy)$. Then we have

$$\nabla_{\check{a}} \otimes \nabla^*_{\mathcal{P}}|_{C_a \times \{\check{y}\}} = d + 2\pi i \sum_{j=1}^{k} (y^j - a^j) dy_j,$$

it has nonzero flat section if and only if

$$f(y_1, \cdots, y_k) = \exp\Big(- 2\pi i \sum_{j=1}^{k} (y^j - a^j) y_j \Big)$$

defines a function on C. This is equivalent to the condition $\check{y} - \check{a} \in C^*_{\mathbb{Z}}$.

We can verify that, for a suitable choice of $a \in V/C_{\mathbb{R}}$, the connection $\check{\nabla}_{\check{E}}$ can be expressed in the form

$$\check{\nabla}_a = d - 2\pi i \sum_{j=k+1}^{n} a_j dy^j,$$

with respect to some trivialization of $\check{E}$.

In general, $\mathcal{B} = (C_a, E)$ is a flat brane in T with E a flat $U(r)$-bundle. Since C_a itself is a torus, (E, ∇) can always be decomposed into a direct sum $\bigoplus_{i=1}^{r}(E_i, \nabla_i)$. Thus, transformation of $\mathcal{B}$ reduces to transformation of (C_a, E_i) and gives r flat branes in T^*.

Finally, we define a tautological section ϕ of $E^* \otimes \check{E} \otimes \mathcal{P}$ over $C_a \times \check{C}_{\check{a}}$ which is used in the transformation of the quantum intersection complex. Over a point $(y, \check{y})$, there is a natural pairing

$$E^*|_y \otimes H^0(E \otimes \mathcal{P}^*|_{\check{y}}) \otimes \mathcal{P}|_{(y,\check{y})} \to \mathbb{C}.$$

This gives an identification of the bundle $E^* \otimes \check{E} \otimes \mathcal{P}$ with the trivial bundle and determines a section ϕ which is identified to the constant function 1. Notice that ϕ is flat with respect to the connection of $E^* \otimes \check{E} \otimes \mathcal{P}$.

The inverse transform $\mathcal{F}^{-1}$ can be defined in a similar way using the dual Poincaré line bundle $\mathcal{P}^*$, and we have

Proposition 3.9. *$\mathcal{F}^{-1} \circ \mathcal{F}(\mathcal{B}) = \mathcal{B}$.*

Remark 3.10. When we carry out the inverse transform $\mathcal{F}^{-1}$ on $\mathcal{F}(\mathcal{B}) = (\check{C}_{\check{a}}, \check{E})$, we get a line bundle $\check{\check{E}}$ over C_a which is defined by $\check{\check{E}}|_y := H^0(\check{E} \otimes \mathcal{P}|_{\{y\}\times \check{C}_{\check{a}}}, \check{\nabla}_{\check{E}} \otimes \nabla_{\mathcal{P}}|_{\{y\}\times \check{C}_{\check{a}}})$. The tautological section ϕ gives an identification of E and $\check{\check{E}}$.

Transform of quantum intersection complex

Given flat branes $\mathcal{B}_i = (C_{i,a_i}, E_i)$ in T, for $i = 1, 2$, we are going to define a Fourier transform

$$\mathcal{F} : \Omega^*(\mathcal{B}_1, \mathcal{B}_2) \to \Omega^*(\mathcal{F}(\mathcal{B}_1), \mathcal{F}(\mathcal{B}_2)).$$

We write $\mathcal{F}(\mathcal{B}_i) = (\check{C}_{i,\check{a}_i}, \check{E}_i)$. For simplicity, we work on the case of line bundles.

Let us consider the product of spaces $M_{1,2} \times \check{M}_{1,2}$ with two natural projections π and $\check{\pi}$ to $M_{1,2}$ and $\check{M}_{1,2}$ respectively. As analogue to the Fourier-Mukai transform, our transform is established by defining some universal objects over the space $M_{1,2} \times \check{M}_{1,2}$.

First, we define a tautological section $\phi_{\mathcal{F}}$ of the bundle

$$\pi^* \operatorname{Hom}_{qu}(E_1, E_2)^* \otimes \check{\pi}^* \operatorname{Hom}_{qu}(\check{E}_1, \check{E}_2)$$

on $M_{1,2} \times \check{M}_{1,2}$. The product of evaluation maps gives

$$\begin{array}{ccc} & M_{1,2} \times \check{M}_{1,2} & \\ {}^{ev_s}\swarrow & & \searrow^{ev_t} \\ C_{1,a_1} \times \check{C}_{1,\check{a}_1} & & C_{2,a_2} \times \check{C}_{2,\check{a}_2}. \end{array}$$

We also have two tautological sections ϕ_i of $E_i^* \otimes \check{E}_i \otimes \mathcal{P}$ on $C_{i,a_i} \times \check{C}_{i,\check{a}_i}$, for $i = 1, 2$. $(ev_s)^*\phi_1^* \otimes (ev_t)^*\phi_2$ is a section of the bundle

$$\pi^* \operatorname{Hom}_{qu}(E_1, E_2)^* \otimes \check{\pi}^* \operatorname{Hom}_{qu}(\check{E}_1, \check{E}_2) \otimes (ev_s)^*\mathcal{P}^* \otimes (ev_t)^*\mathcal{P}.$$

Each point $(\gamma, \check{\gamma}) \in M_{1,2} \times \check{M}_{1,2}$ gives a path in $T \times T^*$ and parallel transport along it gives a natural pairing $((ev_s)^*\mathcal{P}^* \otimes (ev_t)^*\mathcal{P})|_{(\gamma,\check{\gamma})} \to \mathbb{C}$. We define $\phi_{\mathcal{F}}$ to be the section $(ev_s)^*\phi_1^* \otimes (ev_t)^*\phi_2$ after identifying $(ev_s)^*\mathcal{P}^* \otimes (ev_t)^*\mathcal{P}$ with $\mathbb{C}$ using the natural pairing.

Next, we define two canonical forms

$$\begin{array}{ll} G_1 & \in \pi^*\Omega^1(M_{1,2}, \wedge^1 \check{\mathcal{N}}_{1\vee 2}) \\ G_2 & \in \check{\pi}^*\Omega^1(\check{M}_{1,2}, \wedge^1 \mathcal{N}_{1\vee 2}). \end{array}$$

Through the evaluation map $ev_s : M_{1,2} \to C_{1,a_1}$, we identify $T_\gamma M_{1,2}$ with the subspace $(T_{\gamma(0)}C_{1,a_1} \cap P_\gamma^{-1} T_{\gamma(1)}C_{2,a_2})$ of $T_{\gamma(0)}T$. At the path $(\gamma, \check{\gamma})$, using the natural pairing on $T_{\gamma(0)}T \otimes T_{\check{\gamma}(0)}\check{T}$, we obtain an isomorphism between $T_\gamma^* M_{1,2}$ and $\check{\mathcal{N}}_{1\vee 2}|_{\check{\gamma}}$. Hence we have two natural sections G_1, G_2 of the bundle

$$\pi^* T^* M_{1,2} \otimes \check{\pi}^* \check{\mathcal{N}}_{1\vee 2} \oplus \check{\pi}^* T^* \check{M}_{1,2} \otimes \pi^* \mathcal{N}_{1\vee 2}$$

given by restrictions of the metric and the dual metric to corresponding subspaces.

The universal section $\phi_{\mathcal{F}} \otimes e^{G_1+G_2}$ serves as the kernel function for the Fourier transform. Given $\alpha \in \Omega^*(\mathcal{B}_1, \mathcal{B}_2)$, we define

$$\mathcal{F}(\alpha) = (-1)^{m+\check{m}} \check{\pi}_*(\pi^*\alpha \wedge (\phi_{\mathcal{F}} \otimes e^{G_1+G_2})),$$

where m and $\check{m}$ are dimensions of $M_{1,2}$ and $\check{M}_{1,2}$ respectively.

Here we choose unit length trivialization of the top exterior power of N_{1v2} with R using the metric, together with an orientation of $M_{1,2}$ to obtain an integration

$$\int : \Omega^m(M_{1,2}, \wedge^{\check{m}} \mathcal{N}_{1\vee 2}) \to \mathbb{C}.$$

In the case that $\mathcal{B}_1 = \mathcal{B}_2 = T$, equipped with the trivial connection, $\check{\mathcal{B}}$ is the trivial subtorus without affine translation. In this case, we recover the ordinary Fourier transform.

Example 3.11. For simplicity, we let $T = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle with coordinate $\theta \in \mathbb{R}$, and $\check{\theta} \in \mathbb{R}^*$ be the corresponding dual coordinate. We write $\Lambda \simeq \mathbb{Z}$ for the lattice and $\Lambda^* \simeq \mathbb{Z}^*$ for the dual lattice. $\check{\lambda} \in \mathbb{Z}^*$ parametrizes the loop which is a line segment from $\check{0}$ to $\check{\lambda}$ in $\mathbb{R}^*$. The universal connection 1-form is given by $\nabla_{\mathcal{P}} = d + \pi i(\theta d\check{\theta} - \check{\theta} d\theta)$ on $\mathbb{R} \times \mathbb{R}^*$.

We take $C_1 = C_2 = T$, equipped with the trivial connection on the trivial bundle. Then $\check{C}_1 = \check{C}_2 = \check{0}$ is the origin. We parametrize $M_{1,2}$ by T and $\check{M}_{1,2}$ by Λ^*. We only need to compute the parallel transport along the path $\check{\gamma}_{\check{\lambda}}$. Parallel transport of $\mathcal{P}$ along $\check{\gamma}_{\check{\lambda}}$ maps 1 to $e^{-\pi i \check{\lambda}\theta} \cdot 1$ on the universal cover, taking the quotient and changing the trivialization results in a multiplication by $e^{-2\pi i \check{\lambda}\theta}$. Hence the tautological section $\phi_{\mathcal{F}}$ is given by $e^{2\pi i \check{\lambda}\theta}$ and we have $\mathcal{F}(f d\theta)(\check{\lambda}) = -\int_T f(\theta) e^{2\pi i \check{\lambda}\theta} d\theta$, under the natural invariant basis of the bundles.

We give three more examples to illustrate the situation.

Example 3.12. In this example, we consider the standard torus $T = \mathbb{R}^n/\mathbb{Z}^n$, equipped with the standard metric. We use $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ as coordinate and let $\check{y} = (y^1, \ldots, y^n) \in (\mathbb{R}^n)^*$ be the corresponding dual coordinate. We consider two flat branes given by

$$\mathcal{B}_i = \left(\tilde{C}_i = \{y : y_k = 0 \;\; for \;\; k \in I_i\},\ E_i = \mathbb{C} \cdot 1_{E_i},\ \nabla_{\check{a}_i} = d - 2\pi i \sum_{j \notin I_i} a_i^j dy_j \right),$$

with $\tilde{C}_i$ are some liftings of C_i to V. The corresponding dual branes are given by

$$\check{\mathcal{B}}_i = \left(\tilde{\check{C}}_{i,\check{a}_i} = \{y : y^k = a_i^k \;\; for \;\; k \notin I_i\},\ \check{E}_i = \mathbb{C} \cdot \check{1}_{\check{E}_i},\ \check{\nabla} = d \right),$$

for some trivialization $\check{1}_{\check{E}_i}$. We also denote by $\check{a}_i \in \check{C}^{\perp}_{i,\mathbb{R}}$ some fixed liftings of the affine translations if there is no confusion.

In this case, we give explicit parametrization of the spaces $M_{1,2}$ and $\check{M}_{1,2}$. We denote by $\check{C}^{\perp}_{1\vee 2,\mathbb{Z}}$ the lattice $(\check{C}_{1,\mathbb{R}} + \check{C}_{2,\mathbb{R}})^{\perp} \cap \Lambda$. There is an isomorphism

$$(\check{C}_1 \cap \check{C}_2) \times \check{C}^{\perp}_{1\vee 2,\mathbb{Z}} \to \check{M}_{1,2},$$

given by sending $(\check{y}, \check{\lambda}) \in (\check{C}_1 \cap \check{C}_2) \times \check{C}^{\perp}_{1\vee 2,\mathbb{Z}}$ to a path $\check{\gamma}_{\check{y},\check{\lambda}}$ with $\check{\gamma}_{\check{y},\check{\lambda}}(0) = \check{y} + \check{a}_1 + \check{a}_2^T$, $\check{\gamma}'_{\check{y},\check{\lambda}} = \check{\lambda} + (\check{a}_2 - \check{a}_1)^{\perp}$ and $\check{\gamma}_{\check{y},\check{\lambda}}(1) = \check{y} + \check{a}_2 + \check{a}_1^T$, where $\check{a}_1^T$ is the projection of $\check{a}_1$ to $\check{C}_{2,\mathbb{R}}$ (similarly for $\check{a}_2$) and $(\check{a}_2 - \check{a}_1)^{\perp}$ is the projection to $\check{\mathcal{N}}_{1\vee 2} \simeq \check{C}^{\perp}_{1\vee 2,\mathbb{R}}$. The space $M_{1,2} \simeq (C_1 \cap C_2) \times C^{\perp}_{1\vee 2,\mathbb{Z}}$ is parametrized in similar way.

We give an explicit description of the tautological section $\phi_{\mathcal{F}}$. In $T \times T^*$, consider a path $(\gamma_{y,\lambda}, \check{\gamma}_{\check{y},\check{\lambda}})$. In order to compute the sections ϕ_i, we choose liftings $\tilde{C}_1 = C_{1,\mathbb{R}}$ and $\tilde{C}_{2,\lambda} = C_{2,\mathbb{R}} + \lambda$ of C_1 and C_2, liftings $\tilde{\check{C}}_{1,\check{a}_1} = \check{C}_{1,\mathbb{R}} + \check{a}_1$ and $\tilde{\check{C}}_{2,\check{a}_2+\check{\lambda}} = \check{C}_{2,\mathbb{R}} + \check{a}_2 + \check{\lambda}$ of $\check{C}_{1,\check{a}_1}$ and $\check{C}_{2,\check{a}_2}$ respectively. The functions $e^{-\pi i(\check{a}_1, y)}$ on $\tilde{C}_1 \times \tilde{\check{C}}_{1,\check{a}_1}$ and $e^{-\pi i[(\check{a}_2+\check{\lambda}, y) - (\lambda, \check{y})]}$ on

$\tilde{C}_2 \times \tilde{\check{C}}_{2,\check{a}_2+\check{\lambda}}$ descend to trivializations e_1 and e_2 of $\mathcal{P}|_{C_1\times\check{C}_{1,\check{a}_1}}$ and $\mathcal{P}|_{C_2\times\check{C}_{2,\check{a}_2}}$ respectively.

Similar to the previous example, we see that the parallel transport of $\mathcal{P}$ on $V \times V^*$ along $(\gamma_{y,\lambda}, \check{\gamma}_{\check{y},\check{\lambda}})$ is multiplication by a factor $e^{-\pi i[(\check{\lambda}+(\check{a}_2-\check{a}_1)^{\perp},y)-(\lambda,\check{y})]}$. It follows that the parallel transport maps e_1 to e_2.

The connection forms of the Poincaré line bundle over $C_1 \times \check{C}_{1,\check{a}_1}$ and $C_2 \times \check{C}_{2,\check{a}_2}$ (corresponding to the trivializations e_1 and e_2) are written as $d - 2\pi i(\check{a}_1, dy)$ and $d + 2\pi i[(\lambda, d\check{y}) - (\check{a}_2 + \check{\lambda}, dy)]$ respectively. We choose the trivializations of $\check{E}_1$ and $\check{E}_2$ given by $1_{E_1} \otimes e_1^*$ and $e^{2\pi i[(\lambda,\check{y})-(\check{\lambda},y)]}1_{E_2} \otimes e_2^*$, denoted by $\check{1}_{\check{E}_1}$ and $\check{1}_{\check{E}_2}$ respectively. Notice that $\check{1}_{\check{E}_i}$ do not depend on λ and $\check{\lambda}$. As a result, we have $\phi_1 = 1^*_{E_1} \otimes \check{1}_{\check{E}_1} \otimes e_1$ and $\phi_2 = e^{2\pi i[(\check{\lambda},y)-(\lambda,\check{y})]}1^*_{E_2} \otimes \check{1}_{\check{E}_2} \otimes e_2$. Hence we have

$$\phi_{\mathcal{F}} = e^{2\pi i[(\check{\lambda},y)-(\lambda,\check{y})]}1_{E_1} \otimes 1^*_{E_2} \otimes \check{1}^*_{\check{E}_1} \otimes \check{1}_{\check{E}_2}.$$

Next, we give the expression of the canonical forms. Under the identification of $T^*M_{1,2} \simeq \check{\mathcal{N}}_{1\vee 2}$, dy_k is identified with $\frac{\partial}{\partial y^k}$, for $k \in I_1 \cap I_2$. Similarly, dy^k is identified with $\frac{\partial}{\partial y_k}$, for $k \in \bar{I}_1 \cap \bar{I}_2$, where $\bar{I}_i = \{1, \ldots, n\} - I_i$. Then we have

$$G_1 + G_2 = \sum_{k\in I_1\cap I_2} dy_k \otimes \frac{\partial}{\partial y^k} - \sum_{l\in \bar{I}_1\cap \bar{I}_2} \frac{\partial}{\partial y_l} \otimes dy^l.$$

Suppose we have $\alpha \in \Omega^*(B_1, B_2)$ written in the form

$$\alpha(y, \lambda) = f(y, \lambda) dy_J \otimes \frac{\partial}{\partial y_K}.$$

Then we will have

$$\begin{aligned}&\mathcal{F}(\alpha)(\check{y}, \check{\lambda})\\ &= \Big(\int_{C_1\cap C_2} (\sum_{\lambda} f(y,\lambda) e^{-2\pi i(\lambda,\check{y})}) e^{2\pi i(\check{\lambda},y)} dy_{I_1\cap I_2} \otimes \frac{\partial}{\partial y_{\bar{I}_1\cap\bar{I}_2}} \Big) dy^{(\bar{I}_1\cap\bar{I}_2)-K} \otimes \frac{\partial}{\partial y^{(I_1\cap I_2)-J}},\end{aligned}$$

expressed in terms of the corresponding trivialization, up to a sign.

As seen from the above example, the Fourier transform is a homomorphism

$$\mathcal{F} : \Omega^{p,q}(\mathcal{B}_1, \mathcal{B}_2) \to \Omega^{\check{m}-q,m-p}(\check{\mathcal{B}}_1, \check{\mathcal{B}}_2).$$

We can use the Hodge star operator over the trivial bundles $\mathcal{N}_{1\vee 2}$ and $\check{\mathcal{N}}_{1\vee 2}$ to turn $\mathcal{F}$ into a homomorphism

$$\hat{\mathcal{F}} : \Omega^{p,q}(\mathcal{B}_1, \mathcal{B}_2) \to \Omega^{q,p}(\check{\mathcal{B}}_1, \check{\mathcal{B}}_2).$$

After doing that, we have

Proposition 3.13. *$\hat{\mathcal{F}} : \Omega^*(\mathcal{B}_1, \mathcal{B}_2) \to \Omega^*(\check{\mathcal{B}}_1, \check{\mathcal{B}}_2)$ is a chain map.*

Remark 3.14. Indeed it is more convenient to write down the transform using the complex $\breve{\Omega}^*(\mathcal{B}_1, \mathcal{B}_2)$. The advantage of this is that the transform preserves the differential without involving the star operator and the metric tensor. Let I_1 be a section of the bundle $T^*M_{1,2} \otimes \check{\mathcal{N}}^*_{1\vee 2}$ defined using the natural pairing between $\check{\mathcal{N}}_{1\vee 2}$ and $TM_{1,2}$, and let I_2 defined similarly. The transform $\breve{\mathcal{F}}$ is defined by

$$\breve{\mathcal{F}}(\alpha) = (-1)^{m+\check{m}} \check{\pi}_*(\pi^*\alpha \wedge (\phi_{\mathcal{F}} \otimes e^{I_1+I_2})).$$

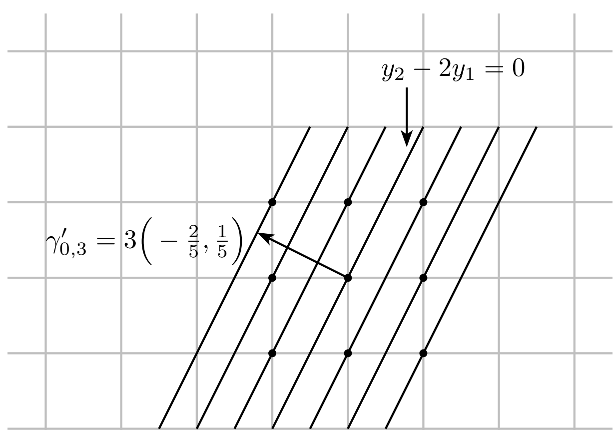
Example 3.15. In this example, we consider an example where the two branes are the same. Let $\mathcal{B}$ be a brane in T^2 with a lifting $\tilde{C}$ of the torus C given by

$$(\tilde{C} = \{y : y_2 - 2y_1 = 0\},\ E = \mathbb{C} \cdot 1_E,\ \nabla = d).$$

The lifting $\tilde{\check{C}}$ of the dual torus with corresponding dual connection is given by

$$(\tilde{\check{C}} = \{\check{y} : 2y^2 + y^1 = 0\},\ \check{E} = \mathbb{C} \cdot \check{1}_{\check{E}},\ \check{\nabla} = d).$$

In this case, we use explicit parametrizations of M and $\check{M}$, with coordinates (y_1, λ) and $(y^2, \check{\lambda})$. A path $\gamma_{y_1,\lambda}$ with coordinate (y_1, λ) has $\gamma(0) = (y_1, 2y_1)$ and $\gamma'_{y_1,\lambda} = \lambda(\frac{-2}{5}, \frac{1}{5})$. A path $\check{\gamma}_{y^2,\check{\lambda}}$ has $\check{\gamma}_{y^2,\check{\lambda}}(0) = (-2y^2, y^2)$ and $\check{\gamma}'_{y^2,\check{\lambda}} = \lambda(\frac{1}{5}, \frac{2}{5})$. For example, $\gamma'_{0,3}$ is shown in the following figure.



We consider a path $(\gamma_{y_1,\lambda}, \check{\gamma}_{y^2,\check{\lambda}})$ in $T \times T^*$. The constant function 1 descends to a trivialization e_1 of the Poincaré bundle over $C \times \check{C}$. We have $\phi_1 = 1_E \otimes \check{1}_{\check{E}} \otimes e_1$.

Then we use the liftings $\tilde{C}_{(0,\lambda)} = C_{\mathbb{R}} + (0,\lambda)$ and $\tilde{\check{C}}_{(\check{\lambda},0)} = \check{C}_{\mathbb{R}} + (\check{\lambda},0)$ to compute ϕ_2. Note that the function $e^{\pi i(\lambda y^2 - \check{\lambda} y_1)}$ on $\tilde{C}_{(0,\lambda)} \times \tilde{\check{C}}_{(\check{\lambda},0)}$ descends to a trivialization e_2 of $\mathcal{P}|_{C_{(0,\lambda)}\times \check{C}_{(\check{\lambda},0)}}$, with corresponding connection form given by $d + 2\pi i(\lambda d\check{y}^2 - \check{\lambda} dy_1)$. We have $\phi_2 = e^{2\pi i(\check{\lambda} y_1 - \lambda y^2)} 1_E \otimes \check{1}_{\check{E}} \otimes e_2$. Parallel transport along $(\gamma_{y_1,\lambda}, \check{\gamma}_{y^2,\check{\lambda}})$ identifies e_1 to e_2. Thus we obtain the tautological section given by

$$\phi_{\mathcal{F}} = e^{2\pi i[(\check{\lambda},y_1)-(\lambda,y^2)]} 1_E \otimes 1_E^* \otimes \check{1}_{\check{E}}^* \otimes \check{1}_{\check{E}},$$

and the Fourier transform for a function $f(y_1,\lambda)$ is given by

$$\hat{\mathcal{F}}(f)(y^2,\check{\lambda}) = \sum_{\lambda\in\mathbb{Z}} (\int_C f(y_1,\lambda) e^{2\pi i[(\check{\lambda},y_1)-(\lambda,y^2)]}),$$

with respect to the chosen basis.

Example 3.16. We take the standard 2-dimensional torus T^2 in this example. We use same notations as last example. The branes $\mathcal{B}_i$ have liftings given by

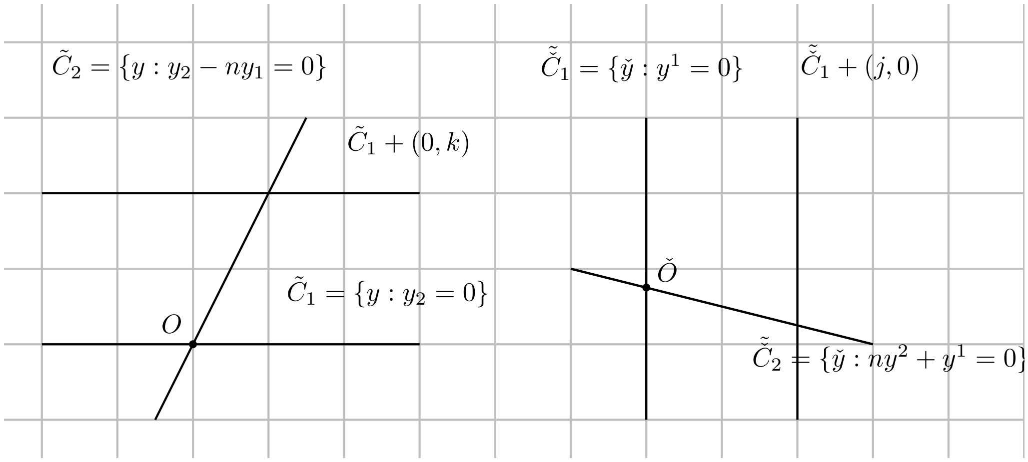
$$(\tilde{C}_1 = \{y : y_2 = 0\},\ E_1 = \mathbb{C}\cdot 1_{E_1},\ \nabla_1 = d)$$
$$(\tilde{C}_2 = \{y : y_2 - ny_1 = 0\},\ E_2 = \mathbb{C}\cdot 1_{E_2},\ \nabla_2 = d).$$

The corresponding dual branes will have the liftings given by

$$(\tilde{\check{C}}_1 = \{\check{y} : y^1 = 0\},\ \check{E}_1 = \mathbb{C}\cdot \check{1}_{\check{E}_1},\ \check{\nabla}_1 = d)$$
$$(\tilde{\check{C}}_2 = \{\check{y} : ny^2 + y^1 = 0\},\ \check{E}_2 = \mathbb{C}\cdot \check{1}_{\check{E}_2},\ \check{\nabla}_2 = d).$$

for some $n \in \mathbb{Z}_{>0}$.

In this case, $M_{1,2}$ and $\check{M}_{1,2}$ consist of a finite number of constant paths. We denote $\gamma_i = (\frac{i}{n}, i)$ and $\check{\gamma}_i = (i, -\frac{i}{n})$, for $i = 0,\cdots,n-1$. The constant function 1 on $\tilde{C}_2 \times \tilde{\check{C}}_2$ descends to a trivialization e_2 of $\mathcal{P}|_{C_2\times\check{C}_2}$. To compute the $\phi_{\mathcal{F}}$ at $(\gamma_k, \check{\gamma}_j)$, we choose $\tilde{C}_{1,(0,k)} = \tilde{C}_1 + (0,k)$ and $\tilde{\check{C}}_{1,(j,0)} = \tilde{\check{C}}_1 + (j,0)$ as liftings of the subtori on the universal cover, as shown in the figure.



The function $e^{\pi i(ky^2 - jy_1)}$ on $\tilde{C}_{1,(0,k)} \times \tilde{\check{C}}_{1,(j,0)}$ descends to a trivialization e_1 of $\mathcal{P}|_{C_1 \times \check{C}_1}$. Hence, we get $\phi_1 = e^{2\pi i(jy_1 - ky^2)} 1^*_{E_1} \otimes \check{1}_{\check{E}_1} \otimes e_1$. At the point $(\gamma_k, \check{\gamma}_j)$, we compute that $\phi_{\mathcal{F}} = \left(e^{-2\pi i \frac{jk}{n}}\right)_{j,k}$, which is an invertible matrix transforming functions on $M_{1,2}$ to functions on $\check{M}_{1,2}$.

The general case is a mixture of the above situations and we claim that for $\hat{\mathcal{F}}^{-1}$ defined similarly, we have

Proposition 3.17.

$$\hat{\mathcal{F}}^{-1} \circ \hat{\mathcal{F}} = id$$

up to a constant.

4. Mirror symmetry without corrections

In this section, we will have a brief review of certain mirror symmetry phenomena for semi-flat Calabi-Yau manifolds [11, 13]. In this case, mirror symmetry is T-duality without any modifications (or so-called quantum corrections). In fact, by performing the classical Fourier transform

$$\mathcal{F}_{cl} : \Omega^*(T) \to \Omega^*(T^*)$$

to each torus fiber, we will see the interchange of complex and symplectic structures on mirror manifolds. Furthermore, we will give a taste of how the Fourier transform on flat branes defined in the last section establishes the correspondence between A-branes and B-branes.

4.1. Semi-flat mirror Calabi-Yau manifolds

Let $N \cong \mathbb{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \mathbb{Z})$ denote the dual lattice of N. We denote by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector spaces spanned by N and M respectively. Let $B \subset M_{\mathbb{R}}$ be a convex domain. (In general, one may consider an affine manifold.) We can construct two (non-compact) Calabi-Yau manifolds X and $\check{X}$ from the tangent and cotangent bundles of B, they are called mirror manifolds.

Construction of $\check{X}$

Firstly, the tangent bundle $TB = B \times iM_{\mathbb{R}}$ is naturally a complex manifold with complex coordinates $z^j = b^j + iy^j$'s, where b^j's and y^j's are the base coordinates on B and corresponding fiber coordinates on $M_{\mathbb{R}}$. TB is equipped with the standard holomorphic volume form $\Omega_{TB} = dz^1 \wedge \cdots \wedge dz^n$. Taking quotient of TB by the lattice $iM \subset iM_{\mathbb{R}}$, each fiber is compactified to a torus. We denote

$$\check{X} := TB/iM = B \times iT_M,$$

where T_M denotes the torus $M_{\mathbb{R}}/M$ and $\check{p} : \check{X} \to B$ is a torus fibration over B. Then the holomorphic volume form Ω_{TB} on TB descends to give the holomorphic volume form $\Omega_{\check{X}} = dz^1 \wedge \cdots \wedge dz^n$ on $\check{X}$.

Furthermore, if ϕ is an elliptic solution of the real Monge-Ampére equation

$$\det\left(\frac{\partial^2 \phi}{\partial b^j \partial b^k}\right) = constant,$$

then the Kähler form

$$\omega_{\check{X}} = 2i\partial\bar{\partial}\phi = \sum_{j,k} \phi_{jk} db^j \wedge dy^k,$$

where ϕ_{jk} is defined to be $\frac{\partial^2 \phi}{\partial b^j \partial b^k}$, determines a Calabi-Yau metric on $\check{X}$. Then $\check{p} : \check{X} \to B$ becomes a special Lagrangian torus fibration, which is called the SYZ fibration.

Construction of X

Next, we consider the cotangent bundle T^*B of B, we have $T^*B = B \times iN_{\mathbb{R}}$. T^*B carries the standard symplectic form $\omega_{T^*B} = \sum_{j=1}^n db^j \wedge dy_j$, where y_j's are fiber coordinates on $N_{\mathbb{R}}$. Taking the quotient of T^*B by the dual lattice $iN \subset iN_{\mathbb{R}}$, each fiber is compactified to be a torus, denoted by

$$X := T^*B/iN = B \times iT_N,$$

where T_N denotes the torus $N_{\mathbb{R}}/N$. Note that T_N is the dual torus of T_M and $p : X \to B$ is the dual torus fibration of $\check{p}$ over B. Then the symplectic form ω_{T^*B} descends to give the symplectic form $\omega_X = \sum_{j=1}^n db^j \wedge dy_j$ on X.

By considering the metric

$$g_X = \sum_{j,k} (\phi_{jk} db^j \otimes db^k + \phi^{jk} dy_j \otimes dy_k),$$

where (ϕ^{jk}) is the inverse matrix of ϕ_{jk}, we obtain a complex structure on X with complex coordinates given by $dz_j = \sum_{k=1}^{n} \phi_{jk} db^k + i dy_j$. Then we have a natural holomorphic volume form which is

$$\Omega_X = dz_1 \wedge \cdots \wedge dz_n = \bigwedge_{j=1}^{n} (\sum_{k=1}^{n} \phi_{jk} db^k + i dy_j).$$

Also $p : X \to B$ becomes a special Lagrangian torus fibration.

4.2. Semi-flat SYZ transform

In this section, we will discuss how the geometric structures of the mirror manifolds $\check{X}$ and X are transformed to each other by performing fiberwise Fourier transformion.

Given a point $x \in B$, $\check{p}^{-1}(x)$ is the torus T_M and $p^{-1}(x)$ is the torus T_N which is the dual torus of T_M. Using the construction in section 2.2, we define the trivial line bundle $\tilde{\mathcal{P}} = N_{\mathbb{R}} \times M_{\mathbb{R}} \times \mathbb{C}$ over $N_{\mathbb{R}} \times M_{\mathbb{R}}$, equipped with the connection 1-form

$$\nabla_{\mathcal{P}} := d + \pi i \sum_{j=1}^{n} (y_j dy^j - y^j dy_j).$$

Taking quotient by the lattice $N \times M$, we obtain the Poincaré line bundle $\mathcal{P}$. The curvature of $\mathcal{P}$ is

$$F = 2\pi i \sum_{j=1}^{n} dy_j \wedge dy^j.$$

We perform this for each fiber torus and dual torus and combine these Poincaré line bundles together to get a relative version of the above picture.

Let $X \times_B \check{X} = B \times i(T_N \times T_M)$ be the fiber product of the fibrations $p : X \to B$ and $\check{p} : \check{X} \to B$, with two natural projections $\pi : X \times_B \check{X} \to X$ and $\check{\pi} : X \times_B \check{X} \to \check{X}$ respectively. We use $\mathcal{P}$ and $F = 2\pi i \sum_{j=1}^{n} dy_j \wedge dy^j \in \Omega^2(X \times_B \check{X})$ again to denote the fiberwise Poincaré line bundle and curvature two form respectively.

Definition 4.1. The semi-flat SYZ transform $\mathcal{F} : \Omega^*(X) \to \Omega^*(\check{X})$ is defined by

$$\mathcal{F}(\alpha) = \check{\pi}_*(\pi^* \alpha \wedge e^{\frac{i}{2\pi} F}) = \int_{T_N} \pi^* \alpha \wedge e^{\frac{i}{2\pi} F}.$$

Remark 4.2. T_N and T_M here should be identified with T and T^* in section 2.

One significant property of this fiberwise Fourier transform is transforming the symplectic structure on X to the complex structure on $\check{X}$ in the sense of the following proposition.

Proposition 4.3.

$$\mathcal{F}(e^{\omega_X}) = \Omega_{\check{X}}.$$

Furthermore, we may define the inverse Fourier transform $\mathcal{F}^{-1} : \Omega^*(\check{X}) \to \Omega^*(X)$ by

$$\mathcal{F}^{-1}(\check{\alpha}) = i^{-n}\pi_*(\check{\pi}^*\check{\alpha} \wedge e^{-\frac{i}{2\pi}F}) = i^{-n}\int_{T_M} \check{\pi}^*\check{\alpha} \wedge e^{-\frac{i}{2\pi}F}.$$

Then, we transform the complex structure on $\check{X}$ to the symplectic structure on X as in the previous proposition.

Proposition 4.4.

$$\mathcal{F}^{-1}(\Omega_{\check{X}}) = e^{\omega_X}.$$

As expected, one can also check that $\mathcal{F}^{-1}(e^{\omega_{\check{X}}}) = \Omega_X$ and $\mathcal{F}(\Omega_X) = e^{\omega_{\check{X}}}$. The propositions above can be proved by direct calculation [9]. For further discussions on the transform and related materials, we refer readers to [11].

4.3. Fourier transform and semi-flat branes

As described in section 3, for A-branes and B-branes which can be regarded as families of flat branes on tori, it is predicted that the mirror correspondence is a fiberwise Fourier-type transform. In this section, we restrict our attention to the case that all bundles are line bundles and establish this correspondence between special classes of A-branes on X and B-branes on $\check{X}$ using the construction given in section 3.

Semi-flat A-branes

As mentioned in section 1, an A-brane is a pair (L, E) where L is a Lagrangian submanifold and E is a flat unitary complex line bundle (in general, vector bundle) on L. As our space X admits a torus fibration $p : X \to B$, we restrict our attention to those semi-flat A-branes which are compatible with the torus fibration structure $p : X \to B$. More precisely, a semi-flat Lagrangian L is a submanifold of $p^{-1}(B_L)$, where B_L is a rational affine submanifold of B, such that $p|_L : L \to B_L$ is a torus fiber bundle over B_L, and the fiber of $p|_L$ at every point b is an affine subtorus of $p^{-1}(b)$.

Definition 4.5. An A-brane (L, E) on X is called semi-flat if L is a semi-flat Lagrangian in X.

As analogue to section 3, we study the quantum intersection theory between two semi-flat A-branes (L_i, E_i), $i = 1, 2$, by considering the space of fiberwise minimal geodesic paths from L_1 to L_2. This space, denoted by $M(L_1, L_2)$ (or simply $M_{1,2}$ if there is no confusion), is a finite dimensional manifold. There are two evaluation maps

$$\begin{array}{ccc} & M(L_1, L_2) & \\ {\scriptstyle ev_s} \swarrow & & \searrow {\scriptstyle ev_t} \\ L_1 & & L_2 \end{array}$$

and $p \circ ev_s = p \circ ev_t : M(L_1, L_2) \to B_{L_1} \cap B_{L_2}$ is a fiber bundle, where the fiber over a point b is the space of instantons $M_b(L_1 \cap p^{-1}(b), L_2 \cap p^{-1}(b))$ defined in section 3. We combine $\mathcal{N}_{1\vee 2,b}$'s on $M_b(L_1 \cap p^{-1}(b), L_2 \cap p^{-1}(b))$ along the base $B_{L_1} \cap B_{L_2}$ to obtain a relative normal bundle of $M_{1,2}$, again denoted by $\mathcal{N}_{1\vee 2}$, equipped with a connection induced from the metric. The natural section $s \in \Gamma(M_{1,2}, \mathcal{N}_{1\vee 2})$ is defined similarly. We define $\Omega^*_A((L_1, E_1), (L_2, E_2))$, with the operator $D = \nabla_{1,2} + (-1)^{p+q+1}\delta$ as in section 3. It is not a complex in general, as $s \in \Gamma(\mathcal{N}_{1,2})$ is not necessary a flat section.

In order to obtain a complex, we use the symplectic form to get an isomorphism $\lrcorner(i\omega_X) : \mathcal{N}_{1\vee 2} \to T^*(B_{L_1} \cap B_{L_2})$ of bundles over $M_{1,2}$. This identification induces a map

$$\Omega^*_A((L_1, E_1), (L_2, E_2)) \;\to\; \Omega^*(M(L_1, L_2), \mathrm{Hom}(ev_s^* E_1, ev_t^* E_2)).$$

The operator D descends to give a cochain complex. The complex

$$(\Omega^*(M(L_1, L_2),\ \mathrm{Hom}(ev_s^* E_1, ev_t^* E_2)),\ D)$$

is defined to be the quantum intersection complex between two semi-flat A-branes.

Semi-flat B-branes

Similarly, we consider semi-flat complex submanifolds on $\check{X}$. A semi-flat complex submanifold $\check{L}$ is a submanifold of $\check{p}^{-1}(B_{\check{L}})$, where $B_{\check{L}}$ is a rational affine submanifold of B, such that $\check{p}|_{\check{L}} : \check{L} \to B_{\check{L}}$ is a torus fiber bundle over $B_{\check{L}}$, and each fiber over b is an affine subtorus of $\check{p}^{-1}(b)$.

Definition 4.6. A B-brane $(\check{L}, \check{E})$ on $\check{X}$ is called semi-flat if $\check{L}$ is a semi-flat complex submanifold and $\check{E}$ is a unitary line bundle on $\check{L}$, with its connection $\nabla_{\check{E}}$ satisfying $(\nabla_{\check{E}}|_{\check{p}^{-1}(b)\cap\check{L}})^2 = 0$ and $(\nabla^{0,1}_{\check{E}})^2 = 0$.

Given two semi-flat B-branes $(\check{L}_i, \check{E}_i)$, $i = 1, 2$, The space of fiberwise minimal geodesic paths, $\check{M}(\check{L}_1, \check{L}_2)$ is naturally a complex manifold. We define $\Omega^*_B((\check{L}_1, \check{E}_1), (\check{L}_2, \check{E}_2))$ using a similar construction. Again it is not a complex, due to the fact that $\check{E}_i$'s are not flat bundles. Notice that $\check{\mathcal{N}}_{1\vee 2}$ is a holomorphic vector bundle over $\check{M}_{1,2}$, with a holomorphic section $\check{s}$ defining the operator $\check{\delta}$. Using the complex structure on $\check{M}_{1,2}$, we obtain a map given by projection to anti-holomorphic forms

$$\Omega^*_B((\check{L}_1, \check{E}_1), (\check{L}_2, \check{E}_2)) \;\to\; \Omega^{0,*}(\check{M}(\check{L}_1, \check{L}_2), \mathrm{Hom}(ev_s^* \check{E}_1, ev_t^* \check{E}_2) \otimes \wedge^* \check{\mathcal{N}}_{1\vee 2}),$$

the operator $\check{D}$ descends to give the operator $\bar{\partial} + (-1)^{p+q+1}\check{\delta}$. The quantum intersection complex between two B-branes is given by

$$(\Omega^{0,*}(\check{M}(\check{L}_1, \check{L}_2),\ \mathrm{Hom}(ev_s^* \check{E}_1, ev_t^* \check{E}_2) \otimes \wedge^* \check{\mathcal{N}}_{1\vee 2}),\ \check{D}).$$

4.4. Transform of semi-flat branes

We describe the transform between A- and B-branes. For simplicity, we transform an A-brane on X to give a B-brane on $\check{X}$. With the two projection maps

$$\begin{array}{ccc} & X \times_B \check{X} & \\ {}^{\pi}\swarrow & & \searrow^{\check{\pi}} \\ X & & \check{X}, \end{array}$$

we denote the Poincaré line bundle on $X \times_B \check{X}$ by $\mathcal{P}$, which is constructed by combining fiberwise Poincaré line bundles into a family.

The Fourier transform is a family version of that given in section 3. Given a semi-flat A-brane (L, E), we define

$$\check{E}|_{\check{x}} = H^0(L \cap p^{-1}(\check{p}(\check{x})), \pi^* E \otimes \mathcal{P}^*|_{L \cap p^{-1}(\check{p}(\check{x}))}).$$

and

$$\check{L} = \{\check{x} \in \check{X} : \check{E}|_{\check{x}} \neq 0\}.$$

$\check{E}$ becomes a complex line bundle over $\check{L}$ and is equipped with a Hermitian metric and a unitary connection. The fact that E is a flat line bundle implies that $\check{L}$ is a semi-flat complex submanifold of $\check{X}$, and L being a Lagrangian submanifold implies that $\check{E}$ is a holomorphic line bundle over $\check{L}$. Furthermore, if L is a special Lagrangian submanifold, then $\check{E}$ is asymptotically a Hermitian Yang-Mills line bundle when $\check{X}$ is approaching a large complex structure limit. We define

$$\mathcal{F}(L, E) = (\check{L}, \check{E})$$

to be the mirror B-brane on $\check{X}$.

Transform of quantum intersection complexes

Given two semi-flat A-branes (L_i, E_i), $i = 1, 2$, with corresponding mirror B-branes $(\check{L}_i, \check{E}_i)$, we denote by $B_i \subset B$ the common base of the fibrations of L_i and $\check{L}_i$. We consider the fiber product

$$\begin{array}{ccc} & M_{1,2} \times_{B_1 \cap B_2} \check{M}_{1,2} & \\ {}^{ev_s}\swarrow & & \searrow^{ev_t} \\ L_1 \times_{B_1} \check{L}_1 & & L_2 \times_{B_2} \check{L}_2. \end{array}$$

The kernel of Fourier transform $\phi_{\mathcal{F}} \otimes e^{G_1 + G_2}$ is defined by patching the fiberwise kernels into a relative version. The Fourier transform $\hat{\mathcal{F}}$ descends as a map

$$\begin{array}{c}\Omega^*(M(L_1,L_2),\mathrm{Hom}(ev_s^*E_1,ev_t^*E_2))\\ \Big\downarrow \hat{\mathcal{F}}\\ \Omega^{0,*}(\check{M}(\check{L}_1,\check{L}_2),\mathrm{Hom}(ev_s^*\check{E}_1,ev_t^*\check{E}_2)\otimes\wedge^*\check{\mathcal{N}}_{1\vee 2}).\end{array}$$

The Fourier transform can be used to study the relation between the quantum intersection complexes of A- and B-branes.

5. Mirror symmetry and quantum corrections

The Lagrangian fibration $p : X \to B$ typically has singular fibers. The base of the fibration will be (possbliy singular) integral affine manifold with corners, with Γ the critical locus (or discriminant locus) of p. Restricting to $B_0 = B - \Gamma$ and $X_0 = p^{-1}(B_0)$, we obtain a Lagrangian torus fiber bundle $p : X_0 \to B_0$. Roughly speaking, the appearance of singular fibers is due to the existence of vanishing cycles, and compactification data is captured by these vanishing cycles. These vanishing cycles are typically represented by holomorphic disks bounded by a loop in a fiber of $p|_{X_0}$.

In [8, 9], there are generating functions defined by "counting" holomorphic disks, or more precisely one-pointed open Gromov-Witten invariants [17, 18, 19], with boundary in fibers of p. These functions are used to construct the mirror manifold $\check{X}$. We give a brief review of these constructions below.

5.1. Mirror symmetry for toric Fano manifolds

Consider a n-dimensional toric Fano manifold X, with $P \subset M_{\mathbb{R}}$ its moment polytope given by

$$P = \bigcap_{j=1}^{d} \{b : (b, \nu_j) - \lambda_j \geq 0\},$$

and $\Gamma = \partial P$ the critical locus. We write

$$\mu : X \to P$$

for the moment map of X under the Hamiltonian T^n-action. The restriction of μ to the open dense orbit $X_0 \subset X$ is a torus bundle over the interior of the polytope P_0. This gives a Lagrangian torus fibration.

Kontsevich and Hori-Vafa [21] predicted that the mirror of a toric manifold X together with its symplectic structure ω_X is given by a Landau-Ginzburg model $(\check{X}, W)$, where $\check{X}$ is the non-compact Kähler manifold $(\mathbb{C}^*)^n$ and $W = \sum_{j=1}^{d} e^{\lambda_j - 2\pi(b + i\check{y}, \nu_j)}$ is a holomorphic function called the superpotential which is ‘mirror’ to information from the toric divisors at infinity. For example, $W = z_1+z_2+\frac{1}{z_1 z_2}$ is the superpotential for $\mathbb{C}P^2$.

In [9], the authors constructed a generating function $\Phi_q(x,v) \in C^\infty(X_0 \times N)$, by counting Maslov index 2 holomorphic disks bounded by the loop parametrized by (x,v) in $X_0 \times N$, where $X_0 \times N$ is interpreted as the space of fiberwise geodesic loops in X_0.

By taking $\check{X}$ to be the dual torus fibration of $\mu|_{X_0}$, fiberwise Fourier transform gives a function $\mathcal{F}(\Phi_q)$ on $\check{X} \times iM$. Since Φ_q is constant along the fibers of $\mu|_{X_0}$, $\mathcal{F}(\Phi_q)$ can be regarded as a function on $\check{X} \cong (\mathbb{C}^*)^n$. The symplectic structure on X can be incorporated to give the holomorphic structure on the Landau-Ginzburg model $(\check{X}, W)$ in the sense that

$$\mathcal{F}(\Phi_q e^{\omega_X}) = e^W \Omega_{\check{X}}.$$

Remark 5.1. The form $\Phi_q e^{\omega_X}$ can be viewed as the symplectic structure modified by quantum corrections from Maslov index 2 holomorphic disks in X with boundary on a Lagrangian torus fiber. The form $e^W \Omega_{\check{X}}$ should be viewed as the holomorphic volume form of the Landau-Ginzburg model $(\check{X}, W)$.

The isomorphism between the quantum cohomology of X and the Jacobian ring of $(\check{X}, W)$ can be established using the Fourier transform. For details, we refer readers to the article [9].

5.2. Mirror symmetry for toric Calabi-Yau manifolds

In [5, 8], a construction of the mirror manifold $\check{X}$ of toric Calabi-Yau manifolds (necessarily non-compact) is described, using the non-toric special Lagrangian fibrations constructed by M. Gross in [22]. The Gross fibration is introduced to investigate the appearance of singular fibers in the interior of the base B. It serves as local model to study a typical fibration of a Calabi-Yau manifold. For example, the Gross fibration for $\mathbb{C}^2$ will have as base B the upper half plane with boundary and a singular fiber over an interior point of B. The local structure for the singular fiber is a focus-focus singularity. We give a brief review for the construction of mirror manifolds given in [8].

The procedure involves constructing coordinate functions of $\check{X}$ by “counting” holomorphic disks emanating from boundary divisors of X. The problem is that in the Gross fibration, B has only one codimension-1 boundary. This is a technical issue and can be solved by modifying the Gross fibration using symplectic cuts. X appears as the limit as the extra divisors move to infinity.

The modified Gross fibration, which we continue to denote by $\mu : X \to B$, is a proper Lagrangian fibration whose base B is a polyhedral set in $M_{\mathbb{R}}$ having at least n distinct codimension-1 faces, which are denoted by Ψ_j for $j = 0, ..., m-1$. The preimage $D_j := \mu^{-1}(\Psi_j)$ of each Ψ_j is assumed to be a codimension-2 submanifold in X. Furthermore, the critical locus besides Ψ_j is assumed to be contained in a hyperplane H. H is called 'the wall' and separates B_0 into two chambers.

Using the semi-flat Lagrangian fibration $\mu : X_0 \to B_0$, $\check{\mu} : \check{X}_0 \to B_0$ is defined by taking the dual torus fibration. The Lagrangian fibration μ defines a lattice bundle Λ over B_0, parametrizing fiberwise geodesic loops of $\mu : X_0 \to B_0$. For each j, a generating function $\mathcal{I}_j$ is defined on $\Lambda|_{B_0 - H}$ by counting holomorphic disks intersecting the submanifold D_j's, with boundary being the loop parametrized by $\Lambda|_{B_0-H}$. The fiberwise Fourier transforms of $\mathcal{I}_j$'s give holomorphic functions $\check{z}^j$ on $\check{\mu}^{-1}(B_0 - H)$. $\check{z}^j$ changes dramatically from one component to another component, and this is called the wall-crossing phenomenon [5]. Let R be the subring of holomorphic functions on $\check{\mu}^{-1}(B_0 - H)$, generated by constant functions, $\check{z}^j$'s and $1/\check{z}^j$'s. The corrected mirror $\check{X}$ is defined to be $\check{X} = Spec(R)$.

For example, the mirror manifold $\check{X}$ for $K_{\mathbb{P}^2}$, is given by

$$\check{X} = \{(u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv = c(q) + z_1 + z_2 + \frac{q}{z_1 z_2}\},$$

where

$$c(q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots,$$

with the coefficients defined by one-pointed open Gromov-Witten invariants.

Applying the above construction, the mirror manifold $\check{X}$ obtained belongs to the Hori-Iqbal-Vafa mirror family given in [23]. Moreover, a striking feature of the quantum-corrected mirror family constructed above is that it is inherently written in flat coordinates over the moduli space of complex structures of $\check{X}$ and gives an explicit description for the mirror map. For details, we refer readers to [8].

5.3. Semi-flat branes and quantum corrections

Finally, we mention an example where we can see how Fourier transform of semi-flat branes incorporates quantum corrections. We consider the case where X is a toric Fano manifold, equipped with moment map torus fibration. We use the same notations as in subsection 5.1.

For each $b \in P_0$, the fiber torus $\{b\} \times iT_N$ is a special Lagrangian submanifold. We consider a semi-flat A-brane $(\{b\} \times iT_N, E)$ where E is a flat unitary line bundle. The corresponding dual B-brane is a point $\check{z} = b + i\check{y} \in \check{X}$, with 1-dimensional vector space $\check{E} = H^0(\{b\} \times iT_N, E)$ over it. Then $M(\{b\} \times iT_N, \{b\} \times iT_N) \simeq T_N$ is the whole fiber torus and $\check{M}(\check{z},\check{z}) \simeq M$ is the dual lattice of $\{b\} \times iT_N$. The quantum intersection complexes associated to $(\{b\} \times iT_N, E)$ and $(\check{z}, \check{E})$ are $\Omega^*(T_N)$ and $\Gamma(M, \wedge^* M_{\mathbb{C}})$, with differential operators given by d and δ respectively.

In [24], the authors showed that the Floer differential m_1 is given by $d + \iota$, where ι is defined by counting holomorphic disks of Maslov index 2. Given a Lagrangian torus fiber $\{b\} \times iT_N$, there are exactly d families of holomorphic disks of Maslov index 2 with boundary in $\{b\} \times iT_N$ (modulo to automorphisms of the domain). If we let $\{D_j\}_{j=1}^d$ be the holomorphic disks, then the operator ι is given by

$$\iota(\psi) = \sum_{j=1}^{d} [e^{-\int_{D_j} \omega_X} hol^{-1}_{\nabla_E}(\partial D_j)](\iota_{\partial D_j^{\#}} \psi),$$

where $\partial D_j^{\#}$ is the vector field generated by the circle action by $\partial D_j \in N$ and $hol_{\nabla_E}(\partial D_j)$ is the holonomy around the loop ∂D_j. The operator ι commutes with d.

Fourier transform gives an isomorphism

$$\hat{\mathcal{F}} : \Omega^*(T_M) \to \Gamma(M, \wedge^* M_{\mathbb{C}}).$$

If we identify $\wedge^* M_{\mathbb{C}} \simeq \wedge^* T^{1,0}_{\check{z}} \check{X}$ and use the result

$$\int_{D_j} \omega_X = 2\pi((b, \nu_j) - \lambda_j)$$

proven in [24], we deduce that $\hat{\mathcal{F}}(\iota)$ is precisely the operator given by contraction with ∂W, where W is the superpotential. This gives a geometric and direct verification of the result in [24].

6. Conclusion

We have seen that the *Fourier transform* plays an important role in various constructions in the SYZ programme, especially when dealing with quantum corrections, as the quantum corrections appear as “higher Fourier modes” for the mirror complex structures. To understand the construction of mirror manifolds and mirror correspondence of branes, it will be important to extend the definition of Fourier transform to flat branes, as quantum corrections for branes is also expected to incorporate with the Fourier transform. Furthermore, a geometric construction of the mirror correspondence of branes can help to explain certain mirror symmetric phenomena.

Acknowledgements: The second author thanks Akbulut for the invitation and being an excellent host for the Gökova/2011 workshop and the author benefited greatly from attending it. The authors thank Kwokwai Chan, Siu-Cheong Lau and Yi Zhang for useful discussions. The work described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Council of the Hong Kong Special Administrative Region, China (Project No. CUHK 401809).

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The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: leung@math.cuhk.edu.hk

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: klchan@math.cuhk.edu.hk

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: ncma@math.cuhk.edu.hk