Nine-dimensional exceptional
quotient singularities exist

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ABSTRACT. We prove that nine-dimensional exceptional quotient singularities exist.

The Fubini–Studi metric on $\mathbb{P}^n$ is known to be Kähler–Einstein. Moreover, it follows from [1] that every Kähler–Einstein metric on $\mathbb{P}^n$ is a pull back of the Fubini–Studi metric (possibly multiplied by a positive real constant) via some automorphism of $\mathbb{P}^n$. However, there are plenty of non-Kähler–Einstein metrics on $\mathbb{P}^n$ whose Kähler forms lie in $c_1(\mathbb{P}^n)$. Let $g = g_{i\bar{j}}$ be such a metric with a Kähler form ω. Then one can try to obtain the Kähler–Einstein metric on $\mathbb{P}^n$ out of the metric g by taking the normalized Kähler–Ricci iterations defined by

$$\begin{cases} \omega_{i-1} = \mathrm{Ric}(\omega_i), \\ \omega_0 = \omega, \end{cases} \qquad (1)$$

where ω_i is a Kähler form such that $\omega_i \in c_1(\mathbb{P}^n)$. Indeed, it follows from [19] that the solution ω_i to (1) exists for every $i \geqslant 1$. However, it is not clear that any solution to (1) converges to the Kähler form of the Kähler–Einstein metric on $\mathbb{P}^n$ in the sense of Cheeger–Gromov (see [15]). Nevertheless, this is known to be true under an additional assumption that we are going to describe.

Let $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^n)$ be a finite subgroup. Suppose, in addition, that the metric g is $\bar{G}$-invariant. Let $\alpha_{\bar{G}}(\mathbb{P}^n)$ be the $\bar{G}$-invariant α-invariant of Tian of $\mathbb{P}^n$ that is introduced in [18].

Theorem 2 ([15]). *If $\alpha_{\bar{G}}(\mathbb{P}^n) > 1$, then any solution to (1) converges to the Kähler form of the Kähler–Einstein metric on $\mathbb{P}^n$ in $C^\infty(X)$-topology.*

It should be mentioned that the original result by Rubinstein proved in [15] is much stronger than Theorem 2 and is valid for any smooth complex manifold with a positive first Chern class. Nevertheless, even in the simplest possible case of $\mathbb{P}^n$, the assertion of Theorem 2 is still very not obvious. Thus, it is natural to ask the following

Question 3 (Rubinstein). *Is there a finite subgroup $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^n)$ such that $\alpha_{\bar{G}}(\mathbb{P}^n) > 1$?*

It came as a surprise that Question 3 is strongly related to the notion of exceptional singularity that was introduced by Shokurov in [16]. Let us recall this notion.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

Definition 4 (Shokurov)**.** *Let* $(V \ni O)$ *be a germ of Kawamata log terminal singularity. Then* $(V \ni O)$ *is said to be* exceptional *if for every effective* $\mathbb{Q}$-*divisor* D_V *on the variety* V *such that* (V, D_V) *is log canonical and for every resolution of singularities* $\pi \colon U \to V$ *there exists at most one* π-*exceptional divisor* $E \subset U$ *such that* $a(V, D_V, E) = -1$, *where the rational number* $a(V, D_V, E)$ *can be defined through the equivalence*

$$K_U + D_U \sim_{\mathbb{Q}} \pi^* \left(K_V + D_V \right) + \sum a\big(V, D_V, F\big) F.$$

The sum above is taken over all f-*exceptional divisors, and* D_U *is the proper transform of the divisor* D_V *on the variety* U.

One can show that exceptional Kawamata log terminal singularities are straightforward generalizations of the Du Val singularities of type $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ (see [16, Example 5.2.3]), which partially justifies the word "exceptional" in Definition 4. It follows from our earlier result [3, Theorem 1.16] that exceptional Kawamata log terminal singularities exist in every dimension. Surprisingly, Question 3 is *almost* equivalent to the following

Question 5. *Are there exceptional* quotient *singularities of dimension* $n + 1$?

Recall that quotient singularities are always Kawamata log terminal. So Question 5 fits well to Definition 4. It follows from [16], [13], [3], and [4] that the answers to both Questions 3 and 5 are positive for every $n \leqslant 5$ (see Theorems 10 and 12). Moreover, it follows from [4] that the answers to both Questions 3 and 5 are "surprisingly" negative for $n = 6$. The purpose of this paper is to show that the answers to both Questions 3 and 5 are again positive for $n = 8$ by proving the following

Theorem 6. *Let* G *be a finite subgroup in* $\mathrm{SL}_9(\mathbb{C})$ *such that* $G \cong 3^{1+4} : \mathrm{Sp}_4(3)$ *(see* [6] *for notation), let* $\phi \colon \mathrm{SL}_9(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^8)$ *be the natural projection. Put* $\bar{G} = \phi(G)$. *Then* $4/3 \geqslant \alpha_{\bar{G}}(\mathbb{P}^8) \geqslant 10/9$ *and the singularity* $\mathbb{C}^9/G$ *is exceptional.*

How to compute $\alpha_{\bar{G}}(\mathbb{P}^n)$? How to show that a given quotient singularity is exceptional? How are Questions 3 and 5 related? How to prove Theorem 6? What are the expected answers to Questions 3 and 5 for $n = 7$ and $n \geqslant 9$? Let us give partial answers to these questions.

Let $\phi \colon \mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^n)$ be the natural projection. Then there exists a finite subgroup G in $\mathrm{GL}_{n+1}(\mathbb{C})$ such that $\phi(G) = \bar{G}$. Put

$$\mathrm{lct}\left(\mathbb{P}^n, \bar{G} \right) = \sup \left\{ \lambda \in \mathbb{Q} \; \middle| \; \begin{array}{l} \text{the log pair } (\mathbb{P}^n, \lambda D) \text{ has log canonical singularities} \\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n} \end{array} \right\}.$$

Theorem 7 (see e.g., [2, Theorem A.3])**.** *One has* $\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^n)$.

The number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ is usually called $\bar{G}$-equivariant *global log canonical threshold* of $\mathbb{P}^n$. Despite the fact that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^n)$, we still prefer to work with the number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ throughout this paper, because it is easier to handle than $\alpha_{\bar{G}}(\mathbb{P}^n)$. For example, it follows immediately from the definition of the number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ that $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ is less

than or equal to $d/(n+1)$ if the group G has a semi-invariant of degree d (a semi-invariant of the group G is a polynomial whose zeroes define a $\bar{G}$-invariant hypersurface in $\mathbb{P}^n$).

Recall that an element $g \in G$ is called a *reflection* (or sometimes a *quasi-reflection*) if there is a hyperplane in $\mathbb{P}^n$ that is pointwise fixed by $\phi(g)$. To answer Question 5 one can always assume that the group G does not contain reflections (cf. [4, Remark 1.16]). On the other hand, one can easily check that there exists a finite subgroup $G' \subset \mathrm{SL}_{n+1}(\mathbb{C})$ such that $\phi(G') = \bar{G}$. So to answer Question 3 one can also assume that $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$, which implies, in particular, that the group G does not contain reflections. Moreover, if the group G does not contain reflections, then the singularity $\mathbb{C}^{n+1}/G$ is exceptional if and only if the singularity $\mathbb{C}^{n+1}/G'$ is exceptional thanks to the following

Theorem 8 ([3, Theorem 3.17])**.** *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Then the singularity $\mathbb{C}^{n+1}/G$ is exceptional if and only if for any $\bar{G}$-invariant effective $\mathbb{Q}$-divisor D on $\mathbb{P}^n$ such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$, the log pair $(\mathbb{P}^n, D)$ is Kawamata log terminal.*

Corollary 9. *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Then*

- *the singularity $\mathbb{C}^{n+1}/G$ is exceptional if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$,*
- *the singularity $\mathbb{C}^{n+1}/G$ is not exceptional if either $\mathrm{lct}(\mathbb{P}^n, \bar{G}) < 1$,*
- *the singularity $\mathbb{C}^{n+1}/G$ is not exceptional if G has a semi-invariant of degree at most $n + 1$,*
- *for any subgroup $G' \subset \mathrm{GL}_{n+1}(\mathbb{C})$ such that G' does not contain reflections and $\phi(G') = \bar{G}$, the singularity $\mathbb{C}^{n+1}/G$ is exceptional if and only if the singularity $\mathbb{C}^{n+1}/G'$ is exceptional.*

The assumption that G does not contain reflections is crucial for Theorem 8 (see [3, Example 1.18]). On the other hand, it follows from [14, Proposition 2.1] that G must be primitive (see for example [3, Definition 1.21]) if $\mathbb{C}^{n+1}/G$ is exceptional. Moreover, for small $n \leqslant 4$, we have the following

Theorem 10 ([13, Theorem 1.2], [3, Theorem 1.22])**.** *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Suppose that $n \leqslant 4$. Then the following conditions are equivalent:*

- *the singularity $\mathbb{C}^{n+1}/G$ is exceptional,*
- $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \geqslant (n + 2)/(n + 1)$,
- *the group G is primitive and has no semi-invariants of degree at most $n + 1$.*

In particular, both Questions 3 and 5 are equivalent for $n \leqslant 4$ and can be expressed in terms of primitivity and absence of semi-invariants of small degree of the group G. It appears that in higher dimensions the latter is no longer true, since there are non-exceptional six-dimensional quotient singularities arising from primitive subgroups without reflections in $\mathrm{GL}_6(\mathbb{C})$ that have no semi-invariants of degree at most 6 (see [3, Example 3.25]). On the other hand, we still believe that both Questions 3 and 5 are equivalent for every n, which can be summarized as

Conjecture 11. *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Then the singularity $\mathbb{C}^{n+1}/G$ is exceptional if and only if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$.*

In fact, Conjecture 11 still holds for $n = 5$, because of

Theorem 12 ([4, Theorem 1.14]). *Let G be a finite subgroup in $\mathrm{SL}_6(\mathbb{C})$. Then the following are equivalent:*
- *the singularity $\mathbb{C}^6/G$ is exceptional,*
- *the inequality $\mathrm{lct}(\mathbb{P}^5, \bar{G}) \geqslant 7/6$ holds,*
- *either $\bar{G}$ is the Hall–Janko sporadic simple group (see [12]), or $G \cong 6.\mathrm{A}_7$ and $\bar{G} \cong \mathrm{A}_7$.*

In particular, both Questions 3 and 5 are equivalent and both have positive answers for $n = 5$. For $n = 6$, both Questions 3 and 5 are also equivalent and both have negative answers due to

Theorem 13 ([4, Theorem 1.16]). *For every finite subgroup G in $\mathrm{GL}_7(\mathbb{C})$, the singularity $\mathbb{C}^7/G$ is not exceptional and $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leqslant 1$.*

To apply Theorem 8 we may assume that $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$, since there exists a finite subgroup $G' \subset \mathrm{SL}_{n+1}(\mathbb{C})$ such that $\phi(G') = \bar{G}$. On the other hand, it is well known that there are at most finitely many primitive finite subgroups in $\mathrm{SL}_{n+1}(\mathbb{C})$ up to conjugation by Jordan's theorem for complex linear groups. Primitive finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are group-theoretic counterparts of Platonic solids and each of them gives rise to an exceptional quotient singularity (see [16, Example 5.2.3]). Similar classification is possible in small dimensions. For example, primitive finite subgroups of $\mathrm{SL}_{n+1}(\mathbb{C})$ for $n \leqslant 6$ have been classified long time ago (see for example [7]). This allowed to obtain the complete list of all finite subgroups in $\mathrm{SL}_{n+1}(\mathbb{C})$ for every $n \leqslant 6$ that give rise to exceptional quotient singularities (see [13], [3], and [4]), which implies, in particular, that the answers to both Questions 3 and 5 are positive for every $n \leqslant 5$ and are negative for $n = 6$ (see Theorem 13). We have no idea right now what are the answers to Questions 3 and 5 in the cases when $n = 7$ and $n \geqslant 9$, but we expect that the answers to both Questions 3 and 5 may still be negative for all $n \gg 0$ due to the following

Theorem 14. *Let G be the finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ such that $\bar{G}$ is a sporadic simple group. Then $\mathbb{C}^{n+1}/G$ is exceptional if and only if $n = 5$ and $\bar{G}$ is the Hall–Janko sporadic simple group.*

Proof. Since $\bar{G}$ is simple, we may assume that G has no quasi-reflections. Explicit computations in GAP (see [8]) imply that G has a semi-invariant of degree at most $n+1$ (and thus $\mathbb{C}^{n+1}/G$ is not exceptional by Theorem 8) unless $n = 5$ and $\bar{G}$ is the Hall–Janko sporadic simple group[1]. If $n = 5$ and $\bar{G}$ is the Hall–Janko sporadic simple group, then the singularity $\mathbb{C}^{n+1}/G$ is exceptional by [4, Theorem 1.14]. $\qquad\square$

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[1]Similarly, one can show that G has a semi-invariant of degree at most n (and thus $\mathbb{C}^{n+1}/G$ is not weakly-exceptional (see [3, Definition 3.7]) by [3, Theorem 3.16]) unless either $n = 5$ and $\bar{G}$ is the Hall–Janko sporadic simple group, or $n = 11$ and $\bar{G}$ is the Suzuki sporadic simple group (see [17]). We expect that in the latter case the corresponding quotient singularity is actually weakly-exceptional.

An indirect evidence that both Questions 3 and 5 may have negative answers for all $n \gg 0$ is given by

Theorem 15 ([5])**.** *Let G be the finite primitive subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$. Suppose that $n \geqslant 12$. Then $|\bar{G}| \leqslant (n+2)!$. Moreover, if $|\bar{G}| = (n+2)!$, then $\bar{G} \cong \mathrm{S}_{n+2}$.*

In fact, Collins obtained the optimal bounds for $|\bar{G}|$ for every $n \leqslant 11$ if G is primitive (his proof uses known lower bounds for the degrees of the faithful representations of each quasisimple group, for which the classification of finite simple groups is required). Moreover, it follows from [16], [13], [5], [3], and [4] that $|\bar{G}|$ reaches its maximum on a subgroup $\bar{G}$ in $\mathrm{Aut}(\mathbb{P}^n)$ with $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$ if $n \leqslant 3$, and this is no longer true for $4 \leqslant n \leqslant 6$. Surprisingly, it follows from Theorem 6 that in the case when $n = 8$, the number $|\bar{G}|$ reaches its maximum if G is isoclinic to a finite subgroup in $\mathrm{GL}_9(\mathbb{C})$ that is mentioned in Theorem 6. For $n = 11$, the number $|\bar{G}|$ reaches its maximum if $\bar{G}$ is the Suzuki sporadic simple group.

In the remaining part of the paper we prove Theorem 6. Let G be a finite subgroup in $\mathrm{SL}_9(\mathbb{C})$ from Theorem 6. Then the embedding $G \hookrightarrow \mathrm{SL}_9(\mathbb{C})$ is given by an irreducible nine-dimensional G-representation[2], which we denote by U.

The outline of the proof of Theorem 6 is as follows. We assume that $\mathrm{lct}(\mathbb{P}^8, \bar{G}) < 10/9$ and seek for a contradiction. There exists a $\bar{G}$-invariant $\mathbb{Q}$-divisor D on $\mathbb{P}^8$ and a positive rational number $\lambda < 10/9$ such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^8}$ and the log pair $(\mathbb{P}^8, \lambda D)$ is strictly log canonical, i.e., log canonical and not Kawamata log terminal. Arguing as in [3] and [4], we apply Nadel–Shokurov vanishing (see [11, Theorem 9.4.8]) and Kawamata subadjunction (see [10, Theorem 1]) to obtain restrictions on the Hilbert polynomial of the minimal center of log canonical singularities of the log pair $(\mathbb{P}^8, \lambda D)$ (see [9, Definition 1.3], [10]). Composing the latter with results coming from representation theory we obtain a contradiction. One of the few new ingredients of the proof is the *binomial trick* (see Lemma 27).

Let us list without proofs some properties of the G-representation U (Lemmas 16, 17, 18, and 19) that can be verified by direct computations. We used GAP (see [8]) to carry them out.

Lemma 16. *The group G does not have semi-invariants of degree $d \leqslant 11$, and there exists a semi-invariant of the group G of degree 12.*

Denote by Δ_k the collection of dimensions of irreducible subrepresentations of $\mathrm{Sym}^k(U^\vee)$. We will use the following notation: writing $\Delta_k = [\ldots, r \times m, \ldots]$, we mean that among the irreducible subrepresentations of $\mathrm{Sym}^k(U^\vee)$ there are exactly r subrepresentations of dimension m (not necessarily isomorphic to each other). Furthermore, denote by Σ_k the set of partial sums of Δ_k, i.e., the set of all numbers $s = \sum r_i' m_i$, where $\Delta_k = [\ldots, r_1 \times m_1, \ldots, r_2 \times m_2, \ldots, r_i \times m_i, \ldots]$ and $0 \leqslant r_i' \leqslant r_i$ for all i. We use the abbreviation m_i for $1 \times m_i$.

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[2]Note that the group G has two irreducible representations of dimension 9, but they differ only by an outer automorphism of G, so that the *subgroup* $G \subset \mathrm{SL}_9(\mathbb{C})$ is defined uniquely up to conjugation.

Lemma 17. *The representation* $\mathrm{Sym}^2(U^\vee)$ *is irreducible (and has dimension* 45*). Futhermore,* $\Delta_3 = [5, 160]$, $\Delta_4 = [45, 180, 270]$, $\Delta_5 = [36, 90, 135, 216, 270, 540]$,

$$\Delta_6 = [4, 15, 24, 2 \times 80, 3 \times 240, 3 \times 480, 640],$$

$$\Delta_7 = [9, 36, 3 \times 135, 3 \times 180, 3 \times 216, 270, 324, 405, 3 \times 540, 720, 2 \times 729],$$

$$\Delta_8 = [36, 4 \times 45, 5 \times 180, 5 \times 270, 2 \times 324, 6 \times 360, 3 \times 405, 7 \times 540, 2 \times 576, 720, 729],$$

$$\Delta_9 = [3 \times 5, 3 \times 20, 3 \times 30, 40, 45, 60, 80, 12 \times 160, 12 \times 240, 10 \times 480, 10 \times 640, 11 \times 720].$$

Lemma 18. *Let* $U_{45} \subset \mathrm{Sym}^4(U^\vee)$ *be the* 45*-dimensional irreducible subrepresentation. Then*

$$U^\vee \otimes U_{45} \cong U_{90} \oplus U_{135} \oplus U_{180}$$

as a G*-representation, where* U_{90}, U_{135} *and* U_{180} *are irreducible* G*-representations of dimensions* 90, 135 *and* 180*, respectively.*

Lemma 19. *Let* $U_{24} \subset \mathrm{Sym}^6(U^\vee)$ *be the* 24*-dimensional irreducible subrepresentation. Then* $U^\vee \otimes U_{24}$ *is an irreducible* G*-representation.*

Now we are ready to prove Theorem 6. It follows from Theorems 7 and 8 that to prove Theorem 6 it is enough to prove that $4/3 \geqslant \mathrm{lct}(\mathbb{P}^8, \bar{G}) \geqslant 10/9$. On the other hand, one has $\mathrm{lct}(\mathbb{P}^8, \bar{G}) \leqslant 4/3$ by Lemma 16. In fact, we believe that $\mathrm{lct}(\mathbb{P}^8, \bar{G}) = 4/3$, but we are unable to prove this now. To complete the proof of Theorem 6, we must prove that $\mathrm{lct}(\mathbb{P}^8, \bar{G}) \geqslant 10/9$. Suppose that $\mathrm{lct}(\mathbb{P}^8, \bar{G}) < 10/9$. Then there is an effective $\bar{G}$-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^8}(9)$, and there is a positive rational number $\lambda < 10/9$ such that $(\mathbb{P}^8, \lambda D)$ is strictly log canonical.

Let S be a minimal center of log canonical singularities of the log pair $(\mathbb{P}^8, \lambda D)$ (see [9, Definition 1.3], [10]), let V be the $\bar{G}$-orbit of the subvariety $S \subset \mathbb{P}^8$, and let r be the number of irreducible components of the subvariety V. Then $\deg(V) = r \cdot \deg(S)$.

Arguing as in the proofs of [9, Theorem 1.10] and [10, Theorem 1], we may assume that the only log canonical centers of the log pair $(\mathbb{P}^8, \lambda D)$ are components of the subvariety V (see [3, Lemma 2.8]). Then it follows from [9, Proposition 1.5] that the components of the subvariety V are disjoint. In particular, if $\dim(V) \geqslant 4$, then $r = 1$. Put $n = \dim(V)$ which also equals $\dim(S)$. Then $n \neq 7$ by Lemma 16.

Let $\mathcal{I}_V$ be the ideal sheaf of the subvariety $V \subset \mathbb{P}^8$, and let Λ be a general hyperplane in $\mathbb{P}^8$. Put $H = \Lambda|_V$, $h_m = h^0(\mathcal{O}_V(mH))$ and $q_m = h^0(\mathcal{O}_{\mathbb{P}^8}(m) \otimes \mathcal{I}_V)$ for every $m \in \mathbb{Z}$. It follows from the Shokurov–Nadel vanishing theorem (see [11, Theorem 9.4.8]) that

$$q_m = h^0\Big(\mathcal{O}_{\mathbb{P}^8}\big(m\big)\Big) - h_m = \binom{8+m}{m} - h_m$$

for every $m \geqslant 1$. In particular, if $n = 0$, then $r = h_1 \leqslant 9$, which is impossible by Lemma 16. Hence, we see that $1 \leqslant n \leqslant 6$.

It follows from [10, Theorem 1] that the variety V is normal and has at most rational singularities. Moreover, it follows from [10, Theorem 1] that for every positive rational

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number $\epsilon > 0$ there is an effective $\mathbb{Q}$-divisor B_V on the variety V such that

$$\left(K_{\mathbb{P}^n} + \lambda D + \epsilon\Lambda\right)\Big|_V \sim_{\mathbb{Q}} K_V + B_V,$$

and (V, B_V) has Kawamata log terminal singularities. In particular, taking ϵ sufficiently small, we may assume that $K_V + B_V \sim_{\mathbb{Q}} (1 - \nu)H$ for some positive $\nu \in \mathbb{Q}$, because $\lambda < 10/9$.

Remark 20. One can show that any irreducible representation W of the group G such that the center $Z(G) \cong \mathbb{Z}_3$ acts non-trivially on W has dimension $\dim(W)$ divisible by 9. Therefore one has $h_i \equiv 0 \mod 9$ for every i not divisible by 3, since $Z(G)$ acts nontrivially on $\mathrm{Sym}^i(U^\vee)$ if i is not divisible by 3.

Remark 21. One has $q_1 = 0$ since $U^\vee$ is irreducible and $q_2 = 0$ by Lemma 17.

Put $d = H^n = \deg(V)$ and $H_V(m) = \chi(\mathcal{O}_V(m))$. Then the Shokurov–Nadel vanishing theorem (see [11, Theorem 9.4.8]) implies that $H_V(m) = h_m$ for every $m \geqslant 1$. Recall that $H_V(m)$ is a Hilbert polynomial of the subvariety V, which is a polynomial in m of degree n with leading coefficient $d/n!$.

Lemma 22. *For any non-negative integer δ one has*

$$d = h_{\delta+n+1} - \binom{n}{1}h_{\delta+n} + \binom{n}{2}h_{\delta+n-1} + \ldots + (-1)^n h_{\delta+1}.$$

Proof. Induction by n. □

Lemma 23. *If $4 \leqslant n \leqslant 5$, then d is divisible by 3. If $n = 6$, then d is divisible by 9.*

Proof. Suppose that $n = 6$. Applying Lemma 22 with $\delta = 0$ and $\delta = 1$, we get

$$h_7 - 6h_6 + 15h_5 - 20h_4 + 15h_3 - 6h_2 + h_1 = d = h_8 - 6h_7 + 15h_6 - 20h_5 + 15h_4 - 6h_3 + h_2, \quad (24)$$

which gives $21h_6 - 21h_3 \equiv 0 \mod 9$ by subtracting two equalities in (24), reducing everything modulo 9, and using Remark 20. Thus $h_6 - h_3 \equiv 0 \mod 3$, and $15h_6 - 6h_3 \equiv 0 \mod 9$. The latter equality combined with (24) implies that $d \equiv 0 \mod 9$.

If $n = 5$, a similar argument shows that $d \equiv 0 \mod 3$.

Finally, suppose that $n = 4$. Applying Lemma 22 for $\delta = 0$, we get

$$h_5 - 4h_4 + 6h_3 - 4h_2 + h_1 = d,$$

which gives $d \equiv 6h_3 \equiv 0 \mod 3$, since h_5, h_4, h_2, and h_1 are divisible by 3 by Remark 20. □

Remark 25. If $n = 6$, then $q_3 = 0$. Indeed, if $q_3 > 0$, then $q_3 > 2$ by Lemma 17, so that $d < 9$, which is impossible by Lemma 23. Similarly, if $n = 6$ and $q_4 > 0$, one has $d = 9$.

Remark 26. One has $h_m \leqslant h_{m+1}$ and $q_m \leqslant q_{m+1}$ for all $m \geqslant 1$.

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be general hyperplanes in $\mathbb{P}^8$. Put $\Pi_j = \Lambda_1 \cap \ldots \cap \Lambda_j$, $V_j = V \cap \Pi_j$, $H_j = V_j \cap H$, and $B_{V_j} = B_V|_{V_j}$ for every $j \in \{1, \ldots, n\}$. Put $V_0 = V$, $B_{V_0} = B_V$, $H_0 = H$, $\Pi_0 = \mathbb{P}^8$. For every $j \in \{0, 1, \ldots, n\}$, let $\mathcal{I}_{V_j}$ be the ideal sheaf of the subvariety $V_j \subset \Pi_j$. Recall that $\Pi_j \cong \mathbb{P}^{8-j}$ and put $q_i(V_j) = h^0(\mathcal{O}_{\Pi_j}(i) \otimes \mathcal{I}_{V_j})$ for every $j \in \{0, 1, \ldots, n\}$.

Lemma 27. *Suppose that $i \geqslant j+1$ and $j \in \{1, \ldots, n\}$. Then*

$$q_i(V_j) = q_i - \binom{j}{1} q_{i-1} + \binom{j}{2} q_{i-2} - \ldots + (-1)^j q_{i-j}.$$

Proof. For every $j \in \{0, 1, \ldots, n\}$, it follows from the adjunction formula that

$$K_{V_j} + B_{V_j} \sim_{\mathbb{Q}} (j+1-\nu)H_j,$$

because $K_V + B_V \sim_{\mathbb{Q}} (1-\nu)H$ and (V_j, B_{V_j}) has at most Kawamata log terminal singularities. Applying the Nadel–Shokurov vanishing theorem to the log pair (V_j, B_{V_j}), we see that $h^1(\mathcal{O}_{V_j}(i)) = 0$ for every $i \geqslant j+1$ and every $j \in \{0, 1, \ldots, n\}$. Thus, we have

$$h^0\Big(\mathcal{O}_{V_j}\big((i+1)H_j\big)\Big) - h^0\Big(\mathcal{O}_{V_j}\big(iH_j\big)\Big) = h^0\Big(\mathcal{O}_{V_{j+1}}\big((i+1)H_{j+1}\big)\Big) \tag{28}$$

for every $i \geqslant j+1$ and every $j \in \{0, \ldots, n-1\}$. Now applying the Nadel–Shokurov vanishing theorem to the log pair $(\Pi_j, \lambda D|_{\Pi_j})$, we see that $h^1(\mathcal{O}_{\Pi_j}(i) \otimes \mathcal{I}_{V_j}) = 0$ for every $i \geqslant j+1$ and every $j \in \{0, 1, \ldots, n\}$. This implies that

$$q_i(V_j) = \binom{8-j+i}{i} - h^0\Big(\mathcal{O}_{V_j}\big(iH_j\big)\Big) \tag{29}$$

for every $i \geqslant j+1$ and every $j \in \{0, 1, \ldots, n\}$. Combining (28) and (29), we have

$q_i(V_{j-1}) - q_{i-1}(V_{j-1})$

$= \binom{9-j}{i} - h^0\Big(\mathcal{O}_{V_{j-1}}\big(iH_{j-1}\big)\Big) - \binom{8-j+i}{i-1} + h^0\Big(\mathcal{O}_{V_{j-1}}\big((i-1)H_{j-1}\big)\Big) =$

$= \binom{9-j}{i} - \binom{8-j+i}{i-1} - h^0\Big(\mathcal{O}_{V_j}\big(iH_j\big)\Big) = \binom{8-j+i}{i} - h^0\Big(\mathcal{O}_{V_j}\big(iH_j\big)\Big) = q_i(V_j)$

for every $i \geqslant j+1$ and every $j \in \{1, \ldots, n\}$. Thus, we see that

$$q_i(V_j) = q_i(V_{j-1}) - q_{i-1}(V_{j-1}) \tag{30}$$

for every $i \geqslant j+1$ and every $j \in \{1, \ldots, n\}$. Iterating (30), we obtain the required equality. $\square$

Lemma 27 allows one to obtain bounds on the numbers q_i.

Remark 31. There are trivial bounds $0 \leqslant q_i(V_j) < \binom{8-j+i}{i}$.

Recall that $q_1 = q_2 = 0$ by Remark 21, and $q_3 = 0$ if $n = 6$ by Remark 25. Therefore, Remark 31 implies

Corollary 32. *If $n = 5$, one has*

$$0 \leqslant q_4 - 3q_3 < 126. \tag{33}$$

If $n = 6$, one has

$$0 \leqslant q_5 - 4q_4 < 126. \tag{34}$$

Playing with the numbers $q_i(V_j)$, we can obtain

Lemma 35. *Suppose that $n \geqslant 4$. Then*

$$\binom{9}{n} - \frac{nd}{2} > q_n(V_{n-1}) \geqslant \binom{9}{n} - nd - 1.$$

Proof. Recall that the variety $V_{n-1} \subset \mathbb{P}^{8-n+1}$ is a smooth curve of degree d, since V is normal. Since $n \geqslant 4$, we see that V_{n-1} is irreducible. Let g be the genus of the curve V_{n-1}. It follows from the adjunction formula that $K_{V_{n-1}} + B_{V_{n-1}} \sim_{\mathbb{Q}} (n - \nu)H_{n-1}$, because $K_V + B_V \sim_{\mathbb{Q}} (1 - \nu)H$. In particular, one has $2g - 2 < dn$.

Applying the Nadel–Shokurov vanishing theorem to the log pair $(\Pi_{n-1}, \lambda D|_{\Pi_{n-1}})$, we see that

$$q_m(V_{n-1}) = \binom{8 - n + 1 + m}{m} - h^0\Big(\mathcal{O}_{V_{n-1}}\big(mH_{n-1}\big)\Big)$$

for every $m \geqslant n$. Since $2g - 2 < dn$, the divisor nH_{n-1} is non-special. Therefore, it follows from the Riemann–Roch theorem that

$$q_n(V_{n-1}) = \binom{9}{n} - nd + g - 1,$$

which implies the required inequalities, since $2g - 2 < dn$ and $g \geqslant 0$. $\qquad\square$

Combining Lemmas 35 and 27 and recalling the trivial bounds from Remark 31, we obtain

Corollary 36. *If $n = 4$, then*

$$\max\big(0, 125 - 4d\big) \leqslant q_4 - 3q_3 \leqslant 125 - 2d. \tag{37}$$

If $n = 5$, then

$$\max\big(0, 125 - 5d\big) \leqslant q_5 - 4q_4 + 6q_3 \leqslant 126 - \frac{5d}{2}. \tag{38}$$

If $n = 6$, then

$$0 \leqslant q_6 - 5q_5 + 10q_4 - 10q_3 \leqslant 83 - 3d. \tag{39}$$

As a by-product of Corollary 36, we get

Corollary 40. *If $n = 4$, then $d \leqslant 62$. If $n = 5$, then $d \leqslant 50$. If $n = 6$, then $d \leqslant 27$.*

The above restrictions reduce the problem to a combinatorial question of finding all polynomials H_V of degree n with a leading coefficient $d/n!$, such that $h_m = H_V(m) \in \Sigma_m$ for sufficiently many $m \geqslant 1$, and such that the numbers h_m and $q_m = h^0(\mathcal{O}_{\mathbb{P}^8}(m)) - h_m$ satisfy the conditions arising from Lemma 23, Corollaries 32, 36 and 40, and Remarks 21, 25 and 26. This can be done in a straightforward way, although the number of cases to be considered is so large that we had to delegate this part of the proof to a simple computer program. Finally, we get the following four lemmas which we leave without proofs.

Lemma 41. *There are no polynomials* $\mathrm{H}(m)$ *of degree* $n \leqslant 3$ *such that the values* $h_m = \mathrm{H}(m)$ *are in* Σ_m *for* $1 \leqslant m \leqslant 6$ *and* $h_i \leqslant h_{i+1}$ *for* $1 \leqslant i \leqslant 5$.

Lemma 42. *If* $\mathrm{H}(m)$ *is a polynomial of degree* $n = 4$ *with a leading coefficient* $d/n!$ *with* $d \leqslant 62$ *and* d *divisible by* 3, *such that the values* $h_m = \mathrm{H}(m)$ *are in* Σ_m *for* $1 \leqslant m \leqslant 6$, *and the numbers* h_m *and* $q_m = \binom{8+m}{m} - h_m$ *satisfy the bounds of Remark 26 and* (37), *then* $d = 36$, $q_1 = q_2 = q_3 = 0$, $q_4 = 45$, $q_5 = 270$.

Lemma 43. *If* $\mathrm{H}(m)$ *is a polynomial of degree* $n = 5$ *with a leading coefficient* $d/n!$ *with* $d \leqslant 50$ *and* d *divisible by* 3, *such that the values* $h_m = \mathrm{H}(m)$ *are in* Σ_m *for* $1 \leqslant m \leqslant 9$, *and the numbers* h_m *and* $q_m = \binom{8+m}{m} - h_m$ *satisfy the bounds of Remark 26,* (38) *and* (33), *then* $d = 45$, $q_1 = q_2 = q_3 = q_4 = q_5 = 0$, $q_6 = 39$, $q_7 = 270$.

Lemma 44. *There are no polynomials* $\mathrm{H}(m)$ *of degree* $n = 6$ *with a leading coefficient* $d/n!$ *with* $d \leqslant 27$ *and* d *divisible by* 9, *such that the values* $h_m = \mathrm{H}(m)$ *are in* Σ_m *for* $1 \leqslant m \leqslant 9$, *and the numbers* h_m *and* $q_m = \binom{8+m}{m} - h_m$ *satisfy the bounds of Remark 26,* (39) *and* (34).

Applying Lemmas 41, 42, 43, and 44 to the Hilbert polynomial $H_V(m)$, we end up with the following two possibilities: either $n = 4$, $d = 36$, $q_1 = q_2 = q_3 = 0$, $q_4 = 45$, $q_5 = 270$, or $n = 5$, $d = 45$, $q_1 = q_2 = q_3 = q_4 = q_5 = 0$, $q_6 = 39$, $q_7 = 270$.

Remark 45. Let W be a G-subrepresentation in $H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(m)\big)$. Then there is a natural map of G-representations $\psi\colon U^\vee \otimes W \to H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(m+1)\big)$, which is obviously a non-zero map.

Let us suppose that $n = 4$. Then $q_4 = 45$, so that by Lemma 17 there is an irreducible 45-dimensional G-subrepresentation $U_{45} \subset H^0(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(4))$. Thus, there is a morphism of G-representations $\psi_5\colon U^\vee \otimes U_{45} \to H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(5)\big)$, which is a non-zero map by Remark 45. On the other hand, $H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(5)\big)$ has no 180-dimensional G-subrepresentations by Lemma 17. Put $q_5' = \dim(\mathrm{Im}\psi_5)$. Keeping in mind the splitting of $U^\vee \otimes U_{45}$ described in Lemma 18, we see that q_5' equals either 90, or 135, or 225. Since $q_5 = 270$, there must exist a (possibly reducible) G-subrepresentation of $H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(5)\big)$ of dimension $q_5 - q_5'$, i.e., of dimension 180, 135 and 45, respectively. Neither of these cases is possible by Lemma 17 (in particular, the second case is impossible since $H^0\big(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(5)\big)$ contains a unique G-invariant subspace of dimension 135). Therefore, one has $n \neq 4$.

Finally, we see that $n = 5$. Then $q_6 = 39$, so that by Lemma 17 there is an irreducible 24-dimensional G-subrepresentation $U_{24} \subset H^0(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(6))$. Therefore there is a morphism of G-representations $\psi_7 \colon U^\vee \otimes U_{24} \to H^0(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(7))$, which is a non-zero map by Remark 45. Since $U^\vee \otimes U_{24}$ is an irreducible 216-dimensional G-representation by Lemma 19, we see that ψ_7 is injective. Since $q_7 = 270$, there must exist a (possibly reducible) G-subrepresentation of $H^0(\mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^8}(7))$ of dimension $270 - 216 = 54$, which is impossible by Lemma 17. The obtained contradiction completes the proof of Theorem 6.

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