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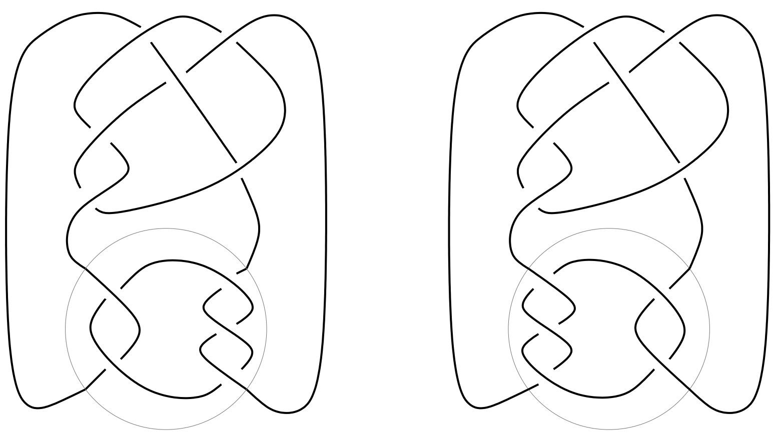
Conway mutation and alternating links

Joshua Evan Greene

Abstract. This paper is a conversational companion to *Lattices, graphs, and Conway mutation* [Gre11], on which I spoke at the 2011 Gökova Geometry/Topology Conference.

1. Introduction.

In his enumeration of small-crossing knots, Conway observed an interesting relationship between a particular pair of knot diagrams [Con70].



At left is the Kinoshita-Terasaka knot, and at right the one that now bears Conway's name. We get from one to the other by (a) excising the indicated disk, (b) rotating it by an angle π about a perpendicular axis piercing its center, and (c) reinserting it. Note that in place of (b) we could have instead rotated the disk about its horizontal or vertical axis to effect a transformation of diagrams; in the first case, the diagrams stay the same, and in the second case, they interchange. This operation is called an *elementary mutation*, and a pair of diagrams are called *mutants* if they are related by a sequence of isotopies and elementary mutations. In less diagrammatic terms, we have a sphere S^2 that meets a link $L \subset S^3$ transversely in four points, which we cut along and reglue by an involution

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that fixes a pair of points disjoint from L and permutes $S^2 \cap L$. Doing so results in a new link $L' \subset S^3$, and a pair of *links* are called mutants if they are related by a sequence of isotopies and such transformations.

Mutant links are notoriously difficult to distinguish, since many invariants take the same values on them. Such invariants include the HOMFLY polynomial (and so the Alexander and Jones polynomials) and the knot signature (see [Lic97]). Other invariants do detect mutation, such as the Seifert genus, which Gabai calculated as 2 for the Kinoshita-Terasaka knot and 3 for the Conway knot [Gab86]. It is always an interesting problem to determine whether a given link invariant detects mutation.

The principal link invariant of interest in this paper is the homeomorphism type of the double-cover of S^3 branched along the link L. This space is called the *branched double cover* of L and is denoted by $\Sigma(L)$. This invariant falls into the category of those insensitive to mutation, as recorded in the following Proposition.

Proposition 1.1 (Viro [Vir76])**.** *If L and L' are mutant links, then $\Sigma(L) \cong \Sigma(L')$.*

Amusingly, Viro was unaware of Conway's work at the time he established this result!

Sketch of proof. It suffices to consider a pair of links L, L' related by mutation in a single Conway sphere S^2. Let τ denote the involution of S^2 relating L and L'. In both $\Sigma(L)$ and $\Sigma(L')$, the Conway sphere lifts to a torus T^2 which splits the space into two pieces whose boundaries get identified by an appropriate gluing map. The two pieces are the same for both $\Sigma(L)$ and $\Sigma(L')$, but the gluing maps differ by the lift of τ to the torus T^2. This lift is conjugate to a translation of T^2 (namely, one of the three translations of order two), so it is isotopic to the identity map. Hence the gluing maps are isotopic, and so $\Sigma(L) \cong \Sigma(L')$. □

Thus, the homeomorphism type of $\Sigma(L)$ is more properly an invariant of the *mutation type of L*. From this perspective, we may ask how well $\Sigma(L)$ (or, indeed, any invariant of mutation type) distinguishes *between* mutation types. In general, $\Sigma(L)$ does not *completely* distinguish between mutation types. For example, the Brieskorn sphere $\Sigma(2,3,7)$ is the branched double cover of both the torus knot $T(3,7)$ and the pretzel knot $P(-2,3,7)$. These knots are not mutants, since their respective Jones polynomials, $-q^{14}+q^8+q^6$ and $-q^{13}+q^{12}-q^{11}+q^7+q^5$, distinguish their mutation types. In fact, it remains a fascinating open problem to describe all pairs of links that have homeomorphic branched double covers. Many interesting constructions and partial results exist, yet no guiding conjecture has emerged.

However, for two-bridge links, $\Sigma(L)$ *is* a complete invariant of the mutation type and, moveover, the isotopy type. The essential reason is that an invariant of manifolds, the *Reidemeister-Franz torsion*, provides a complete invariant of the homeomorphism type of lens spaces. Indeed, two-bridge links and lens spaces are parametrized by pairs of relatively prime integers $p > q > 0$. The two-bridge link $S(p,q)$ has branched double cover $L(p,q)$, and there exists an isotopy $S(p,q) \simeq S(p,q')$ whenever $qq' \equiv 1 \pmod{p}$. Using the torsion invariant, one shows that $L(p,q) \cong L(p',q')$ iff $p = p'$ and either

$q = q'$ or $qq' \equiv 1 \pmod p$. This gives the stated result. Later, Bonahon and Hodgson-Rubinstein established the stronger result that a lens space admits a unique genus one Heegaard splitting up to isotopy. Furthermore, Hodgson-Rubinstein showed that if $\Sigma(L)$ is homeomorphic to a lens space, then L is isotopic to a two-bridge link.

Now, there is a mantram in knot theory which goes as follows.

Mantram 1.2. Generalize all questions and results about two-bridge links to alternating links.

Thus, in the present setting, we are led to ask whether $\Sigma(L)$ is a complete invariant of the mutation type of an alternating link, and whether the Reidemeister-Franz torsion distinguishes the homeomorphism types of these spaces. It turns out that there is a related invariant of 3-manifolds which is easier to manipulate for this broader class of manifolds: the *d-invariant in Heegaard Floer homology.* For the manifolds under consideration, the d-invariant recovers the torsion invariant.

Our main result states that, in fact, within the family of alternating links, the homeomorphism type of $\Sigma(L)$ is a complete invariant of the mutation type. Furthermore, the d-invariant is a complete invariant of the homeomorphism types of the spaces $\Sigma(L)$.

Theorem 1.3. *Given a pair of connected, reduced alternating diagrams D, D' for a pair of links L, L', the following assertions are equivalent:*

(1) *D and D' are mutants;*
(2) *L and L' are mutants;*
(3) *$\Sigma(L) \cong \Sigma(L')$; and*
(4) *the d-invariants of $\Sigma(L)$ and $\Sigma(L')$ are the same.*

(The equivalence (1) $\Longleftrightarrow$ (2) is originally due to Menasco [Men84].)

One consequence of Theorem 1.3 is that an alternating link without an essential Conway sphere has a unique reduced, alternating diagram up to isotopy. This, of course, follows from the Menasco-Thistlethwaite theorem (formerly, the Tait flyping conjecture), but our argument in this case is substantially different. In fact, Theorem 1.3 motivates the following question.

Question 1.4. Is there a natural Floer-theoretic invariant that distinguishes isotopy types of alternating links?

We emphasize that Theorem 1.3 says *nothing* about the case of a non-alternating link. In fact, we make the following conjecture.

Conjecture 1.5. *If a pair of links have homeomorphic branched double-covers, then either both are alternating or both are non-alternating.*

The Hodgson-Rubinstein result implies the validity of Conjecture 1.5 in the case that one of the links is a two-bridge link. Besides that result, our strongest evidence in support of Conjecture 1.5 is the lack of any counterexample! If true, its proof would undoubtedly require techniques outside the scope of Heegaard Floer homology. Indeed, there exist infinitely many examples of prime links L, L' where L is alternating and L' is non-alternating, yet $\Sigma(L)$ and $\Sigma(L')$ have identical Floer invariants.

In the next section, we describe the d-invariant and how it is calculated for the manifolds of interest. We close this Introduction by remarking that we do not know how to establish the equivalence (2) $\iff$ (3) of Theorem 1.3 without the use of the d-invariant. Is there some more topological argument, a là Bonahon and Hodgson-Rubinstein, that could establish it, and perhaps resolve Conjecture 1.5 as well?

2. The d-invariant.

2.1. The input from Floer theory.

All the major definitions and results in this section are due to Ozsváth-Szabó. Throughout, let Y denote a closed, oriented, rational homology sphere. The d-invariant of Y is an invariant derived from the Heegaard Floer homology of Y. Since its introduction, it has played an instrumental role in many applications of Heegaard Floer homology to low-dimensional topology, including problems pertaining to knot concordance, Dehn surgery, the unknotting number, and, as we have here, a classification result.

In short, it is a mapping $d : \mathrm{Spin}^c(Y) \to \mathbb{Q}$, where $\mathrm{Spin}^c(Y)$ denotes the set of spin^c structures on Y. The set of spin^c structures on a 3- or 4-manifold forms a torsor over the integral second cohomology group, and we shall see that they are quite easy to manipulate, if a little tricky to define (as we shall avoid doing). Spin^c structures occur naturally in Heegaard Floer homology. The hat version of this theory assigns to a pair $(Y, \mathfrak{t})$, $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, a finite-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$ denoted $\widehat{HF}(Y, \mathfrak{t})$. This group is graded by rational numbers, and the invariant $d(Y, \mathfrak{t}) \in \mathbb{Q}$ records a distinguished grading on this group.

For us, the most important property of the d-invariant is its relationship with the four-dimensional theory. Suppose that X is a smooth manifold with boundary Y and a positive[1] definite intersection pairing. Suppose, furthermore, that $H_1(X; \mathbb{Z}) = 0$, so that every spin^c structure on Y extends to one on X (by analogy to the case for cohomology). Then, for all $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, we have an inequality

$$d(Y, \mathfrak{t}) \leq \min \left\{ \frac{c_1(\mathfrak{s})^2 - b_2(X)}{4} \;\middle|\; \mathfrak{s} \in \mathrm{Spin}^c(X),\, \mathfrak{s}|Y = \mathfrak{t} \right\}, \tag{1}$$

where we make use of the *first Chern class* mapping $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$. The group $H^2(X; \mathbb{Z})$ comes equipped with a symmetric, bilinear, $\mathbb{Q}$-valued pairing induced by the cup product, thereby making sense out of the term $c_1(\mathfrak{s})^2$ (much more on this in Subsection 2.3).

We call the manifold X *sharp* when inequality (1) is an equality for every $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. Sharp 4-manifolds are a prized possession of anyone attempting to calculate the d-invariant. The d-invariant of a space Y is typically quite complicated to compute, but when Y is

[1] Practitioners of Floer homology typically orient their definite manifolds to be *negative* definite, but as we shall focus on the resulting algebraic/combinatorial story, we prefer positive definite ones.

known to bound a sharp 4-manifold X, we can extract it in terms that depend solely on the intersection pairing on $H_2(X;\mathbb{Z})$.

2.2. Branched double covers and graph lattices.

As luck would have it, the branched double cover of a non-split alternating link L bounds a very natural sharp 4-manifold. Begin with an alternating diagram D of L. The diagram splits the plane into regions, which we may color black and white in checkerboard fashion (see Figure 1 in page 38). The union of the black regions, together with a half-twist at each crossing, produces a (typically non-orientable) spanning surface F with boundary L. Place the link $L \subset S^3$ in the boundary of a 4-ball D^4, and push the interior of F slightly into the interior of D^4. Then the double cover of D^4, branched along the pushed-in copy of F, is a sharp 4-manifold X_D with boundary $\Sigma(L)$. (We are cheating a little bit on orientations here, but never mind).

The intersection pairing on $H_2(X_D)$ has a very nice description in terms of the combinatorics of D. Construct a planar graph G by placing a vertex in each white region and an edge between a pair of vertices whenever their regions touch at a crossing (Figure 2). This graph is called the *Tait graph* of the diagram D, and the intersection pairing on $H_2(X_D)$ is captured by the *lattice of flows* on the Tait graph, $\mathcal{F}(G)$.

Let us elaborate on what this means. Given a graph G, we orient its vertices and edges arbitrarily so as to give it the structure of a 1-dimensional CW complex. Letting $e_1, \ldots, e_m$ denote its oriented edges, we give the chain group $C_1(G;\mathbb{Z})$ the structure of a lattice by declaring that these edges form an orthonormal basis for it. Now the cycle space $Z_1(G;\mathbb{Z}) = \ker(\partial_1 : C_1(G;\mathbb{Z}) \to C_0(G;\mathbb{Z}))$ inherits the structure of a sublattice of $C_1(G;\mathbb{Z})$, and its isomorphism type is the lattice of flows $\mathcal{F}(G)$. We point out that an alternate choice of orientation on the vertices and edges of G does not affect the isomorphism type of $\mathcal{F}(G)$.

The lattice of flows on a graph has a natural companion called the *lattice of cuts* $\mathcal{C}(G)$. This is the sublattice of $C_1(G;\mathbb{Z})$ arising as the image of the adjoint mapping $\partial_1^* : C_0(G;\mathbb{Z}) \to C_1(G;\mathbb{Z})$. Together, $\mathcal{F}(G)$ and $\mathcal{C}(G)$ form a pair of complementary sublattices of $C_1(G;\mathbb{Z})$ that interact in a very nice way that we utilize in Subsection 3.2.

2.3. Lattices and intersection pairings.

Returning to our earlier level of generality, let X denote a smooth, compact 4-manifold with $H_1(X;\mathbb{Z}) = 0$ and boundary a rational homology sphere Y. The intersection pairing on homology gives the group $H_2(X;\mathbb{Z})$ the structure of an *integral lattice*, that is, a free abelian group equipped with a symmetric, bilinear, non-degenerate, integer-valued pairing. A portion of the long exact sequence in integral cohomology reads

$$0 \to H_2(X) \to H^2(X) \to H^2(Y) \to 0.$$

Here we have substituted the term $H_2(X)$ for $H^2(X, \partial X)$ using Poincaré duality. The Hom-pairing endows the group $H^2(X)$ with the structure of the *dual* lattice to $H_2(X)$, so endowing it with a symmetric, bilinear, *rational*-valued pairing. The quotient of $H^2(X)$

by $H_2(X)$ is isomorphic to the finite, abelian group $H^2(Y)$. In purely lattice-theoretic terms, the set-up is modeled by the sequence

$$0 \to \Lambda \to \Lambda^* \to \overline{\Lambda} \to 0,$$

where Λ denotes an integral lattice, Λ^* the dual lattice $\{\lambda^* \in \Lambda \otimes \mathbb{Q} \mid \langle \lambda^*, \lambda \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\}$, and $\overline{\Lambda}$ the *discriminant group* (or *dual quotient group*) Λ^*/Λ.

To pull spin^c structures into the picture, we make two constructions. The dual lattice Λ^* contains a distinguished subset

$$\mathrm{Char}(\Lambda) = \{\chi \in \Lambda^* \mid \langle \chi, \lambda \rangle \equiv |\lambda| \,(\mathrm{mod}\ 2),\ \forall\, \lambda \in \Lambda\},$$

the set of *characteristic elements* for Λ. In terms of a fixed basis $\{\lambda_1, \dots, \lambda_n\}$ for Λ, this set is expressed with respect to the dual basis as $\{\sum \chi_i \lambda_i^* \in \Lambda^* \mid \chi_i \equiv |\lambda_i| \,(\mathrm{mod}\ 2), \forall i\}$. We see that $\mathrm{Char}(\Lambda)$ is a coset of $2\Lambda^*$ in Λ^*, so comes with a natural free, transitive action by Λ^*: an element λ^* shifts χ by *twice* λ^*. In other words, $\mathrm{Char}(\Lambda)$ forms a torsor over Λ^*. Lastly, we define

$$C(\Lambda) = \mathrm{Char}(\Lambda) \,(\mathrm{mod}\ 2\Lambda).$$

This set forms a torsor over $\overline{\Lambda}$.

The significance of these constructions in the present setting is as follows. The first Chern class mapping sets up a one-to-one correspondence

$$c_1 : \mathrm{Spin}^c(X) \xrightarrow{\sim} \mathrm{Char}(\Lambda) \subset \Lambda^*,$$

where $\Lambda = H_2(X)$. Furthermore, there is a correspondence $C(\Lambda) \to H^2(Y)$, under which the restriction mapping $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ is the reduction $\mathrm{Char}(\Lambda) \to C(\Lambda)$. Furthermore, the torsor structure $\mathrm{Spin}^c(X)/H^2(X)$ is modeled by $\mathrm{Char}(\Lambda)/\Lambda^*$, and $\mathrm{Spin}^c(Y)/H^2(Y)$ is modeled by $C(\Lambda)/\overline{\Lambda}$.

Example. Let X_k denote the D^2-bundle over S^2 with Euler number $k \neq 0$. Thus, $H_1(X_k;\mathbb{Z}) = 0$, and $\partial X_k \cong -L(k,1)$, a lens space. Write Λ_k for the lattice $H_2(X_k)$, so Λ_k is generated by a single element λ with norm k. It follows that Λ^* is generated by a single element λ^* such that $\langle \lambda^*, \lambda \rangle = 1$. This element has norm $1/k$. We have $\mathrm{Char}(\Lambda_k) = \{j\lambda^* \mid j \equiv k \,(\mathrm{mod}\ 2)\}$. The classes in $C(\Lambda)$ take the form $(j + 2k\mathbb{Z})\lambda^*$ for the k different values $j \,(\mathrm{mod}\ 2k)$, $j \equiv k \,(\mathrm{mod}\ 2)$. This set forms a torsor over the discriminant group $\overline{\Lambda_k} \cong \mathbb{Z}/k\mathbb{Z}$.

Now suppose that the pairing on Λ is positive definite. Given a class $[\chi] \in C(\Lambda)$, we define

$$d_\Lambda([\chi]) = \min \left\{ \frac{|\chi'| - \mathrm{rk}(\Lambda)}{4} \;\middle|\; \chi' \in [\chi] \right\}.$$

Thus, we obtain a mapping $d_\Lambda : C(\Lambda) \to \mathbb{Q}$, the *$d$-invariant of the lattice* Λ. This definition is obviously modeled on the d-invariant in Heegaard Floer homology. Let Λ be $H_2(X)$, then the correspondence $C(\Lambda) \xrightarrow{\sim} \mathrm{Spin}^c(Y)$, $[\chi] \mapsto \mathfrak{t}$ gives us $d_\Lambda([\chi]) \leq d(Y,\mathfrak{t})$. Furthermore, if X is sharp, then $d_\Lambda([\chi]) = d(Y,\mathfrak{t})$, for all $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. Based on this definition, we call an element $\chi \in \mathrm{Char}(\Lambda)$ *short* if its norm is minimal in $[\chi]$, and denote the set of short elements by $\mathrm{Short}(\Lambda)$.

Example (continued). We are going to calculate the d-invariant of the lens spaces $-L(k,1)$, $k \geq 1$. When $k \geq 1$, the lattice Λ_k is positive definite. Each class in $C(\Lambda_k)$ contains a unique representative $j\lambda^*$ with $|j| \leq k$, except for the class containing $\pm k\lambda^*$, which contains these two. Altogether, these $k+1$ elements constitute $\mathrm{Short}(\Lambda_k)$, and it follows that the d-invariant of Λ takes the values $(j^2/k-1)/4$ on these respective classes. The manifold X_k is sharp for all $k \geq 1$, so under the correspondence between $\mathrm{Spin}^c(-L(k,1))$ and $\{-k < j \leq k,\, j \equiv k \,(\mathrm{mod}\ 2)\}$, we have $d(-L(k,1),j) = (j^2/k-1)/4$.

We conclude this subsection on lattices with a very important fact. Note that for the case of the lattice $\mathbb{Z} = \Lambda_1$, the d-invariant takes the value 0 on the unique class in $C(\mathbb{Z})$. This is the case as well for the lattice $\mathbb{Z}^n$, which has an orthonormal basis $\{e_1, \ldots, e_n\}$: in this case, $\mathrm{Char}(\mathbb{Z}^n) = \{\sum \chi_i e_i \,|\, \chi_i \equiv 1 \,(\mathrm{mod}\ 2),\, \forall i\}$, and $\mathrm{Short}(\mathbb{Z}^n)$ consists of $\{\sum \chi_i e_i \,|\, |\chi_i| = 1,\, \forall i\}$, so indeed $d_{\mathbb{Z}^n}([\chi]) = (n-n)/4 = 0$. A powerful result due to Elkies asserts the converse to this observation for the case of a unimodular lattice Λ (i.e., $\Lambda = \Lambda^*$).

Theorem 2.1 (Elkies [Elk95]). *If Λ is a rank n, unimodular, integral, positive definite lattice with d-invariant 0, then $\Lambda \cong \mathbb{Z}^n$, i.e., Λ admits an orthonormal basis.* □

3. Diagrams, graphs, and lattices.

The groundwork laid over the course of the preceding section can be summarized in the following result.

Theorem 3.1 (Ozsváth-Szabó [OSz05], Thm. 3.4). *Let L denote a non-split alternating link and G the Tait graph of an alternating diagram of L. Then the d-invariant of $\Sigma(L)$ is calculated as* minus *the d-invariant of $\mathcal{F}(G)$.* □

Where does this result put us? We would like to show that the d-invariant of $\Sigma(L)$ determines the mutation type of a reduced, alternating diagram D for L. In light of Theorem 3.1, our task now is to first recast mutation of alternating diagrams in terms of some equivalence relation on graphs, and then to characterize the equivalence classes under this relation in terms of the d-invariant on the flow lattice.

3.1. Mutation and 2-isomorphism.

Let us consider the effect that mutation has on the Tait graph. Figure 1 displays four different link diagrams, each pair of which are related by an appropriate mutation in the indicated disk. Figure 2 displays their Tait graphs, which are distinctly embedded planar graphs. However, both the top two and the bottom two diagrams give rise to isomorphic Tait graphs as *abstract* graphs. We say that their two planar embeddings differ by a *flip*. This happens when a circle in the plane meets the planar graph G in a two-vertex cutset, and we simply reflect the embedding of G interior to the disk it bounds. The Tait graphs top and bottom are actually distinct from one another. They are related by a *planar switch*. This happens when a circle in the plane meets a planar graph G in a two-vertex cutset, and we rotate its interior by 180° (possibly composing with a reflection).

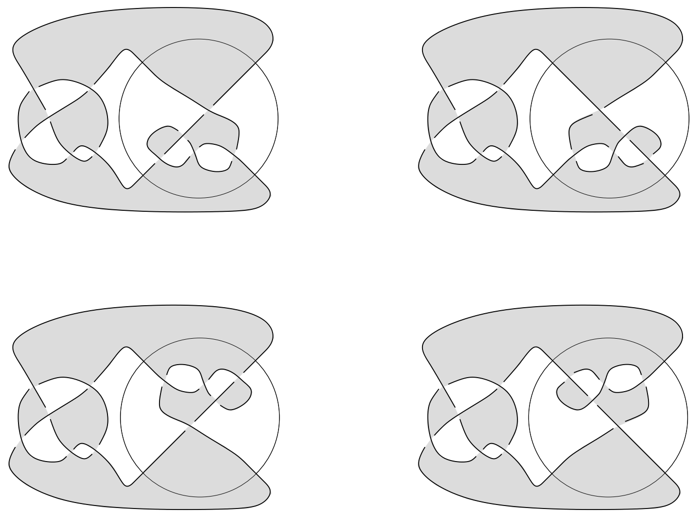

FIGURE 1. Four alternating link diagrams related by mutation.

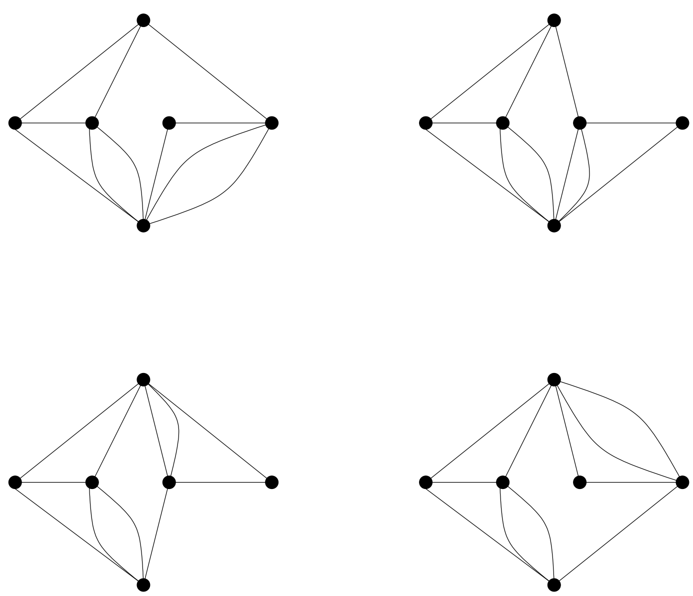

FIGURE 2. Four Tait graphs related by flips and switches.

Thus, mutant diagrams give rise to embedded Tait graphs that are related by a sequence of flips and planar switches.

At the level of abstract graphs, a planar switch is an instance of an *abstract switch*. This happens when we have a pair of graphs H_1, H_2, each with a pair of distinguished vertices $v_i, w_i \in V(H_i)$, $i = 1, 2$. We can glue together the vertices between the pairs $\{v_1, w_1\}$ and $\{v_2, w_2\}$ in either of two ways to form two graphs G, G', and these two are said to differ by a switch. A switch between a pair of graphs G and G' gives rise to a bijection between the edge sets $E(G) \xrightarrow{\sim} E(G')$. This bijection is far from arbitrary: it preserves edge sets of cycles, as one can check in Figure 2. A cycle-preserving bijection between the edge sets of a pair of graphs is called a *2-isomorphism*.

Now, a fundamental result due to Whitney asserts that a 2-isomorphism between a pair of abstract graphs is a composition of switches. Another fundamental result (essentially) due to Whitney asserts that any two planar drawings of a planar graph are related by a sequence of flips. Collecting these facts with the preceding observations, we have the following graph-theoretic characterization of mutation type.

Theorem 3.2 (Whitney $+\,\epsilon$ [Whi32, Whi33])**.** *The Tait graph construction establishes a 1-1 correspondence*

$$\left\{ \frac{\text{alternating diagrams}}{\text{mutation}} \right\} \xrightarrow{\sim} \left\{ \frac{\text{planar graphs}}{\text{2-isomorphism}} \right\}.$$

(More precisely, the alternating diagrams should be connected and reduced, and the planar graphs should be 2-connected.)

3.2. 2-isomorphism and the d-invariant.

In view of Theorem 3.2, we now want to say that the d-invariant of the lattice $\mathcal{F}(G)$ determines the 2-isomorphism type of G. This is the content of the following result.

Theorem 3.3. *The following are equivalent for a pair of 2-edge-connected graphs G, G':*

(1) *G and G' are 2-isomorphic;*
(2) *$\mathcal{F}(G) \cong \mathcal{F}(G')$; and*
(3) *$\mathcal{F}(G)$ and $\mathcal{F}(G')$ have isomorphic d-invariants.*

Equipped with Theorem 3.3, we can quickly wrap up the proof of Theorem 1.3 as follows. Given a pair of reduced, alternating diagrams D, D' representing links L, L' such that $\Sigma(L)$ and $\Sigma(L')$ have the same d-invariant, the corresponding Tait graphs G, G' are 2-isomorphic by Theorems 3.1 and 3.3, and so we conclude that the diagrams D, D' are mutants by Theorem 3.2.

Thus, Theorem 3.3 is the combinatorial heart of the matter. There are three principal ingredients that go into its proof: the *lattice gluing* construction, Elkies's Theorem 2.1, and the complementary nature of the lattices $\mathcal{F}(G)$ and $\mathcal{C}(G)$.

The instance of lattice gluing that we utilize goes as follows. Suppose that we have an isomorphism of discriminant groups $\varphi : \overline{\Lambda_1} \xrightarrow{\sim} \overline{\Lambda_2}$, where Λ_1, Λ_2 denote a pair of integral lattices. Then we can form a new lattice

$$\Lambda = \Lambda_1 \oplus_\varphi \Lambda_2 := \{\lambda_1^* + \lambda_2^* \in \Lambda_1^* \oplus \Lambda_2^* \,|\, \varphi(\overline{\lambda_1^*}) + \overline{\lambda_2^*} = 0\}.$$

The glue lattice Λ is integral and unimodular, it contains $\Lambda_1 \oplus \Lambda_2$ as a sublattice, and it will be definite provided that Λ_1 and Λ_2 are.

If we ask for more than just an isomorphism φ, then we get a corresponding payoff. Thus, suppose that Λ_1 and Λ_2 are definite lattices, and we have an isomorphism of torsors $C(\Lambda_1) \xrightarrow{\sim} C(\Lambda_2)$ that carries d_{Λ_1} into $-d_{\Lambda_2}$. Then we can carry out the above construction, and in this case the glue lattice Λ that results will have a vanishing d-invariant: hence $\Lambda \cong \mathbb{Z}^n$, thanks to Elkies's Theorem! Furthermore, it turns out that every $\chi_1 \in \mathrm{Short}(\Lambda_1)$ adds with some $\chi_2 \in \mathrm{Short}(\Lambda_2)$ to produce $\chi_1 + \chi_2 \in \mathrm{Short}(\mathbb{Z}^n)$.

The setup of the preceding paragraph applies, in particular, to the pair of lattices $\mathcal{F}(G)$ and $\mathcal{C}(G)$. (This is the nice interaction between these two to which we alluded at the end of Subsection 2.2.) But gluing them together just results in the lattice $C_1(G;\mathbb{Z})$ with $\mathcal{F}(G) \oplus \mathcal{C}(G)$ embedded standardly within it, so we have not really learned anything.

But now let us suppose that G and G' are graphs for which $\mathcal{F}(G)$ and $\mathcal{F}(G')$ have the same d-invariant. Then it follows that the preceding setup applies to the pair of lattices $\mathcal{F}(G)$ and $\mathcal{C}(G')$, and we can apply the gluing construction to *this* pair of lattices and actually get somewhere. Thus, we get an embedding $\mathcal{F}(G) \oplus \mathcal{C}(G') \hookrightarrow \mathbb{Z}^n$ with the property every element in $\mathrm{Short}(\mathcal{F}(G))$ adds with some element in $\mathrm{Short}(\mathcal{C}(G'))$ to give an element in $\mathrm{Short}(\mathbb{Z}^n)$. A subtle combinatorial argument then implies that the embedding $\mathcal{F}(G) \hookrightarrow \mathbb{Z}^n$ factors as the composite $\mathcal{F}(G) \hookrightarrow C_1(G;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^n$ for a suitable isomorphism between the latter, and similarly for $\mathcal{C}(G')$ [Gre11, Prop.2.8]. Now the isomorphisms $C_1(G;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^n \xleftarrow{\sim} C_1(G';\mathbb{Z})$ set up a bijection between the edge sets of G and G'. As one might guess, this bijection establishes the desired 2-isomorphism, and arguing that it is a 2-isomorphism is actually quite easy!

This sketch establishes the implication (3) $\Longrightarrow$ (1) of Theorem 3.3, which suffices for the desired application to Theorem 1.3. We point out that the implication (2) $\Longrightarrow$ (3) is immediate, and that (1) $\Longrightarrow$ (2) is a nice exercise (see [BdlHN97, Prop.5]).

4. Conclusion.

The proof sketch we have given of Theorem 1.3 seems tailor-made to the case of alternating links. Is there any natural, broader class of links to which it applies? It seems essential that the spaces $\Sigma(L)$ bound a sharp 4-manifold with both orientations. This motivates a question.

Question 4.1. Suppose that Y is a rational homology sphere, and Y bounds a sharp 4-manifold with both orientations. Does it follow that $Y \cong \Sigma(L)$ for some alternating link L?

If this were the case, and in addition Conjecture 1.5 were true, then we would obtain a non-diagrammatic characterization of alternating links, albeit in very round-about terms.

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Department of Mathematics, Boston College, Chestnut Hill, MA 02467
E-mail address: `joshua.greene@bc.edu`