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Crowell's state space is connected

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Abstract. We study the set of Crowell states for alternating knot projections and show that for prime alternating knots the space of states for a reduced projection is connected, a result similar to that for Kauffman states. As an application we give a new proof of a result of Ozsváth and Szabó characterizing $(2, 2n + 1)$ torus knots among alternating knots.

1. Introduction

Some of the many definitions of the Alexander polynomial of a knot is stated through state sums. Kauffman has described and studied a state sum model for the Alexander polynomial in great detail [Kau83]. In an earlier paper Crowell has described another state sum model for the Alexander polynomial for the subclass of alternating knots ([Cro59], Theorem 2.12).

In the next section we will recall the definition of Crowell states and examine some of their properties. In Section 3 we will prove that

Theorem 1.1. *If K is an alternating prime knot and D is a reduced knot diagram for K, then any two states differ by a finite sequence of terminal edge exchanges.*

This theorem is similar in nature to the Clock Theorem of Kauffman [Kau83] which states that any two Kauffman states differ by a finite sequence of clockwise and counterclockwise moves, which was also proven in the language of graphs in section 4 of [GL86]. This work is independent of those mentioned because of the simple reason that Kauffman states and Crowell states do not correspond to each other in any natural way as observed from the fact that the space of Crowell states do not form a lattice in general (see Proposition 2.6).

In section 4, as an application we will give an alternative proof that $(2, 2n + 1)$ torus knots are characterized by their Alexander polynomials among alternating knots, which was originally proven by Ozsváth and Szabó (Proposition 4.1 in [OS05]).

2. A state model

In this section we will review the definition of the state sum model for alternating links given by Crowell [Cro59] and investigate some properties of the states.

Key words and phrases. State sum, Alexander polynomial, spanning trees.

Given a knot K and an oriented alternating diagram D of K with n crossings we obtain a weighted labeled directed planar graph $G(D)$ as follows: replace a small neighborhood of each crossing by a degree 4 vertex according to the following figure (k is the vertex label):

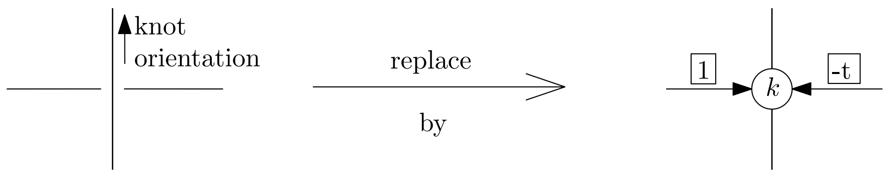

FIGURE 1. From a knot diagram to a directed graph.

Proposition 2.1. *This definition orients all edges and these orientations are compatible with orientations coming from a checkerboard coloring of the regions in the complement of D.*

Proof. Each edge gets an orientation since while traveling around a region the strands traveled alternate between under strand and over strand. Hence at each crossing if an edge is coming in, the boundary of the region will continue along the over strand, which becomes an under strand at its other end, hence we get an orientation on that edge as well, consistent with the previous edge. These orientations are compatible with a checkerboard coloring since crossing a region to another across an edge we get opposite orientations in the plane. □

Choose a vertex v_0 of $G(D)$, called the root. Directed paths starting at v_0 that don't contain cycles will be called rooted simple paths. Trees with edges oriented away from v_0 will be called rooted trees. Spanning rooted trees will be called states. Let $Tr(v_0)$ be the space of states and $w(T)$ be the product of weights of all edges in a state T. According to [Cro59, Theorem 2.12] we get the renormalized Alexander polynomial as a sum of monomials corresponding to each state by

$$\Delta_K(t) = (-t)^m \cdot \sum_{T \in Tr(v_0)} w(T) \tag{1}$$

where m is chosen so that the term with the least power of t is a positive constant.

Proposition 2.2. *For any vertex v, there is a rooted simple path from v_0 to v in $G(D)$.*

Proof. Since K is a knot, $G(D)$ as an unoriented graph is connected. Pick an unoriented path $e_1, e_2, \ldots, e_m$ starting at v_0, ending at v. If e_i is not oriented away from the root, pick edges $e_{i,1}, e_{i,2}, \ldots e_{i,m_i}$ that go around one of the two regions adjacent to e_i. Due to Proposition 2.1, we get compatible orientations on this sequence of edges. At the end we get a rooted path which might visit some vertices more than once. For each vertex that is visited more than once, remove all edges between the first and last visits. □

Corollary 2.3. *Any rooted tree T can be extended to a rooted spanning tree $\tilde{T}$.*

Proof. For any vertex v not in T, find a rooted simple path α from v_0 to v using Proposition 2.2. Add enough of the final segment of α to the current tree so that the union will be connected without creating cycles. □

Proposition 2.4. *Given a reduced alternating diagram for a prime knot, any edge in $G(D)$ except those ending at v_0 appear as a teminal edge in at least one state.*

Proof. Call the given edge e_0 starting at vertex w_0, ending at vertex w_1. We will find a rooted simple path from v_0 to w_0, add the edge e_0, and extend this path into a rooted spanning tree.

Use Proposition 2.2 to construct a rooted simple path Γ from v_0 to w_0. There are two possible cases:
(a) Γ doesn't go through w_1: Do nothing extra.
(b) Γ goes through w_1 before reaching w_0: Adding e_0 to Γ produces a cycle. To avoid this problem we will go around as follows. Assume Γ contains an edge e' going from v' to w_1 and an edge e'' going from w_1 to v''. We need to connect v' to v'' by a directed simple path avoiding e' and e''. To achieve this, let R be the region bounded on two sides by e' and e'' and $\hat{R}$ be the union of regions adjacent to R along edges other than e' and e''. Then starting at v', following edges on the boundary of $\hat{R}$ that are not on the boundary of R, we get a directed path α ending at v''. Since K is prime and D is reduced, this new path α does not include e' and e'' since otherwise we could draw a separating circle passing through the e' or e'' and another common edge of R and $\hat{R}$. Replacing e' and e'' by α in Γ, we get a rooted path from v_0 to w_0 avoiding w_1. It could include cycles, which can be eliminated as in the proof of Proposition 2.2.

Next we need to extend the rooted simple path $\Gamma \cup e_0$ to a rooted spanning tree, keeping e_0 a terminal edge. Consider the two edges coming out from w_1, call them e_1' and e_1'', with terminal vertices w_1' and w_1''. Using a similar argument as in the proof of case (b) above, consider R being the region bounded by e_0 and e_1', and $\hat{R}$ the union of regions adjacent to R except along e_0 and e_1', and removing cycles we get a directed path β from w_0 to w_1' avoiding w_1. Starting at w_1', add enough edges from the final segment of β so that w_1' is connected to a vertex in $\Gamma \cup e_0$ without creating a cycle. Do similarly for w_1''.

Now, use Corollary 2.3 to extend this rooted tree to a rooted spanning tree. During this process e_0 stays a terminal edge since adding e_1' or e_1'' would create a cycle. □

The state sum in Equation 1 resembles the state sum defined by Kauffman [Kau83]. Kauffman has studied an operation called a *clock move* (transposition) that transforms a state to another that differs only at two crossings and showed that all states differ from one another by a sequence of clock moves. With that in mind we define the following operation for reduced alternating diagrams:

Definition 2.1. A state T_2 is obtained from a state T_1 by a *terminal edge exchange* move (edge exchange for short) if replacing a terminal edge in T_1 by the other incoming edge at the terminal vertex gives T_2.

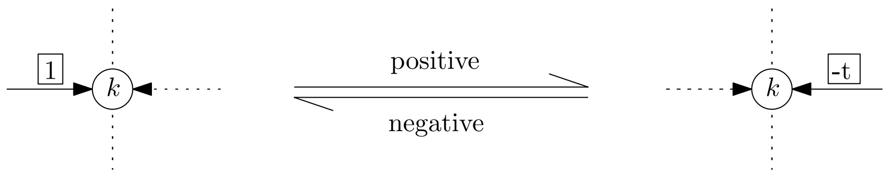

FIGURE 2. Terminal edge exchange.

Proposition 2.5. *At any terminal edge, an edge exchange gives a new state, except at edges ending at a kink.*

Proof. If there is a kink at v, there is a unique edge that connects to v in any spanning tree, since the opposite edge is a loop. At any other vertex, it is easy to check that one still gets a rooted spanning tree. □

Edge exchange gives a partial order on the set of states by defining the covering relation of the partial order as T_1 comes immediately before T_2 if T_2 is obtained from T_1 by one positive edge exchange.

Comparing these states with the black trees for Kauffman states, even though states are rooted spanning trees in both models, in Kauffman states the orientations on the edges are chosen after a spanning tree of the black graph is chosen, so the same edge can inherit different orientations in different states. Furthermore, consider the graph whose vertices are Crowell states and any two vertices are connected by an edge if there is a terminal edge exchange that takes one state to the other. The following proposition shows that edge exchanges do not correspond to clock moves under any bijection between the Crowell and Kauffman states since Kauffman states form a distributive lattice [Kau83].

Proposition 2.6. *The space of Crowell states is not a lattice in general with any choice of a partial order compatible with terminal edge exchanges even for reduced alternating diagrams for prime alternating knots.*

Proof. Figure 3 illustrates the graph of Crowell states for the knot 7_6 in Rolfsen's table. Let us assume that there is a partial order compatible with this graph, i.e., an edge between two states exist if one is an immediate successor of the other. Then any degree one vertex is either a local maximum or a local minimum. Since this graph has three degree 1 vertices and in a finite lattice there is only one local maximum and only one local minimum, this particular graph can not be a lattice. □

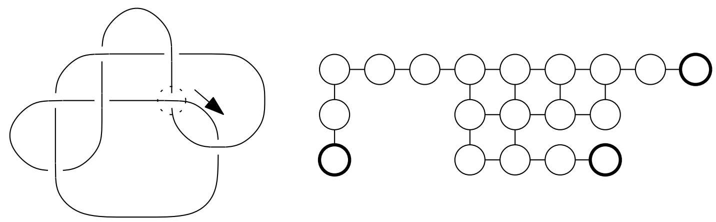

FIGURE 3. The knot 7_6, the chosen root, and its space of states.

3. Proof of Theorem 1.1

In this section we will assume that K is a prime knot and D is a reduced alternating projection for K. Choose a root vertex v_0 in $G(D)$. We will provide an algorithm to go from one rooted spanning tree T_1 to another rooted spanning tree T_2 through a sequence of edge exchanges. We will label vertices v of $G(D)$ with distinct integers.

An initial segment $IS(v,T)$ of a rooted spanning tree T is the sequence of vertices on the unique rooted path from the root to v in T. For $v \neq v_0$, let $\phi(v,T)$ denote the vertex that points to v through an edge not in T. Let $Bel(v,T)$ be a small neighborhood of the set of vertices below v in T, i.e., those that can be reached from v via directed paths in T, the edges between them (not necessarily in T) and the elementary regions surrounded by those edges. Let $Bel_1(w,T)$ be the connected component of $Bel(w,T)$ containing the successor of w in T with the smaller label. When the tree T is obvious from the context, we will suppress T from these notations. The rooted meet of two rooted trees T_1 and T_2 is the connected component of the root in $T_1 \cap T_2$ and will be denoted by $T_1 \cap_r T_2$.

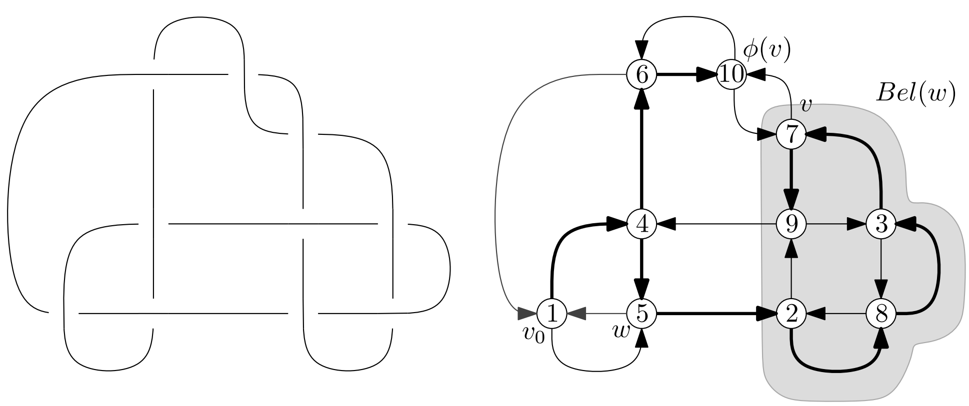

FIGURE 4. The knot diagram D and a state T in $G(D)$ marked by thick edges, and the region $Bel(w)$ for the vertex labeled by 5.

In order to prove Theorem 1.1, we will show that for any given two states T_1 and T_2, we can persistently enlarge $T_1 \cap_r T_2$.

Lemma 3.1. *Given a state T and a vertex w other than the root v_0, if $\phi(w,T)$ is not in $Bel(w,T)$, then there is a sequence of edge exchanges that converts the incoming edge for w into a terminal edge, removing edges only below w.*

Lemma 3.2. *Under the conditions of Lemma 3.1, there is a vertex $w' \in Bel(w,T)$ with $\phi(w') \notin Bel(w,T)$.*

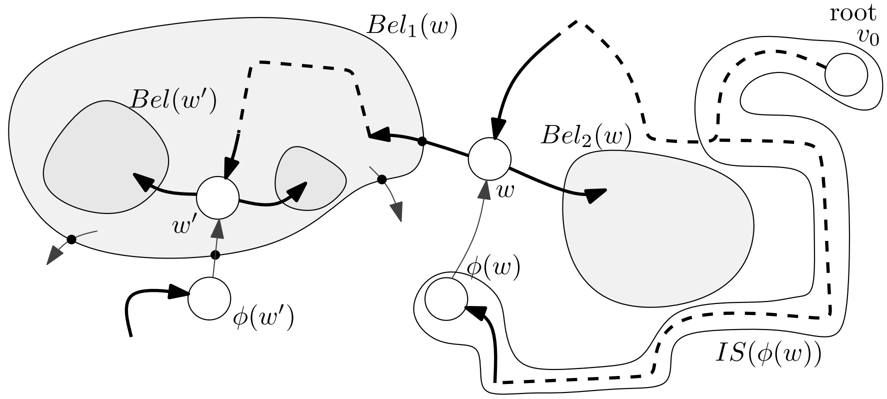

FIGURE 5. Thick edges belong to the spanning tree T.

Proof of Lemma 3.2. Assume $Bel(w)$ is nonempty. Given w, consider $D \cap \partial Bel_1(w)$. Since K is prime and D is reduced, $\partial Bel_1(w)$ is not a separating circle, hence there are at least 4 intersections. Since the orientations of adjacent regions alternate, we get at least two edges entering into $Bel_1(w)$.

Pick a vertex w' among all terminal vertices of edges in G entering into $Bel_1(w)$ not originating from w. This choice implies that $\phi(w') \notin Bel_1(w)$. Furthermore the union $IS(w) \cup \{w \to \phi(w)\} \cup IS(\phi(w))$ contains an unoriented circuit of edges and vertices that separate $Bel_1(w)$ from $Bel_2(w)$ (see Figure 5), hence going from w' to a vertex in $Bel_2(w)$ would take at least two edge exchanges. We conclude that $\phi(w') \notin Bel_2(w)$ as well. $\square$

Proof of Lemma 3.1. If $Bel(w)$ is empty, then w is already a terminal vertex. Otherwise, we will use induction on the depth d of the tree $Bel(w)$.

For $d = 1$, $Bel(w)$ could contain up to two vertices. If there is only one vertex, it is a terminal vertex and an edge exchange empties $Bel(w)$. If there are two vertices, $Bel(w)$ has two components, which as in the proof of Lemma 3.2, are not adjacent to one another.

Lemma 3.2 tells that an edge exchange at either vertex decreases the size of $Bel(w)$, and we are led to the case of one vertex.

Assume the hypothesis is true for all trees $Bel(w)$ of depth d and less. If $Bel(w)$ has depth $d+1$, use Lemma 3.2 to find a w' with the property $\phi(w') \notin Bel(w)$, in particular $\phi(w') \notin Bel(w')$. Therefore by the induction hypothesis w' becomes a terminal vertex after a finite sequence of edge exchanges only removing edges in $Bel(w')$. Then performing an edge exchange at w' decreases the size of $Bel(w)$. Hence repeating this process w becomes a terminal vertex while only edges below w being removed throughout the process. □

Proof of Theorem 1.1. Given two distinct rooted spanning trees T_1 and T_2, pick a vertex w adjacent to $T_1 \cap_r T_2$ along an edge in T_2. Note that $w \neq v_0$ since $v_0 \in T_1 \cap_r T_2$ and once w is a terminal edge, after an edge exchange w will be connected to v_0 along the same rooted Hamiltonian path as in T_2.

Let v_1, v_2 be the vertices that lead to w in T_1 and T_2 respectively. By definition, $v_2 = \phi(w, T_1)$ and $v_2 \in T_1 \cap_r T_2$. Hence $IS(\phi(w, T_1), T_1) \subset T_1 \cap_r T_2$, but $w \notin T_1 \cap_r T_2$, hence $w \notin IS(\phi(w, T_1), T_1)$, which means $\phi(w, T_1) \notin Bel(w, T_1)$.

Applying Lemma 3.1, we get a sequence of edge exchanges that ends in a state where w is a terminal vertex without removing any edges from the rooted meet. Now perform an edge exchange at w, this enlarges the rooted meet. Since the rooted meet only enlarges during this process, in finitely many repetitions of this process we reach T_2. □

4. An application to $(2, 2n+1)$ torus knots

In this section we will provide a different proof of the following result originally proved by Ozsváth and Szabó:

Theorem 4.1. *The $(2, 2n+1)$ torus knots are characterized among alternating knots by the Alexander polynomial.*

Proof. Let D be a reduced alternating projection for a knot K with Alexander polynomial $\Delta_K(t) = 1 + (-t) + (-t)^2 + ... + (-t)^{2n}$. Since all coefficients of powers of $-t$ are $+1$, each state has a different weight and $\Delta_K(t)$ is not a product of two alternating knot polynomials (cf. [OS05, Prop. 4.1]), hence K is prime.

Let T_0 be the state with the least t power. Since K is prime and the fact that each edge exchange changes the power of $-t$ by ± 1, using Theorem 1.1 we get a linear ordering on the $2n+1$ states starting at T_0, reaching each next state by exchanging an edge of weight $+1$ with an edge of weight $-t$.

According to Proposition 2.5 and due to this linear order, T_0 and the top state T_{2n} have only one terminal edge each, hence they have no branching, whereas intermediate states have 2 terminal edges.

Since Proposition 2.4 tells that each edge v (except the two that point to v_0) can be extended to a state having v as a terminal edge, and since we can reach that state from T_0 by positive edge exchanges, we see that all edges in T_0 have weight $+1$. We conclude that T_0 has $2n$ edges since each edge of weight $+1$ is used only once in an edge exchange and no new edges emerge with weight $+1$ as we go from T_0 to T_{2n}.

Edge orientations and weights do not depend on the choice of the root vertex, hence, after moving the root from v_0 to v_1, we still get a space of states with the same properties, in particular, there will be a new state T_0' containing a linear directed chain of $2n$ vertices starting at v_1, ending at v_0. Hence we get a cycle of length $2n+1$ of edges of weight $+1$. Similarly, all remaining edges have weight $-t$, form a cycle and are used in T_{2n}, except the one pointing at the root.

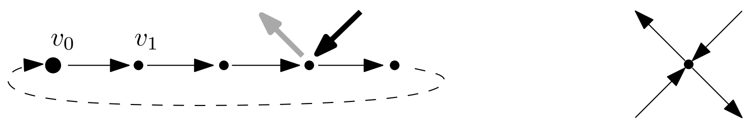

FIGURE 6. Thick edges have weight $-t$. On the right, orientations at a typical node.

This information tells us that if there is an incoming edge of weight $-t$ at a vertex v, the next edge of weight $-t$ has to be on the same side of the cycle of $+1$ edges due to the cyclic alternating orientation of edges at a vertex. Since these edges with weight $-t$ form a cycle as well, they have to go between consecutive vertices. This gives us the diagram for the $(2, 2n+1)$ torus knot. □

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