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Landau–Ginzburg models — old and new

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Abstract. In the last three years a new concept — the concept of wall crossing has emerged. The current situation with wall crossing phenomena, after papers of Seiberg–Witten, Gaiotto–Moore–Neitzke, Vafa–Cecoti and seminal works by Donaldson–Thomas, Joyce–Song, Maulik–Nekrasov–Okounkov–Pandharipande, Douglas, Bridgeland, and Kontsevich–Soibelman, is very similar to the situation with Higgs Bundles after the works of Higgs and Hitchin — it is clear that a general "Hodge type" of theory exists and needs to be developed. Nonabelian Hodge theory did lead to strong mathematical applications —uniformization, Langlands program to mention a few. In the wall crossing it is also clear that some "Hodge type" of theory exists — Stability Hodge Structure (SHS). This theory needs to be developed in order to reap some mathematical benefits — solve long standing problems in algebraic geometry. In this paper we look at SHS from the perspective of Landau–Ginzburg models and we look at some applications. We consider simple examples and explain some conjectures these examples suggest.

1. Introduction

Mirror symmetry is a physical duality between $N = 2$ superconformal field theories. In the 1990's Maxim Kontsevich reinterpreted this concept from physics as an incredibly deep and far-reaching mathematical duality now known as Homological Mirror Symmetry (HMS). In a famous lecture in 1994, he created a frenzy in the mathematical community which lead to synergies between diverse mathematical disciplines: symplectic geometry, algebraic geometry, and category theory. HMS is now the cornerstone of an immense field of active mathematical research.

In the last three years a new concept — the concept of wall crossing has emerged. The current situation with wall crossing phenomena, after papers of Seiberg–Witten, Gaiotto–Moore–Neitzke, Vafa–Cecoti and seminal works by Donaldson–Thomas, Joyce–Song, Maulik–Nekrasov–Okounkov–Pandharipande, Douglas, Bridgeland, and Kontsevich–Soibelman, is very similar to the situation with Higgs Bundles after the works of Higgs and Hitchin — it is clear that a general "Hodge type" of theory exists and needs to be developed. Nonabelian Hodge theory did lead to strong mathematical applications

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— uniformization, Langlands program to mention a few. In the wall crossing it is also clear that some "Hodge type" of theory needs to be developed in order to reap some mathematical benefits — solve long standing problems in algebraic geometry.

The foundations of these new Hodge structures, which we call *Stability Hodge Structures (SHS)* will appear in a paper by the first author, Kontsevich, Pantev and Soibelman (see [1]). In this paper we will look at SHS from the perspective of Landau–Ginzburg models and we will also look at some applications. We will consider simple examples and explain some conjectures these examples suggest. Further elaboration and examples will appear in [1] and [2].

We start with the classical interpretation of wall crossings in Landau–Ginzburg models. After that we describe a hypothetical program of "Stability Hodge Theory" which combines Nonabelian and Noncommutative Hodge theory. We consider some possible applications in this paper. First we consider an approach to the conjecture that the universal covering of a smooth projective variety is holomorphically convex. This is a classical question in algebraic geometry proven by the first author and collaborators for linear fundamental groups [3]. It was believed that for nonresidually finite fundamental groups one needs a different approach and in this paper we outline a procedure of extending the argument to the nonresidually finite case based on SHS. We also outline possible applications to Hodge structures with many filtrations and to Sarkisov's theory.

Stability Hodge Structure is a notion which originates from functions of one complex variable and combinatorics — gaps, polygons, and circuits. We give these classical notions a new read through HMS and category theory, dressing them up with some cluster varieties and integrable systems. After that we enhance these data additionally with some basic nonabelian Hodge theory in order to get a property we need — strictness. In the same way as moduli spaces of Higgs bundles parameterize spectral coverings, the moduli space of deformed stability conditions parameterizes Landau–Ginzburg models.

We believe this is only the tip of the iceberg and this very rich motivic conglomerate of ideas will play an important role in the studies of categories and of algebraic cycles. In particular we suggest that the categorical notion of spectra can be seen as a Hodge theoretic notion related to the "homotopy type of a category".

The paper is organized as follows. In Sections 2 and 3 we describe the classical approach to Landau–Ginzburg models and wall crossings. After that in Sections 4, 5, 6 we define Stability Hodge Structures and build a parallel with Simpson's nonabelian Hodge theory. We also discuss possible applications in Sections 7, 8, 9.

2. Classical Landau–Ginzburg models and wall crossings

In this section we recall the "classical" way of interpreting wall crossing in the case of Landau–Ginzburg models. We will establish a certain combinatorial framework on which we later base our constructions.

We recall the notion of Landau–Ginzburg models from the Laurent polynomials point of view. For more details see, say, [4] and references therein.

Let X be a smooth Fano variety of dimension N. We can associate *a quantum cohomology ring* $QH^*(X) = H^*(X,\mathbb{Q}) \otimes \Lambda$ to it, where Λ is the Novikov ring for X. The multiplication in this ring, the so called *quantum multiplication*, is given by *(genus zero) Gromov–Witten invariants* — numbers counting rational curves lying in X. Given these data one can associate *a regularized quantum differential operator* Q_X (the second Dubrovin connection) — the regularization of an operator associated with connection in the trivial vector bundle given by a quantum multiplication by the canonical class K_X. In "good" cases such as we consider (for Fano threefolds or complete intersections) the equation $Q_X I = 0$ has a unique normalized analytic solution $I = 1 + a_1 t + a_2 t^2 + \ldots$.

Definition 2.1. *A toric Landau–Ginzburg model* is a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm}]$ such that:

- **Period condition:** The constant term of $f^i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm}]$ is a_i for any i (this means that I is a period of a family $f\colon (\mathbb{C}^*)^n \to \mathbb{C}$, see [4]).
- **Calabi–Yau condition:** Any fiber of $f\colon (\mathbb{C}^*)^n \to \mathbb{C}$ after some fiberwise compactification has trivial dualizing sheaf.
- **Toric condition:** There is an embedded degeneration $X \rightsquigarrow T$ to a toric variety T whose fan polytope (the convex hull of generators of its rays) coincides with the Newton polytope (the convex hull of non-zero coefficients) of f. A Laurent polynomial without the toric condition is called *a weak Landau–Ginzburg model*.

Toric Landau–Ginzburg models for complete intersections can be derived from the Hori–Vafa suggestions (see, say, [5]).

Definition 2.2. Let X be a general Fano complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$ in $\mathbb{P}^n$. Let $d_0 = n - d_1 - \ldots - d_k$ be its index. Then a Laurent polynomial

$$f_X = \frac{(x_{1,1} + \ldots + x_{1,d_1-1} + 1)^{d_1} \cdot \ldots \cdot (x_{k,1} + \ldots + x_{k,d_k-1} + 1)^{d_k}}{\prod x_{ij}} + x_{01} + \ldots + x_{0d_0-1}.$$

we call *of Hori–Vafa type*.

Theorem 2.3 (Proposition 9 in [5] and Theorem 2.2, [6])**.** *The polynomial f_X is a toric Landau–Ginzburg model for X.*

Definition 2.4. Let f be a Laurent polynomial in $\mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then a (non-toric birational) symplectomorphism is called *of cluster type* if it is a composition of toric change of variables and symplectomorphisms of type

$$y_0 = x_0 \cdot f_0(x_1, \ldots, x_i)^{\pm 1}, \quad y_1 = x_1, \ldots, y_n = x_n,$$

for some Laurent polynomial f_0 and under this change of variables f goes to a Laurent polynomial for which a Calabi–Yau condition holds.

It is called *elementary of cluster type* if (up to toric change of variables)

$$f_0 = x_1 + \ldots + x_i + 1.$$

It is called *of linear cluster type* if it is a composition of elementary symplectomorphisms of cluster type and toric change of variables.

Remark 2.5. For all examples in the rest of the paper the Calabi–Yau condition holds for all considered cluster type transformations.

Proposition 2.6. *Let f be a weak Landau–Ginzburg model for X. Let f' be a Laurent polynomial obtained from f by symplectomorphism of cluster type. Then f' is a weak Landau–Ginzburg model for X.*

Proof. A period giving constant terms of Laurent polynomials is, up to proportion, an integral of the (depending on $\lambda \in \mathbb{C}$) form $\frac{1}{1-\lambda f}\prod \frac{dx_i}{x_i}$ over a standard n-cycle on the torus $|x_1| = \ldots = |x_n| = 1$. This integral does not change under cluster type symplectomorphisms. □

Example 2.7. Let X be a quadric threefold. There are two types of degenerations of X to normal toric varieties inside the space of quadratic forms. That is,

$$T_0 = \{x_1x_2 = x_2^3\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5]$$

and

$$T_1 = \{x_1x_2 = x_3x_4\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5].$$

Let

$$f_0 = \frac{(x+1)^2}{xyz} + y + z$$

be a weak Landau–Ginzburg model of Hori–Vafa type for X. Let

$$f_1 = \frac{(x+1)}{xyz} + y(x+1) + z$$

be its cluster-type transformation given by the change of variables

$$\frac{y}{(x+1)} \mapsto y.$$

One can see that $T_0 = T_{f_0}$ and $T_1 = T_{f_1}$.

Remark 2.8. One can see that applying the same change of variables a second time to f_1 gives back (up to toric change of variables) f_0.

Example 2.9. Let X be a cubic threefold. There are two types of degenerations of X to normal toric varieties inside the space of cubic forms. That is,

$$T_0 = \{x_1x_2x_3 = x_4^3\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5]$$

and

$$T_1 = \{x_1x_2x_3 = x_4^2x_5\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5].$$

Let

$$f_0 = \frac{(x+y+1)^3}{xyz} + z$$

be a weak Landau–Ginzburg model of Hori–Vafa type for X. Let

$$f_1 = \frac{(x+y+1)^2}{xyz} + z(x+y+1)$$

be its cluster-type transformation given by the change of variables

$$\frac{z}{(x+y+1)} \mapsto z.$$

One can see that $T_0 = T_{f_0}$ and $T_1 = T_{f_1}$.

Remark 2.10. Applying this change of variables a second time to f_1 we get (up to toric change of variables) f_1 again and applying it a third time we get f_0 back.

Example 2.11. Let X be a cubic fourfold. There are three types of degenerations of X to normal toric varieties inside the space of cubic forms. That is,

$$T_{00} = \{x_1x_2x_3 = x_4^3\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5 : x_6],$$

$$T_{10} = \{x_1x_2x_3 = x_4^2x_5\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5 : x_6],$$

and

$$T_{11} = \{x_1x_2x_3 = x_4x_5x_6\} \subset \mathbb{P}[x_1 : x_2 : x_3 : x_4 : x_5 : x_6],$$

Let

$$f_{00} = \frac{(x+y+1)^3}{xyzt} + z + t$$

be a weak Landau–Ginzburg model of Hori–Vafa type for X. Let

$$f_{10} = \frac{(x+y+1)^2}{xyzt} + z(x+y+1) + t$$

be its cluster-type transformation given by the change of variables

$$\frac{z}{(x+y+1)} \mapsto z$$

and let

$$f_{11} = \frac{(x+y+1)}{xyzt} + z(x+y+1) + t(x+y+1)$$

be the cluster-type transformation of f_{10} given by the change of variables

$$\frac{t}{(x+y+1)} \mapsto t.$$

One can see that $T_{00} = T_{f_{00}}$, $T_{10} = T_{f_{10}}$, and $T_{11} = T_{f_{11}}$.

Remark 2.12. Applying the first change of variables a second time to f_{10} we get (up to toric change of variables) f_{10} again, applying it once more we get f_{00}, and applying any change of variables to f_{11} we get f_{10}.

Example 2.13. Consider quadrics in $\mathbb{P} = \mathbb{P}(1,1,1,1,2)$. Denote the coordinates in $\mathbb{P}$ by x_0, x_1, x_2, x_3, x_4, where the weight of x_4 is 2. The general quadric is

$$T_1 = \{F_2(x_0, x_1, x_2, x_3) + \lambda x_4 = 0\},$$

where F_2 is a quadratic form and $\lambda \in \mathbb{C} \setminus 0$. Projection on the hyperplane generated by $x_0, \ldots, x_3$ gives an isomorphism of T_1 with $\mathbb{P}^3$. The general variety with $\lambda = 0$ is a toric variety

$$T_2 = \{x_0 x_1 = x_2 x_3\}.$$

It degenerates to

$$T_3 = \{x_1 x_2 = x_0^2\}.$$

One can see that T_3 is an image of $\mathbb{P}(1,1,2,4)$ under the Veronese map v_2.

Consider the following 3 weak Landau–Ginzburg models for $\mathbb{P}^3$:

$$f_1 = x + y + z + \frac{1}{xyz},$$

$$f_2 = x + \frac{y}{x} + \frac{z}{x} + \frac{1}{xy} + \frac{1}{xz},$$

$$f_3 = \frac{(x+1)^2}{xyz} + \frac{y}{z} + z.$$

Changing toric variables one can rewrite f_1 as

$$f_1' = z(x+1) + y + \frac{1}{xyz^2},$$

$$f_1'' = z(x+1) + \frac{y}{z} + \frac{1}{xyz}.$$

The cluster-type change of variables

$$x \mapsto x, \quad y \mapsto y, \quad z(x+1) \mapsto z$$

sends f_1' to a Laurent polynomial that differs from f_3 by a toric change of variables and f_1'' to a polynomial

$$z + \frac{(x+1)y}{z} + \frac{(x+1)}{xyz},$$

which differs from f_2 by toric change of variables.The cluster-type change of variables

$$x \mapsto x, \quad y(x+1) \mapsto y, \quad z \mapsto z$$

sends the last expression to f_3.

One can see that $T_1 = T_{f_1}$, $T_2 = T_{f_2}$, and $T_3 = T_{f_3}$.

Theorem 2.14 (Hacking–Prokhorov, [7])**.** *Let X be a degeneration of $\mathbb{P}^2$ to a $\mathbb{Q}$-Gorenstein surface with quotient singularities. Then $X = \mathbb{P}(a^2, b^2, c^2)$, where (a, b, c) is any solution of the Markov equation $a^2 + b^2 + c^2 = 3abc$.*

Remark 2.15. All Markov triples are obtained from the basic one $(1, 1, 1)$ by a sequence of *elementary transforms*

$$(a, b, c) \mapsto (a, b, 3ab - c).$$

Proposition 2.16 (S. Galkin)**.** *Let (a,b,c) be a Markov triple and let f be a weak Landau–Ginzburg model for $\mathbb{P}^2$ such that $T_f = \mathbb{P}(a^2,b^2,c^2)$. Then there is an elementary cluster-type transformation such that for the image f' of f under this transformation $T_{f'} = \mathbb{P}(a^2,b^2,(3ab-c)^2)$.*

Sketch of the proof (S. Galkin). Consider $d \geq c$ such that $3ad = b \pmod{c}$. One can check that we can choose toric coordinates x, y such that in these coordinates vertices of the Newton polytope of f are (d,c), $(d-c,c)$, and $(-\frac{d(3ab-c)-b^2}{c}, -3ab+c)$. Let p be the k-th integral point from the end of an edge of integral length n of the Newton polytope of f. Then the coefficient of f at p is $\binom{n}{k}$ (this can be proved by induction). This means that

$$f = x^{d-c}y^c(x+1)^c + \frac{1}{x^{\frac{d(3ab-c)-b^2}{c}}y^{3ab-c}} + \sum_r y^{-n_r} f_r(x),$$

where n_i's are non-negative and f_i's are some Laurent polynomials in x. One can check that the change of variables of cluster type

$$y' = y(x+1), \quad x' = x$$

sends f to a weak Landau–Ginzburg model f' such that $T_{f'} = \mathbb{P}(a^2,b^2,(3ab-c)^2)$. □

We extend observed connection between degenerations and birational transformations further to a general connection between geometry of moduli space of Landau–Ginzburg models, birational and symplectic geometry. We summarize this connection in Table 1 and we will investigate it (mainly conjecturally) in the sections that follow.

	Fano variety X	Landau–Ginzburg model $LG(X)$
A side	$\mathsf{Fuk}(X)$: symplectomorphisms and general degenerations	$FS(LG(X))$: degenerations
B side	$D^b_{sing}(LG(X))$: phase changes	$D^b(X)$: birational transformations

Table 1. Wall crossings.

3. Minkowski decompositions and cluster transformations

Definition 3.1. Let $N \cong \mathbb{Z}^n$ be a lattice. Denote $N_\mathbb{R} = N \otimes \mathbb{R}$. *A polytope* $\Delta \subset N_\mathbb{R}$ is a convex hull of finite number of points in $N_\mathbb{R}$. A polytope is called *integral* iff these points lie in $N \otimes 1$. A polytope is called *primitive* if it is integral and its vertices are primitive. A Laurent polynomial is called *primitive* if its Newton polytope is primitive.

Definition 3.2. *The Minkowski sum* $\Delta_1 + \ldots + \Delta_k$ of polytopes $\Delta_1, \ldots, \Delta_k$ is the polytope $\{v_1 + \ldots + v_k | v_i \in \Delta_i\}$. An integral polytope is called *irreducible* if it can't be presented as a Minkowski sum of two non-trivial integral polytopes.

Remark 3.3. A Minkowski sum of integral polytopes is integral.

Definition 3.4. Consider an integral polytope $\Delta \subset \mathbb{Z}^n$. *A Minkowski presentation* of Δ is a presentation of each of its faces as a Minkowski sum of irreducible integral polytopes such that if a face Δ' lies in a face Δ then the intersections of Minkowski summands for Δ with Δ' give a presentation for Δ'.

Consider a Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$. For any face Δ of Δ_f denote the sum of all monomials of f lying in Δ by f_Δ. The polynomial f is called a *Minkowski polynomial* if there exists a Minkowski presentation such that, for any face Δ of Δ_f with given Minkowski sum expansion $\Delta = \Delta_1 + \ldots + \Delta_k$, there are Laurent polynomials $f_{\Delta_i} \in \mathbb{C}[\mathbb{Z}^n]$ such that the coefficients of f_{Δ_i} at vertices of Δ_i are 1's and $f_\Delta = f_{\Delta_1} \cdot \ldots \cdot f_{\Delta_k}$.

Remark 3.5. Let e be an edge of a Minkowski Laurent polynomial of integral length n. Its unique Minkowski expansion to irreducible summands is the expansion to n segments of integral length 1. Thus the coefficient of the monomial associated to the i'th integral point of e (from any end) is $\binom{n}{i}$.

Remark 3.6. Toric Landau–Ginzburg models of Hori–Vafa type or toric Landau–Ginzburg models from [5] are Minkowski Laurent polynomials.

Example 3.7 (Ilten–Vollmert construction, [8])**.** Consider an integral polytope $\Delta \subset N = \mathbb{Z}^n$. Let the origin of N lie strictly inside Δ. Let $X = T_\Delta$ be the toric variety whose fan is the face fan for Δ. Denote the dual lattice to N by $M = N^\vee$. Put $N' = N \oplus \mathbb{Z}$, $M' = M \oplus \mathbb{Z}$. Let C be the cone generated by $(\Delta, 1)$. Then $X = Proj\, \mathbb{C}[C^\vee \cap M']$ with grading given by $d = (0,1) \in M'$. For any primitive $r \in M'$, consider the map $r\colon N' \to \mathbb{Z}$. Let $L_r = ker(r)$. Let s_r be a retract (cosection) of the inclusion $i\colon L_r \to N'$, that is, a map $N' \to L_r$ such that $s_r i = Id_{L_r}$. It is unique up to translations along L_r. Let $C^+ = s_r(\{p \in C | \langle p, r\rangle = 1\})$, and $C^- = s_r(\{p \in C | \langle p, r\rangle = -1\})$ be two "slices" of C cut out by evaluating function at r.

Choose r such that $r = (r_0, 0) \in M'$ and such that C^- is a cone with its single vertex a lattice point. Consider a Minkowski decomposition $C^+ = C_1 + C_2$ to (possibly rational) polytopes such that for any vertex v of C^+, at least one of the corresponding vertices in C_1 and C_2 is a lattice point. Let D be the cone in $L_r \oplus \mathbb{Z}$ generated by $(C^-, 0)$, $(C_1, 1)$, and $(C_2, -1)$. Denote $X' = Proj\, \mathbb{C}[D^\vee \cap (L_r \oplus \mathbb{Z})^\vee)]$ where the grading is now given by $(s_r(d), 0)$.

Proposition 3.8 (Remark 1.8 and Theorem 4.4 in [8])**.** *There is an embedded degeneration of X' to X.*

Example 3.9 (Ilten)**.** Let $\Delta \subset \mathbb{Z}^2$ be the convex hull of the points $(-1, 2)$, $(1, 2)$, and $(0, -1)$. Then $X = \mathbb{P}(1, 1, 4)$. Let $r = (0, 1, 0)$. Then s_r is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We are in the setup of Example 3.7 (see Figure 1). The vertex of $\{p \in C | \langle p, r\rangle = -1\}$ is $(0, -1, 1)$ and goes to a vertex $(0, 0)$ under s_r and the vertices of $\{p \in C | \langle p, r\rangle = 1\}$

are $(\pm\frac{1}{2}, 1, \frac{1}{2})$ and goes to vertices $(\pm\frac{1}{2}, \frac{3}{2})$ under s_r. That is, we have a Minkowski decomposition drawn on Figure 2.

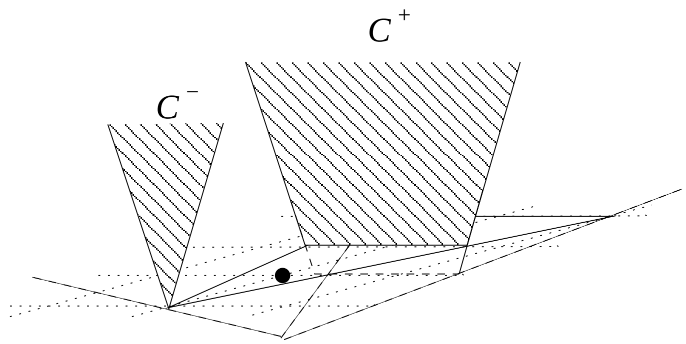


FIGURE 1. Deformation of $\mathbb{P}(1,1,4)$.

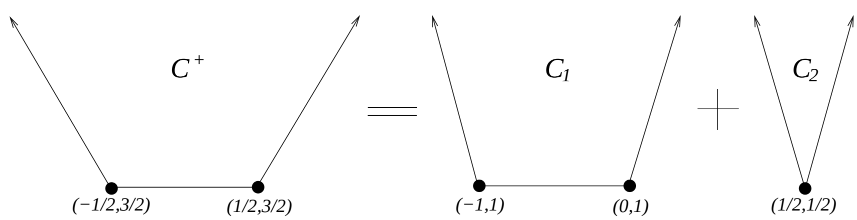


FIGURE 2. Decomposition of C^+.

The polytope for X' is a convex hull of points $(-1,1)$, $(0,1)$, and $(1,-2)$ since the second coordinate becomes to be equal to 1 not on $(C_2,-1)$ but on $(2C_2,-2)$. Its face fan is a fan of $\mathbb{P}^2$. Thus we get a deformation of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$.

The following proposition shows that the degenerations given by Example 3.7 give cluster transformations for Minkowski polynomials.

Proposition 3.10. *Let $\Delta = \Delta_f$ be the Newton polytope of a Minkowski polynomial f. Let Δ' be a polytope obtained from Δ by the procedure described in Example 3.7 given by integral Minkowski summands agreeing with the Minkowski decompositions of the faces of Δ. Then $\Delta' = \Delta_{f'}$ for some Minkowski polynomial f'.*

Proof. Let $f \in \mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. After toric changes of variables we can assume that s_r is the projection on coordinates $x_1, \ldots, x_n$. Then

$$f = f_+(x_1, \ldots, x_n)x_0 + f_0(x_1, \ldots, x_n) + \frac{f_-(x_1, \ldots, x_n)}{x_0}.$$

As f is a Minkowski polynomial we have $f_+ = f_1 f_2$. Thus after change of variables $x_0 \to x_0/f_2$ we get a Minkowski polynomial

$$f' = f_1(x_1, \ldots, x_n)x_0 + f_0(x_1, \ldots, x_n) + \frac{f_-(x_1, \ldots, x_n)f_2(x_1, \ldots, x_n)}{x_0}$$

with Newton polytope Δ'. □

Remark 3.11. Example 3.9 shows that the statement of Proposition 3.10 holds for non-integral case as well. This example is the first non-trivial cluster transformation given by Proposition 2.16.

Example 3.12. Let Δ be the convex hull of points $(-1,1)$, $(1,1)$, and $(0,-1)$. Then X is a quadratic cone $\mathbb{P}(1,1,2)$. (A unique) Minkowski polynomial for Δ is

$$f = \frac{(x+1)^2 y}{x} + \frac{1}{y}.$$

After cluster change of variables $y \to \frac{y}{x+1}$ we get a polynomial

$$\frac{(x+1)y}{x} + \frac{x+1}{y}.$$

It is (a unique) Minkowski polynomial for the polytope Δ' — the convex hull of points $(-1,-1)$, $(0,-1)$, $(1,-1)$, and $(0,-1)$. These points generate the fan of a smooth quadric X'.

4. Degenerations and wall crossings

In the previous section we have established certain combinatorial structures — cluster transformations connected to wall crossings. We will relate these combinatorial structures to the moduli space of stability conditions. We do this in two steps:

Step 1. First we relate the combinatorial structures to the "moduli space of Landau–Ginzburg models".

Step 2. Next we describe hypothetically how the "moduli space of Landau–Ginzburg models" fits in a "twistor family" with generic fiber the moduli space of stability conditions of a Fukaya–Seidel category.

We start with step one — collecting all Landau–Ginzburg models in a moduli space. The idea is to record wall crossings as relations in the mapping class group and then relations between relations and so on. This suggests a connection with Hodge theory and higher category theory. We will start with a rather simple approach which we will enhance later in order to serve our purposes. Nearly ten years ago it was discovered that, while the symplectic mapping class group of a curve equals the ordinary (oriented) mapping class group, these two groups differ greatly for higher dimensional symplectic manifolds. Understanding the structure of these groups has been a goal of many researchers in symplectic geometry. The initial purpose of the construction below was to obtain a presentation of the symplectic mapping class group of toric hypersurfaces. Along the way we have obtained a characterization of the zero fiber of a Stability Hodge Structure.

To explain our approach, we recall some notation and constructions. Assume $A \subset \mathbb{Z}^d$ is a finite set, X_A is the polarized toric variety associated to A with ample line bundle $\mathcal{L}$. In [9], the secondary polytope $\mathrm{Sec}(A)$ parameterizing regular subdivisions was constructed and shown to be the Newton polytope of the E_A determinant (a type of discriminant). We realize the toric variety associated to $\mathrm{Sec}(A)$ as the coarse moduli space of a stack $\mathcal{X}_{\mathrm{Sec}(A)}$ defined in [10]. We observe that the stack $\mathcal{X}_{\mathrm{Laf}(A)}$ constructed in [10] has a proper map π to $\mathcal{X}_{\mathrm{Sec}(A)}$ whose fibers are degenerations of X_A, and we constructed a polytope $\mathrm{Laf}(A)$ which is dual to the fan defining $\mathcal{X}_{\mathrm{Laf}(A)}$. The zero set $\mathcal{H}_{\mathrm{Sec}(A)}$ of a section of the associated line bundle parameterizes sections of $\mathcal{L}$ and degenerated sections are hypersurfaces in the associated degenerated toric variety. Upon restriction, we obtain a proper map $\pi : \mathcal{H}_{\mathrm{Sec}(A)} \to \mathcal{X}_{\mathrm{Sec}(A)}$ with non-singular fibers symplectomorphic to any non-degenerate section of $\mathcal{L}$.

Since $\pi : \mathcal{H}_{\mathrm{Sec}(A)} \to \mathcal{X}_{\mathrm{Sec}(A)}$ is a proper map, we may consider symplectic parallel transport of the non-singular fibers along paths in the complement of the zero set Z_A of the E_A determinant. Denote by H_p the fiber of π. We observe that the subset of the fibers meeting the toric boundary H are horizontal in the sense that if $q \in \partial H_p$ then the symplectic orthogonal $(T_q H_p)^{\perp_\omega} \subset T_q(\partial \mathcal{H}_{\mathrm{Sec}(A)})$, where ω corresponds to a restriction of Fubini–Study metric. That is, parallel transport is a symplectomorphism that preserves the boundary of the hypersurfaces. Choosing a base point p of $\mathcal{X}_{\mathrm{Sec}(A)} \setminus Z_A$, we obtain a map from the based loop space $\rho : \Omega(\mathcal{X}_{\mathrm{Sec}(A)} \setminus Z_A) \to \mathrm{Symp}^{\partial}(H_p)$ and a group homomorphism

$$\rho_* : \pi_1(\mathcal{X}_{\mathrm{Sec}(A)} \setminus Z_A) \to \pi_0(\mathrm{Symp}^{\partial}(H_p)),$$

where $\pi_0(\mathrm{Symp}^{\partial}(H_p))$ is a mapping class group. However, from a field theory perspective, this homomorphism in imprecise; one should consider not only symplectomorphisms preserving the boundary, but also those that preserve the normal bundle of the boundary. In this way, we can glue two hypersurfaces together without creating an ambiguity in the symplectomorphism groups. We call such a symplectomorphism *boundary framed morphism* and denote the corresponding group $\mathrm{Symp}^{\partial,\mathrm{fr}}(H_p)$. For toric hypersurfaces, this group is a central extension of $\mathrm{Symp}^{\partial}(H_p)$. It is not generally the case, however, that parallel transport preserves the framing, but the change in framing can be controlled by keeping track of the homotopies in $\Omega(\mathcal{X}_{\mathrm{Sec}(A)} \setminus Z_A)$ or by passing to the loop space of an auxiliary real torus bundle $\mathcal{E} \to \mathcal{X}_{\mathrm{Sec}(A)} \setminus Z_A$, giving a homomorphism

$$\tilde{\rho}_* : \pi_1(\mathcal{E}) \to \pi_0(\mathrm{Symp}^{\partial,\mathrm{fr}}(H_p)).$$

In many cases, this homomorphism is surjective.

The stack $\mathcal{X}_{\mathrm{Sec}(A)}$ is as complicated combinatorially as the secondary polytope $\mathrm{Sec}(A)$, which is computationally expensive to describe. While the Newton polytope of E_A was found in [9], Z_A is far from smooth and there are open questions about its singular structure. We bypass these difficulties by considering only the lowest dimensional boundary strata of $\mathcal{X}_{\mathrm{Sec}(A)}$ where non-trivial behavior occurs. Thus the first and main case we examine are the one dimensional boundary strata of $\mathcal{X}_{\mathrm{Sec}(A)}$. Combinatorially, these are known as circuits.

A circuit A is a collection of $d+2$ points in $\mathbb{Z}^d$, such that there are exactly two coherent triangulations (see [9]) of A, so the secondary polytope is a line segment and the secondary stack a weighted projective line $\mathbb{P}(a,b)$. Z_A is either two or three points; two of the points are the equivariant orbifold points $\{0,\infty\}$ and the possible third is an interior point. Both the constants, a,b and the number of points in Z_A depends on the convex hull and affine positioning of A — for more details see [2]. When Z_A consists of three points, their complement retracts onto a figure eight and the fundamental group is free on two letters. In this case, we have the based loops $\delta_1, \delta_2, \delta_3 = \delta_2^{-1}\delta_1^{-1}$ encircling the three points. The symplectic monodromy $T_i = \tilde{\rho}_*(\delta_i)$ is computable from known results in symplectic geometry as either spherical Dehn twists or as twists about a tropical decomposition. The image via $\tilde{\rho}_*$ gives the relation

$$T_1T_2T_3 = T_{\partial H_p}, \tag{1}$$

where $T_{\partial H_p}$ is the central element determined by twisting the framing about the toric boundary. One of the most elementary examples is $X_A = \mathbb{P}^1\times\mathbb{P}^1$ with polarization $\mathcal{O}(1,1)$, and the circuit is the four vertices of a unit square with the two diagonal triangulations. Here the hypersurface is $\mathbb{P}^1$ with four boundary points and the relation obtained above yields a classical relation in the mapping class group called the *Lantern relation*.

When Z_A consists of two points, one is an orbifold point and the other is a point with trivial stabilizer. If δ_1, δ_2 are based paths encircling Z_A and T_1, T_2 are the associated symplectomorphisms, we obtain a relation

$$(T_1T_2)^a = T_{\partial H_p}. \tag{2}$$

A basic example of this relation arises as the homological mirror to $\mathbb{P}^2$ which is the set $A = \{(0,0),(1,0),(0,1),(-1,-1)\}$. The constant a occurring above is 3 and the relation is in fact another classical mapping class group relation known as the *star relation*.

We call the boundary framed, symplectic mapping class group relation occurring in equations 1 and 2 *the circuit relation*. In general, any complex line in $\mathcal{X}_{\mathrm{Sec}(A)}$ yields a relation in $\mathrm{Symp}^{\partial,\mathrm{fr}}(H_p)$ by homotoping the product of all the loops around the intersections with Z_A to the identity. However, each such line can be degenerated to a chain of equivariant lines which are precisely circuits supported on A. Thus every relation obtained this way can be thought of as arising from a composition of circuit relations. As we saw in the previous two sections Landau–Ginzburg mirrors of Fano manifolds are fibrations of Calabi–Yau hypersurfaces. Therefore the above simple examples generalize to

Theorem 4.1 ([2]). *Landau–Ginzburg mirrors of Fano manifolds can be obtained by a superposition of circuits described above.*

Interpreting Landau–Ginzburg models as lines in the secondary stack we get

Theorem 4.2 ([2]). $\mathcal{X}_{Sec(A)}$ *can be seen as moduli space of Landau–Ginzburg models. In particular some wall crossings correspond to passing through* Z_A.

These two theorems complete Step 1.

5. Wall crossings and Stability Hodge Structures

We move to Step 2, building a "twistor family" with generic fiber the moduli space of stability conditions for Fukaya–Seidel categories — see [11].

Stability Hodge Structures. We start with Stability Hodge Structures, an artifact of Donaldson–Thomas (DT) invariants. We will mainly consider Fukaya–Seidel categories but discussion in this section applies in general.

The theory of Donaldson–Thomas invariants and wall crossing has become a central subject of Geometry and Physics. In a nutshell DT invariants are virtual numbers of stable objects in three dimensional Calabi–Yau category. Kontsevich and Soibelman suggested Donaldson–Thomas invariants applicable to triangulated category and Bridgeland stability conditions — a refined version of so called *motivic Donaldson–Thomas invariants* — MDT. The wall crossing formulae (WCF) of MDT are expressed in terms of factorization of quantum torus. A connection with nonabelian Hodge structures comes naturally here. WCF for the Hitchin system is connected to ODE with small parameter and its asymptotic behavior. In fact the WCF relates to Stokes data at infinity for this ODE and connects with the work of Ecalle and Voros on resurgence.

We will introduce a new geometric structure which seems to be present in many of above considerations — Stability Hodge Structures. These structures seem to have a huge potential of geometric applications some of which we discuss.

The moduli space of stability conditions of a category C is very complicated with possibly fractal boundary. In the case of derived category of Calabi–Yau manifolds of dimension three and higher there is not any hypothetical description. Still HMS predicts that the moduli space of mirror dual Calabi–Yau manifolds is embedded in a locally closed cone in the moduli space of stability conditions of a category C. So it is a big open question how to characterize Hodge structures corresponding to mirror duals. Classically the moduli space of pure Hodge structures has a compactification by Mixed Hodge Structures (MHS). So it is natural to study limiting Donaldson–Thomas invariants and relate to WCF.

In the case of three-dimensional Calabi–Yau manifolds there are different types of MHS. The cusp case — the deepest degeneration — corresponds to t-structures which is an extension of Tate motives. As a result we take a generating series of Donaldson–Thomas rank one torsion free invariants. It is known that in this case this generating series (modulo change of coordinates) is the classical Gromov–Witten series which satisfies the holomorphic anomaly equation. This translates into automorphic property for the DT generating function. We expect that automorphic property holds for higher ranks and plan to study it and show that WCF is necessary to assemble limiting data.

A different MHS corresponds to conifold points and non-maximal degeneration points. The wall crossings and DT data give a family of Integrable Systems in the following way. The vanishing cycles Γ_{short} and the monodromy define a quotient category $\mathcal{T}/\mathcal{A}$ with the following sequence on the level of K–theory:

$$\Gamma_{short} \to K_0(\mathcal{T}) \to K_0(\mathcal{T}/\mathcal{A}).$$

Using the Kontsevich–Soibelman noncommutative torus approach we define a superscheme

$$\mathbb{G} = \oplus_{p \in \Gamma_{short}} \mathbb{G}_p \to T_{non}.$$

Consider the zero grade $\mathbb{G}_0$ of $\mathbb{G}$ over $\mathbb{Z}$. The global sections of $\mathbb{G}_0$ define the Betti moduli space — an integrable system

$$\Gamma(G_0) = \oplus \mathcal{O}(M_j).$$

In order to consider the interaction with the rest of the category we include global WCF. In this case we obtain a torus action, which produces a stack over the Betti moduli space:

$$X/(\mathbb{C}^*)^{\times n} \to M_1 \times M_2 \times \ldots \times M_k.$$

All these stacks fit in a constructible sheaf.

To summarize we give a provisional definition, which covers the cases of Bridgeland, geometric (volume forms), and generalized (log forms) stability conditions:

Definition 5.1. A *Stability Hodge Structure (SHS)* for a Fukaya–Seidel category $\mathcal{F}$ is the following data:

i) The moduli space of stability conditions S for $\mathcal{F}$.

ii) Divisor D at infinity giving a partial compactification of S and parametrizing the degenerated limiting stability conditions — stability conditions for quotient categories, the category factored by the objects (vanishing cycles) on which stability conditions vanish.

iii) Besides the degeneration we record the WCF — all recorded together. Over each point of D we put the Betti moduli space locally produced by WCF. All these moduli space fit in a constructible sheaf over S.

Let us illustrate these structures through two examples. We start with the category $\widetilde{\mathbb{A}}_2$ — the Fukaya category of the conic bundle $\{uv = y^2 - x^3 - ax - b\}$, $a, b \in \mathbb{C}$. In this case, the Stability Hodge Structure is a sheaf over $\mathbb{C}^2$ with coordinates a, b.

The points of the discriminant parameterize limiting stability conditions. The fibers are Betti moduli spaces of vanishing cycles which generically over the discriminant are the affine surface $z(1 - xy) = 1$. The special fiber over the cusp is the moduli space $M_{0,5}$ of rank two bundles over the projective line with one irregular singularity and five Stokes directions at infinity (see Figure 3).

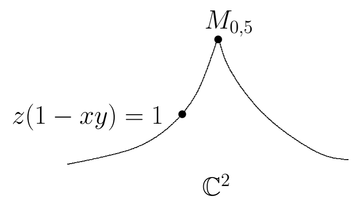


FIGURE 3. Compactification of moduli space of stability conditions for a category $\widetilde{\mathbb{A}}_2$.

A different example is the Fukaya–Seidel category $\mathbb{A}_4$. We start with a generic polynomial $p \in \mathbb{C}[z]$ of degree 5. It defines a Riemann surface $C = \{p(z) = w\}$ and 5:1-covering $\varphi\colon C \to \mathbb{C}$. The ramification locus for φ are 4 points $p_1, \ldots, p_4$— roots of p'. Consider 4 paths $l_1, \ldots, l_4$ from p_i's to infinity. The polynomial p is generic, so the ramification is as simple as it can be and $\varphi^{-1}(l_i)$ are thimbles covering l_i's 2:1. They generate a Fukaya category for C and correspond to vertices of the $\mathbb{A}_4$ quiver. "Neighbor" thimbles intersect at infinity: i-th one intersects $(i+1)$-th at one point. These intersections correspond to arrows between vertices in the quiver.

In this example the divisor D at infinity parameterizes the semiorthogonal decompositions of the $\mathbb{A}_4$ category. The fibers of the constructible sheaf are moduli spaces of stability conditions for $\mathbb{A}_3 \times \mathbb{A}_1$ categories. Similarly on the singular points of D we get as fibers moduli spaces of stability conditions for $\mathbb{A}_2 \times \mathbb{A}_2$ categories. This leads to a rich mixed Hodge theory structure associated with D and monodromy action around it. In the next section we will see that in the limit the stability conditions behave as coverings so the above picture fits. This monodromy relates to the wall-crossings changes. In particular it sends the preferred set of thimbles generating the $\mathbb{A}_4$ category from a generator consisting of the sum of 4 thimbles $G = L_1 + L_2 + L_3 + L_4$ (with $Hom(L_i, L_{i+1})$ of rank 1) to $G' = L' + L_1 + L_3 + L_4$ by a mutation. This mutation reduces the generation time (see Section 7) from $t(G) = 3$ to $t(G') = 2$. We will represent it as an invariant of of Stability Hodge Structures in Section 7.

In the next section we build a twistor type of family where the generic fiber is a SHS.

6. Higgs bundles and stability conditions — analogy

In this section we proceed describing the analogy between Nonabelian and Stability Hodge Structures. We build the "twistor" family so that the fiber over zero is the "moduli space" of Landau–Ginzburg models and the generic fiber is the Stability Hodge Structure defined above.

Noncommutative Hodge theory endows the cohomology groups of a dg-category with additional linear data — the noncommutative Hodge structure — which records important information about the geometry of the category. However, due to their linear nature, noncommutative Hodge structures are not sophisticated enough to codify the full geometric information hidden in a dg-category. In view of the homological complexity of such categories it is clear that only a subtler non-linear Hodge theoretic entity can adequately capture the salient features of such categorical or noncommutative geometries. In this section by analogy with "classical nonabelian Hodge theory" we construct and study from such a perspective a new type of entity of exactly such type — the Stability Hodge Structure associated with a dg-category.

As the name suggests, the SHS of a category is related to the Bridgeland stabilities on this category. The moduli space Stab_C of stability conditions of a triangulated dg-category C is, in general, a complicated curved space, possibly with fractal boundary. In the special case when C is the Fukaya category of a Calabi–Yau threefold, the space Stab_C admits a natural one-parameter specialization to a much simpler space S_0. Indeed, HMS predicts that the moduli space of complex structures on the Calabi–Yau threefold maps to a Lagrangian subvariety $\mathsf{Stab}_C^{\mathrm{geom}} \subset \mathsf{Stab}_C$. (Recall the holomorphic volume form and integrating it defines a stability condition and its charges.) The idea is now to linearize Stab_C along $\mathsf{Stab}_C^{\mathrm{geom}}$, i.e., to replace Stab_C with a certain discrete quotient S_0 of the total space of the normal bundle of $\mathsf{Stab}_C^{\mathrm{geom}}$ in Stab_C. Specifically, by scaling the differentials and higher products in C, one obtains a one parameter family of categories C_λ with $\lambda \in \mathbb{C}^*$, and an associated family $\mathsf{S}_\lambda := \mathsf{Stab}_{C_\lambda}$, $\lambda \in \mathbb{C}^*$ of moduli of stabilities. Using holomorphic sections with prescribed asymptotic at zero one can complete the family $\{\mathsf{S}_\lambda\}_{\lambda\in\mathbb{C}^*}$ to a family $\mathsf{S} \to \mathbb{C}$ which in a neighborhood of $\mathsf{Stab}_C^{\mathrm{geom}}$ behaves like a standard deformation to the normal cone. The space S_0 is the fiber at 0 of this completed family and conjecturally $\mathsf{S} \to \mathbb{C}$ is one chart of a twistor-like family $\mathcal{S} \to \mathbb{P}^1$ which is by definition *the Stability Hodge Structure associated with* C.

Stability Hodge Structures are expected to exist for more general dg-categories, in particular for Fukaya–Seidel categories associated with a superpotential on a Calabi–Yau space or with categories of representations of quivers. Moreover, for special non-compact Calabi–Yau 3-folds, the zero fiber S_0 of a Stability Hodge Structure can be identified with the Dolbeault realization of a nonabelian Hodge structure of an algebraic curve. This is an unexpected and direct connection with Simpson's nonabelian Hodge theory which we exploit further suggesting some geometric applications.

We briefly recall nonabelian Hodge theory settings. According to Simpson we have one parametric twistor family such that the fiber over zero is the moduli space of Higgs bundles and the generic fiber is the moduli space of representations of the fundamental group — M_{Betti}.

In this section we state that we expect similar behavior of moduli space of stability conditions. The moduli space of stability conditions of a Fukaya–Seidel category can be included in a one parameter twistor family, and we describe the fiber over zero in details in the next subsection.

We give an example:

Example 6.1 ("twistor" family for Stability Hodge Structures for the category $\mathbb{A}_n$)**.** We will give a brief explanation the calculation of the "twistor" family for the SHS for the category $\mathbb{A}_n$. We start with the moduli space of stability conditions for the category $\mathbb{A}_n$, which can be identified with differentials $e^p dz$, where $p \in \mathbb{C}[z]$ is a generic polynomial of degree $n+1$, see [1].

Let us denote one particular holomorphic form $e^p dz$ by Vol. Locally there exists a holomorphic coordinate w such that $Vol = dw$. Geodesics in the metric $|Vol|^2$ are the straight real lines in the coordinate w, the same as real lines on which Vol has constant phase. Therefore they are special Lagrangians for Vol (and in fact for any real symplectic structure).

Observe that these geodesics are asymptotic to infinity because the integral of $|Vol| = e^{Re(p)}|dz|$ absolutely converges on them hence $Re(p)$ approaches infinity, as these lines are noncompact in the uncompactified plane z, and therefore $|z|$ goes to infinity. To compensate infinite length in the usual metric $|dz|^2$ we use the fact that $e^{Re(p)}$ converges to zero iff $Re(p)$ converges to minus infinity.

So after completion in the metric defined above (so the vertices are in the finite part now) we enhance the polygon by assigning angles and lengths. These enhanced polygons record our stability conditions. Indeed we have $(2(n+1)-3)$-dimensional space of polygons plus one global angle — it is a real $2n$ dimensional space. In Example 6.2 we give a simple example illustrating the polygons for the category $\mathbb{A}_2$ and a wall crossing phenomenon. The stable objects correspond to edges and diagonals. In the picture in the example we lose one stable object while crossing a wall.

Example 6.2 (stability for $\mathbb{A}_2$)**.** For $\mathbb{A}_2$ category we have $\deg p = 3$. The left part of Figure 4 represents two of the stable objects for the $\mathbb{A}_2$ category. The third stable object is the third edge of the triangle. The wall crossing makes the angle between the first two edges bigger then π and as a result the third edge is not a stable object any more.

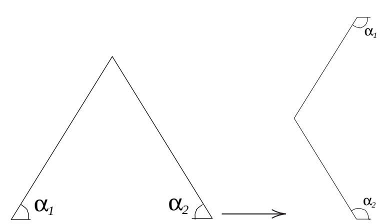


FIGURE 4. Stability conditions for the $\mathbb{A}_2$ category.

Now we consider the "twistor" family — the limit of $e^{p(z)/u}dz$, where u is a complex number tending to 0. Geometrically limit differential can be identified with graphs — see Example 6.3.

Example 6.3 (limit of $e^{p/u}dz$)**.** Take a limit of $e^{p/u}dz$ with u tending to zero. The limits of polygons are graphs. We record the length, angle, and monodromy and this defines a covering of the complex plane. Thus this construction identifies a limit of moduli space of stability conditions for $\mathbb{A}_n$ category with some Hurwitz subspace — a subscheme of coverings. In particular, these two spaces have the same number of components. Figure 5 represents a procedure of associating the monodromy of the covering to the vertices of the graph.

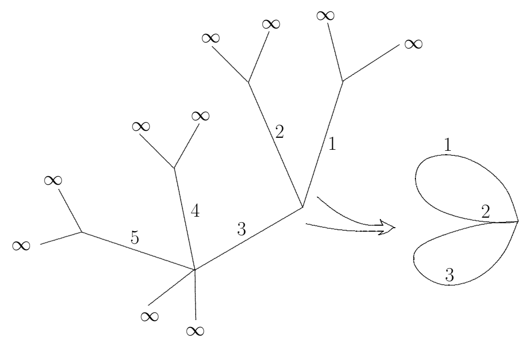


Figure 5. Building coverings out of limit.

Remark 6.4. Similarly one can compute the "twistor" family for the equivariant $\mathbb{A}_n$ category and see appearance of gaps in spectra in connection with the weight filtration of completions of special local rings — see Section 7. Observe that the idea of coverings brings the Fukaya category of a Riemann surface of genus g very close to the $\mathbb{A}_{2g+1}$ category. Also product of Fukaya categories of curves in combination with Luttinger surgeries gives many opportunities for stability conditions with many components as well as many possibilities for the behavior of gaps and spectra. The interplay between coverings and stability conditions suggests that one can have symplectic manifolds with the same Fukaya categories but different moduli of stability conditions. We conjecture that the moduli spaces of coverings obtained near different cusps being different algebraically should imply that this different manifolds are nonsymplectomorphic.

The fiber over zero. The fiber over zero (described in what follows) plays an analogous role to the moduli space of Higgs bundles in Simpson's twistor family in the theory of nonabelian Hodge structures. Constructing it amounts to a repetition of our construction in Section 4 from a new perspective and enhanced with more structure.

The $\mathbb{A}_n$ example considered above is a simple example of more general Fukaya–Seidel categories that arise in Homological Mirror Symmetry. Stability conditions associated to the Fukaya–Seidel category are closely related to the complex deformation parameters, i.e., the moduli space of Landau–Ginzburg models. We begin by recalling the general setup in the case of Landau–Ginzburg models. The prescription given by Batyrev, Borisov, Hori, Vafa in [12], [13] to obtain homological mirrors for toric Fano varieties is perfectly explicit and provides a reasonably large set of examples to examine. We recall that if Σ is a fan in $\mathbb{R}^n$ for a toric Fano variety X_Σ, then the homological mirror to the B model of X_Σ is a Landau–Ginzburg model $w : (\mathbb{C}^*)^n \to \mathbb{C}$ where the Newton polytope Q of w is the convex hull of generators of rays of Σ. In fact, we may consider the domain $(\mathbb{C}^*)^n$ to occur as the dense orbit of a toric variety X_A, where A is $Q \cap \mathbb{Z}^n$ and X_A indicates the polytope toric construction. In this setting, the function w occurs as a pencil $V_w \subset H^0(X_A, L_A)$ with fiber at infinity equal to the toric boundary of X_A. Similar construction works for generic non-toric Fanos. In this paper we work with the directed Fukaya category associated to the superpotential w — Fukaya–Seidel categories. To build on the discussion above, we discuss here Fukaya–Seidel categories in the context of stability conditions. The fiber over zero corresponds to the moduli of complex structures. If X_A is toric, the space of complex structures on it is trivial, so the complex moduli appearing here are a result of the choice of fiber $H \subset X_A$ and the choice of pencil w respectively. The appropriate stack parameterizing the choice of fiber contains the quotient $[U/(\mathbb{C}^*)^n]$ as an open dense subset where U is the open subset of $H^0(X_A, L_A)$ consisting of those sections whose hypersurfaces are nondegenerate (i.e., smooth and transversely intersecting the toric boundary) and $(\mathbb{C}^*)^n$ acts by its action on X_A. To produce a reasonably well-behaved compactification of this stack, we borrow from the works of Alexeev ([14]), Gelfand, Kapranov, and Zelevinsky ([9]), and Lafforgue ([10]) to construct the stack $\mathcal{X}_{\mathrm{Sec}(A)}$ with universal hypersurface stack $\mathcal{X}_{Laf(A)}$. We quote the following theorem which describes much of the qualitative behavior of these stacks:

Theorem 6.5 ([2])**.**

i) *The stack $\mathcal{X}_{Sec(A)}$ is a toric stack with moment polytope equal to the secondary polytope $Sec(A)$ of A.*
ii) *The stack $\mathcal{X}_{Laf(A)}$ is a toric stack with moment polytope equal to the Minkowski sum $Sec(A) + \Delta_A$ where Δ_A is the standard simplex in $\mathbb{R}^A$.*
iii) *Given any toric degeneration $F : Y \to \mathbb{C}$ of the pair (X_A, H), there exists a unique map $f : \mathbb{C} \to \mathcal{X}_{Sec(A)}$ such that F is the pullback of $\mathcal{X}_{Laf(A)}$.*

We note that in the theorem above, the stacks $\mathcal{X}_{Laf(A)}$ and $\mathcal{X}_{\mathrm{Sec}(A)}$ carry additional equivariant line bundles that have not been examined extensively in existing literature, but are of great geometric significance. The stack $\mathcal{X}_{\mathrm{Sec}(A)}$ is a moduli stack for toric degenerations of toric hypersurfaces $H \subset X_A$. There is a hypersurface $\mathcal{E}_A \subset \mathcal{X}_{\mathrm{Sec}(A)}$ which parameterizes all degenerate hypersurfaces. For the Fukaya category of hypersurfaces in X_A, the complement $\mathcal{X}_{\mathrm{Sec}(A)} \setminus \mathcal{E}_A$ plays the role of the classical stability conditions, while including $\mathcal{E}_A$ incorporates the compactified version where MHS come into effect. We predict that the walls of the stability conditions occurring in this setup are seen as components of the tropical amoeba defined by the principal A-determinant E_A.

To find the stability conditions associated to the directed Fukaya category of (X_A, w), one needs to identify the complex deformation parameters associated to this model. In fact, these are precisely described as the coefficients of the superpotential, or in our setup, the pencil $V_w \subset H^0(X_A, w)$. Noticing that the toric boundary is also a toric degeneration of the hypersurface, we have that the pencil V_w is nothing other than a map from $\mathbb{P}^1$ to $\mathcal{X}_{\mathrm{Sec}(A)}$ with prescribed point at infinity. If we decorate $\mathbb{P}^1$ with markings at the critical values of w and ∞, then we can observe such a map as an element of $\mathcal{M}_{0,Vol(Q)+1}(\mathcal{X}_{\mathrm{Sec}(A)},[w])$ which evaluates to $\mathcal{E}_A$ at all points except one and ∂X_A at the remaining point. We define the cycle of all stable maps with such an evaluation to be $\mathcal{W}_A$ and regard it as the appropriate compactification of complex structures on Landau–Ginzburg A-models. Applying techniques from fiber polytopes we obtain the following description of $\mathcal{W}_A$:

Theorem 6.6 ([2])**.** *The stack $\mathcal{W}_A$ is a toric stack with moment polytope equal to the monotone path polytope of $Sec(A)$.*

The polytope occurring here is not as widely known as the secondary polytope, but occurs in a broad framework of so called iterated fiber polytopes introduced by Billera and Sturmfels.

In addition to the applications of these moduli spaces to stability conditions, we also obtain important information on the directed Fukaya categories and their mirrors from this approach. In particular, the above theorem may be applied to computationally find a finite set of special Landau–Ginzburg models $\{w_1,\ldots,w_s\}$ corresponding to the fixed points of $\mathcal{W}_A$ (or the vertices of the monotone path polytope of $Sec(A)$). Each such point is a stable map to $\mathcal{X}_{\mathrm{Sec}(A)}$ whose image in the moment space lies on the 1-skeleton of the secondary polytope. This gives a natural semiorthogonal decomposition of the directed Fukaya category into pieces corresponding to the components in the stable curve which is the domain of w_i. After ordering these components, we see that the image of any one of them is a multi-cover of the equivariant cycle corresponding to an edge of $Sec(A)$. These edges are known as circuits in combinatorics (see [2]).

Now we put this moduli space as a “zero fiber” of the “twistor” family of moduli family of stability conditions.

We do this in two steps:

1. The following theorem suggests the existence of a formal moduli space M of Landau–Ginzburg models $f\colon \overline{Y}\to\mathbb{CP}^1$.

Theorem 6.7 (see [1])**.** *There exists a formal moduli space M determined by the solutions of the Maurer–Cartan equations for the following dg-complex:*

$$\begin{array}{ccccccccc}\cdots \longleftarrow & \Lambda^3 T_{\overline{Y}} & \longleftarrow & \Lambda^2 T_{\overline{Y}} & \longleftarrow & T_{\overline{Y}} & \longleftarrow & \mathcal{O}_{\overline{Y}} & \longleftarrow 0\\ & -3 & & -2 & & -1 & & 0 & \end{array}$$

In the above complex the differential is df and we can restate it by saying that this complex determines deformations of the Landau–Ginzburg model, and these deformations are unobstructed. We also have a $\mathbb{C}^*$-action on M with fixed points corresponding to limiting stability conditions — see [1].

Over the moduli space M defined above we have a variation of Hodge structures defined by the cohomologies of the perverse sheaf of vanishing cycles over Y. This defines local system V over M and its compactification.

Conjecture 6.8 (see [1])**.** *The relative completion with respect of V in the fixed points of the $\mathbb{C}^*$-action on the compactification of M has a mixed Hodge structure.*

2. The above moduli space is too big. So we will cut its dimension down to the moduli space of stability conditions. We introduce a new moduli space which embeds in M.

We study deformations of $\overline{Y} \to \mathbb{CP}^1$ with "fixing the fiber at infinity". Deformation of a smooth variety $\overline{Y}$ with fixed $\mathbb{CP}^1$ is controlled by the following sheaf of dg Lie algebras on $\overline{Y}$:

$$T_{\overline{Y}} \to f^* T_{\mathbb{CP}^1}$$

(the differential is the tangent map).

By fixing the fiber at infinity we get a subsheaf of dg Lie algebras

$$T_{\overline{Y}, Y_\infty} \to f^* T_{\mathbb{CP}^1, \infty}.$$

Theorem 6.9 ([1])**.** *The subsheaf of dg Lie algebras*

$$T_{\overline{Y}, Y_\infty} \to f^* T_{\mathbb{CP}^1, \infty}$$

determines a smooth moduli stack. Its dimension is equal to the dimension of the moduli space of stability conditions.

A geometric realization of this moduli space, which embeds in M was described above. We will denote it by $M(\mathbb{P}^1, CY)$ (or $M(\mathbb{P}^k, CY)$ for multipotential Landau–Ginzburg models).

Remark 6.10. We can consider a bigger moduli space by fixing the vector fields only over a part of the divisor at infinity. This corresponds to taking a Landau–Ginzburg model through a point of non maximal degeneration. This defines a bigger moduli space of stability conditions with more stable objects.

Remark 6.11. The moduli spaces we discuss could have many components. Such a phenomenon would have many interesting implications. It produces possibilities of many new birational and symplectic invariants.

In the same way as the fixed point set under the $\mathbb{C}^*$-action plays an important role in describing the rational homotopy types of smooth projective varieties we study the fixed points of the $\mathbb{C}^*$-action on F and derive information about the homotopy type of a category. In the rest of the paper we will denote $\mathcal{W}_A$ by $M(\mathbb{P}^1, \mathcal{X}_{\mathrm{Sec}(A)})$ (or $M(\mathbb{P}^k, CY)$) in order to stress the connection with Landau–Ginzburg models (here CY denotes the moduli space of Calabi–Yau mirrors to the anticanonical section of the Fano manifold we consider).

7. Spectra and holomorphic convexity

In this section we explain briefly how Orlov spectra are related to Stability Hodge Structures.

Recall that noncommutative Hodge structures were introduced by Kontsevich and Katzarkov and Pantev [15] as means of bringing the techniques and tools of Hodge theory into the categorical and noncommutative realm. In the classical setting, much of the information about an isolated singularity is recorded by means of the Hodge spectrum, a set of rational eigenvalues of the monodromy operator. The Orlov spectrum (defined below), is a categorical analogue of this Hodge spectrum appearing in the work of Orlov and Rouquier. The missing numbers in the spectra are called gaps.

Let $\mathcal{T}$ be a triangulated category. For any $G \in \mathcal{T}$ denote by $\langle G\rangle_0$ the smallest full subcategory containing G which is closed under isomorphisms, shifting, and taking finite direct sums and summands. Now inductively define $\langle G\rangle_n$ as the full subcategory of objects, B, such that there is a distinguished triangle, $X \to B \to Y \to X[1]$, with $X \in \langle G\rangle_{n-1}$ and $Y \in \langle G\rangle_0$, and direct summands of such objects.

Definition 7.1. Let G be an object of a triangulated category $\mathcal{T}$. If there is an n with $\langle G\rangle_n = \mathcal{T}$, we set

$$t(G) := \min \{n \geq 0 \mid \langle G\rangle_n = \mathcal{T}\}.$$

Otherwise, we set $t(G) := \infty$. We call $t(G)$ the *generation time* of G. If $t(G)$ is finite, we say that G is a *strong generator*. The *Orlov spectrum* of $\mathcal{T}$ is the union of all possible generation times for strong generators of $\mathcal{T}$. The *Rouquier dimension* is the smallest number in the Orlov spectrum. We say that a triangulated category $\mathcal{T}$, has a *gap* of length s, if a and $a+s+1$ are in the Orlov spectrum but r is not in the Orlov spectrum for $a < r < a+s+1$.

The first connection to Hodge theory appears in the form of the following theorem:

Theorem 7.2 ([16]). *Let X be an algebraic variety possessing an isolated hypersurface singularity. The Orlov spectrum of the category of singularities of X is bounded by twice the embedding dimension times the Tjurina number of the singularity.*

After this brief review of the theory of spectra and their gaps we connect them with SHS. Let $SHS(X)$ be the Stability Hodge Structure of $D^b(X)$ for a given Fano variety X, $M(\mathbb{P}^1, CY)$ be its zero fiber.

Conjecture 7.3 ([1]). *Let p be a point of the divisor D at infinity of the compactification of $M(\mathbb{P}^1, CY)$. The mixed Hodge structures on the completion of the local ring O_p, where p runs over all commponnents of D, determines the spectrum of $D^b(X)$.*

Remark 7.4. The above considerations suggests the existence of a Riemann–Hilbert correspondence for $SHS(X)$ for a Fano variety X as well as deep and interesting analytical interpretation of it by analogy with Yang–Mills–Higgs equations.

As a consequence of the above conjecture we have that SHS satisfy two important properties — functoriality and strictness. We arrive at:

Conjecture 7.5. *The infinite chain condition (*[3]*) can be ruled out for the universal coverings of smooth projective surfaces.*

This is the strongest obstruction to Shafarevich conjecture [3] and SHS gives an approach proving that universal coverings of smooth projective varieties are holomorphically convex.

Observe that the "twistor" family of compactified SHS depends on the choice of Landau–Ginzburg model and still computes some purely categorical invariants. It is natural to ask whether this family rigidifies the data. In particular we pose:

Question 7.6. Does the "twistor" family of compactified SHS of bounded derived category of coherent sheaves of a smooth projective variety X recover the fundamental group of X?

8. Multipotential Landau–Ginzburg models and Hodge structures.

In this section we extend the correspondence among categories and Stability Hodge Structures further. We underscore the idea that rich geometry of the Landau–Ginzburg models gives a possibility of constructing interesting Stability Hodge Structures with many filtrations.

8.1. Multipotential Landau–Ginzburg model for cubic fourfold

We describe fiberwise compactifications of multipotential Landau–Ginzburg models for the cubic fourfold X. This example is representative and illustrates what we mean by a multipotential Landau–Ginzburg model in general.

The Hori–Vafa toric Landau–Ginzburg for X is

$$w = \frac{(x+y+1)^3}{xyt_1t_2} + t_1 + t_2.$$

The cubic fourfold is of index 3. So there are two decompositions of its anticanonical divisor: $3H = H + H + H$ and $3H = 2H + H$. Multipotential Landau–Ginzburg models correspond to such decompositions.

First we describe compactification for the first decomposition. We have the family

$$\frac{(x+y+1)^3}{xyt_1t_2} = w_1, \qquad t_1 = w_2, \qquad t_2 = w_3,$$

where w_i's are complex parameters. After compactifying we get the family

$$(x+y+z)^3 = w_1w_2w_3xyz$$

of elliptic curves over $\mathbb{C}^3$. After blowing up the point $(0,0,0)$ we get a divisor over this point. After that we resolve the rest of the singularities. The restriction of our family to planes $w_j = \mathit{const} \neq 0$ is the Landau–Ginzburg model for cubic threefold so we get the following configuration of singularities.

(1) Ordinary double points along the surface $w_1w_2w_3 = 27$.
(2) 7 lines forming a diagram of type $\widetilde{\mathbb{E}}_6$ over planes $w_1 = 0$, $w_2 = 0$, and $w_3 = 0$.
(3) 5 surfaces over axes w_1, w_2, w_3.
(4) A divisor over (0,0,0).

After projection on the diagonal $\mathbb{C}^3 \to \mathbb{C}$ we get a fiberwise open part of the usual Landau–Ginzburg model for cubic fourfold. Its fiber over zero consists of the divisor described above and an elliptic fibration over the plane passing through the origin and orthogonal to the diagonal. The intersection of these divisors is an elliptic K3 surface with 3 fibers of type $\widetilde{\mathbb{E}}_6$ corresponding to intersections of this orthogonal plane with planes $w_1 = 0$, $w_2 = 0$, and $w_3 = 0$.

Now we describe multipotential Landau–Ginzburg model for the second case $2H + H$. We have the family

$$\frac{(x+y+z)^3}{xyzt_1t_2} = w_1, \qquad t_1 + t_2 = w_2.$$

In other words,

$$(x+y+z)^3 = w_1(w_2 - t)txyz$$

(we denote t_1 by t for simplicity).

This family of surfaces can be obtained from the decomposition $H + H + H$ by a projection along $w_2 + w_3 = 0$. Indeed, the equation of this family over $\mathbb{C}^2$ can be obtained from the equation for the family over $\mathbb{C}^3$ by the coordinate change $w_2 + w_3 \to w_2$, $w_3 \to t$.

So the singularities are the following.

(1) Ordinary double points along a curve.
(2) 5 surfaces over the axis $w_2 = 0$.
(3) 17 surfaces over the axis $w_1 = 0$. Their configuration can be described as follows: configuration of curves of type $\widetilde{\mathbb{E}}_6$ multiplied by a line and two examples of configuration of 5 surfaces described above. Each of them are glued by intersection of "pages" with a line of multiplicity 3 on $\widetilde{\mathbb{E}}_6 \times pt$.
(4) A divisor over (0,0,0).

The restriction of this family to the line $w_1 = const \neq 0$ is (up to a multiplication of a potential by a constant) an open part of Landau–Ginzburg model for the cubic threefold. Indeed,

$$(x+y+z)^3 - w_1(w_2 - t)txyz = (x+y+z)^3 - (\sqrt{w_1}w_2 - (\sqrt{w_1}t))(\sqrt{w_1}t)xyz = (x+y+z)^3 - (w - t_1)t_1xyz,$$

where $w = \sqrt{w_1}w_2$ and $t_1 = \sqrt{w_1}t$.

The restriction to the line $w_2 = const \neq 0$ is an open part of Landau–Ginzburg model for the threefold complete intersection of a quadric and a cubic. Indeed,

$$(x+y+z)^3 - w_1(w_2 - t)txyz = (x+y+z)^3 - (w_1w_2^2)\left(1 - \frac{t}{w_2}\right)\left(\frac{t}{w_2}\right)xyz = (x+y+z)^3 - w(1 - t_1)t_1xyz,$$

where $w = w_1w_2^2$ and $t_1 = t/w_2$.

On the other hand, compactified (singular) Landau–Ginzburg model for the intersection of a quadric and a cubic is

$$(t_1+t_2)^2(x+y+z)^3 - wt_1t_2xyz = t_0^2(x+y+z)^3 - w(t_0-t_1)t_1xyz,$$

where $t_0 = t_1 + t_2$. In the local chart $t_0 = 1$ we get the family written down before.

Thus, after compactifying fibers of the family corresponding to $2H + H$ we get 4 additional surfaces over the w_2 axis and all together $21 = 17 + 4$ surfaces.

8.2. Hodge structures with many filtrations

We now utilize above construction of multipotential Landau–Ginzburg models from the point of view of "twistor" families. This part of the paper is highly speculative.

It is expected that Fukaya–Seidel categories with many potentials can be defined similarly to Fukaya–Seidel categories with one potential. In this case we have a divisor S of singular fibers and thimbles involved reflect not only the geometry of the fibers but the geometry of S as well. In a similar way we can associate to a Fukaya–Seidel category with many potentials a Stability Hodge Structure with a formal scheme over $M(\mathbb{P}^k, CY)$ as a fiber over zero. The following conjecture (briefly explained in Table 2) suggests a way of constructing Hodge structures with multiple filtrations.

Conjecture 8.1 (see [1])**.** *The mixed Hodge structure over formal scheme over $M(\mathbb{P}^k, CY)$ as fiber over zero is a mixed Hodge structure with many filtrations.*

Landau–Ginzburg moduli spaces	Nonabelian Hodge structures
$M(\mathbb{P}^1, CY)$ Landau–Ginzburg model with one potential: The fiber over zero is a formal scheme over $M(\mathbb{P}^1, CY)$, generic fibers are Stab.	Twistor family — Nonabelian Hodge Structure with one weight filtration.
Landau–Ginzburg models with k potentials	Generalized twistor families with k parameters.
The zero fiber is a formal scheme over $M(\mathbb{P}^k, CY)$, fibers (over a point in $\mathbb{C}^k$) are Stab.	Generalized multi twistor family over a k-simplex.
Extensions $M(\mathbb{P}^1, CY) \boxtimes M(\mathbb{P}^1, CY)$	Extending filtrations $u_i \boxtimes u_j$.

Table 2. Creating Hodge structures with multiple filtrations.

9. Birational transformations and Poisson varieties

Discussion from previous sections suggests that there is a connection between the moduli space of Landau–Ginzburg models, generators and birational geometry.

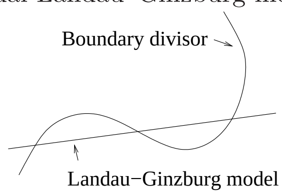
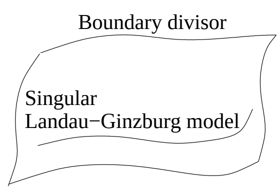
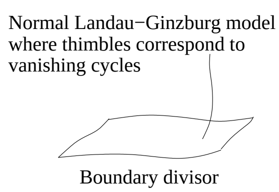
Landau–Ginzburg model	Stability
Usual Landau–Ginzburg model 	Ω^3_X
	$\Omega^3_{X\setminus D}$ D is a divisor with stratification of singular set.
	Ω_D $Sing\, D$ stability conditions of the vanishing cycles on D.

TABLE 3. Stability Clemens–Schmidt sequence.

Table 3 gives a version of noncommutative Clemens–Schmidt sequence for geometric stability conditions — log 3-forms. This table treats the case of three-dimensional Calabi–Yau manifolds (four dimensional Landau–Ginzburg models) but the situation in general should be rather similar. In the case at hand (three-dimensional Calabi–Yau manifold) — the stability conditions are just holomorphic 3-forms. For the quotient category (the category which produces stability conditions of the compactification) we get stability conditions to be holomorphic 3-forms vanishing in a stratified way over a divisor D. The vanishing cycles define a subcategory with its own moduli space of stability conditions and the relative (with respect to this subcategory) WCF defining an integrable system (in general a Poisson variety). The corresponding Landau–Ginzburg models can be seen as follows:

(1) The Landau–Ginzburg models associated with quotient categories are given by monotonic maps passing through an intersection of many boundary divisors in $M(\mathbb{P}^k, CY)$.
(2) The local categories of vanishing cycles are given by Landau–Ginzburg models totally within intersections of divisors.

From the perspective of generators the above splitting corresponds to splitting of the generators into the union of generators associated with the subcategory of vanishing cycles and the quotient category. In fact we get a sequence of splittings — a flag parallel to Okounkov polytopes.

These observations suggest the following conjecture, treated in [2].

Conjecture 9.1. *One-parameter families of Landau–Ginzburg models parameterize Sarkisov links.*

Recall that Sarkisov links [17] are birational maps (birational cobordisms) connecting two Mori fibrations. In our interpretation Sarkisov links become families connecting circuits. In fact we have a more general picture on the connections between moduli spaces of Landau–Ginzburg models and birational geometry. Namely we conjecture that the geometry of moduli spaces of Landau–Ginzburg models for the mirror of Fano manifold X determines its birational geometry. In particular we see a connection with relations between Sarkisov links and then relations between relations and so on. We summarize our picture in Table 4. For more details see [2], [18], [19], [20].

Sarkisov programs	Changes in the spaces of stability conditions
Commutative Sarkisov program: Sarkisov faces.	Wall crossings inside a component of stability conditions.
Non-commutative Sarkisov program: non-commutative cobordisms.	Passing from one component of stability conditions to another one.

Table 4. Birational geometry.

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