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Lefschetz fibrations on cotangent bundles of two-manifolds

Joe Johns

Abstract. In this paper our aim is to explain an explicit and relatively simple construction of some symplectic Lefschetz fibrations on the disk cotangent bundle of an arbitrary compact two dimensional manifold. Morally, the construction is inspired by the idea of complexifying a given Morse function on the 2-manifold. The paper is meant to give a treatment of a simple case in [J09], with extra focus on examples and concrete constructions.

1. Introduction

Let N denote a compact two dimensional manifold without boundary. Let $D(T^*N)$ denote the closed unit disk bundle of the cotangent bundle of N with respect to some metric on the bundle T^*N. We equip $D(T^*N)$ with the canonical 1-form θ which makes it into an exact symplectic manifold. The goal of this paper is to construct a Lefschetz fibration

$$\pi : D(T^*N) \longrightarrow D^2$$

with an explicit description of the regular fiber M and the vanishing cycles $V_1, \dots, V_k$ in M.

The basic motivation is that we obtain an explicit presentation for the $4-$manifold $D(T^*N)$ in terms of the regular fiber M (which is a 2-manifold with boundary) and the vanishing cycles $V_1, \dots, V_k$ (which are some circles in M). For the reader unfamiliar with these things, we explain in §2 the definition of a Lefschetz fibration, the regular fiber, and the vanishing cycles; and in §3 we explain two examples in some detail.

One way we might try to proceed is to start with a Morse function $f : N \longrightarrow \mathbb{R}$ and then "complexify" it in some sense to get a Lefschetz fibration $f_{\mathbb{C}} : D(T^*N) \longrightarrow \mathbb{C}$. For example, if $N = \mathbb{R}^2$, and $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a real polynomial which is a Morse function, then the obvious extension $p_{\mathbb{C}} : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a Lefschetz fibration on $\mathbb{C}^2 = T^*(\mathbb{R}^2)$ which we call the complexification of p. In this case, there is indeed a beautiful explicit description of the regular fiber and vanishing cycles of $p_{\mathbb{C}}$, given by A'Campo in [AC99].

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This suggests a two step approach:

(i) Choose a Morse function $f : N \longrightarrow \mathbb{R}$, and complexify f in some sense to get a Lefschetz fibration $f_{\mathbb{C}} : D(T^*N) \longrightarrow \mathbb{C}$.

(ii) Describe the fiber M and vanishing cycles $V_1, \ldots, V_k$ of $f_{\mathbb{C}}$ in terms of the Morse theory of f.

Unfortunately, both steps (i) and (ii) seem to be difficult to carry out if N is an arbitrary 2-manifold. In step (i), it is difficult to define the "complexification" $f_{\mathbb{C}}$ so as to obtain Lefschetz fibration on $D(T^*N)$. In step (ii), even if we succeed in step (i) in some cases, it is not obvious how to describe the fiber and vanishing cycles explicitly.

In this paper, we will therefore take slightly different approach. Namely, we carry out the following four steps.

(1) We choose a Morse function f on N and a metric g such that (f, g) is Morse-Smale (which means the stable and unstable manifolds of $-\nabla_g f$ intersect transversely). Then we take the handle decomposition of N induced by (f, g).

(2) Given the handle decomposition of N, we give an explicit construction of an exact symplectic 2-manifold manifold M with contact boundary, and we specify some exact Lagrangian spheres (circles) $V_1, \ldots, V_k \subset M$. Here, there is one V_j for each critical point of f.

(3) We make use of a well-known construction which allows us to produce an exact symplectic Lefschetz fibration with any prescribed regular fiber and vanishing cycles (see Theorem 2.2). In this case we obtain a Lefschetz fibration

$$\pi : E \longrightarrow D^2$$

with regular fiber equal to M and with vanishing cycles $V_1, \ldots, V_k$. Here, E is some exact symplectic $4-$manifold with codimension 2 corners, which is determined by the construction.

(4) We prove that E is conformally exact symplectomorphic to the disk cotangent bundle $D(T^*N)$ (after we smooth the corners of E). See Theorem 1.1 below.

Thus, the key idea is to make an educated guess in step (2) for what the regular fiber M and vanishing cycles $V_1, \ldots, V_k$ ought to be. This guess is then verified to be correct in step (4). The source of inspiration for the construction in step (2) is A'Campo's paper [AC99] which we mentioned earlier, about complexifications of real polynomials.

The more precise result corresponding to step (4) is the following Theorem; the proof is sketched in §6.

Theorem 1.1. *Let N be a closed 2-manifold, equipped with a Morse function $f : N \longrightarrow \mathbb{R}$ and let g be a metric such that (f, g) is Morse-Smale. Let $\pi : E \longrightarrow D^2$ be a Lefschetz fibration constructed according to steps (1) to (3) above. Then (E, π) has the following properties:*

- *There is an exact Lagrangian embedding* $N \subset E$.
- $Crit(\pi) \subset N$, $\pi(N) \subset \mathbb{R}$, *and* $\pi|N = f : N \longrightarrow \mathbb{R}$ *(up to reparameterizing* N *and* $\mathbb{R}$ *by diffeomorphisms).*
- E *is conformally exact symplectomorphic to the disk cotangent bundle* $D(T^*N)$ *(after we smooth the corners of* E*).*

Remark 1.2. Steps (1) to (4), and Theorem 1.1, work perfectly well in the case where N has boundary, but to keep things simple we stick to the case where N is without boundary in this paper. But see §5.4 for some simple examples of how the construction works in the case where N has boundary.

Remark 1.3. We make some technical remarks. First, smoothing the corners of E is done in a standard way - see [S08] lemma 7.6. Second, we recall that a conformal exact symplectomorphism between exact symplectic manifolds is by definition a map

$$\phi : (E_1, \omega_1, \theta_1) \longrightarrow (E_2, \omega_2, \theta_2)$$

such that $\phi^*\theta_2 = \lambda\theta_1 + df$ for some smooth function $f : E_1 \longrightarrow \mathbb{R}$, and some $\lambda \in \mathbb{R}$ with $\lambda > 0$. (Here, $\omega_i = d\theta_i$ is an exact symplectic structure on E_i.) In particular $\phi^*\omega_2 = \lambda\omega_1$, so ϕ is a conformal symplectomorphism. The basic example of a conformal exact symplectomorphism is given by integrating the Liouville vector field of T^*N to give a map from a neighborhood of N in T^*N onto a smaller neighborhood of N in T^*N, with strictly less symplectic volume.

In some cases it *is* possible to carry out steps (i) and (ii), and in such cases it is natural to compare the result with the Lefschetz fibrations constructed as in steps (1) to (4). In §3 we carry out steps (i) and (ii) for $D(T^*\mathbb{R}P^2)$ and $D(T^*T^2)$ (in these examples, we can make use of the embeddings $D(T^*\mathbb{R}P^2) \subset \mathbb{C}P^2$ and $D(T^*T^2) \subset (\mathbb{C}^*)^2$). In §5 we explain how to construct the regular fiber vanishing cycles as in step (2) for $D(T^*T^2)$ and $D(T^*\mathbb{R}P^2)$. Interestingly, we find the *same* regular fiber and vanishing cycles that we got via steps (i) and (ii) for $D(T^*T^2)$ and $D(T^*\mathbb{R}P^2)$ in §3. In general, it remains an interesting problem to carry out steps (i) and (ii) for more general manifolds and compare the result to our construction in steps (1) to (4).

We conclude the introduction with a quick summary of the contents of the paper. Note that §2 and §3 are of a more pedagogical nature, and they play only a peripheral role in the rest of the paper.

In §2 we give the definition of a Lefschetz fibration, and review some of the basic theory. In §3 we explain two examples of Lefschetz fibrations, on $D(T^*\mathbb{R}P^2)$ and $D(T^*T^2)$, which come from algebraic geometry. In §4 we explain step (2) above. That is, we describe how to construct M and $V_1, \dots, V_k \subset M$ for an arbitrary closed 2-manifold N. We also specify the basis of vanishing paths we want to use. In §5 we illustrate how the construction of M and $V_1, \dots, V_k \subset M$ works in some examples, in particular, for the cases $N = \mathbb{R}P^2$ and $N = T^2$. In §6 we sketch the proof of Theorem 1.1.

2. A quick lesson on Lefschetz fibrations and Picard-Lefschetz theory

In this section we discuss some of the basic theory of exact symplectic Lefschetz fibrations. As general references we recommend [AGZV88, GS99, L81] for a classical point of view on Lefschetz fibrations and [S08, S03A, AS04] for a symplectic point of view. We remark that §2.8 describes the main result on constructing Lefschetz fibrations that will be used later in the paper.

2.1. The definition of an exact symplectic Lefschetz fibration

Let M be a manifold with boundary. An exact symplectic structure on M is a $1-$form θ such that $\omega = d\theta$ is a symplectic form; and in addition the Liouville vector field X_θ defined by $\omega(X_\theta, \cdot) = \theta$ must be transverse to the boundary of M and point outwards; this makes the boundary into a contact manifold, using θ as a contact form.

The simplest example of an exact symplectic Lefschetz fibration is a trivial fiber bundle

$$\pi : E = M \times D^2 \longrightarrow D^2$$

where the fiber is given by an exact symplectic manifold M, and the base $D^2 \subset \mathbb{C}$ has the standard exact symplectic structure. Roughly speaking, a nontrivial Lefschetz fibration is similar except it is allowed to have finitely many *singular fibers* of a certain type. That is, there are finitely many points $p_1, \ldots, p_k \in Int(E)$ which are isolated critical points of π of Morse type. We give the precise definition in moment. For now we just observe that the total space of an exact symplectic Lefschetz fibration is naturally a manifold with codimension 2 corners, because the fiber and base both have boundary.

Let E be a manifold with codimension 2 corners. An exact symplectic structure on E is a $1-$form θ such that $\omega = d\theta$ is a symplectic form, and such that the Liouville vector field X_θ is transverse to each boundary stratum of codimension 1, and points outwards; this makes each boundary stratum into a contact manifold, using θ as a contact form.

Let E be an exact symplectic manifold with codimension 2 corners. An exact symplectic Lefschetz fibration on E is a map

$$\pi : E \longrightarrow D^2$$

such that the following conditions are satisfied (following [S08]):

- There are finitely many points $p_1, \ldots, p_k \in Int(E)$ such that, for each p_j, there are complex coordinates $(z_1, \ldots, z_n)$ near p_j such that p_j corresponds to 0; the standard complex structure $J_0 = i$ on $\mathbb{C}^n$ is compatible with ω; and

$$\pi = z_1^2 + \ldots + z_n^2 + \pi(p_j)$$

in these coordinates. The points p_j are called the *singular points or critical points* of π; this type of singularity is called a Morse type singularity.

- For each $z \in D^2 \setminus \{c_1, \dots, c_k\}$, where $c_j = \pi(p_j)$, we require that $M_z = \pi^{-1}(z)$ is an exact symplectic submanifold of E, where $\theta_z = \theta|M_z$ makes it into an exact symplectic submanifold with boundary. For each c_j, $\pi^{-1}(c_j) \setminus \{p_j\}$ is a (noncompact) exact symplectic manifold with boundary.
- Set $\partial_v E = \pi^{-1}(\partial D^2)$ and let $\partial_h E$ denote the boundary of all the fibers (including singular ones). Then we require that $\partial E = \partial_h E \cup \partial_v E$, where ∂E has two boundary strata $\partial_h E$ (the horizontal part) and $\partial_v E$ (the vertical part) which meet at a codimension 2 corner. Furthermore, $\pi|\partial_h E \longrightarrow \partial D^2$ and $\pi|\partial_h E \longrightarrow D^2$ are required to be surjective fiber bundles.
- For each $x \in E$ consider the splitting
$$T_x E = T_x^h E \oplus T_x^v E,$$
where $T_x^v E = Ker(D\pi_x)$ and $T_x^h E$ is the symplectic complement in $T_x E$. This splitting always exists; our requirement is
$$T_x^h E = T_x(\partial_h E) \text{ for all } x \in \partial_h E.$$

2.2. Parallel transport

The purpose of the last condition in the definition of a Lefschetz fibration (we drop the adjective "exact symplectic" from now on) is to ensure that parallel transport is well-defined, as follows. Set $M_z = \pi^{-1}(z)$ for any $z \in D^2$. Given any two regular values $x, y \in D^2 \setminus \{c_1, \dots, c_k\}$, and any path $\gamma : [0, 1] \longrightarrow D^2 \setminus \{c_1, \dots, c_k\}$ from x to y which avoids the critical values, we define a *parallel transport map*

$$\tau_\gamma : M_x \longrightarrow M_y$$

by using the connection given by $T^h E$. The last condition ensures that the vector field given by the horizontal lifts of $\gamma'(t)$ for each t can be integrated for all $t \in [0, 1]$ and that τ_γ maps the boundaries of the fibers into themselves.

2.3. The regular fiber

Let us now give a more intuitive description of a Lefschetz fibration. As we saw above, if x, y are two regular values τ_γ gives an exact symplectic isomorphism from M_x to M_y. Thus, a Lefschetz fibration is, roughly speaking, a fiber-bundle with one common fiber M, except that finitely many of the fibers have isolated singular points (modeled locally by $z_1^2 + \dots + z_n^2 = 0$ in $\mathbb{C}^n$). Thus we speak of "the regular fiber" M, which is defined to be $M = \pi^{-1}(b)$ for some fixed regular value b.

Remark 2.1. The basic reason Lefschetz fibrations on E are useful for studying E is that they give rise to a dimensional reduction: One can shift focus from E to the regular fiber M which is two dimensions less (and then, potentially, one can repeat this process for M and so on ...). So, for example, if E is a 4-manifold, we can instead focus on a 2-manifold M. In principle, this philosophy may be applicable to many different questions about the symplectic topology of E, such as the symplectomorphism type (classification and construction of exotic structures, see [M09, MS09, AS10]), the symplectomorphism group of E and M (see [AMP05, S06]), and Lagrangian submanifolds of E and M (see [S03B, S08, FSS08]).

2.4. Vanishing paths

For the rest of this section we fix some notation, in order to discuss some general properties of Lefschetz fibrations.

- Let $\pi : E \longrightarrow D^2$ be a Lefschetz fibration.
- Let $p_1, \ldots, p_k \in E$ denote the critical points of π.
- Let $c_1, \ldots, c_k \in Int(D^2)$ denote the critical values of π.
- Fix a base point $b \in D^2 \setminus \{c_1, \ldots, c_k\}$.
- Set $M = \pi^{-1}(b)$, set $M_x = \pi^{-1}(x)$, for any $x \in D^2 \setminus \{c_1, \ldots, c_k\}$, and set $M_j = \pi^{-1}(c_j)$ for all j.
- For each j we pick a path $\gamma_j : [0,1] \longrightarrow D^2$ from b to c_j which avoids all other $c_i \neq c_j$. And assume that $\gamma_i((0,1]) \cap \gamma_j((0,1]) = \emptyset$ for all $i \neq j$.

If $(\gamma_1, \ldots, \gamma_k)$ is a collection of paths satisfying the last condition, we call it a *basis of vanishing paths* for (E, π).

2.5. Vanishing cycles and Lefschetz thimbles

Associated to each vanishing path γ_j there is an exact Lagrangian sphere $V_j = V_{\gamma_j}$ in the regular fiber M. This is called the *vanishing cycle*, or vanishing sphere, associated to γ_j. Roughly speaking, if we follow the parallel transport map along γ_j from M to the singular fiber M_j then V_j collapses down to a point in M_j, which is the singular point p_j. This gives a nice description of each singular fiber M_j.

More precisely, we have the following lemma (for a proof see [S03A] §1). Since $T_x^h E = 0$ for $x = p_j$, we first have to say what we mean by the transport map $\tau_{\gamma_j} : M \longrightarrow M_j$. We define

$$\tau_{\gamma_j}(x) = \lim_{t \longrightarrow 1} \tau_{\gamma_j|_{[0,t]}}(x).$$

Lemma 1. Let $\pi : E \longrightarrow D^2$ be a Lefschetz fibration with critical points $p_1, \ldots, p_k$. Let $(\gamma_1, \ldots, \gamma_k)$ be a basis of vanishing paths. Set $M = \pi^{-1}(b)$ and $M_j = \pi^{-1}(c_j)$ for each j. Then for each j there is an exact Lagrangian sphere $V_j \subset M$ such that the parallel transport map

$$\tau_{\gamma_j} : M \longrightarrow M_j$$

is such that

$$\tau_{\gamma_j}|_{M \setminus V_j} : M \setminus V_j \longrightarrow M_j \setminus \{p_j\},$$

is an exact symplectic isomorphism, and satisfies $\tau_{\gamma_j}(V_j) = \{p_j\}$. Moreover, V_j comes equipped with a diffeomorphism

$$\phi_j : S^{n-1} \longrightarrow V_j$$

which is determined as a well-defined element $[\phi_j] \in \pi_0(\mathrm{Diff}(S^{n-1})/O(n))$.

To each vanishing path γ_j there is also an associated Lagrangian disk

$$\Delta_j = \Delta_{\gamma_j} \subset Int(E),$$

called the Lefschetz thimble of γ_j. Roughly speaking, Δ_j is the trace of the vanishing cycle V_j as it is transported over the path γ_j and eventually collapses to the critical point p_j. More precisely, for $t \in [0,1)$, let $V_j(t)$ denote the vanishing cycle in $\pi^{-1}(\gamma_j(t))$ corresponding to the restricted path $\gamma_j|_{[t,0]}$ (so $V_j(1) = V_j$, in particular). Then

$$\Delta_j = \big(\bigcup_{t\in[0,1)} V_j(t)\big) \cup \{p_j\}.$$

In particular Δ_j has the following properties: $\pi(\Delta_j) = \gamma_j([0,1])$, $\partial\Delta_j = V_j$, and p_j is in $Int(\Delta_j)$. We give one more perspective: If $\gamma_j([0,1]) \subset \mathbb{R}$ and $\gamma_j : [0,1] \longrightarrow \mathbb{R}$ is an embedding, then we can consider the Morse function $f = Re(\pi) : Int(E) \longrightarrow \mathbb{R}$ and Δ_j is just the unstable manifold of $\pm\nabla_g f$ at $p_j \in Crit(f)$ with respect to any metric g on $Int(E)$. (More precisely, Δ_j is the part of the unstable manifold of p_j lying in $f^{-1}([a,b])$, where $[a,b] = \gamma_j([0,1])$.)

2.6. An example: the standard local model

The simplest example of a Lefschetz fibration with at least one critical point comes from the map $q : \mathbb{C}^n \longrightarrow \mathbb{C}$, where

$$q(z_1,\dots,z_n) = z_1^2 + \ldots + z_n^2.$$

This example is of course very important because it serves as the local model near any critical point of an arbitrary Lefschetz fibration. We now summarize some key features of q. For a detailed discussion we refer to [S03A], §1.

Every regular fiber $q^{-1}(z)$ is exact symplectomorphic to T^*S^{n-1}. To see that, we note that for any $s > 0$,

$$q^{-1}(s) = \{x + iy \in \mathbb{C}^n : |x|^2 - |y|^2 = s, x \cdot y = 0\};$$

and we realize T^*S^n as

$$T^*S^{n-1} = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n : |u| = 1, u \cdot v = 0\}$$

equipped with the restriction of the standard exact symplectic structure on $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{C}^n$. Now we define a map

$$\sigma_s : q^{-1}(s) \longrightarrow T^*S^{n-1}$$

by the formula $\sigma_s(x+iy) = (u,v)$, where $u = \frac{x}{|x|}$, $v = -|x|y$; this is an exact symplectomorphism.

To make q into an exact symplectic Lefschetz fibration, we cut down the fiber from

T^*S^{n-1} to the disk bundle $D(T^*S^{n-1})$, and we cut down the base from $\mathbb{C}$ to D^2, as follows. For $z \in \mathbb{C}^n$, set

$$k(z) = \frac{1}{4}(|z|^4 - |q(z)|^2).$$

Then $k \geq 0$ and the sub-level sets of k precisely cut down the fibers of q to disk bundles with respect to the standard metric:

$$\sigma_s(q^{-1}(s) \cap \{k \leq r\}) = D_r(T^*S^{n-1}).$$

Let $r, s > 0$ and set

$$E_{r,s} = \{z \in \mathbb{C}^n : |q(z)| \leq s, |k(z)| \leq r\}, \, \pi_{r,s} = q|E_{r,s}.$$

Then it is easy to see that the symplectic complement to $Ker(Dq_z)$ is precisely $\overline{z}\mathbb{C} \subset \mathbb{C}^n = T_z(\mathbb{C}^n)$, and that Dk_z is zero on $\overline{z}\mathbb{C}$. Thus

$$\partial_h E_{r,s} = \{k = r\} \cap E_{r,s} \text{ and } \partial_v E_{r,s} = \{|q| = s\} \cap E_{r,s}$$

and $T^h_x(E) = T_x(\partial_h E)$ for $x \in \partial_h E$, because dk is zero on $T^h_x(E)$.

Take $s = 1$ and set $E_0 = E_{r,1}$, $\pi_0 = \pi_{r,1}$. Then (E_0, π_0) is a Lefschetz fibration with regular fiber isomorphic to $D_r(T^*S^{n-1}) = \{(u, v) \in T^*S^{n-1} : |v| \leq r\}$. The vanishing cycle can be described as follows. For *any* path γ in D^2 from $b = 1$ to $c = 0$, the vanishing cycle V_γ is given by the zero section $S^{n-1} \subset D_r(T^*S^{n-1})$ (under the isomorphism $\sigma_1 : \pi_0^{-1}(1) \longrightarrow D_r(T^*S^{n-1})$).

Because an arbitrary Lefschetz fibration $\pi : E \longrightarrow D^2$ has the same local form given by $\pi_0 : E_0 \longrightarrow D^2$ near every critical point, we obtain the same picture for π near each critical point: A neighborhood of the critical point in E corresponds to a neighborhood of the origin in $\mathbb{C}^n$ (where the neighborhood of $0 \in \mathbb{C}^n$ can be taken to be $E_{r,s} \subset \mathbb{C}^n$) and a neighborhood of each vanishing cycle V_j in M corresponds to a neighborhood of S^{n-1} in T^*S^{n-1}.

2.7. Monodromy

The main classical result about Lefschetz fibrations is the Picard-Lefschetz theorem. We will not need this result, so we will not give the precise statement, but the rough idea is the following. Take a loop λ_j in D from b to b which winds counter-clockwise around c_j, and suppose λ_j does not wind around any $c_i \neq c_j$. Then the Picard-Lefschetz theorem asserts that the monodromy map $\tau_{\lambda_j} : M \longrightarrow M$ is isotopic to a Dehn twist around the vanishing sphere $V_j \subset M$. If $dimM = 2$, then V_j is a circle, and a Dehn twist is the familiar map from geometric topology. If $dimM > 2$ there is a generalization of the notion of Dehn twist in any symplectic manifold. See [S08] §16c for more details. Since $\pi_1(D^2 \setminus \{c_1, \ldots, c_k\})$ is generated by $\lambda_1, \ldots, \lambda_k$, the Picard-Lefschetz theorem can also be used to describe the monodromy map $\tau_\gamma : M \longrightarrow M$ up to isotopy, for any loop γ from b

to b which avoids $c_1, \dots, c_k$. The corresponding map

$$\Theta : \pi_1(D^2 \setminus \{c_1, \dots, c_k\}) \longrightarrow \pi_0(Symp(M))$$

is called the monodromy homomorphism.

2.8. Constructing Lefschetz fibrations

Given a Lefschetz fibration $\pi : E \longrightarrow M$, we call the data

$$(M, V_1, \dots, V_k, \gamma_1, \dots, \gamma_k),$$

Picard-Lefschetz data for (E, π), where $(\gamma_1, \dots, \gamma_k)$ is a basis of vanishing paths, and $(V_1, \dots, V_k)$ is the family of parameterized vanishing spheres determined by $(\gamma_1, \dots, \gamma_k)$. Here, each V_j is *parameterized*, which means that each V_j comes with a diffeomorphism $\phi_j : S^{n-1} \longrightarrow V_j$, more precisely, an element $[\phi_j] \in \pi_0(\mathrm{Diff}(S^{n-1})/O(n))$, as in lemma 1.

In this paper, the main result we will need is the following theorem that we quote from [S08, lemma 16.9]. It says that any desired Picard-Lefschetz data can be realized by some exact symplectic Lefschetz fibration; and in fact there is an explicit construction of the desired Lefschetz fibration. The more precise statement goes as follows:

Theorem 2.2. *Let M be any exact symplectic manifold, and let $V_1, \dots, V_k$ be any choice of parameterized exact Lagrangian spheres in M. Let $b \in D^2$, and let $c_1, \dots, c_k \in Int(D^2)$ be any points with $b \neq c_j$ for all j. Let $(\gamma_1, \dots, \gamma_k)$ be any choice of paths $\gamma_j : [0,1] \longrightarrow D^2$ satisfying the conditions of a basis of vanishing paths, that is: $\gamma_j(0) = b, \gamma_j(1) = c_j$, $\gamma_j(t) \neq c_i$ for all $t \in [0,1]$, and for all $i \neq j$, and $\gamma_i((0,1]) \cap \gamma_j((0,1]) = \emptyset$ for all $i \neq j$. Then, there exists a Lefschetz fibration*

$$\pi : E \longrightarrow D^2$$

equipped with a canonical isomorphism $\pi^{-1}(b) \cong M$ such that π has critical values $c_1, \dots, c_k$ and the vanishing cycles corresponding to $(\gamma_1, \dots, \gamma_k)$ are precisely $(V_1, \dots, V_k)$, under the identification $\pi^{-1}(b) \cong M$.

The proof of this theorem follows from an explicit construction. The basic idea is start with the trivial fibration $M \times D^2 \longrightarrow D^2$ and cut and paste in the local model $\pi_0 : E_0 \longrightarrow D^2$ to produce a Lefschetz fibration E_j with exactly one vanishing cycle, $j = 1, \dots, k$. Then we fiber-connect sum each $E_1, \dots, E_k$ onto one more copy of the trivial fibration to get a Lefschetz fibration $\pi : E \longrightarrow D^2$.

We remark that there is also a definition of Lefschetz fibrations $\pi : X \longrightarrow \mathbb{C}P^1$, where the total space and fiber are closed symplectic manifolds, as in [AS04] for example. But there is no analogue of Theorem 2.2 in that setting.

3. Two examples of complexifications in algebraic geometry

In this section we discuss two examples of complexifications in algebraic geometry. The first is a Lefschetz fibration

$$\pi : D(T^*\mathbb{R}P^2) \longrightarrow D^2$$

which arises from the embedding $\mathbb{R}P^2 \subset \mathbb{C}P^2$. The second example is a Lefschetz fibration

$$\pi : D(T^*T^2) \longrightarrow D^2$$

which arises from the embedding $T^2 = S^1 \times S^1 \subset \mathbb{C}^* \times \mathbb{C}^*$.

3.1. A complexification of a Morse function $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$ using a classical Lefschetz pencil on $\mathbb{C}P^2$

The first example will be based on a Lefschetz fibration on $D(T^*\mathbb{R}P^2)$ using the algebraic geometry of a simple Lefschetz pencil on $\mathbb{C}P^2$. (This is a modification of the example in [AS04, p. 39].) Let $s_0 = x_1^2 + x_2^2$ and $s_1 = x_0^2 - x_2^2$ be two real homogeneous polynomials of degree 2. For each $\alpha \in \mathbb{C}$ we consider the subsets of $\mathbb{C}P^2$

$$C_\alpha = \{s_1 + \alpha s_0 = 0\}, \text{ and } C_\infty = \{s_0 = 0\}.$$

For $\alpha \in \mathbb{C}$, we have $s_1 + \alpha s_0 = 0$ iff $\frac{s_1}{s_0} = -\alpha$ iff $[s_0, s_1] = [1, -\alpha]$ in $\mathbb{C}P^1$. Thus we can think of C_α as a family of subsets parameterized by $\alpha \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, where $\alpha = \infty$ just means $C_\infty = \{s_0 = 0\}$.

The collection C_α, $\alpha \in \mathbb{C}P^1$ is called a *Lefschetz pencil* of curves in $\mathbb{C}P^2$. We call C_α the "fiber" of the pencil lying above $\alpha \in \mathbb{C}P^1$. (The word "pencil" is a bad translation of the French word "pinceau", meaning "brush".)

For each α, if C_α is smooth then it follows from standard facts in algebraic geometry relating degree to Euler characteristic that C_α is diffeomorphic to S^2. If all the curves C_α were smooth, and if they were mutually disjoint, then they would fiber $\mathbb{C}P^2$ into a fiber bundle $\mathbb{C}P^2 \longrightarrow \mathbb{C}P^1$ with fiber S^2. However, this picture is not correct for two reasons: first, not all C_α are smooth, and so not all are diffeomorphic to S^2; and second, the curves C_α are not mutually disjoint. Let us discuss the second issue first.

The *base locus* of the pencil is by definition

$$B = \{s_0 = s_1 = 0\}$$

which in this case consists of four points

$$B = \{[1, 1, i], [1, -1, i], [1, 1, -i], [1, -1, -i]\}$$

The meaning of B is that any two of our curves C_α and $C_{\alpha'}$ meet in B.

Next, there are three singular fibers, each of which is the union of two lines in $\mathbb{C}P^1$:

$$C_\infty = \{s_0 = 0\} = \{x_1 = -ix_2\} \cup \{x_1 = ix_2\}$$

$$C_0 = \{s_1 = 0\} = \{x_0 = x_1\} \cup \{x_0 = -x_1\}$$

$$C_1 = \{s_1 + s_0 = 0\} = \{x_0 = -ix_1\} \cup \{x_0 = ix_1\}.$$

It is straight-forward to check that for $\alpha \notin \{0, 1, \infty\}$, that C_α is regular. (For this let $F_\alpha = s_1 + \alpha s_0$ and check that the complex rank of DF_α is 1 on C_α in each of the three standard charts $\psi_0(y_1, y_2) = [1, y_1, y_2]$, etc.)

Now we delete one regular curve from $\mathbb{C}P^2$, say $C_i = \{s_1 + is_0 = 0\}$. Then what remains,

$$X = \mathbb{C}P^2 \setminus C_i,$$

is the union of a family of curves $C_\alpha \setminus B$, $\alpha \in \mathbb{C}P^1 \setminus \{i\}$ which are mutually disjoint. There are three singular fibers and every regular fiber is diffeomorphic to a four times punctures sphere $S^2 \setminus \{\text{four points}\}$. We define

$$\pi : X \longrightarrow \mathbb{C}P^1 \setminus \{i\} \cong \mathbb{C}$$

to be the map which sends $p \in C_\alpha$ to α. Thus

$$\pi = -\frac{s_1}{s_0} : X \longrightarrow \mathbb{C}P^1 \setminus \{i\} \cong \mathbb{C}.$$

This defines a *Lefschetz fibration* on X in the sense of classical complex algebraic geometry. (This also yields an exact symplectic Lefschetz fibration with codimension 2 corners if we restrict π to a compact subset of X in such a way that we down the base to D^2 and we cut down the regular fiber to $S^2 \setminus \{$ four small open disks $\}$.)

Notice that our three singular fibers are related to the four base points in the following way. Take the line through two of the base points and take the other line through the other two base points. The union of these two lines yields one of the singular fibers. There are three ways to do this and in this way we obtain all three singular fibers.

Now, this actually tells us what the three vanishing cycles corresponding to the three singular fibers look like, as follows. Fix one regular fiber, which is $S^2 \setminus B$. Now take the singular fiber consisting of the line through $p_1, p_2 \in B$ and the line through $q_1, q_2 \in B \setminus \{p_1, p_2\}$. Then the corresponding vanishing cycle $V \subset S^2 \setminus B$ must be such that when we collapse V to a point the result consists of $S^2 \setminus \{p_1, p_2\}$ and $S^2 \setminus \{q_1, q_2\}$ meeting at one point. Thus, V must have divided B into two halves $\{p_1, p_2\}$ and $\{q_1, q_2\}$. There are therefore three vanishing cycles which divide B into pairs in all three possible ways. See figure 1.

Now consider $\mathbb{R}P^2 \subset \mathbb{C}P^2$. Notice that three singular fibers have critical points (where the two lines intersect) $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ which all lie in $\mathbb{R}P^2$. Also notice that $\pi(\mathbb{R}P^2) \subset \mathbb{R}$ and in fact $f = \pi|_{\mathbb{R}P^2}$ is the standard Morse function on $\mathbb{R}P^2$ with three critical points.

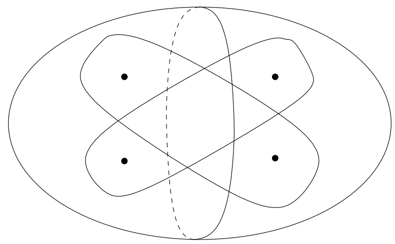


FIGURE 1. The regular fiber of the pencil $\pi = s_1/s_0$ (with the four base points deleted) and the three vanishing spheres.

Let $B_\epsilon(i)$ denote a small open disk around i such that $\epsilon < 1/2$ so that $\pi(\mathbb{R}P^2) \subset \mathbb{R}$ has a neighborhood disjoint from $B_\epsilon(i)$. Now set

$$D = \mathbb{C}P^1 \setminus B_\epsilon(i) \cong D^2$$

so that D is diffeomorphic to a compact disk and it contains $\pi(\mathbb{R}P^2)$ in its interior. Now instead of deleting just C_i from $\mathbb{C}P^2$ let us delete $U = \pi^{-1}(B_\epsilon(i))$, which we view as a small neighborhood of the fiber C_i which is disjoint from $\mathbb{R}P^2 \subset \mathbb{C}P^2$. Set

$$X_0 = \mathbb{C}P^2 \setminus U, \text{ and } \pi_0 = \pi|_{X_0} : X_0 \longrightarrow D.$$

Now for $\alpha \in D$, the fiber $\pi_0^{-1}(\alpha) = C_\alpha \setminus U$ consists of S^2 minus a small neighborhood of B, which means we get

$$\pi_0^{-1}(\alpha) = S^2 \setminus \{\text{four small disks}\}$$

with the same vanishing cycles.

In fact X is symplectomorphic to $D(T^*\mathbb{R}P^2)$ so that $\pi_0 : X_0 \longrightarrow D$ is a Lefschetz fibration on $D(T^*\mathbb{R}P^2)$ which we can view as a complexification (extension, in particular) of the standard Morse function $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$.

3.2. A complexification of a Morse function $f : T^2 \longrightarrow \mathbb{R}$ using a Laurent polynomial on $(\mathbb{C}^*)^2$

Realize the torus as

$$T^2 = S^1 \times S^1 \subset (\mathbb{C}^*)^2$$

Now define a Lefschetz fibration

$$W : (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}$$

by the formula

$$W = x + \frac{1}{x} + y + \frac{1}{y}.$$

One can easily check this has four critical points $(x, y) = (\pm 1, \pm 1)$ and the complex Hessian D^2W is nondegenerate there; thus W is indeed a Lefschetz fibration.

Notice that

$$W|_{T^2} = 2\cos\theta_1 + 2\cos\theta_2$$

where we parameterize T^2 by

$$x = \cos\theta_1 + i\sin\theta_1,$$
$$y = \cos\theta_2 + i\sin\theta_2,$$
$$\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}.$$

Thus $f = W|_{T^2}$ is the standard Morse function on T^2 (f is the sum of two copies of the height function on S^1). f has four critical points given by $\theta_1, \theta_2 \in \{0, \pi\}$, that is, $(x, y) = (\pm 1, \pm 1) \in S^1 \times S^1$. Thus all the critical points of W lie on $T^2 \subset (\mathbb{C}^*)^2$.

To describe the topology of the regular fiber of W, notice that

$$W(-1,-1) = -4, \;\; W(1,1) = 4, \;\; W(-1,1) = W(1,-1) = 0$$

so $b = 1$ is a regular value. Let us take the equation

$$x + \frac{1}{x} + y + \frac{1}{y} = 1$$

and multiply by xy to get

$$x^2y + y + y^2x + x - xy = 0.$$

Now let us homogenize this equation in the coordinates $[x, y, z] \in \mathbb{C}P^2$ to get an equation in $\mathbb{C}P^2$:

$$M = \{x^2y + yz^2 + y^2x + xz^2 - xyz = 0\} \subset \mathbb{C}P^2$$

Since this is a regular cubic in $\mathbb{C}P^2$ standard results in algebraic geometry relating the Euler characteristic to the degree tell us that

$$M \cong T^2.$$

Now to put M back into $(\mathbb{C}^*)^2$ we should delete the lines $x = 0$, $y = 0$, and $z = 0$. We note that

$x = 0$ implies $yz^2 = 0$, so $y = 0$ or $z = 0$;
$y = 0$ implies $xz^2 = 0$, so $x = 0$ or $z = 0$;
$z = 0$ implies $x^2y = y^2x$, so $y = 0$, or $x = 0$, or $y = x$.

Thus the intersection of M with the three lines consists of four points

$$(M \cap \{x = 0\}) \cup (M \cap \{y = 0\}) \cup (M \cap \{z = 0\})$$
$$= \{[0,0,1], [0,1,0], [1,0,0], [1,1,0]\}.$$

Thus the fiber of W is diffeomorphic to T^2 minus four points. We do not determine the vanishing cycles. But in §5.2 we shall construct a Lefschetz fibration

$$\pi : D(T^*T^2) \longrightarrow D^2$$

which has four critical points all lying on $T^2 \subset D(T^*T^2)$ and such that the restriction $\pi|_{T^2} = f : T^2 \longrightarrow \mathbb{R}$ is the standard Morse function above. The fiber of π is T^2 minus four small disks (which is compact version of the fiber of W). Thus the natural conjecture is that the vanishing cycles of W coincide with those of π.

4. The construction of the fiber and vanishing cycles

In this section we explain how to carry out step (2) in our four step plan in the introduction.

4.1. The construction of the fiber M

Let N be a compact 2-manifold without boundary and let $f : N \longrightarrow \mathbb{R}$ be a Morse function. Take a metric g such that (f, g) is Morse-Smale. Assume that f has just one maximum and minimum and denote the critical points by x_0, x_1^j, x_2, where the subscript indicates the Morse index. (This assumption is for convenience; see section 4.3 for a discussion of how to remove this assumption if desired.) Take the handle-decomposition of N induced by (f, g). Let us denote the handles by h_0, h_1^j, h_2, where the subscript indicates the index.

The fiber M will be constructed as follows. First let $V_0 = S^1$. And set

$$M_0 = D(T^*V_0).$$

This can be thought of as the first approximation to our fiber; it contains only one vanishing cycle, namely V_0, which we think of as the vanishing cycle corresponding to x_0.

To finish constructing the fiber M we will attach $2k$ 1-handles to the boundary of

$$M_0 = D(T^*V_0) \cong S^1 \times [-1, 1].$$

Remark 4.1. As usual, the manifold resulting from each handle attachment must be smoothed afterwards to get a smooth manifold with boundary. We are working with exact symplectic 2-manifolds, so in our case we use the method of handle-attachments described by Weinstein [W91], so that the manifold resulting from attaching a handle has a canonical exact symplectic structure inherited form the one on $D(T^*V_0)$ plus the exact symplectic structure on the handle (there is a standard model of a handle in this context).

For each 1-handle h_1^j in the handle decomposition of N, we take two 1-handles $\tilde{h}_1^j$ and $\hat{h}_1^j$ which will be attached to $D(T^*V_0)$.

Each 1-handle h_1^j is attached to $h_0 = D^2$ at a pair of points

$$\{a_j, b_j\} \subset \partial h_0 = S^1,$$

called the attaching sphere (0-sphere) of h_1^j. Now we identify

$$V_0 = \partial h_0 = S^1.$$

In this way V_0 inherits the attaching spheres $\{a_j, b_j\} \subset V_0$, $j = 1, \ldots, k$. For each j we will attach attach two 1-handles, $\tilde{h}_1^j$ and $\hat{h}_1^j$, to the following four points in the boundary of $D(T^*V_0) \cong S^1 \times [-1, 1]$:

$$\{(-1, a_j), (-1, b_j), (1, a_j), (1, b_j)\} \subset \partial(S^1 \times [-1, 1]) \cong \partial(D(T^*V_0)).$$

The way the handles are attached will depend on the framing used to attach h_1^j to h_0, as follows. (One thing to keep in mind is the result M is supposed to be a symplectic 2-manifold, so it must be orientable.)

(1) First consider the case where h_1^j is attached to h_0 *with a twist* (for example, there is a twist in the 1-handle of the standard handle decomposition of $\mathbb{R}P^2$). In this case $\tilde{h}_1^j$ will be attached to $(a_j, 1)$ and $(b_j, 1)$ while $\hat{h}_1^j$ will be attached to $(a_j, -1)$ and $(b_j, -1)$. For both of the handles, the framing will be such that there is *no twist* in the handle. This means, more precisely, that there are small tubular neighborhoods

$$[a_j - \epsilon, a_j + \epsilon] \times \{1\} \text{ and } [b_j - \epsilon, b_j + \epsilon] \times \{1\}$$

such that the handle attachment is equivalent to identifying these neighborhoods while *preserving* the orientations of the intervals. (And the same statement holds with 1 replaced by -1.)

(2) Next consider the case where h_1^j is attached to h_0 *without a twist* (for example, both 1-handles in the standard handle decomposition of T^2 have no twists). In this case $\tilde{h}_1^j$ will be attached to $(a_j, 1)$ and $(b_j, -1)$ while $\hat{h}_1^j$ will be attached to $(a_j, -1)$ and $(b_j, 1)$. For both of the handles, the framing will be such that there is *no twist* in the handle. This means, more precisely, that there are small tubular neighborhoods

$$[a_j - \epsilon, a_j + \epsilon] \times \{1\} \quad \text{and} \quad [b_j - \epsilon, b_j + \epsilon] \times \{-1\}$$

such that the handle attachment is equivalent to identifying these neighborhoods while *preserving* the orientations of the intervals. (And the same statement holds with 1 and -1 exchanged.)

Once we have attached these $2k$ 1-handles to $D(T^*V_0)$, this completes the construction of the fiber, M.

Remark 4.2. Here is an alternative way of thinking of the construction of M which is conceptually appealing. Start with $D(T^*V_0)$ as before. Now for each $1-$handle h_1^j take a copy of the circle, denoted by

$$V_1^j = S^1.$$

This will be the vanishing cycle corresponding to the critical point x_1^j. Now for each j, do a *plumbing* of $D(T^*V_0)$ and $D(T^*V_1^j)$ such that in the plumbing

$$V_0 \cap V_1^j = \{a_j, b_j\}$$

and such that the result is the same as attaching the two 1-handles $\tilde{h}_1^j$ and $\hat{h}_1^j$ as before. See for example figure 3 for how this plumbing looks like. The conceptual advantage of plumbing is that the vanishing cycles V_1^j corresponding to x_1^j are apparent from the beginning.

4.2. Construction of the vanishing cycles

We have constructed our fiber M. It remains to describe the vanishing cycles in M. First, V_0, the vanishing cycle corresponding to x_0, is apparent. Next we define V_1^j, the vanishing cycle corresponding to x_1^j, to be union of the two segments:

$$(\{a_j\} \times [-1,1]) \cup (\{b_j\} \times [-1,1]) \ \subset \ V_0 \times [-1,1] \cong D(T^*V_0)$$

and the core of $\tilde{h}_1^j$ and the core of $\hat{h}_1^j$. See figures 3 and 5 for example. (Alternatively, see remark 4.2 at the end of the last section for another construction of M where V_1^j is quite manifest.)

The last vanishing cycle V_2 corresponding to the critical point x_2 is the most interesting one. It is obtained by doing the Lagrangian surgery of V_0 with all V_1^j simultaneously for $j = 1, \dots, k$ (see [P91] for a general discussion of Lagrangian surgery). There is also a choice of "left" or "right" Lagrangian surgery; we have chosen a "left" surgery $V_0 \# V_1^j$, which means as we move along V_0 towards the surgery point the surgered curve moves to the left, see figure 4.

The reason we choose the left surgery is because of our choice of vanishing paths which we describe now. Our choice of left surgery corresponds to the fact that our vanishing paths (see figure 2) lie in the lower half plane $\{z \in \mathbb{C} : Re(z) \leq 0\}$. (If the vanishing paths were reflected to the upper half plane then the right surgery would have been appropriate in the definition of V_2). We choose the critical values in the unit disk $D^2 \subset \mathbb{C}$ to lie on the real line:

$$c_2 = 3/4, \, c_1 = 0, \, c_0 = -3/4.$$

And we choose the base point to be

$$b = -1/2.$$

Let γ_0, $\gamma_1 = \gamma_1^j$, $j = 1, \dots, k$, and γ_2 denote the vanishing paths as in figure 2. Note that we chose all γ_1^j, $j = 1, \dots, k$ to be all the same, equal to γ_1:

$$\gamma_1^1 = \ldots = \gamma_1^k = \gamma_1.$$

This is not a problem since the vanishing cycles V_1^j, $j = 1, \dots, k$ are mutually disjoint.

4.3. General Morse functions

In this section we briefly discuss how to remove the assumption that there is only one critical point index 0, and one of index 2. (This discussion is not essential for the rest of the paper, so the reader is free to skip it, or come back to it later.) In the general case

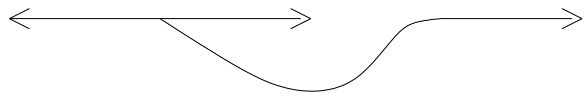


FIGURE 2. Here we have the vanishing paths γ_0, $\gamma_1 = \gamma_1^j$, $j = 1, \ldots, k$, and γ_2 in $\mathbb{C}$. Here, γ_0 goes along the interval $[-3/4, -1/2]$ from $b = -1/2$ to $c_0 = -3/4$; γ_1 goes along the interval $[-1/2, 0]$ from $b = -1/2$ to $c_1 = 0$; and γ_2 goes from $b = -1/2$ to $c_2 = 3/4$ by going around $c_1 = 0$ via an arc in the lower half plane, and then along the interval $[1/2, 3/4]$.

we proceed as follows. Let $x_0^1, \ldots x_0^l$ denote the critical points of index 0; let $x_1^1, \ldots x_1^k$ denote the critical points of index 1; and let $x_2^1, \ldots x_2^m$ denote the critical points of index 2. Corresponding to this we have the handle decomposition of f with 0-handles h_0^i, 1-handles h_1^j, and 2-handles h_2^p, $i = 1, \ldots, l$, $j = 1, \ldots, k$, $p = 1, \ldots, m$.

To construct M, we start with l disjoint copies of $D(T^*S^1)$, denoted $D(T^*V_0^i)$, $i = 1, \ldots, l$. So we set $M_0 = \cup_i D(T^*V_0^i)$ as our first approximation to M. Now we identify each V_0^i with the boundary of h_0^i, that is $V_0^i = \partial h_0^i$. In this way $\cup_i V_0^i$ inherits several embedded copies of S^0, say $K^j = S^0$, $j = 1, \ldots k$, which are the attaching spheres of the 1-handles h_1^j, $j = 1, \ldots, k$. As before, each K_j gives rise to four points in the boundary of $\cup_i D(T^*V_0^i)$. That is, each pair of points $a_j, b_j \in V_0^i = S^1$ gives rise to $(a_j, \pm 1)$ and $(b_j, \pm 1)$ in $D(T^*V_0^i) = S^1 \times [-1, 1]$. To construct M we attach a pair of 1-handles $\widetilde{h}_1^j$, $\widehat{h}_1^j$ at each quadruple of points $(a_j, \pm 1)$, $(b_j, \pm 1)$ according to the same rules as before (that is, rule (1) or (2) in §4.1, depending whether the 1-handle h_1^j was attached with or without a twist in the handle decomposition of N).

The vanishing cycles V_0^i and V_1^j are evident as before. Now V_2^p, $p = 1, \ldots, m$ arise by doing the (left) Lagrangian surgery of $\cup_i V_0^i$ and $\cup_j V_1^j$ in M at each point where they meet. There will result from all these surgeries precisely m circles which are by definition V_2^p, $p = 1, \ldots, m$.

The reason we will obtain the correct number of circles (i.e., m) is that the Lagrangian surgery of $\cup_i V_0^i$ and $\cup_j V_1^j$ in M perfectly mirrors the surgery that happens in N as we pass between level sets,

$$\text{from the level set } \partial[\cup_i h_0^i], \text{ to the level set } \partial[(\cup_i h_0^i) \cup (\cup_j h_1^j)].$$

As an example/exercise, we suggest considering $N = S^2$, where $f : S^2 \longrightarrow \mathbb{R}$ has two maxima, two minima, and two critical points of index 1. Before trying this exercise the reader may first wish to look at the next two sections, where we will carry out the examples $N = \mathbb{R}P^2$ and $N = T^2$ where the Morse function has only one maximum and one minimum in each case.

5. Some examples of the construction

In this section we explain how the construction described in §4 works in some examples, in particular, we consider the cases $N = T^2$ and $N = \mathbb{R}P^2$.

5.1. Example 1: $N = \mathbb{R}P^2$.

First, we consider the example $N = \mathbb{R}P^2$. In this case we have a handle decomposition with three handles h_0, h_1, h_2. To construct the fiber M of our Lefschetz fibration $\pi : D(T^*\mathbb{R}P^2) \longrightarrow D^2$, we start with $D(T^*V_0)$ and then attach two 1-handles $\tilde{h}_1$ and $\hat{h}_1$ to $D(T^*V_0)$. To begin, identify $\partial h_0 = S^1 = V_0$ and let $\{a, b\}$ denote the two points where h_1 is attached to h_0 (i.e., $\{a, b\}$ is the attaching 0-sphere). Thus we obtain two points $\{a, b\} \subset V_0$. Now identify

$$D(T^*V_0) = V_0 \times [-1, 1]$$

and consider the following four points in the boundary of $D(T^*V_0)$:

$$\{(a, -1)(a, 1), (b, 1), (b, -1)\} \ \subset \ \partial(V_0 \times [-1, 1]).$$

In the handle decomposition of $N = \mathbb{R}P^2$ the 1-handle h_1 is attached with a twist. Therefore, according to the instructions in §4.1 (1), we must attach $\tilde{h}_1$ to $(a, 1), (b, 1)$ and $\hat{h}_1$ to $(a, -1), (b, -1)$. The result is shown in figure 3 (notice that the 1-handles are attached without twists so that M is orientable).

There are three vanishing cycles. Two, V_0 and V_1 are quite obvious: In figure 3, V_0 is the horizontal segment, and V_1 is the circle which meets V_0 vertically; V_1 is the core of the evident annulus plumbed on to $D(T^*V_0)$.

Remark 5.1. Notice that V_0 and V_1 intersect in two points. These two points correspond exactly to the two flow lines from x_1 to x_0, which are represented by the two attaching points, a, b, where h_1 attached to h_0.

The last vanishing cycle V_2 is the Lagrangian surgery of V_0 and V_1. It is shown in figure 4: V_2 coincides with V_0 and V_1 away from a neighborhood of $V_0 \cap V_1$, and inside a neighborhood of $V_0 \cap V_1$ it coincides with the four curved arcs in figure 4.

Notice that the resulting fiber M is diffeomorphic to S^2 with four small disks removed. This agrees with the example we did from classical algebraic geometry in §3.1. Furthermore, V_0, V_1 and V_2 divide the four punctures in all three possible ways. Thus the vanishing cycles also agree with the example from classical algebraic geometry.

5.2. Example 2: $N = T^2$.

Next, we consider the example of the torus, $N = T^2$. In this case we have a handle decomposition with four handles h_0, h_1^1, h_1^2, h_2. To construct the fiber M of our Lefschetz fibration $\pi : D(T^*T^2) \longrightarrow D^2$, we start with $D(T^*V_0)$ and then attach four 1-handles to $D(T^*V_0)$ in pairs, denoted $\tilde{h}_1^1$, $\hat{h}_1^1$ and $\tilde{h}_1^2$, $\hat{h}_1^2$. To begin, consider the points $\{a_j, b_j\} \subset \partial h_0$

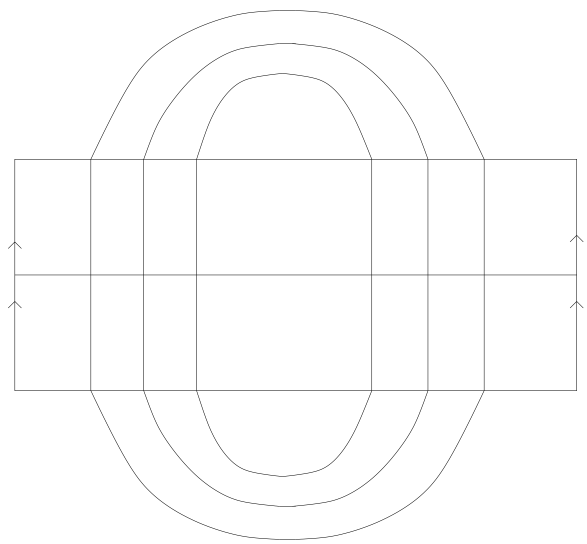


FIGURE 3. This is the fiber M in the case $N = \mathbb{R}P^2$. The two vertical edges marked by >> are identified. The vanishing cycle V_0 is represented by the horizontal line in the middle. The vanishing cycle V_1 is the circle meeting V_0 vertically. The last vanishing cycle V_2 is not shown (for V_2 see figure 4).

where h_1^j is attached to h_0, $j = 1, 2$. Now identify $S^1 = V_0 = \partial h_0$. Thus we obtain four points in V_0:

$$a_1, b_1, a_2, b_2 \subset V_0.$$

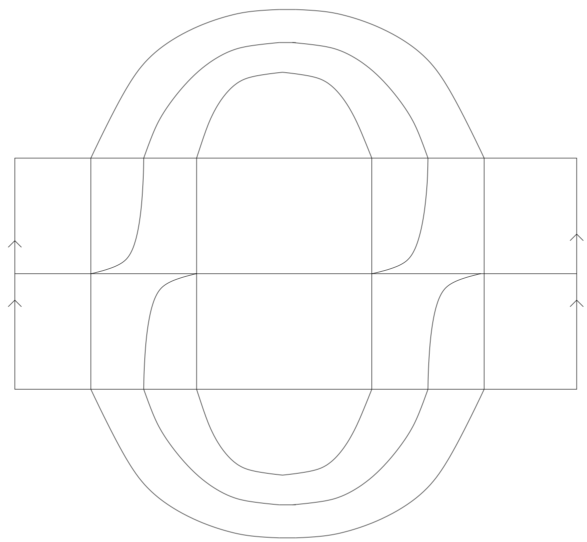
The cyclic order in V_0 is: a_1, a_2, b_1, b_2, which we see from the handle decomposition of T^2. Now identify $D(T^*V_0) = V_0 \times [-1, 1]$ and for $j = 1, 2$, consider the following four points in the boundary of $D(T^*V_0)$:

$$\{(a_j, -1)(a_j, 1), (b_j, 1), (b_j, -1)\} \subset \partial(V_0 \times [-1, 1]).$$

In the handle decomposition of $N = T^2$ both 1-handles h_1^j, $j = 1, 2$ are attached with no twist. Therefore, according to the instructions in §4.1 (2), we must attach $\tilde{h}_1^j$ to $(a_j, 1), (b_j, -1)$ and $\hat{h}_1^j$ to $(a_j, 1), (b_j, -1)$. The result is shown in figure 5 (notice that the 1-handles are attached without twists so that M is orientable).

There are four vanishing cycles V_0, V_1^1, V_1^2, V_2. Three of them, V_0, V_1^1, and V_1^2 are quite obvious, see figure 5.

FIGURE 4. This is the fiber M in the case $N = \mathbb{R}P^2$. The two vertical edges marked by >> are identified. The vanishing cycle V_2 is shown (it is curvy); it is the Lagrangian surgery of V_0 and V_1 which are shown in figure 3.



Remark 5.2. Notice that V_0 and V_1^j intersect in two points. These two points correspond exactly to the two flow lines from x_1^j to x_0, which are represented by the two attaching points, a_j, b_j, where h_1^j is attached to h_0.

The last vanishing cycle V_2 is the Lagrangian surgery of V_0 and $V_1^1 \cup V_1^2$. It is shown in figure 6: Outside of a neighborhood of the four intersection points,

$$(V_0 \cap V_1^2) \cup (V_0 \cap V_1^2) = \{a_1, b_1, a_2, b_2\},$$

V_2 coincides with V_0, V_1^1, and V_2; and inside a neighborhood of $\{a_1, b_1, a_2, b_2\}$, V_2 coincides with the eight curved arcs shown in figure 6. It is instructive to trace through the figure and verify that we indeed get a copy of S^1, which is V_2 by definition.

By inspecting the boundary carefully we see there are 4 boundary components. The Euler characteristic is -4 (to see this, we can retract M onto the 1-skeleton, which has one 0-cell and five 1-cells). Thus we conclude M is diffeomorphic to T^2 with 4 small disks removed. (Here we use the formula for the Euler characteristic of a compact 2-manifold Σ given by $\chi(\Sigma) = (2 - 2g) - b$, where $g = genus(\Sigma)$, and b is the number of boundary components of Σ.)

Thus M agrees with the fiber in the example of the Lefschetz fibration on $(\mathbb{C}^*)^2 \cong T^*T^2$ we

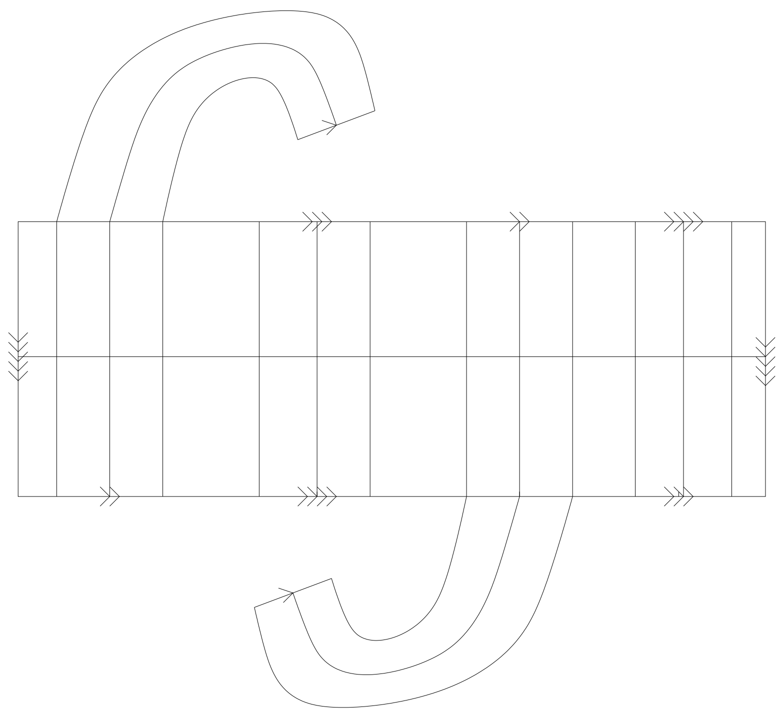


FIGURE 5. The fiber M in the case $N = T^2$. The two vertical edges indicated by >>>>> are identified to form $S^1 \times [-1,1] = D(T^*V_0)$. The other marked edges, in pairs, indicate where the four 1−handles are attached. We have drawn in part of one of the 1-handles (up to identifying > and >) to indicate how a handle would look. Three of the four vanishing cycles are visible: V_0 is the horizontal segment, and V_1^1 is represented by two of the vertical segments (joining the edges marked > and >>) and V_1^2 is represented by the other two vertical segments (joining the edges marked >>> and >>>>). Because of the identifications indicated, these vertical segments join together to form two copies of S^1. See figure 6 for the final vanishing cycle V_2.

considered in §3.2. In that example we had four critical points all lying on $(S^1)^2 \subset (\mathbb{C}^*)^2$ which corresponds to $T^2 \subset T^*T^2$. This agrees with the fact that we have four vanishing cycles in M. The natural conjecture is that the vanishing cycles of the example in §3.2 are the same as the ones in M.

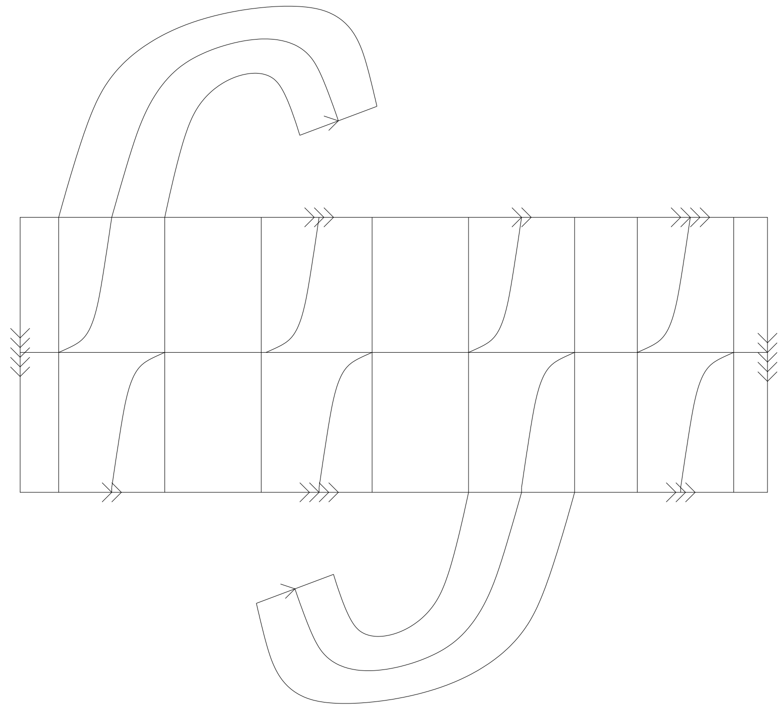


FIGURE 6. The fiber M in the case $N = T^2$. Here V_2 is shown; it is the Lagrangian surgery of V_0 and V_1^2, V_1^2. See figure 5 for a picture of V_0 and V_1^2, V_1^2.

5.3. The genus of the fiber in general

Let N be the closed oriented surface of genus g. Take the standard handle decomposition of N with one $0-$handle, one $2-$handle, and $2g$ $1-$handles. We describe without proof the fiber M in this case: If we construct M as in §4.1, and reason in a way similar to the example of the torus, then we find the fiber M is equal to a genus g surface with $2+2g$ small disks removed, and with $2g+2$ vanishing cycles. Theorem 1.1 implies there is a Lefschetz fibration on $D(T^*N)$ with this regular fiber and vanishing cycles. This answer for M makes sense in view of the general fact that the total space of a Lefschetz fibration on a 4-manifold is homotopy equivalent to the fiber M with a 2-cell attached at each vanishing cycle. Indeed, in this case M is equal to N with $2g+2$ small disks removed, so it is plausible that attaching $(2g+2)$ 2-cells to the vanishing cycles should recover N as expected. (This is not a complete argument that E is homotopy equivalent to N because the the position of the vanishing cycles has not taken into account.)

5.4. Two quick examples where N has boundary

Let

$$N_1 = T^2 \setminus D,$$

where D is an embedded disk, and let

$$N_2 = \mathbb{R}P^2 \setminus D',$$

where D' is an embedded disk. Then N_1 and N_2 admit Morse functions and handle decompositions which are the same as the ones we had in §5.1, 5.2 except we remove a neighborhood of the maximum, and correspondingly we omit h_2 from the handle decompositions. In these cases the regular fibers are exactly the same as before, as in §5.1, 5.2, and we have the same vanishing cycles, except we omit V_2 in both cases.

6. Sketch of $E \cong D(T^*N)$

Let N be a 2-manifold and construct M and $V_0, V_1^j, V_2 \subset M$ as described in §4. Take the vanishing paths $\gamma_0, \gamma_1^j, \gamma_2$ as described in §4.2. Now we invoke Theorem 2.2 and obtain a Lefschetz fibration

$$\pi : E \longrightarrow D^2$$

which, by construction, has the Picard-Lefschetz data

$$(M, V_0, V_1^j, V_2, \gamma_0, \gamma_1^j, \gamma_2).$$

In this section we sketch the proof of the following theorem:

Theorem 6.1. (1) *There is an exact Lagrangian embedding $N \subset E$.*
(2) *$Crit(\pi) \subset N$, $\pi(N) = [a, b] \subset \mathbb{R}$, and $\pi|N = f : N \longrightarrow \mathbb{R}$ (up to reparameterizing N and $\mathbb{R}$ by diffeomorphisms).*
(3) *E is conformally exact symplectomorphic to the disk cotangent bundle $D(T^*N)$ (after we smooth the corners of E).*

The main step is (1). Then (2) follows easily by construction of N. The last step (3) is accomplished by describing a retraction of E onto a small Weinstein neighborhood of N in E, symplectomorphic to $D(T^*N)$. The retraction is obtained by using the parallel transport map with varying time along some fixed paths; that is why it is a conformally exact symplectomorphism. See remark 1.3 for more about conformally exact symplectomorphisms. As a model example one should imagine some neighborhood of the zero section in T^*N retracting onto the disk bundle $D_\epsilon(T^*N)$ for some small $\epsilon > 0$ via the Liouville flow along the cotangent fibers, with varying time. This is obviously not symplectic because it distorts the symplectic volume, but it is a conformal exact symplectomorphism, which is the best one could hope for in this situation, and still useful.

We will discuss the proof in the case $N = \mathbb{R}P^2$, as in §5.1. The proof in the general case involves no new ideas. But see remark 6.2 for a brief discussion of the general case.

The regular fiber M and vanishing spheres V_0, V_1, V_2 are as in figure 8 below. The critical values in D^2 are chosen to be

$$\begin{aligned} \pi(x_2) &= c_2 = -3/4, \\ \pi(x_1) &= c_1 = 0, \\ \pi(x_0) &= c_0 = 3/4. \end{aligned}$$

And the base point is

$$b = -1/2.$$

Let γ_0, γ_1, γ_2 denote the vanishing paths in figure 2.

Now let $b' = 1/2$ and consider the obvious reflection of the vanishing paths through the origin, γ_0', γ_1', γ_2'. Then let

$$M' = \pi^{-1}(b').$$

We will suppress how M and M' are identified. (Roughly speaking they are related by a "half twist operation", and then transport around the half circle from 1/2 to -1/2 gives an exact symplectomorphism from M' to M which can be understood explicitly. See Lemma 6.1 and Lemma 7.2 in [J09] for details.)

If we identify $M = M'$ then the vanishing spheres V_0',V_1',V_2' in M' appear as in figure 7. Under the identification $M' = M$, we have $V_1' = V_1$, $V_2' = V_0$ and V_0' looks similar to V_2, but the surgery is the *right* surgery of V_0' and V_1', which means as you move along V_0' towards the surgery region, the curve moves to the right. Compare with figure 8 which shows $V_0, V_1, V_2 \subset M$.

At this point we need to go into the construction of (E, π) in a bit more detail (see [J09] for full details). Let $(E_{r,s}, \pi_{r,s})$ denote the standard local model (see §2.6) with fiber $D_r(T^*S^n)$, $r > 0$; but use $q_1 = z_1^2 - z_2^2 + c_1$ rather than the usual $q = z_1^2 + z_2^2$. Let

$$\phi_1 : D_r(T^*S^1) \longrightarrow M$$

denote an exact Lagrangian embedding such that $\phi_1(S^1) = V_1$ (such an embedding exists by Weinstein's theorem).

Let E_1 denote the Lefschetz fibration over $D_s(c_1) = \{z \in \mathbb{C} : |z - c_1| \leq s\}$ obtained in the following way. Take the trivial fibration

$$F = M \times D_s(c_1),$$

and consider the subset obtained by deleting a neighborhood of V_1 in every fiber:

$$F_0 = (M \setminus \phi_1(D_{r/2}(T^*S^1)) \times D_s(c_1).$$

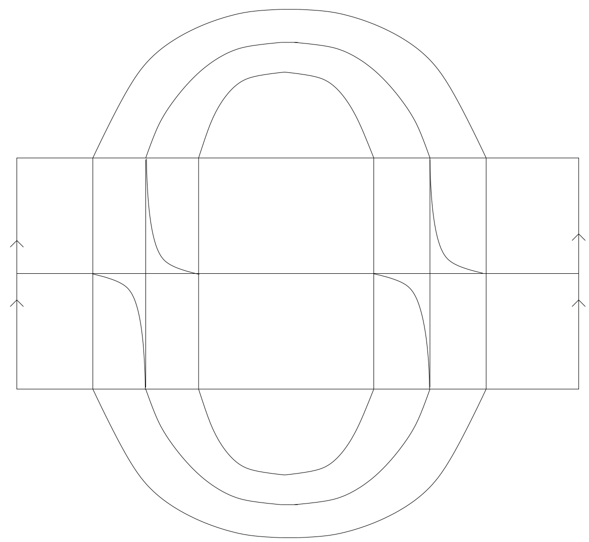


FIGURE 7. In the case $N = \mathbb{R}P^2$: The fiber M' at $b' = 1/2$

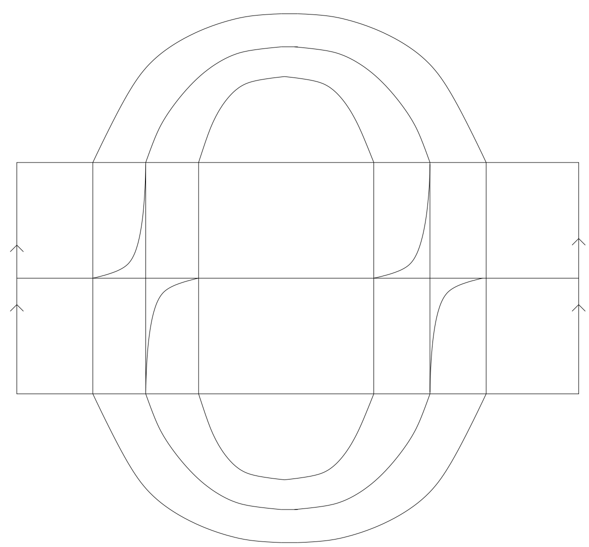


FIGURE 8. In the case $N = \mathbb{R}P^2$: The fiber M at $b = -1/2$

Now use the fact that we can trivialize $E_{r,s} \cap \{z \in E_{r,s} : r/2 < k(z) \le r\}$ (where $k(z) = \frac{1}{4}(|z|^4 - |q_1(z)|^2)$ as in §2.6) by radial parallel transport, yielding an exact symplectomorphism

$$\rho : E_{r,s} \cap \{r/2 < k \le r\} \longrightarrow \phi_1(D_{(r/2,r]}(T^*S^1)) \times D_s(c_1),$$

where $D_{(r/2,r]}(T^*S^1) = \{(u,v) \in T^*S^1 : |v| \in (r/2, r]\}$. We define

$$E_1 = F_0 \cup_\rho E_{r,s}$$

where we glue F_0 and $E_{r,s}$ using the map ρ. The map $\pi_1 : E_1 \longrightarrow D_s(c_1)$ is defined to be $\pi_{r,s} = z_1^2 - z_2^2 + c_1$ on $E_{r,s}$ and the projection map to $D_s(c_1)$ on F_0 (these two maps agree on the overlap using the gluing map ρ).

Now, the Lefschetz fibration $\pi : E \longrightarrow D^2$ is constructed such that the restriction of π to $\pi^{-1}(D_s(c_1))$ agrees with (E_1, π_1). Let

$$N_{r,s} = E_{r,s} \cap \mathbb{R}^2 \subset \mathbb{C}^2.$$

Set

$$f_1(x_1, x_2) = x_1^2 - x_2^2 + c_1,$$

so f_1 is the standard Morse function of index 1 on $\mathbb{R}^2$. Note that

$$\pi_{r,s}|N_{r,s} = f_1, \text{ and}$$

$$N_{r,s} = \{x \in \mathbb{R}^2 : |f_1(x)| \le s, \, |x|^4 - f_1(x)^2 \le r\}.$$

Note that $N_{r,s}$ is diffeomorphic to an 8 sided polygon as in figure 9 (i.e., $N_{r,s}$ is the same as N_1^{loc} in figure 9).

When we construct the embedding $N \subset E$ (see §6.1 below), we will write N as the union of several over-lapping pieces. These pieces correspond to something like a handle-decomposition of N. $N_{r,s}$ will play the role of the 1-handle. Let

$$I_+ = N_{r,s} \cap \{f_1 = s\} \quad \text{and} \quad I_- = N_{r,s} \cap \{f_1 = -s\}.$$

Then $I_\pm \cong S^0 \times D^1$. I_- corresponds to the part of the 1-handle which attaches to the 0-handle, and I_+ corresponds to the part of the 1-handle which meets the boundary of the 2-handle. We identify I_- with $S^0 \times D^1$ explicitly using:

$$\psi_- : S^0 \times [-a, a] \longrightarrow I_-,$$

$$\psi_-(\pm 1, \theta) = (\pm\sqrt{s}\sinh(\theta), \pm\sqrt{s}\cosh(\theta)),$$

where $a > 0$ is chosen suitably.

Using an explicit formula for ρ it is easy to check (see the proof of Lemma 7.2 in [J09]) that under the trivialization

$$\rho : E_{r,s} \cap \{r/2 < k \le r\} \longrightarrow \phi_1(D_{(r/2,r]}(T^*S^1)) \times D_s(c_1),$$

we have $\rho(I_- \cap \{r/2 < k \leq r\})$ is equal to

$$D_{(r/2,r]}(T_a^* S^1) \cup D_{(r/2,r]}(T_b^* S^1)$$

for some $K_- = \{a, b\} \subset S^1$, $K_- \cong S^0$. We assume that ϕ_1 is chosen such that $\phi_1(K_-)$ is $V_0 \cap V_1$ in M and $\phi_1(D_r(T_a^* S^1) \cup D_r(T_b^* S^1))$ is equal to a neighborhood of $V_0 \cap V_1$ in V_0. The main technical ingredient for our construction of $N \subset E$ is the following lemma.

Lemma 2. Let $0 < a_0 < a$ be the unique number such that

$$\psi_-(S^0 \times ([-a, -a_0) \cup (a_0, a])) = I_- \cap \{r/2 < k \leq r\}.$$

Let ψ_-^0 denote the restriction $\psi_-|_{(S^0 \times ([-a,-a_0) \cup (a_0,a]))}$. Then under the trivialization

$$\rho : E_{r,s} \cap \{r/2 < k \leq r\} \longrightarrow \phi_1(D_{(r/2,r]}(T^* S^1)) \times D_s(c_1),$$

we have that

$$\phi_1 \circ \rho \circ \psi_-^0 : S^0 \times ([-a, -a_0) \cup (a_0, a]) \longrightarrow V_0$$

agrees with the framing that is used to attach the 1-handle in the given handle decomposition of N. More precisely, if $h : S^0 \times [-a, a] \longrightarrow \partial h_0$ is the attaching map of the 1-handle in the given handle decomposition of N, then the restriction $h|(S^0 \times ([-a, -a_0] \cup [a_0, a]))$ is equal to $\phi_1 \circ \rho \circ \psi_-^0$, up to isotopy.

The proof involves inspecting each map, and we find that rules (1) and (2), which we used for constructing M in §4.1, are precisely what we need to get this result.

6.1. Sketch of the exact Lagrangian embedding $N \subset E$

We describe $N \subset E$ as the union of several overlapping pieces. These correspond to something like a handle-decomposition of N. In fact, this type of decomposition is used in Milnor's book on the h-cobordism theorem [M65, pages 27-32]. For $N = \mathbb{R}P^2$ our decomposition will have four pieces as in figure 9 below. We will call this a Milnor type handle decomposition.

In the Milnor type handle decomposition of $\mathbb{R}P^2$ shown in figure 9 we have $N_0 = D^2$ and $N_2 = D^2$, which are the same as the usual 0- and 2-handles. Then there is

$$N_1^{loc} = \{x \in \mathbb{R}^2 : |f_1(x)| \leq \delta,\ |x|^4 - f_1(x)^2 \leq \epsilon\},$$

where $\delta, \epsilon > 0$ are some small numbers and $f_1(x) = x_1^2 - x_2^2$. Here, N_1^{loc} plays the role of the 1-handle, but it is diffeomorphic to polygon with eight edges (as opposed to a standard 1-handle, which is diffeomorphic to $D^1 \times D^1$). For the last piece, suppose that the 1−handle (in the usual handle-decomposition) is attached using an embedding

$$\phi : S^0 \times [-\epsilon, \epsilon] \longrightarrow S^1 = \partial N_0.$$

Then the last piece is

$$N_1^{triv} = [S^1 \setminus \phi(S^0 \times (-\epsilon/2, \epsilon/2))] \times [-1, 1].$$

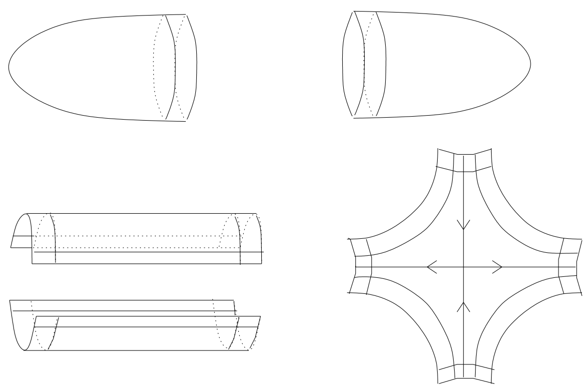


FIGURE 9. In the case $N = \mathbb{R}P^2$, the pieces N_0,N_2 (top), N_1^{triv} (bottom left), N_1^{loc} (bottom right). The overlap regions are also indicated.

This last piece has no analogue in a usual handle-decomposition; roughly, it fills in the rest of the space in N after N_0, N_2, N_1^{loc} are glued together. Now we realize four embeddings

$$(*)\ \ N_0, N_1^{loc}, N_1^{triv}, N_2 \subset E$$

such that the four pieces overlap just like they do in the handle decomposition of $N = \mathbb{R}P^2$ (as in figure 9). Fix some small $\delta > 0$. Here is how we define the four embeddings $(*)$:

(1) N_0 is defined to be the Lefschetz thimble $\Delta_{[-3/4,-\delta]}$ with respect to the vanishing path $[-3/4, -\delta]$.
(2) N_2 is defined to be the Lefschetz thimble $\Delta_{[\delta,3/4]}$ with respect to the vanishing path $[\delta, 3/4]$.
(3) To define N_1^{loc}, first take a neighborhood U of the critical point x_1, which lies over the middle critical value c_1. By construction of π (see above) we may assume $U = E_{\epsilon,2\delta}$ (so $r = \epsilon, s = 2\delta$), and

$$\pi|E_{\epsilon,2\delta} = \pi_{\epsilon,2\delta},\ \text{where}\ \pi_{\epsilon,2\delta}(z_1, z_2) = z_1^2 - z_2^2 + c_1.$$

Set

$$N_1^{loc} = N_{\epsilon,2\delta} := E_{\epsilon,2\delta} \cap \mathbb{R}^2,\ \text{and}$$

$$f_1 = \pi_{\epsilon,2\delta}|N_1^{loc},\ f_1(x_1, x_2) = x_1^2 - x_2^2 + c_1.$$

Explicitly,

$$N_1^{loc} = \{x \in \mathbb{R}^2 : |f_1(x)| \leq 2\delta,\ |x|^4 - f_1(x)^2 \leq \epsilon\}$$

for some small $\delta, \epsilon > 0$.
(4) We define N_1^{triv}. Consider $V_0 \subset M$ and let $\widetilde{V}_0$ denote the result of deleting a small neighborhood of $V_1 \cap V_0$ from V_0. Thus $\widetilde{V}_0$ is diffeomorphic to the disjoint union of two closed intervals: $\widetilde{V}_0 \cong [a, b] \cup [c, d]$. Now let N_1^{triv} be the result of parallel

transporting $\widetilde{V}_0$ over the interval $[-3/4+\sigma, 3/4-\sigma]$, where $\sigma > 0$ is small enough that $b = -1/2$ and $b' = 1/2$ lie in the interior $(-3/4+\sigma, 3/4-\sigma)$. Obviously,

$$N_1^{triv} \cong ([a,b] \cup [c,d]) \times [-3/4+\sigma, 3/4-\sigma].$$

Note that the critical point x_1 does not cause a singularity to arise, since we deleted V_1 from V_0 to form $\widetilde{V}_0$.

As we discussed just before Lemma 2, $f_1^{-1}(-\delta) \cap N_1^{loc}$ corresponds precisely to a neighborhood of $V_0 \cap V_1$ in V_0 (where V_0 here is understood to be transported from $M = \pi^{-1}(-1/2)$ to $\pi^{-1}(-\delta)$ along $[-1/2, -\delta]$). Moreover, Lemma 2 asserts that the embedding

$$f_1^{-1}(-\delta) \cap N_1^{loc} \longrightarrow V_0$$

precisely *agrees* with the framing used to attach h_1 to h_0 in the handle decomposition of $\mathbb{R}P^2$.

Similarly, there is an embedding

$$f_1^{-1}(\delta) \cap N_1^{loc} \longrightarrow V_0' \subset \pi^{-1}(\delta).$$

So, summarizing we can say there is are embeddings

$$f_1^{-1}(-\delta) \cap N_1^{loc} \longrightarrow \partial N_0$$

$$f_1^{-1}(\delta) \cap N_1^{loc} \longrightarrow \partial N_2$$

and the first one has the correct framing (agreeing with the one used to attach h_1 to h_0 in the standard handle decomposition of $\mathbb{R}P^2$). Now, N_1^{triv} simply fills in the gap from ∂N_0 to ∂N_1. We should define it so that it overlaps with N_0, N_2 and N_1^{loc}.

To conclude, the union $N_0 \cup N_1^{loc} \cup N_1^{triv} \cup N_2$ is diffeomorphic to N because it reproduces the Milnor style handle decomposition of N; the key point being that the correct framing is used for N_1^{loc} as it overlaps N_0. Moreover, N is an exact Lagrangian submanifold because for each piece θ restricts to be zero.

Remark 6.2. In the general case, i.e., if there are several critical points of index 1, then we have the same three vanishing paths, with $V_1^1, \ldots, V_1^k$ all having the same vanishing path $\gamma_1 = \gamma_1^j$, $j = 1, \ldots, k$. This is not a problem since all V_1^j are mutually disjoint. We make the same argument locally near each critical point of index 1. Of course N_1^{triv} is diffeomorphic to $(S^1 \setminus J) \times [a,b]$, where J is the union of several copies of $S^0 \times D^1$, disjointly embedded in $V_0 = S^1$.

6.2. Sketch proof of part 2 of Theorem

Part 2 of Theorem 6.1 essentially follows from inspection of the construction of $N \subset E$ given above. For instance $\pi(N_0) = [-3/4, -\delta]$ and $\pi|N_0$ obviously coincides (up to smooth reparameterization) with the usual model for an index 0 critical point on $\mathbb{R}^2$ given by the model $f_0 = -x_1^2 - x_2^2$.

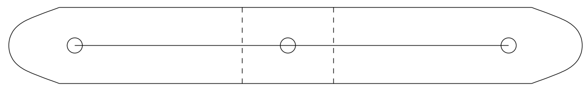


FIGURE 10. Here we have $A \subset \mathbb{C}$, a small neighborhood of $\pi(N) \subset \mathbb{R}$. Also A_0 is the region between the two dotted lines; A_- is the part of A to the left of A_0, and A_+ is to the right of A_0.

6.3. Sketch proof of $E \cong D(T^*N)$

First, use radial parallel transport to retract E onto $\pi^{-1}(A)$, where A is a small tubular neighborhood of $\pi(N) = [-3/4, 3/4] \subset D^2$. Let's split A into three pieces:

$$\begin{aligned} A_- &= A \cap \{x + iy : x < -\delta\} \\ A_+ &= A \cap \{x + iy : x > \delta\} \\ A_0 &= A \cap \{x + iy : -2\delta < x < 2\delta\}. \end{aligned}$$

See figure 10. Now let $E_- = \pi^{-1}(A_-)$. We construct (E, π) such that E_- consists of a trivial piece

$$F_- = (M \setminus D_{\epsilon/2}(T^*V_0)) \times A_-$$

with a copy of the local model with fiber $D_\epsilon(T^*V_0)$ glued on. Now consider the subset ${E_-}' \subset E_-$ given by

$${E_-}' = (M \setminus D_\epsilon(T^*V_0)) \times A_- \subset F_-.$$

Now, let

$$\begin{aligned} A_0^- &= A \cap \{x + iy : -2\delta < x < -\delta\} \subset A_0, \\ A_0^+ &= A \cap \{x + iy : \delta < x < 2\delta\} \subset A_0. \end{aligned}$$

And set

$$E_0^\pm = \pi^{-1}(A_0^\pm).$$

Now we transport almost the whole region ${E_-}'$ into $E_0^- = \pi^{-1}(A_0^-)$ along paths parallel to the real line, so that the points corresponding to $M \setminus D_\epsilon(T^*V_0)$ move onto the corresponding points in the fibers of $E_0 = \pi^{-1}(A_0^-)$. See figure 11 for a schematic picture.

We say we move "almost" all of ${E_-}'$ because what we do more precisely is the following. Fix some small fixed $a > 0$. Let $p = (p', z) \in {E_-}' = (M \setminus D_\epsilon(T^*V_0)) \times A_-$. If $p' \in M \setminus D_\epsilon(T^*V_0)$ has distance $> a$ from $D_\epsilon(T^*V_0)$ then we transport p, over a path parallel to the real line, into $E_0^- = \pi^{-1}(A_0^-)$ (for a time depending only on $Re(z)$, where $z = \pi(p) \in A_-$). But if $p' \in M \setminus D_\epsilon(T^*V_0)$ has distance $t \in [0, a]$ from $D_\epsilon(T^*V_0)$ then we multiply the old transport time by a cut-off function $\phi(t)$, so as to taper down to the identity map. This means not quite all of ${E_-}'$ is transported out of E_- into E_0. But the part that remains can be retracted in the fiber direction (by a Liouville flow) onto the local model over A_- with fiber $D_\epsilon(T^*V_0)$.

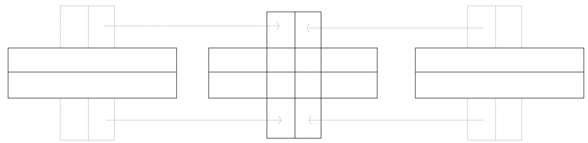


FIGURE 11. Here we have a schematic drawing of the retraction map. The three crosses represent the regular fibers over A_-, A_0, and A_+. The dotted parts of the crosses on the left and right correspond to the part of the fiber corresponding to $M \setminus D(T^*V_0)$. The idea of the retraction is to parallel transport those dotted parts of the fibers over A_- and A_+ onto the corresponding parts of the fibers over A_0.

We do a similar procedure to E_+, transporting (most) of a set $E_+{}' \subset E_+$ into E_0^+, and then retracting what remains in the fiber direction onto the local model over A_+ with fiber $D_\epsilon(T^*V_2')$. (Here, (E,π) is constructed such that the part over A_+ consists of a trivial piece

$$F_+ = (M' \setminus D_{\epsilon/2}(T^*V_2')) \times A_+$$

with a copy of the local model with fiber $D_\epsilon(T^*V_2')$ glued on.)

So what we have reduced to so far in E_+ and E_- is an arbitrarily small neighborhood of N_0 and N_2. Indeed, in the base direction we began by retracting onto a small neighborhood of $\pi(N) = [-3/4, 3/4]$ and in the fiber direction, we are inside the local models with fiber $D_\epsilon(T^*V_0)$ over A_- and with fiber $D_\epsilon(T^*V_2)$ over A_+. Thus, in E_+ and E_- we have retracted (by a Liouville type flow) onto small neighborhoods of N_0 and N_1 which can be modeled by $D(T^*N_0)$ and $D(T^*N_2)$.

To finish our discussion, let's take a look at E_0. As in the discussion before Lemma 2, (E,π) is constructed such that E_0 consists of a trivial piece

$$F_0 = (M \setminus D_{\epsilon/2}(T^*V_1)) \times A_0$$

with a copy of the local model with fiber $D_\epsilon(T^*V_1)$ glued on.

The part of E_0 given by the local model over A_0 with fiber $D_\epsilon(T^*V_1)$ is a neighborhood of N_1^{loc} in $\mathbb{C}^2$ which is isomorphic to $D(T^*N_1^{loc})$. Next, note that N_1^{triv} is given by

$$N_1^{triv} = (V_0 \setminus D_{\epsilon/2}(T^*V_1)) \times A_0 \subset F_0.$$

The trivial part of E_0 given by F_0 can be retracted to an arbitrarily small neighborhood N_1^{triv} as follows. First we deal with the fiber direction by shrinking the radius of $D(T^*V_0) \subset M$ enough (by a fiberwise retraction). Then, in the base direction we make A_0 close enough to $\pi(N_1^{triv}) = [-2\delta, 2\delta]$. In this way F_0 retracts onto a small neighborhood of N_1^{triv} which can be modeled by $D(T^*N_1^{triv})$.

To conclude we have retracted E (using a Liouville type flow) onto a union of neighborhoods of the form $D(T^*N_0)$, $D(T^*N_2)$, $D(T^*N_1^{triv})$, $D(T^*N_1^{loc})$, which together form a Weinstein neighborhood of N in E, isomorphic to $D(T^*N)$.

References

[AS10] M. Abouzaid, and P. Seidel, Altering symplectic manifolds by homologous recombination, arXiv preprint, arXiv:1007.3281.

[AC99] N. A'Campo, Real deformations and complex topology of plane curve singularities, *Ann. Fac. Sci. Toulouse Math. (6)* **8** (1999), 5–23.

[AGZV88] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals*, Monographs in Mathematics, vol. 83, Birkhauser, 1988.

[AMP05] D. Auroux, V. Muñoz, and F. Presas, Lagrangian submanifolds and Lefschetz pencils, *J. Symplectic Geom.*, **3** (2005), no. 2, 171–219.

[AS04] D. Auroux and I. Smith, Lefschetz pencils, branched covers and symplectic invariants, in Lect. Notes in Math., no. 1938, Springer, 2008, 1–53.

[FSS08] K. Fukaya, P. Seidel, and I. Smith, Exact Lagrangian submanifolds in simply-connected cotangent bundles, *Invent. Math.*, **172** (2008), no. 1, 1–27.

[GS99] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Math. Vol. 20, AMS, 1999.

[J09] J. Johns, The Picard-Lefschetz theory of Complexifications of Morse functions, arXiv preprint, arXiv:0906.1218

[L81] K. Lamotke, The topology of complex projective varieties after S. Lefschetz., *Topology*, **20** (1981), no. 1, 15–51.

[M09] M. Maydanskiy, Exotic symplectic manifolds from Lefschetz fibrations, arXiv preprint, arXiv:0906.2224.

[MS09] M. Maydanskiy and P. Seidel, Lefschetz fibrations and exotic symplectic structures on cotangent bundles of spheres, *J. Topol.*, **3** (2010), no. 1, 157-180.

[M65] J. Milnor, *Lectures on the h-cobordism theorem,* Princeton Mathematical Notes, Princetion University Press, 1965.

[P91] L. Polterovich, The surgery of Lagrange submanifolds, *Geom. Funct. Anal.*, **1** (1991), no. 2, 198–210.

[S03A] P. Seidel, A long exact sequence for Floer cohomology, *Topology*, **42** (2003), no. 5, 1003–1063.

[S03B] P. Seidel, Homological mirror symmetry for the quartic surface, arXiv preprint, arXiv:math.SG/0310414.

[S06] P. Seidel, Lectures on four-dimensional Dehn twists, in Symplectic 4-manifolds and algebraic surfaces, 231–267, Lecture Notes in Math., no. 1938, Springer, Berlin, 2008

[S08] P. Seidel, *Fukaya categories and Picard-Lefschetz theory,* Zurich Lectures in Advanced Mathematics, EMS, 2008.

[W91] A. Weinstein, Contact surgery and symplectic handlebodies, *Hokkaido Math. J.*, **20** (1991), no. 2, 241–251.

Max Planck Institute, Inselstr. 22, Leipzig 04103, Germany

E-mail address: johns@mis.mpg.de