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Tube formula for self-similar fractals with non-Steiner-like generators

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Abstract. We give a pointwise proof for the tube formula of Lapidus-Pearse for self-similar fractals, where we extend the types of generators beyond the piecewise polynomial case. We illustrate the new approach on several examples.

1. Introduction

M. Lapidus and E. Pearse proved in [5] a tube formula for higher dimensional fractals, extending the earlier work of Lapidus and his coworkers [4]. They associate certain tilings with fractals and express the volume of the inner ε-neighborhood of the tiling in terms of residues of a certain associated function, called the tubular ζ-function. The residues thereby are taken at the complex dimensions of the fractal, a notion introduced and elaborated by Lapidus and his coworkers.

By a "tube formula" it is understood a formula giving the sum of Lebesgue measures of the inner ε-neighborhoods of the open sets constituting a tiling associated with a fractal as explained in Section 2. (The union of the inner ε-neighborhoods is also regarded as "the inner ε-neighborhood of the tiling".) It should be noted that this Lebesgue measure of the inner ε-neighborhood of the tiling is generally different from the Lebesgue measure of the ε-neighborhood of the fractal itself. But a recent work of E. Pearse and S. Winter [8] clarifies the relationship between the inner ε-neighborhood of the tiling and the genuine ε-neighborhood of the fractal in a very satisfactory way: If the boundary of the convex hull of the fractal is a subset of the fractal, then the volume of the ε-neighborhood of the fractal is the sum of the volumes of the inner ε-neighborhood of the tiling and the outer ε-neighborhood of the convex hull of the fractal.

As the volume of the outer ε-neighborhood of the convex hull is rather trivial, the tube formula in terms of residues gives effectively the true volume of the ε-neighborhood of the fractal if the Pearse-Winter condition is fulfilled. This circumstance attributes a higher value to the utility of the tube formula.

The original proof of the tube formula of Lapidus-Pearse was distributional, but thereafter pointwise proofs have been given in [2],[3] and [6].

Key words and phrases. Self-similar fractals, tube formula, complex dimensions, self-similar tiling, zeta functions.

In [2], we generalized part of their theory to the graph-directed fractals, defined complex dimensions for them and gave a general scheme of summation yielding tube formulas for self-similar as well as graph-directed fractals.

The original setting of Lapidus-Pearse assumed the generators of the tiling to be pluriphase, i.e., the volumes of the inner tubes of the generators should be piecewise polynomial. In [3] and [6] the theory has been extended to more general types of generators, but the conditions and the terminology differ in the two approaches. We consider generators of the following type: Volumes of the inner tubes of generators should be polynomial on a range between zero and a certain value less than the inradius and then arbitrarily piecewise continuously differentiable on the range between this intermediary value and the inradius. Historically, the adjective "Steiner" refers to the theorem that the volume of the ε-neighborhood of a convex body in $\mathbb{R}^d$ is a polynomial of ε. We prefer to use the term "Steiner-like" for a generator with a piecewise polynomial inner tube volume, as a synonym of "pluriphase" and reserve the term "non-Steiner-like" for other types of generators. We note that in [6] the term "Steiner-like" is used essentially for all (bounded) open sets. Our goal is to give a pointwise proof of a tube formula for self-similar fractals with non-Steiner-like generators of the above type and illustrate it on several examples.

2. The Tube Formula for Self-Similar Tilings

Let

$$F = \bigcup_{j=1}^{J} \varphi_j(F) =: \Phi(F) \subset \mathbb{R}^d$$

be a self-similar fractal, where $\varphi_j : \mathbb{R}^d \to \mathbb{R}^d$ are similitudes with scaling ratios $0 < r_j < 1$, $j = 1, \ldots, J$. Let $C := [F]$ be the convex hull of the fractal (for which we assume $\dim C = d$). We assume that the system $\{\varphi_j\}$ satisfies the so-called "tileset condition" of [8]: The open set condition with $\operatorname{int} C$ a feasible open set. Additionally we assume the non-triviality condition of [8]:

$$\operatorname{int} C \not\subseteq \Phi(C) = \bigcup_{j=1}^{J} \varphi_j(C).$$

This condition amounts to $\operatorname{int} F = \emptyset$, [8, Proposition 2.11].

Now define $T_1 = \operatorname{int}(C \setminus \Phi(C))$ and its iterates $T_n = \Phi^{n-1}(T_1)$, $n = 2, 3, \ldots$ (see [7]). The tiling of the self-similar system is given by

$$\mathcal{T} := \{T_n\}_{n=1}^{\infty}$$

and the volume of the inner ε-neighborhood of the tiling $\mathcal{T}$ is defined by

$$V_{\mathcal{T}}(\varepsilon) := \sum_{n=1}^{\infty} V_{T_n}(\varepsilon),$$

where $V_{T_n}(\varepsilon)$ is the volume of the inner ε-neighborhood of T_n.

To state the tube formula we need some additional assumptions and definitions. Assume that T_1 is the union of finitely many (connected) components, $T_1 = G_1 \cup \cdots \cup G_Q$, called the generators of the tiling. In the original work of Lapidus-Pearse (L-P) they assume the generators to be Steiner-like in the following sense:

A bounded, open set $G \subset \mathbb{R}^d$ is called (monophase) Steiner-like if the volume $V_G(\varepsilon)$ of the inner ε-neighborhood of G admits an expression of the form

$$V_G(\varepsilon) = \sum_{i=0}^{d-1} \kappa_i(G)\varepsilon^{d-i}, \qquad \text{for } \varepsilon \leq g,$$

where g denotes the inradius of G, i.e., supremum of the radii of the balls contained in G.

For $\varepsilon > g$ we have $V_G(\varepsilon) = \text{volume}(G)$ which is denoted by $-\kappa_d(G)$, the negative sign being conventional [5].

To be precise, L-P consider also the "pluriphase" case, where $V_G(\varepsilon)$ is given piecewise by different polynomials in the region $0 < \varepsilon < g$. But this generalization brings no essential complication.

Lapidus-Pearse introduce the following "scaling ζ-function":

Definition 2.1. The scaling ζ-function of the self-similar fractal is defined by

$$\zeta(s) = \sum_{k=0}^{\infty} \sum_{w \in W_k} r_w^s,$$

where W_k is the set of words $w = w_1 w_2 \cdots w_k$ of length k (with letters from $\{1, 2, \ldots, J\}$) and $r_w = r_{w_1} r_{w_2} \ldots r_{w_k}$.

The above series can be shown to converge for $\text{Re}(s) > D$, where D is the *similarity dimension* of the system (i.e., the unique real root of the Moran equation $1 - \sum_{j=1}^J r_j^s = 0$ which coincides with the Minkowski and Hausdorff dimensions if the open-set condition holds). A simple calculation shows that $\zeta(s)$ can be expressed as [4, Theorem 2.4]

$$\zeta(s) = \frac{1}{1 - \sum_{j=1}^J r_j^s} \qquad \text{for } \text{Re}(s) > D. \tag{1}$$

$\zeta(s)$ can then be meromorphically extended to the whole complex plane. We will denote this extension also by $\zeta(s)$.

Definition 2.2. The set $\mathfrak{D} := \{\omega \in \mathbb{C} \mid \zeta(s) \text{ has a pole at } \omega\}$ is called the set of complex dimensions of the self-similar fractal.

Lapidus-Pearse define a second type of "ζ-function" associated with the tiling and related to the geometry of the (monophase) Steiner-like generators. As the case of multiple generators does not bring additional complications, we express their notion of tubular ζ-functions and their tube formula for the case of a single generator G.

Definition 2.3. The tubular ζ-function $\zeta_{\mathcal{T}}(s,\varepsilon)$ associated with the generator G is defined by

$$\zeta_{\mathcal{T}}(s,\varepsilon) := \zeta(s)\varepsilon^{d-s}\sum_{i=0}^{d}\frac{g^{s-i}}{s-i}\kappa_i(G).$$

The formula of Lapidus-Pearse for $V_{\mathcal{T}}(\varepsilon)$ now reads as follows:

Theorem 2.1 (Tube formula for tilings of self-similar fractals, [5])**.**

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathfrak{D}_{\mathcal{T}}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega), \tag{2}$$

where $\mathfrak{D}_{\mathcal{T}} = \mathfrak{D}\cup\{0,1,\dots,d-1\}$.

Remark 2.1. A distributional proof for this formula is given in [5] that also holds for fractal sprays which is a more general concept than self-similar fractals. It is shown in [2] that the above formula holds pointwise for self-similar as well as graph-directed fractals. Pointwise proofs were also given in [3] and [6].

The aim of this work is to extend the Lapidus-Pearse tube formula to the case where the generators are non-Steiner-like in the sense below. The paper [6] deals with the same question where the authors use other conditions on the generators and a different terminology.

We shall assume the volume $V_G(\varepsilon)$ of the inner ε-neighborhood of a generator G to be a continuous function of the following form:

$$V_G(\varepsilon) = \begin{cases} \displaystyle\sum_{i=0}^{d-1}\kappa_i(G)\varepsilon^{d-i} & \text{for } 0<\varepsilon<h \\ \lambda_G(\varepsilon) & \text{for } h\le\varepsilon\le g \\ -\kappa_d(G) & \text{for } \varepsilon>g, \end{cases} \tag{3}$$

where λ_G is a piecewise continuously differentiable function on $[h,g]$ (g denotes the inradius throughout). Vol(G) is denoted again by $-\kappa_d(G)$.

For additional simplicity we also assume in our setting that there is a single generator G. (In the case of multiple generators we have to apply the formula to each generator separately and add them up.)

Remark 2.2. For any (not self-intersecting) polygon P in the plane, the volume of the inner ε-neighborhood of P is given by a quadratic polynomial in ε for sufficiently small ε. (A polygon has finitely many sides.) Let the polygon P have the vertices $A_1, A_2, \dots, A_n$ (with $A_{n+1} = A_1$) in a certain successive ordering. Denote the length of the edge A_iA_{i+1} with a_i and the inner angle at A_i by α_i ($i = 1,2,\dots,n$), see Fig.1. Then it can be shown that for small ε

$$V_P(\varepsilon) = \left(\sum_{i=1}^{n} a_i\right)\varepsilon - \left(\sum_{i=1}^{n}\delta_i\right)\varepsilon^2,$$

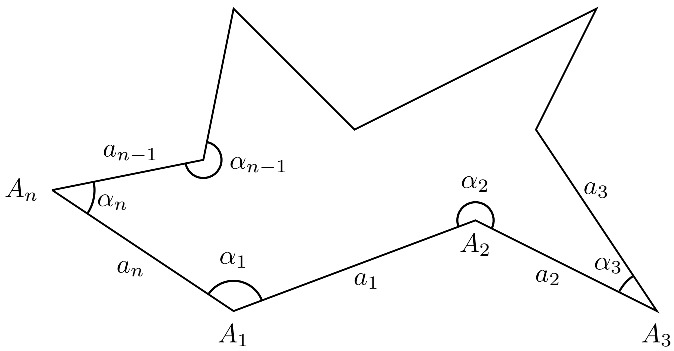

Figure 1. Polygon with n vertices.

where

$$\delta_i = \begin{cases} \cot \frac{\alpha_i}{2} & \text{for } 0 < \alpha_i < \pi \\ \frac{\pi}{2} - \frac{\alpha_i}{2} & \text{for } \pi \leq \alpha < 2\pi. \end{cases}$$

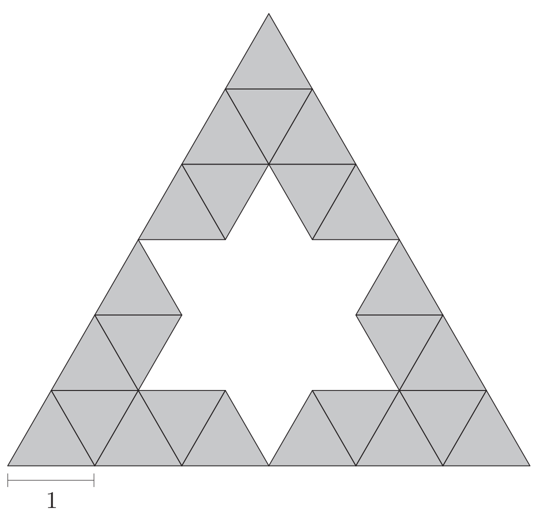

Figure 2. Maps of Example 2.3.

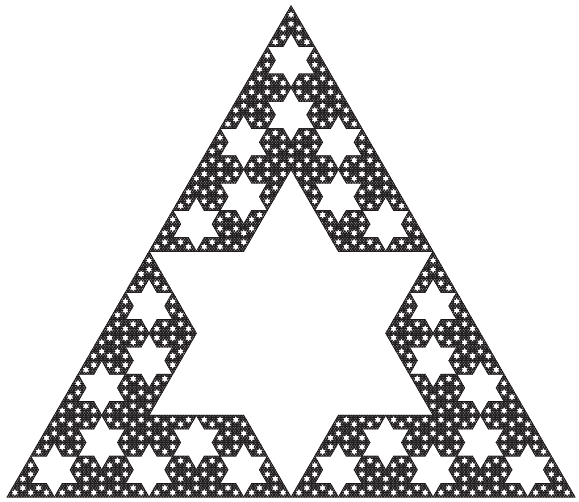

FIGURE 3. The attractor of Example 2.3.

Example 2.3. Consider the iterated function system $\Phi = \{\varphi_j\}_{j=1}^{24}$ on $\mathbb{R}^2$ with scaling ratios $\frac{1}{6}$ as indicated in Figure 2. The associated self-similar fractal is shown in Figure 3. This system satisfies the Pearse-Winter condition. The generator is a non-convex polygon G (with $a_i = 1$, $i = 1, 2, \ldots, 24$; $\alpha_i = \frac{\pi}{3}$ for odd i and $\alpha_i = \frac{4\pi}{3}$ for even i), see Figure 4. In Figure 5 the inner $\frac{1}{\sqrt{3}}$-neighborhood of G is shown.

By the formula above we have $V_G(\varepsilon) = 12\varepsilon - (6\sqrt{3} - \pi)\varepsilon^2$ for small ε. More precisely one can compute

$$V_G(\varepsilon) = \begin{cases} 12\varepsilon - (6\sqrt{3} - \pi)\varepsilon^2 & \text{for } 0 < \varepsilon < \frac{1}{\sqrt{3}} \\ \lambda_G(\varepsilon) & \text{for } \frac{1}{\sqrt{3}} \le \varepsilon \le 1 \\ 3\sqrt{3} & \text{for } \varepsilon > 1, \end{cases}$$

where $\lambda_G(\varepsilon) = \frac{3\sqrt{3}}{2} + \frac{3}{2}\sqrt{4\varepsilon^2 - 1} + (2\pi - 6\arccos\frac{1}{2\varepsilon})\varepsilon^2$. (Note that $\kappa_0(G) = \pi - 6\sqrt{3}$, $\kappa_1(G) = 12$ and $\kappa_2(G) = -3\sqrt{3}$.) It can be checked that $V_G(\varepsilon)$ is continuously differentiable on $[0, \infty)$ (see Figure 6).

We still use the notion of scaling ζ-function and the associated complex dimensions in the sense of L-P. Since the tubular ζ-function depends on the geometry of the generators, we want to define a new tubular ζ-function taking into account the type of generators satisfying (3).

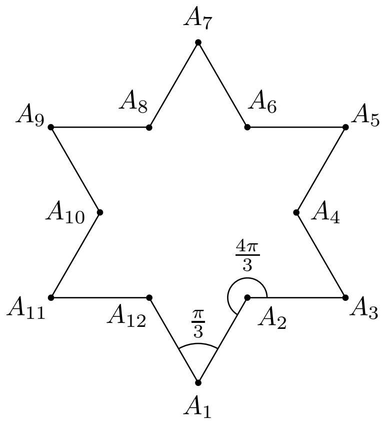

FIGURE 4. Generator G of Example 2.3.

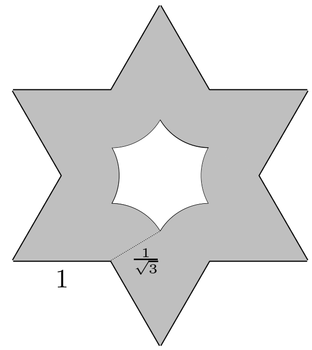

FIGURE 5. The inner $\varepsilon = \frac{1}{\sqrt{3}}$-neighborhood of G.

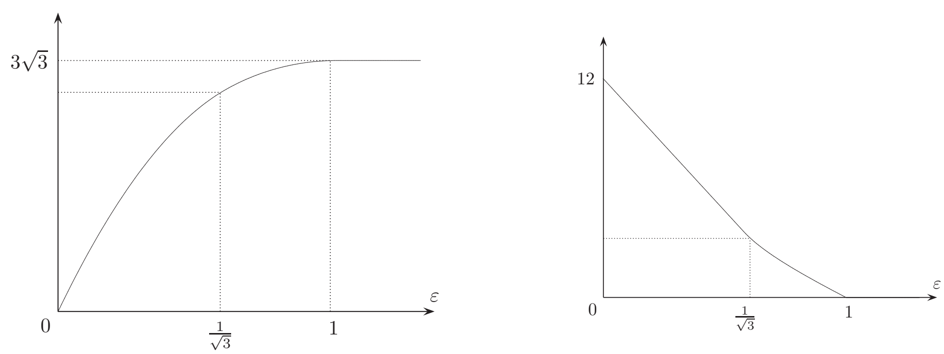

FIGURE 6. The graph of $V_G(\varepsilon)$ (left) and $V'_G(\varepsilon)$ (right).

Definition 2.4. The tubular ζ-function $\zeta_{\mathcal{T}}(s, \varepsilon)$ associated with the generator G satisfying (3) is defined by

$$\zeta_{\mathcal{T}}(s, \varepsilon) := \zeta(s)\varepsilon^{d-s} \left(\sum_{i=0}^{d-1} \frac{h^{s-i}}{s-i} \kappa_i(G) + \frac{g^{s-d}}{s-d} \kappa_d(G) + \Lambda(s) \right),$$

where $\Lambda(s)$ is an entire function given by

$$\Lambda(s) = \int_h^g u^{s-d-1} \lambda_G(u) du. \tag{4}$$

Now we can state our version of the L-P tube formula for generators satisfying condition (3):

Theorem 2.2 (Tube formula for tilings of self-similar fractals with non-Steiner-like generators)**.**

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathfrak{D}_{\mathcal{T}}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) \quad \text{for } \varepsilon < h,$$

where $\mathfrak{D}_{\mathcal{T}} = \mathfrak{D} \cup \{0, 1, \dots, d-1\}$.

Remark 2.4. We note that Theorem 2.2 (as well as Theorem 2.1) holds for the more general situation of [8] and even for more general sprays [4]. The specific Pearse-Winter condition (that the boundary of the convex hull of the fractal be a subset of the fractal) is not needed. But if this beautiful condition holds, then the tube formula yields a formula for the volume of the genuine ε-neighborhood of the fractal as explained in the introduction.

Remark 2.5. Note that d is not included in the set $\mathfrak{D}_{\mathcal{T}}$. The reason of this exclusion will be clear from the proof of the theorem.

Lemma 2.3. *Let the volume of the inner* ε*-neighborhood of a generator* $G \subset \mathbb{R}^2$ *be given as in (3):*

$$V_G(\varepsilon) = \begin{cases} \kappa_0 \varepsilon^2 + \kappa_1 \varepsilon & \text{for } 0 < \varepsilon < h \\ \lambda_G(\varepsilon) & \text{for } h \le \varepsilon \le g \\ -\kappa_2 & \text{for } \varepsilon > g. \end{cases}$$

We assume additionally $\lambda_G(\varepsilon) \in C^3[h, g]$. *Then, using the notation*

$$\begin{aligned} \Delta^i(x) &= \text{jump of the } i\text{-th derivative of } V_G \text{ at } x \\ &= V_G^i(x+) - V_G^i(x-), \end{aligned}$$

the tubular ζ*-function*

$$\zeta_{\mathcal{T}}(s,\varepsilon) = \zeta(s)\varepsilon^{2-s} \left(\frac{h^s}{s}\kappa_0 + \frac{h^{s-1}}{s-1}\kappa_1 + \frac{g^{s-2}}{s-2}\kappa_2 + \int_h^g u^{s-3}\lambda_G(u)du \right)$$

can be expressed as follows:

$$\begin{aligned} \zeta_{\mathcal{T}}(s,\varepsilon) &= \zeta(s)\varepsilon^{2-s} \left[\frac{1}{(s-1)(s-2)} \left(\Delta^1(h)h^{s-1} + \Delta^1(g)g^{s-1} \right) \right. \\ &\left. - \frac{1}{s(s-1)(s-2)} \left(\Delta^2(h)h^s + \Delta^2(g)g^s + \int_h^g u^s \lambda_G'''(u)du \right) \right] \end{aligned}$$

Proof. Apply integration by parts. □

Remark 2.6. Similar formulas can be obtained if $\lambda_G(\varepsilon)$ is piecewise C^3 and also for higher dimensions.

Example 2.3 (continued). The scaling ζ-function of the system Φ of Example 2.3 is given by

$$\zeta(s) = \frac{1}{1 - \sum_{j=1}^{24} r_j^s} = \frac{1}{1 - 24(\frac{1}{6})^s}.$$

The set of complex dimensions is $\mathfrak{D} = \{D + \mathbb{i}np \mid n \in \mathbb{Z}\}$ where $D = \log_6 24$ and $p = \frac{2\pi}{\log 6}$.

The tubular ζ-function is given by

$$\zeta_{\mathcal{T}}(s,\varepsilon) = \zeta(s)\varepsilon^{2-s}\left(\frac{h^s}{s}(\pi - 6\sqrt{3}) + \frac{h^{s-1}}{s-1}\,12 + \frac{g^{s-2}}{s-2}(-3\sqrt{3}) + \Lambda(s)\right).$$

Applying the above lemma we obtain:

$$\zeta_{\mathcal{T}}(s,\varepsilon) = \zeta(s)\varepsilon^{2-s}\frac{-4\sqrt{3}}{s(s-1)(s-2)}\left(1 + \sqrt{3}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{u^{s-1}}{(4u^2-1)^{3/2}}du\right).$$

Thus by Theorem 2.2 we get

$$\begin{aligned}
V_{\mathcal{T}}(\varepsilon) &= \sum_{\omega\in\mathfrak{D}\cup\{0,1\}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) \\
&= \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);0) + \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);1) + \sum_{n\in\mathbb{Z}}\operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);D+\mathbb{i}np) \\
&= \frac{6\sqrt{3}-\pi}{23}\,\varepsilon^2 - 4\varepsilon \\
&\quad -\frac{4\sqrt{3}}{\log 6}\sum_{n\in\mathbb{Z}}\frac{\varepsilon^{2-D-\mathbb{i}np}}{(D+\mathbb{i}np)(D-1+\mathbb{i}np)(D-2+\mathbb{i}np)}\left(1+\sqrt{3}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{u^{D-1+\mathbb{i}np}}{(4u^2-1)^{3/2}}du\right).
\end{aligned}$$

3. Proof of Theorem 2.2

Our goal is to find a closed expression for $V_{\mathcal{T}}(\varepsilon) = \sum_{n=1}^{\infty} V_{T_n}(\varepsilon)$ as stated in Theorem 2.2. As we assumed a single generator for simplicity, the volume of the inner ε-tube of the tiling $\mathcal{T}$ is, by the tileset condition, the sum of the volumes of the inner ε-neighborhoods of all the scaled copies of G appearing in the tiling:

$$V_{\mathcal{T}}(\varepsilon) = \sum_{k=0}^{\infty}\sum_{w\in W_k} V_{r_w G}(\varepsilon),$$

where W_k and r_w are as in Definition 2.1 and r_wG is a copy of G scaled by r_w. A simple calculation shows that, if $V_G(\varepsilon)$ is given as in (3), then

$$V_{xG}(\varepsilon) = \begin{cases} \sum_{j=0}^{d-1} \kappa_j(G) x^j \varepsilon^{d-j} & \text{for } \varepsilon < xh \\ x^d \lambda_G(\frac{\varepsilon}{x}) & \text{for } xh \leq \varepsilon \leq xg \\ -x^d \kappa_d(G) & \text{for } \varepsilon > xg. \end{cases}$$

It will be more convenient for us to regard $V_{xG}(\varepsilon)$ as a two-variable function of x and ε:

$$V_G(x, \varepsilon) := V_{xG}(\varepsilon) = \begin{cases} -x^d \kappa_d(G) & \text{for } 0 < x < \frac{\varepsilon}{g} \\ x^d \lambda_G(\frac{\varepsilon}{x}) & \text{for } \frac{\varepsilon}{g} \leq x \leq \frac{\varepsilon}{h} \\ \sum_{j=0}^{d-1} \kappa_j(G) x^j \varepsilon^{d-j} & \text{for } x > \frac{\varepsilon}{h}. \end{cases}$$

Recall that the Mellin transform $\mathcal{M}[f; s]$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{M}[f; s] = \widetilde{f}(s) = \int_0^\infty x^{s-1} f(x) dx.$$

For fixed ε, we take the Mellin transform of $V_G(x, \varepsilon)$ as a function of x:

$$\begin{aligned} \widetilde{V}_G(s, \varepsilon) = \int_0^{\frac{\varepsilon}{g}} x^{s-1}(-x^d \kappa_d(G)) dx + \int_{\frac{\varepsilon}{g}}^{\frac{\varepsilon}{h}} x^{s-1} x^d \lambda_G(\frac{\varepsilon}{x}) dx \\ + \int_{\frac{\varepsilon}{h}}^\infty x^{s-1} \left(\sum_{j=0}^{d-1} \kappa_j(G) x^j \varepsilon^{d-j} \right) dx \\ = -\varepsilon^{s+d} \left(\frac{g^{-s-d}}{s+d} \kappa_d(G) + \sum_{j=0}^{d-1} \frac{h^{-s-j}}{s+j} \kappa_j(G) \right) + \int_h^g \varepsilon^{s+d} u^{-s-d-1} \lambda_G(u) du, \end{aligned}$$

for $-d < \text{Re}(s) < 1 - d$. (In the second integral we made the change of variables $\frac{\varepsilon}{x} \to u$.)

$$\begin{aligned} \widetilde{V}_G(s, \varepsilon) &= -\varepsilon^{s+d} \left(\frac{g^{-s-d}}{s+d} \kappa_d(G) + \sum_{j=0}^{d-1} \frac{h^{-s-j}}{s+j} \kappa_j(G) - \int_h^g u^{-s-d-1} \lambda_G(u) du \right) \\ &= -\varepsilon^{s+d} \left(\frac{g^{-s-d}}{s+d} \kappa_d(G) + \sum_{j=0}^{d-1} \frac{h^{-s-j}}{s+j} \kappa_j(G) - \Lambda_G(-s) \right), \end{aligned} \tag{5}$$

where $\Lambda(s) = \int_h^g u^{s-d-1} \lambda_G(u) du$.

Since $V_G(x,\varepsilon)$ is piecewise continuously differentiable as a function of x and therefore of bounded variation, we can take the inverse Mellin transform of $\widetilde{V}_G(s,\varepsilon)$ to obtain ([9, Theorem 28, p.46])

$$V_G(x,\varepsilon) = \mathcal{M}^{-1}\left[\widetilde{V}_G(s,\varepsilon);x\right] = \frac{1}{2\pi\mathbb{i}}\int_{-c-\mathbb{i}\infty}^{-c+\mathbb{i}\infty} x^{-s}\widetilde{V}_G(s,\varepsilon)ds,$$

where $d-1<c<d$. For an additional purpose below, we choose c such that $D<c<d$. (Note that the non-triviality condition implies $D<d$, [8, Corollary 2.13].)

We shall now insert the above expression for $V_G(x,\varepsilon)$ into the sum

$$V_{\mathcal{T}}(\varepsilon) = \sum_{k=0}^{\infty}\sum_{w\in W_k} V_{r_wG}(\varepsilon),$$

but for the ease of the computation, we order the scaling coefficients r_w into a sequence $\{x_m\}_{m=1}^{\infty}$:

$$V_{\mathcal{T}}(\varepsilon) := \sum_{m=1}^{\infty} V_{x_mG}(\varepsilon),$$

so that we get

$$V_{\mathcal{T}}(\varepsilon) = \sum_{m=1}^{\infty}\frac{1}{2\pi\mathbb{i}}\int_{-c-\mathbb{i}\infty}^{-c+\mathbb{i}\infty} x_m^{-s}\widetilde{V}_G(s,\varepsilon)ds. \tag{6}$$

Changing the order of the sum and the integral (a detailed verification of this can be found in [3, p.12-15]), we obtain

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \frac{1}{2\pi\mathbb{i}}\int_{-c-\mathbb{i}\infty}^{-c+\mathbb{i}\infty}\sum_{m=1}^{\infty} x_m^{-s}\widetilde{V}_G(s,\varepsilon)ds \\ &= \frac{1}{2\pi\mathbb{i}}\int_{-c-\mathbb{i}\infty}^{-c+\mathbb{i}\infty}\zeta(-s)\widetilde{V}_G(s,\varepsilon)ds \quad \text{by Def.2.1.}\end{aligned}$$

Changing the variable of the integral by $s'=-s$ we find

$$V_{\mathcal{T}}(\varepsilon) = \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}\zeta(s)\widetilde{V}_G(-s,\varepsilon)ds.$$

By (5) we get

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}\zeta(s)\varepsilon^{d-s}\left(\frac{g^{s-d}}{s-d}\kappa_d(G) + \sum_{j=0}^{d-1}\frac{h^{s-j}}{s-j}\kappa_j(G) + \Lambda(s)\right)ds \\ &= \frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}\zeta_{\mathcal{T}}(s,\varepsilon)ds.\end{aligned}$$

The poles of $\zeta(s)$ are contained in a horizontally bounded strip $D_\ell \le \mathrm{Re}(s) \le D$, for some real number D_ℓ, which can be assumed to be negative also (see [4, Theorem. 2.17]).

Let c_ℓ be chosen such that

$$\text{i)}\quad c_\ell < D_\ell \tag{7}$$

$$\text{ii)}\quad \zeta(s) \text{ is bounded for } \mathrm{Re}(s) \le c_\ell. \tag{8}$$

(The second property is possible because $|\zeta(s)| \to 0$ as $\mathrm{Re}(s) \to -\infty$.)

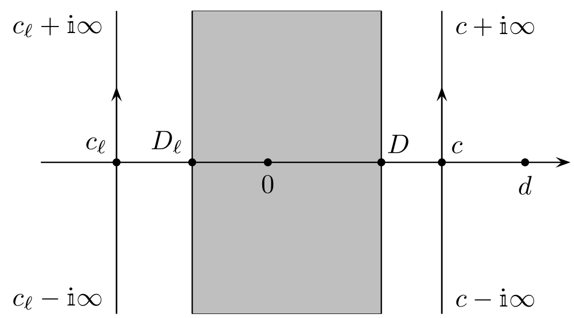

FIGURE 7. All the poles of $\zeta_\mathcal{T}$ are on the shaded strip.

We now proceed to show that (see Fig.7)

$$\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} \zeta_\mathcal{T}(s,\varepsilon)ds = \frac{1}{2\pi\mathbb{i}}\int_{c_\ell-\mathbb{i}\infty}^{c_\ell+\mathbb{i}\infty} \zeta_\mathcal{T}(s,\varepsilon)ds + \sum_{\omega\in\mathfrak{D}\cup\{0,1,\ldots,d-1\}} \mathrm{res}(\zeta_\tau(s,\varepsilon);\omega) \tag{9}$$

and

$$\frac{1}{2\pi\mathbb{i}}\int_{c_\ell-\mathbb{i}\infty}^{c_\ell+\mathbb{i}\infty} \zeta_\mathcal{T}(s,\varepsilon)ds = 0 \qquad \text{for } \varepsilon < h. \tag{10}$$

Clearly, this will complete the proof of the theorem.

We begin with (9): First note that there exists a positive increasing sequence $\{\tau_n\}_{n=1}^\infty$ with $\{\tau_n\} \to \infty$ such that

$$|\zeta(\sigma \pm \mathbb{i}\tau_n)| \le M \text{ for } c_\ell \le \sigma \le d \text{ and for all } n, \tag{11}$$

where M is some positive constant. (see [4, Theorem 3.26].)

Denote the rectangle with the corners $c - \mathbb{i}\tau_n, c + \mathbb{i}\tau_n, c_\ell + \mathbb{i}\tau_n, c_\ell - \mathbb{i}\tau_n$ (and with the edges $L_{1,n}, L_{2,n}, L_{3,n}, L_{4,n}$) by S_n (see Figure 8). By the residue theorem

$$\frac{1}{2\pi\mathbb{i}}\int_{\partial S_n} \zeta_\mathcal{T}(s,\varepsilon)ds = \sum_{\omega\in(\mathfrak{D}\cup\{0,1,\ldots,d-1\})\cap S_n} \mathrm{res}(\zeta_\mathcal{T}(s,\varepsilon);\omega).$$

(Note that d lies outside the rectangles S_n and does not contribute to the above sum.)

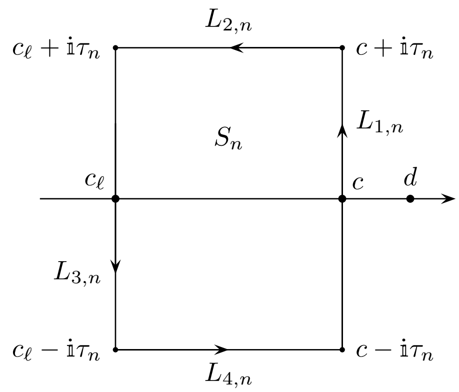

FIGURE 8. The rectangle S_n.

Therefore

$$\frac{1}{2\pi\mathbb{i}}\left(\int_{L_{1,n}}\zeta_{\mathcal{T}}(s,\varepsilon)ds+\int_{L_{2,n}}\zeta_{\mathcal{T}}(s,\varepsilon)ds+\int_{L_{3,n}}\zeta_{\mathcal{T}}(s,\varepsilon)ds+\int_{L_{4,n}}\zeta_{\mathcal{T}}(s,\varepsilon)ds\right) \tag{12}$$
$$=\sum_{\omega\in(\mathfrak{D}\cup\{0,1,\ldots,d-1\})\cap S_n}\operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega).$$

Let us consider first the integral over $L_{2,n}$. For $s\in L_{2,n}$ we have $s=\sigma+\mathbb{i}\tau_n$, $c_\ell\le\sigma\le c$.

$$\begin{aligned}
|\zeta_{\mathcal{T}}(s,\varepsilon)| &= \left|\zeta(s)\varepsilon^{d-s}\left(\sum_{j=0}^{d-1}\frac{h^{s-j}}{s-j}\kappa_j(G)+\frac{g^{s-d}}{s-d}\kappa_d(G)+\Lambda(s)\right)\right| \\
&\le M\varepsilon^{d-\sigma}\left(\sum_{j=0}^{d-1}\frac{h^{\sigma-j}}{|\sigma+\mathbb{i}\tau_n-j|}|\kappa_j(G)|+\frac{g^{\sigma-d}}{|\sigma+\mathbb{i}\tau_n-d|}|\kappa_d(G)|+|\Lambda(s)|\right) \\
&\le M\varepsilon^{d-\sigma}\left(\sum_{j=0}^{d-1}\frac{h^{\sigma-j}|\kappa_j(G)|}{\tau_n}+\frac{g^{\sigma-d}|\kappa_d(G)|}{\tau_n}\right)+M\varepsilon^{d-\sigma}|\Lambda(s)| \\
&\le \frac{MM'}{\tau_n}+M\varepsilon^{d-\sigma}|\Lambda(s)|,
\end{aligned}$$

where $M'=\max_{c_\ell\le\sigma\le c}\left\{\varepsilon^{d-\sigma}\left(\sum_{j=0}^{d-1}h^{\sigma-j}|\kappa_j(G)|+g^{\sigma-d}|\kappa_d(G)|\right)\right\}$, and we used (11) to obtain the first inequality.

Now we consider the term $M\varepsilon^{d-\sigma}|\Lambda(s)|$: Recall that

$$\Lambda(s)=\int_h^g u^{s-d-1}\lambda_G(u)du.$$

Integrating by parts,

$$\Lambda(s) = \frac{u^{s-d}}{s-d}\lambda_G(u)\Big|_h^g - \int_h^g \frac{u^{s-d}}{s-d}\lambda'_G(u)du, \tag{13}$$

whence we can write $\varepsilon^{d-\sigma}|\Lambda(s)| \le \frac{M''}{|s-d|} \le \frac{M''}{\tau_n}$ by the assumption of continuity and piecewise continuous differentiability of $\lambda_G(u)$ and $c_\ell \le \sigma \le c$. Thus we obtain

$$|\zeta_{\mathcal{T}}(s,\varepsilon)| \le \frac{MM'}{\tau_n} + \frac{MM''}{\tau_n}.$$

Therefore,

$$\left|\int_{L_{2,n}} \zeta_{\mathcal{T}}(s,\varepsilon)\ ds\right| \le \frac{M(M'+M'')}{\tau_n}(c-c_\ell) \to 0 \text{ for } n \to \infty,$$

since $\tau_n \to \infty$ as $n \to \infty$.

Similarly the integral over $L_{4,n}$ tends to 0 as $n \to \infty$. Thus letting $n \to \infty$ in (12) gives (9).

Now, we will show (10): Let L_n be the line segment $L_n(t) = c_\ell + \mathbb{i}t$, $-n \le t \le n$; C_n be the semicircle $C_n(t) = c_\ell + ne^{\mathbb{i}t}$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$ and $\Gamma_n = L_n + C_n$ (see Figure 9). By the choice of c_ℓ, $\zeta_{\mathcal{T}}(s,\varepsilon)$ is analytic on and inside Γ_n and therefore

$$\int_{L_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds = -\int_{C_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds.$$

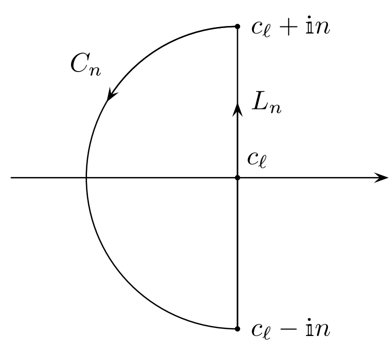

FIGURE 9. The contour $\Gamma_n = L_n + C_n$.

We will be done when we show that the right-hand side tends to 0 as $n \to \infty$. Using the parametrization of C_n, we obtain

$$\begin{aligned}\left|\int_{C_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds\right| &= \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \zeta(c_\ell + ne^{\mathrm{i}t})\varepsilon^{d-c_\ell-ne^{\mathrm{i}t}} \left(\sum_{j=0}^{d-1} \frac{h^{c_\ell+ne^{\mathrm{i}t}-j}}{c_\ell + ne^{\mathrm{i}t} - j}\kappa_j(G)\right.\right. \\ &\quad \left.\left. + \frac{g^{c_\ell+ne^{\mathrm{i}t}-d}}{c_\ell + ne^{\mathrm{i}t} - d}\kappa_d(G) + \Lambda(c_\ell + ne^{\mathrm{i}t})\right) \mathrm{i}ne^{\mathrm{i}t}dt\right|.\end{aligned}$$

By condition (8), $\zeta(s)$ is bounded for $\mathrm{Re}(s) \le c_\ell$, say $|\zeta(s)| \le K$:

$$\begin{aligned}\left|\int_{C_n} \zeta_{\mathcal{T}}(s,\varepsilon)ds\right| &\le K\varepsilon^{d-c_\ell}\left(\sum_{j=0}^{d-1} \frac{n\,h^{c_\ell-j}}{n-|c_\ell-j|}|\kappa_j(G)| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{h}{\varepsilon}\right)^{n\cos t} dt\right. \\ &\quad + \frac{n\,g^{c_\ell-d}}{n-|c_\ell-d|}|\kappa_d(G)| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{g}{\varepsilon}\right)^{n\cos t} dt \\ &\quad \left. + n\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left|\Lambda(c_\ell + ne^{\mathrm{i}t})\right| \varepsilon^{-n\cos t}dt\right).\end{aligned}$$

Let us denote the right-hand side by $I_1 + I_2 + I_3$. We have

$$I_1 = K\varepsilon^{d-c_\ell}\sum_{j=0}^{d-1} \frac{n\,h^{c_\ell-j}}{n-|c_\ell-j|}|\kappa_j(G)| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{h}{\varepsilon}\right)^{n\cos t} dt \le K_1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{h}{\varepsilon}\right)^{n\cos t} dt,$$

where K_1 is some constant.

The well-known Jordan Lemma [1] states that

$$\lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} a^{n\cos t}dt = 0,$$

for any fixed $a > 1$. Since we assumed $\varepsilon < h$ for Theorem 2.2, we thus get $I_1 \to 0$ as $n \to \infty$.

Similarly $I_2 \to 0$ as $n \to \infty$, since $\varepsilon < g$ as $h < g$.

To deal with the term I_3, recall that by (13)

$$\Lambda(s) = \left.\frac{u^{s-d}}{s-d}\lambda_G(u)\right|_h^g - \int_h^g \frac{u^{s-d}}{s-d}\lambda_G'(u)du.$$

By continuity and piecewise continuous differentiability of λ_G, we can write

$$|\Lambda(s)| \le \frac{1}{|s-d|}\left(K_1\,g^{\mathrm{Re}(s)-d} + K_2\,h^{\mathrm{Re}(s)-d}\right) + \frac{1}{|s-d|}\int_h^g u^{\mathrm{Re}(s)-d}\,K_3\,du$$

with some constants K_1, K_2, K_3.

For $s = c_\ell + ne^{\mathrm{i}t}, (\frac{\pi}{2} \leq t \leq \frac{3\pi}{2})$ we have $\mathrm{Re}(s) = c_\ell + n\cos t < 0$ and $u^{\mathrm{Re}(s)-d}$ is a decreasing function on $[h, g]$. Thus,

$$\begin{aligned} |\Lambda(s)| &\leq K_4 \frac{1}{|s-d|} h^{\mathrm{Re}(s)-d} + K_3 \frac{1}{|s-d|}(g-h)h^{\mathrm{Re}(s)-d} \\ &\leq K_5 \frac{1}{|c_\ell + ne^{\mathrm{i}t} - d|} h^{c_\ell + n\cos t - d} \\ &\leq K_6 \frac{h^{n\cos t}}{n - |c_\ell - d|}. \end{aligned} \tag{14}$$

Now we show that $I_3 \to 0$ as $n \to \infty$. Since

$$I_3 = n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left|\Lambda\left(c_\ell + ne^{\mathrm{i}t}\right)\right| \varepsilon^{-n\cos t} dt,$$

using (14)

$$I_3 \leq \frac{K_6\, n}{n - |c_\ell - d|} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{h}{\varepsilon}\right)^{n\cos t} dt \to 0 \quad \text{as } n \to \infty,$$

by Jordan Lemma. Hence the claim (10) is verified and thus the proof of Theorem 2.2 is completed.

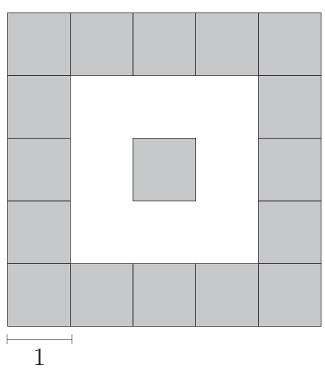

FIGURE 10. Maps of Example 4.1.

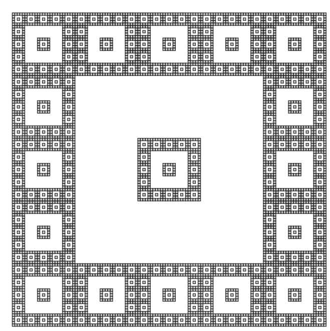

FIGURE 11. The attractor of Example 4.1.

4. Further Examples

4.1. An example with non-simply connected generator

Consider the iterated function system $\Phi = \{\varphi_j\}_{j=1}^{17}$ on $\mathbb{R}^2$ with scaling ratios $\frac{1}{5}$ as indicated in Figure 10. The associated self-similar fractal is shown in Figure 11. This

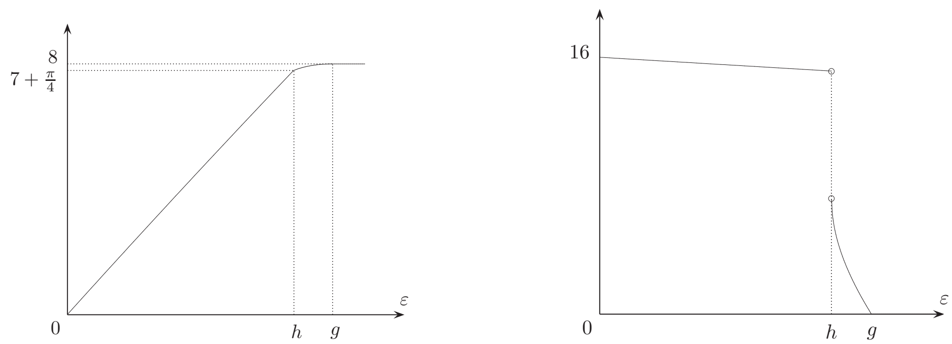

FIGURE 12. The graph of $V_G(\varepsilon)$ (left) and $V_G'(\varepsilon)$ (right) of the generator G of Example 4.1.

system satisfies the Pearse-Winter condition. The generator G is a non-simply connected set, where the volume of the inner ε-tube is given by the following function:

$$V_G(\varepsilon) = \begin{cases} 16\varepsilon + (\pi-4)\varepsilon^2 & \text{for} \quad 0 < \varepsilon < h \\ \lambda_G(\varepsilon) & \text{for} \quad h \le \varepsilon \le g \\ 8 & \text{for} \quad \varepsilon > g, \end{cases}$$

where

$$\lambda_G(\varepsilon) = 4 + 8\varepsilon + (\pi-4)\varepsilon^2 + 4(1-\varepsilon)\sqrt{2\varepsilon-1} - 4\varepsilon^2 \arctan\left(\frac{\sqrt{2\varepsilon-1}}{1-\varepsilon}\right),$$

and $h = \frac{1}{2}$, $g = 2-\sqrt{2}$. Note that $\kappa_0(G) = \pi - 4$, $\kappa_1(G) = 16$, $\kappa_2(G) = -8$. We remark that the function $V_G(\varepsilon)$ is not differentiable (at $\varepsilon = \frac{1}{2}$) (see Figure 12). The scaling ζ-function of the system Φ is given by

$$\zeta(s) = \frac{1}{1-\sum_{j=1}^{17} r_j^s} = \frac{1}{1-17\left(\frac{1}{5}\right)^s}.$$

The set of complex dimensions is $\mathfrak{D} = \{D + \mathbb{i}np \mid n \in \mathbb{Z}\}$, where $D = \log_5 17$ and $p = \frac{2\pi}{\log 5}$. The tubular ζ-function is given by

$$\zeta_{\mathcal{T}}(s,\varepsilon) = \zeta(s)\varepsilon^{2-s}\left(\frac{h^s}{s}(\pi-4) + \frac{h^{s-1}}{s-1}16 + \frac{g^{s-2}}{s-2}(-8) + \Lambda(s)\right),$$

where $\Lambda(s)$ is as in (4).

Applying Theorem 2.2 we get

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \sum_{\omega \in \mathfrak{D} \cup \{0,1\}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);\omega) \\ &= \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);0) + \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon);1) + \sum_{n\in\mathbb{Z}} \operatorname{res}(\zeta_{\mathcal{T}}(s,\varepsilon); D + \mathbb{i}np) \\ &= \frac{4-\pi}{16}\varepsilon^2 - \frac{20}{3}\varepsilon + \frac{1}{\log 5} \sum_{n\in\mathbb{Z}} \varepsilon^{2-D-\mathbb{i}np} \left(\frac{h^{D+\mathbb{i}np}}{D+\mathbb{i}np}(\pi - 4) + \frac{h^{D-1-\mathbb{i}np}}{D-1-\mathbb{i}np} 16 + \right. \\ &\qquad \left. + \frac{g^{D-2-\mathbb{i}np}}{D-2-\mathbb{i}np}(-8) + \Lambda(D+\mathbb{i}np) \right). \end{aligned}$$

4.2. A non-lattice example

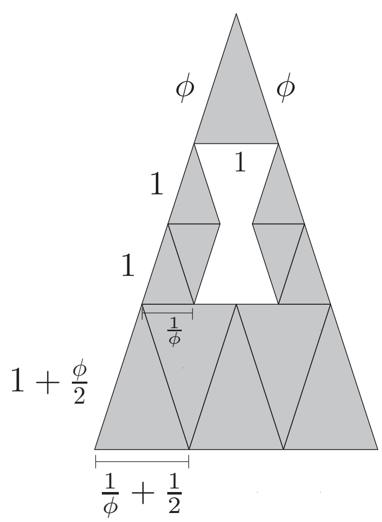

FIGURE 13. Maps of Example 4.2.

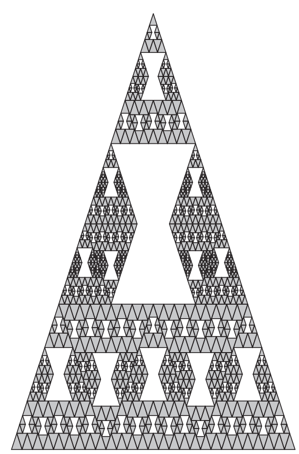

FIGURE 14. The attractor of Example 4.2.

A self-similar fractal F with scaling ratios r_j, $(j = 1, \ldots, J)$ is called of lattice type if the additive subgroup $\sum_{j=1}^{J} (\log r_j)\mathbb{Z} \subset \mathbb{R}$ is discrete and otherwise of non-lattice type.

Consider the iterated function system $\Psi = \{\psi\}_{j=1}^{12}$ on $\mathbb{R}^2$ with scaling ratios

$$r_1 = \frac{1}{3\left(\frac{1}{\phi}+\frac{1}{2}\right)} = \frac{2\phi}{3\phi+6},$$

$$r_2 = \frac{\frac{1}{\phi}}{3\left(\frac{1}{\phi}+\frac{1}{2}\right)} = \frac{2}{3\phi+6}, \qquad (\phi \text{ being the golden ratio})$$

and $r_3 = \frac{1}{3}$ as indicated in Figure 13. The associated self-similar fractal is shown in Figure 14. This system satisfies the Pearse-Winter condition. The generator is a non-convex set, with the following volume function for the inner ε-tube:

$$V_G(\varepsilon) = \begin{cases} 6\varepsilon - \left(\frac{4\sqrt{4\phi+3}}{2\phi-1} - \frac{\pi}{5}\right)\varepsilon^2 & \text{for} \quad 0<\varepsilon<h \\ \lambda_G(\varepsilon) & \text{for} \quad h \le \varepsilon \le g \\ \frac{1}{2}\frac{(2\phi-1)\sqrt{4\phi+3}}{\phi+1} & \text{for} \quad \varepsilon > g, \end{cases}$$

where

$$\lambda_G(\varepsilon) = \begin{cases} 6\varepsilon - \left(\frac{4\sqrt{4\phi+3}}{2\phi-1} - \frac{\pi}{5}\right)\varepsilon^2 - 2\varepsilon^2\arccos\left(\frac{\phi-1}{2\phi\varepsilon}\right) + \frac{\phi-1}{\phi}\sqrt{\varepsilon^2 - \frac{2-\phi}{4\phi+4}}, & \text{if } h \le \varepsilon < h' \\ \frac{1}{2}\frac{(\phi-2)\sqrt{4\phi+3}}{\phi+1} + 2\varepsilon\sqrt{2\phi+1} - 2\varepsilon^2\frac{8\phi+5}{\sqrt{4\phi+3}}, & \text{if } h' \le \varepsilon \le g, \end{cases}$$

and $h = \frac{\phi-1}{2\phi}$, $h' = \frac{\phi-1}{\sqrt{4\phi+3}}$ and $g = \frac{\sqrt{4\phi+3}}{4\phi+2}$.

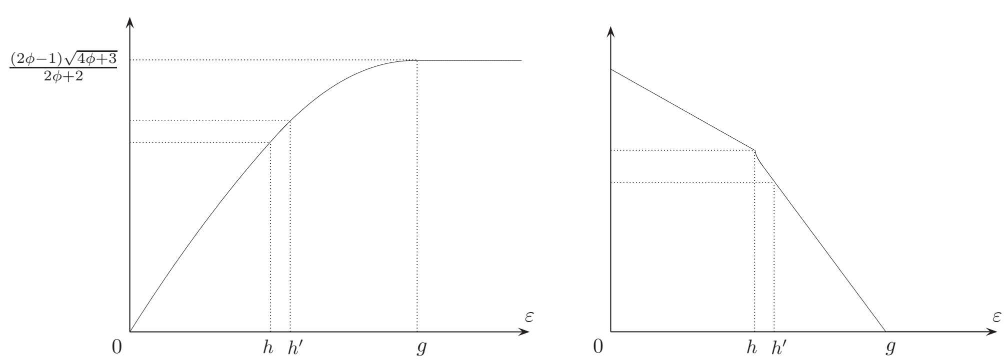

FIGURE 15. The graph of $V_G(\varepsilon)$ (left) and $V_G'(\varepsilon)$ (right) of the generator G of Example 4.2.

We remark that the function $V_G(\varepsilon)$ is differentiable (see Figure 15). The scaling ζ-function of the system is given by

$$\zeta(s)=\frac{1}{1-\left(\frac{2\phi}{3\phi+6}\right)^s-6\left(\frac{2}{3\phi+6}\right)^s-5\left(\frac{1}{3}\right)^s}.$$

The dimension D of this example is approximately 1.8689. As this example is of non-lattice type, we can not exactly compute the complex dimensions [4, p.66], but we give several approximations of the set of the complex dimensions in Figure 17 along the lines of computation of [4].

The tubular ζ-function is given by

$$\zeta_{\mathcal{T}}(s,\varepsilon)=\zeta(s)\varepsilon^{2-s}\left(\frac{h^s}{s}\left(\frac{4\sqrt{4\phi+3}}{2\phi-1}-\frac{\pi}{5}\right)+\frac{h^{s-1}}{s-1}6+\frac{g^{s-2}}{s-2}k+\Lambda(s)\right),$$

where $k=\frac{1}{2}\frac{(2\phi-1)\sqrt{4\phi+3}}{\phi+1}$ and $\Lambda(s)$ is as in (4).

Appendix

To determine the complex dimensions of a self-similar fractal, we have to solve the Moran equation in the complex domain. As it is not possible to solve this equation in the non-lattice case exactly, we can only approximate solutions. We want to explain briefly the approach we used to plot the approximate complex dimensions for the last example. The Moran equation in this case is

$$r_1^s+6r_2^s+5r_3^s=1,$$

where $r_1=\frac{2\phi}{3\phi+6}$, $r_2=\frac{2}{3\phi+6}$, $r_3=\frac{1}{3}$.

We can rewrite this equation as follows:

$$\begin{aligned} e^{s\log r_1}+6e^{s\log r_2}+5e^{s\log r_3} &= 1\\ e^{-s\log 3(-\log_3 r_1)}+6e^{-s\log 3(-\log_3 r_2)}+5e^{-s\log 3} &= 1. \end{aligned}$$

Now, we choose some appropriate rational approximations to $\log_3 r_1$ and $\log_3 r_2$, then solve the resulting polynomial equation and find approximate complex dimensions. We used for the pair $(-\log_3 r_1,-\log_3 r_2)$ the following pairs of approximations:

$$\left(\frac{10}{9},\frac{3}{2}\right),\left(\frac{11}{10},\frac{17}{11}\right),\left(\frac{65}{59},\frac{20}{13}\right),\left(\frac{76}{69},\frac{97}{63}\right).$$

To detail the first case, the pair $\left(\frac{10}{9},\frac{3}{2}\right)$ leads to the equation

$$e^{-(s\log 3)\left(\frac{10}{9}\right)}+6e^{-s\log 3\left(\frac{3}{2}\right)}+5e^{-s\log 3}=1.$$

Setting $z=e^{-\frac{s}{18}\log 3}$, we get the polynomial equation

$$z^{20}+6z^{27}+5z^{18}=1.$$

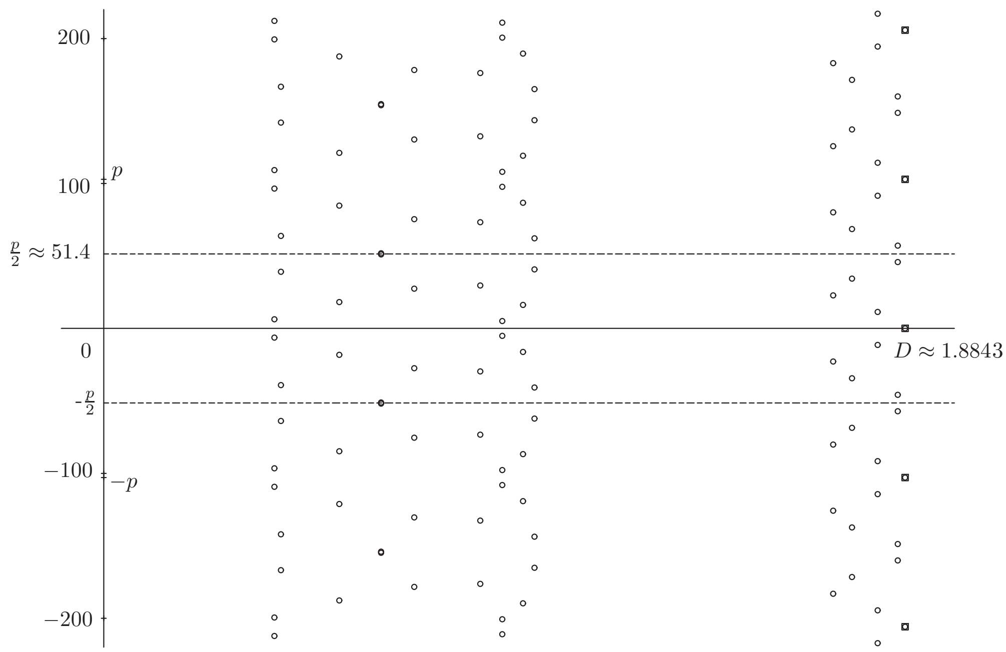

FIGURE 16. Approximate complex dimensions of Example 4.2 for $(-\log_3 r_1, -\log_3 r_2) \approx \left(\frac{10}{9}, \frac{3}{2}\right)$.

We solved this equation using Maple. There are 27 distinct roots, one of which is real. In terms of these roots, we can express the approximate complex dimensions as

$$s = -\frac{18 \log z}{\log 3} + \frac{18}{\log 3} 2\pi \mathbb{i} n, \qquad n \in \mathbb{Z}.$$

The real root $z \approx 0.8913$ gives for $n = 0$ the approximate similarity dimension $D \approx 1.8843$.

The pairwise-conjugate 13 root pairs give rise to 13 vertical lines, so that all approximate complex dimensions are located on 14 vertical lines with a period $\frac{18}{\log 3} 2\pi \mathbb{i}$ (see Figure 16).

The other approximations lead to the polynomials

$$z^{121} + 6z^{170} + 5z^{110} = 1$$
$$z^{845} + 6z^{1180} + 5z^{767} = 1$$
$$z^{4788} + 6z^{6693} + 5z^{4347} = 1$$

and the corresponding approximate complex dimensions are plotted in Figure 17.

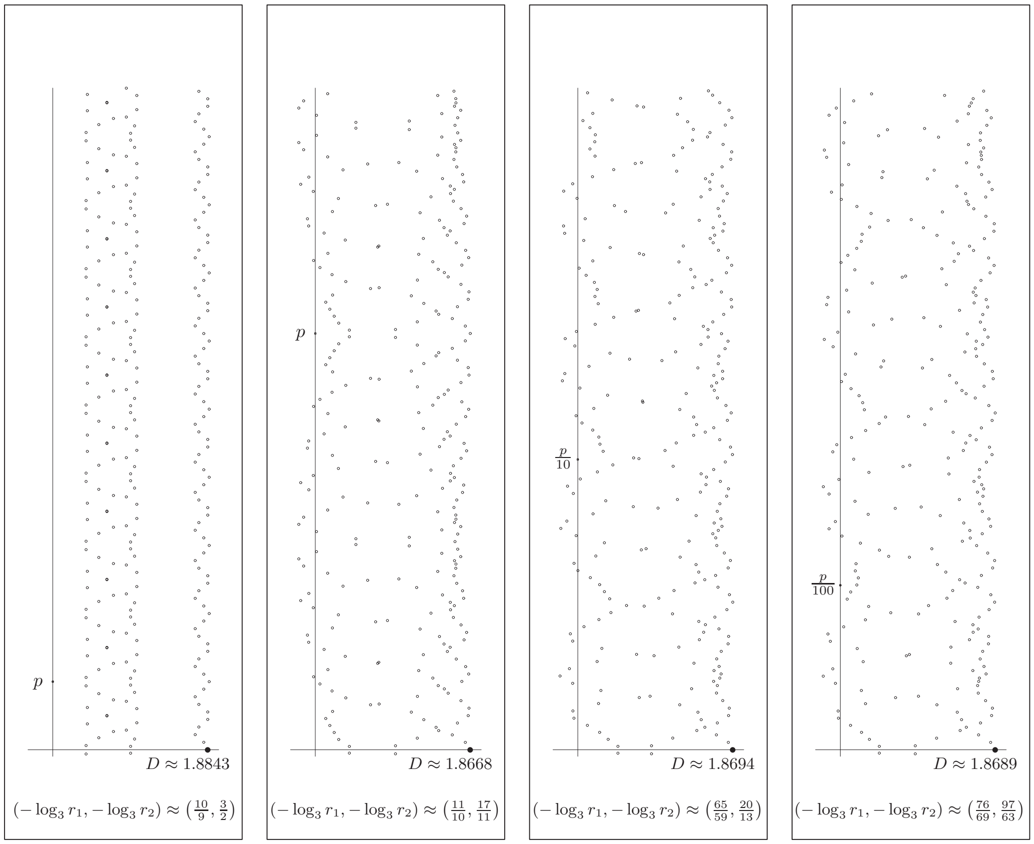

FIGURE 17. Consecutive approximations of complex dimensions of Example 4.2.

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