Proceedings of 19th Gökova
Geometry-Topology Conference
pp. 111 – 122

Locally conformally flat and self-dual structures on simple 4-manifolds

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Abstract. This is a survey article on the existence of locally conformally flat (LCF) and self-dual (SD) metrics on various basic 4-manifolds like simply-connected ones or product types.

1. Introduction

A Riemannian manifold (M,g) is called *locally conformally flat (LCF)* if there is a neighborhood U of any point, and a strictly positive smooth function f such that $\tilde{g} = fg$ is a metric of zero sectional curvature everywhere on $U \subset M$. Sometimes the more concise terminology *conformally flat* is also used since globally conformally flat manifolds admit flat metrics. They are quotients of $\mathbb{R}^n$ so that there is no need to give them a new name. Alternating and symmetry properties imply that we can think of the $(0,4)$ Riemann curvature tensor R as an element of the space $S^2\Lambda^2M \subset \otimes^4 T^*M$. It also satisfies the algebraic Bianchi identity, hence it lies in the kernel of the *Bianchi symmetrization map*

$$b : S^2\Lambda^2M \to S^2\Lambda^2M, \qquad b(T)(x,y,z,t) := \frac{1}{3}T((x,y,z),t).$$

Since $b^2 = b$ and b is $GL(T^*M)$-equivariant, we have the equivariant decomposition $S^2\Lambda^2M = \operatorname{Ker} b \oplus \operatorname{Im} b$, where we call this kernel as the space of *curvature-like tensors*. Thinking $\operatorname{Ker} b$ as an invariant $O(g)$-module, we have a unique irreducible decomposition, according to this the curvature tensor decomposes as $R = U \oplus Z \oplus W$. The components can be computed as

$$U = \frac{s}{2n(n-1)} g \owedge g \quad \text{and} \quad Z = \frac{1}{n-2} \overset{\circ}{\operatorname{Ric}} \owedge g$$

where s is the scalar curvature, $\overset{\circ}{\operatorname{Ric}} = \operatorname{Ric} - \frac{s}{n}g$ is the trace-free Ricci tensor, "$\owedge$" is the *Kulkarni-Nomizu product* defined by,

$$A \owedge B\,(X,Y,Z,T) := \begin{vmatrix} A(X,Z) & B(X,T) \\ A(Y,Z) & B(Y,T) \end{vmatrix} + \begin{vmatrix} B(X,Z) & A(X,T) \\ B(Y,Z) & A(Y,T) \end{vmatrix},$$

Key words and phrases. Locally conformally flat manifolds, self-dual manifolds, Kähler manifolds.

which is commutative and multiplies two symmetric 2-tensors to produce a curvature-like 4-tensor. Finally W is the *Weyl tensor* which is defined to be what is left over from the first two pieces.

Next, assume that we are in dimension $n = 4$ and our manifold is oriented. Then the metric together with the orientation determines a unique volume form ω_g. In this case we define the *Hodge star involution* $*_g : \Lambda^2 \to \Lambda^2$ pointwise by imposing the equality $\langle \alpha, \beta \rangle \omega_g = \alpha \wedge *\beta$. This yields the ± 1 eigenspace decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ of the 2-forms. These 3-dimensional eigenspaces are interchanged if one works with the reversed orientation. By an appropriate change of indices, we consider $W : \Lambda^2 \to \Lambda^2$ as an operator. One can show that the mixed parts $W^-_+ : \Lambda^2_+ \to \Lambda^2_-$ and $W^+_- : \Lambda^2_- \to \Lambda^2_+$ vanish so that the Weyl tensor decomposes as $W = W^+_+ \oplus W^-_-$. Abbreviating $W_\pm = W^\pm_\pm$, we say that the Riemannian manifold is *self-dual* if $W_- \equiv 0$, *anti-self-dual* if $W_+ \equiv 0$ respectively. We also call each of these two cases as *half-conformally flat* if we do not want to specify any orientation. This terminology actually comes from the interpretation that W is a conformally invariant tensor if one considers it as a $(1,3)$ tensor. One can show that the manifold is conformally flat if and only if the tensor $W \equiv 0$ for dimensions $n \geq 4$. See [Kü] for a proof. In dimension $n = 3$ local conformal flatness is determined by the Schouten tensor, and in dimension $n = 2$ all manifolds are locally conformally flat. Basic examples of LCF manifolds are constant sectional curvature spaces, e.g., S^n and T^n, their products with S^1 or $\mathbb{R}$, and products of two Riemannian manifolds with constant sectional curvature 1 and -1 respectively. See [Bes] for further details. Note that SD metrics are only defined on orientable 4-manifolds. Basic SD 4-manifolds are first of all LCF spaces, e.g. S^4, T^4, $S^1 \times S^3$ (see Theorem 3.10), among non-LCF ones we have $\mathbb{CP}_2$ with its Fubini-Study metric (see Corollary 3.1) and scalar-flat-Kähler (SFK) surfaces (see Theorem 2.1) in particular the K3 surface. SD 4-manifolds are first considered by [Pen] and later put on a firm mathematical foundation in [AHS].

In this survey, we analyze LCF and SD structures on various simple 4-manifolds like product type or simply-connected. These results are spread out various places. Some of them are not written, if so not in detail. We hope that it is a good public service to accumulate these results in an article. Interested reader may consult to the resources [AK] and [AKO] for some recent progress on this type of geometry. In section 2 we introduce the basic tools, in section 3 we apply these tools on the manifolds and finally in the appendix we present a partial result along with an open problem.

Acknowledgements. Thanks to S. Finashin, C. Koca and M. Korkmaz for useful discussions.

2. Tools

In this section we develop the main tools to analyze our spaces. Let M be a closed, oriented 4-manifold. We have the following two basic identities which connect quantities related to curvature with topological numbers.

$$\chi(M) = \frac{1}{8\pi^2}\int_M \frac{s^2}{24} - \frac{|\overset{\circ}{\mathrm{Ric}}|^2}{2} + |W_+|^2 + |W_-|^2\omega_g \tag{1}$$

$$\tau(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2\omega_g \tag{2}$$

The first one is called the *generalized Gauss-Bonnet theorem* [AW, ST] which can also be generalized to all even dimensions. The second one is called the *signature formula* which is specific to dimension 4. It is obtained through the Hirzebruch signature theorem [Hi] $\tau(M) = \frac{1}{3}p_1[M]$ by expanding the Pontrjagin class with Chern-Weil theory [ST]. See also second volume of [KN]. These two equations relate geometric information with the topological ones. As an immediate application for example if a 4-manifold admits a locally conformally flat metric then both of the self-dual and anti-self-dual Weyl curvatures vanish since $W = W_+ \oplus W_- = 0$. Hence the signature formula (2) implies that the signature $\tau(M) = 0$. Perhaps this signature condition is the most important topological obstruction for admitting LCF metrics. We start with a very common way of producing ASD metrics on a Kähler manifold. Recall that a Riemannian manifold is called *scalar-flat* if its scalar curvature is zero everywhere.

Theorem 2.1. *A Kähler surface is scalar-flat(SF) iff anti-self-dual(ASD).*

Proof. We follow [Bes] and [LS]. The alternating and symmetry properties of the curvature tensor implies that it is a symmetric section of the bundle $\wedge^2 \otimes \wedge^2$. Furthermore, since we are on a Kähler manifold we have the identities [KN],

$$R(JX, JY)Z = R(X, Y)Z \quad \text{and} \quad R(X, Y)JZ = JR(X, Y)Z.$$

These imply respectively the J-invariance of the first and the second pair of components of the curvature tensor hence it is a type $(1, 1)$ real 2-form in both of these components. So we can think of the Kähler curvature tensor as a symmetric section of $\wedge^{1,1} \otimes \wedge^{1,1}$ or after dualizing the second component, as a symmetric element of $\mathrm{End}(\wedge^{1,1})$.

Since we are on a complex manifold, we have the Dolbeault decomposition of 2-forms $\wedge^2_{\mathbb{C}} = \wedge^{2,0} \oplus \wedge^{1,1} \oplus \wedge^{0,2}$ according to their type. This decomposition is orthogonal with respect to the action of the Hodge star operator $*_g$. We also have the orthogonal eigenspace decomposition in dimension 4 as explained in the introduction. In the Kähler case, if one complexifies these eigenspaces, we claim to have the following:

$$\wedge^2_{+\mathbb{C}} = \mathbb{C}\omega \oplus \wedge^{2,0} \oplus \wedge^{0,2}$$

$$\wedge^2_{-\mathbb{C}} = \wedge^{1,1}_0$$

Here, the set of *primitive* $(1, 1)$ *forms* $\wedge^{1,1}_0$ can be defined to be the orhogonal complement of ω in $\wedge^{1,1}$. Take a unitary coframe $\{dz^1, dz^2\}$ at a point. Then the Kähler form is

$$\omega = \frac{i}{2}\sum_{i=1}^{2} dz^i \wedge d\bar{z}^i = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$$

and the volume form is computed as

$$\omega_g = \frac{\omega^2}{2!} = \frac{-1}{4}\, dz^{1\bar{1}2\bar{2}} = dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2.$$

Since $*_g(dx^i \wedge dy^i) = dx^{3-i} \wedge dy^{3-i}$ we have $*_g\omega = \omega$, i.e., the Kähler form is self-dual. This observation links to the following interpretation of the primitive $(1,1)$ forms. We can write $\wedge_0^{1,1} = \operatorname{Ker} L$, for the *Lefschetz operator* $L : \wedge^{1,1} \to \wedge^{2,2}$ defined by $L(\alpha) = \alpha \wedge \omega$ since

$$\langle \alpha, \omega \rangle = 0 \Leftrightarrow 0 = \langle \alpha, \omega \rangle \omega_g = \alpha \wedge *_g\omega = \alpha \wedge \omega = L(\alpha).$$

The forms $\omega_5 = dz^1 \wedge dz^2$ and $\omega_6 = d\bar{z}^1 \wedge d\bar{z}^2$ pointwise generate the complex bundles $\wedge^{2,0}$ and $\wedge^{0,2}$ respectively. One easily checks that their real parts are self-dual. Hence ω, ω_5, $\omega_6 \in \wedge^2_{+\mathbb{C}}$ are orthogonal and their real parts are elements of the real rank 3 vector bundle of self-dual 2-forms $\wedge^2_+$. On the other hand the $(1,1)$ forms $\omega_2 = dz^1 \wedge d\bar{z}^2$, $\omega_3 = dz^2 \wedge d\bar{z}^1$ and $\omega_4 = dz^1 \wedge d\bar{z}^1 - dz^2 \wedge d\bar{z}^2$ are orthogonal and their real parts are anti-self-dual. For example $*_g \Re\omega_2 = *_g(dx^{12} + dy^{12}) = -(dx^{12} + dy^{12}) = -\Re\omega_2$. Moreover they are all orthogonal to ω, ω_5, $\omega_6 \in \wedge^2_{+\mathbb{C}}$.

Now, since the curvature operator $\mathcal{R}$ is in $\operatorname{End}(\wedge^{1,1})$, its upper left piece $\mathcal{R}^+_+ = \mathcal{W}_+ + \frac{s}{12} I$ is an element of $\operatorname{End}(\mathbb{C}\omega)$. So suppose $\mathcal{R}^+_+ = f\,\omega \otimes \omega^\sharp$ for some function $f : M \to \mathbb{R}$.

$\mathcal{R}$	$\mathbb{C}\omega \oplus \wedge^{2,0} \oplus \wedge^{0,2}$	$\wedge_0^{1,1}$
$\mathbb{C}\omega \oplus \wedge^{2,0} \oplus \wedge^{0,2}$	$\mathcal{W}_+ + \frac{s}{12} I$	$\mathring{\mathrm{Ric}}$
$\wedge_0^{1,1}$	$\mathring{\mathrm{Ric}}$	$\mathcal{W}_- + \frac{s}{12} I$

TABLE 1. Curvature operator for Kähler surfaces

$\mathcal{R}$	$\mathbb{C}\omega$	$\wedge_0^{1,1}$
$\mathbb{C}\omega$	$s/4\cdot$	$\mathring{\mathrm{Ric}}$
$\wedge_0^{1,1}$	$\mathring{\mathrm{Ric}}$	$\mathcal{W}_- + \frac{s}{12} I$

$\mathcal{R}^+_+$	$\mathbb{C}\omega$	$\wedge^{2,0}$	$\wedge^{0,2}$
$\mathbb{C}\omega$	$s/4\cdot$	0	0
$\wedge^{2,0}$	0	0	0
$\wedge^{0,2}$	0	0	0

TABLE 2. Curvature operator and its self-dual part for Kähler surfaces

To figure out the function f we need to compute some inner products. First compute the norm of the Kähler form ω. Since the volume form $\omega_g = \omega^2/2!$ we have

$$\langle \omega, \omega \rangle\, \omega_g = \omega \wedge *\omega = \omega \wedge \omega = 2\,\omega_g.$$

Hence we get $|\omega| = \sqrt{2}$. [1] Secondly we want to compute the inner product $\langle \rho, \omega \rangle$ where ρ is the *Ricci form* defined by $\rho(\cdot,\cdot) = \mathrm{Ric}(J\cdot,\cdot)$. The above trick does not work in this case. We need to use some tensor analysis. Start with fixing a convention for the complex structure. Suppose $J = J_i{}^k\, dx^i \otimes \partial_k$. Then the basic property $J^2 = -Id$ reads $J_i{}^k J_k{}^j = -\delta_i^j$ [2] in terms of tensors. Keep in mind that the ω and J are skew symmetric tensors. We compute the following:

$$\omega^{ij} = \omega(dx^i, dx^j) = g(Jdx^i, dx^j) = g(J_k{}^i dx^k, dx^j) = J_k{}^i g^{kj} = J^{ji} = -J^{ij}.$$

$$\rho_{ij} = \rho(\partial_i, \partial_j) = \mathrm{Ric}(J\partial_i, \partial_j) = J_i{}^k \mathrm{Ric}(\partial_k, \partial_j) = J_i{}^k R_{kj}.$$

And then,

$$\begin{aligned}\langle \rho, \omega \rangle &= \frac{1}{2!}\,\rho_{ij}\,\omega^{ij} = \frac{1}{2}\,J_i{}^k R_{kj}(-J^{ij}) = \frac{-1}{2}\,J_i{}^k R_{kj} J_k{}^j g^{ki} \\ &= \frac{-1}{2}\,(-\delta_i^j) R_{kj} g^{ki} = \frac{1}{2}\,R_{ki} g^{ki} = \frac{s}{2}.\end{aligned}$$

Now, writing $\mathcal{R}\omega = f\omega + g\omega^{\perp}$, multiplying both sides with the Kähler form and using $\mathcal{R}\omega = \rho$, we get the following:

$$\begin{aligned}\langle \mathcal{R}\omega, \omega \rangle &= f\,\langle \omega, \omega \rangle \\ \langle \rho, \omega \rangle &= f\,\langle \omega, \omega \rangle \\ s/2 &= f \cdot 2 \\ s/4 &= f.\end{aligned}$$

Hence $\mathcal{W}_+$ is a multiple of the scalar curvature s. Therefore we have $s = 0$ if and only if $\mathcal{W}_+ = 0$. ☐

This result has an immediate corollary.

Corollary 2.2. *For Kähler metrics on a complex surface we have the pointwise identity*

$$|W_+|^2 = \frac{s^2}{24}. \tag{3}$$

Proof. Since in the Kähler case the term $\mathcal{R}_+^+ = W_+ + \frac{s}{12} I$ acts by the following

$$\frac{s}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

as in the Table 2, we have

$$W_+ = \frac{s}{12}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

[1] In general dimension n, writing in local orthonormal frame $\omega = e_1 \wedge e_2 + \ldots + e_{2n-1} \wedge e_{2n}$ we compute $|\omega| = \sqrt{n}$. Then from $|\omega|^2 \omega_g = \omega \wedge *\omega$ we get $*\omega = |\omega|^2 \omega^{n-1}/n!$ so $*\omega = \omega^{n-1}/(n-1)!$

[2] If one fixes the alternative convention $J = J^k_{\ i}\, \partial_k \otimes dx^i$ one gets $J^j_{\ k} J^k_{\ i} = -\delta_i^j$ and $\omega_{ij} = -J_{ij}$, a negative sign.

Taking the norms of both sides yields the result. □

Theorem 2.3. *If a 4-manifold M admits a Kähler-Einstein (KE) metric g then $\chi(M) = 3\tau(M)$ if and only if g is self-dual (SD).*

Proof. For Kähler metrics we have the pointwise identity (3)

$$|W_+|^2 = \frac{s^2}{24}.$$

For Einstein metrics, by definition the trace-free Ricci tensor $\overset{\circ}{\mathrm{Ric}} = 0$. Plugging these two identities into the Gauss-Bonnet formula (1) we get

$$8\pi^2\chi = 2\,\|W_+\|^2 + \|W_-\|^2. \tag{4}$$

Eliminating $\|W_+\|$ from (2) and (4) we get the following equality

$$8\pi^2(\chi - 3\tau) = 3\,\|W_-\|,$$

which yields the result. □

Finally we state the celebrated theorems of Kuiper. See also [Ho] for a recent exposition and improvement.

Theorem 2.4 ([Kui]). *Let (M^n, g) be a simply connected, LCF n-manifold of class C^1. Then there is a conformal immersion $f : M \to S^n$. If in addition M is compact, then this map is a conformal diffeomorphism.*

Here comes another very useful theorem of Kuiper. See also [Kob].

Theorem 2.5 ([Kui2]). *Universal cover of a compact, LCF space with an infinite Abelian fundamental group must be $\mathbb{R}^n$ or $\mathbb{R} \times S^{n-1}$.*

3. Simple 4-manifolds

We start with a basic space, the complex projective space with its standart Fubini-Study metric. This can be though as the metric quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by unit complex scalar multiplication. Alternatively on $\mathbb{C}^n$ take the following complex metric coefficients:

$$g_{FS_{i\bar{j}}} = g_{FS}(\partial_i, \partial_{\bar{j}}) := \frac{(1 + |z|^2)\delta_{i\bar{j}} - z_i z_{\bar{j}}}{(1 + |z|^2)^2}.$$

Taking the completion of this space gives the complex projective space $\mathbb{CP}_n$. As an application of the theorems in the previous section we obtain the following.

Corollary 3.1. $(\mathbb{CP}_2, g_{FS})$*, the complex projective space with its Fubini-Study metric is self-dual.*

Proof. One can easily compute $\chi(\mathbb{CP}_2) = 3$. Since the intersection form $Q_{\mathbb{CP}_2} = [1]$ we have $\tau = 1$. These satisfy the equality in Theorem 2.3. □

Theorem 3.2. *The underlying smooth manifold of $\mathbb{CP}_2$, the complex projective space does not admit any LCF metrics.*

Proof. The basic obstruction signature $\tau(\mathbb{CP}_2) = 1$ is nontrivial. $\square$

Next we work on the 4-manifold $S^2 \times S^2$, the product of two spheres. The spheres $S^2 \times q$ and $p \times S^2$ generating the homology have self-intersection zero, and $+1$ with each other. So

$$Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

called the hyperbolic matrix and denoted by H. It has eigenvalues ± 1 and hence the signature $\tau = 0$. [3] Hence an obstruction vanishes for locally conformally flatness. However, this turns out to be not sufficient as follows.

Theorem 3.3. *The 4-manifold $S^2 \times S^2$ does not admit any LCF nor even SD metrics.*

Proof. Since by Kuiper's theorem [Kui], any compact, simply-connected, LCF Riemannian 4-manifold is conformally equivalent to the round 4-sphere, $S^2 \times S^2$ does not admit any LCF metric. Suppose it does have a self-dual metric. Then $W_- = 0$ and by the signature formula (2) the integral $0 = \int_M |W_+|^2 \omega_g$. So, pointwise $W_- = 0$ and $W = 0$. This yields a LCF metric which is already a contradiction by the previous argument. $\square$

This example also illustrates the fact that the product of LCF manifolds may not be LCF.

Corollary 3.4. *The 4-manifold $K3$ is not LCF, but SD.*

Proof. Since the intersection form of a K3 surface is $Q = 2E_8 \oplus 3H$, the signature $\tau(K3)$ is -16 which is nonzero so that it can not be LCF. Since the first Chern class $2\pi c_1(K3)$ is zero, the zero form is a $(1,1)$-form representing this class. Since $K3$ is compact and Kähler by Yau's resolution to the Calabi's problem [Yau], there is a Kähler metric g in the same class with Ricci form $\rho_g \equiv 0$. Hence g is Ricci flat, so scalar flat. SFK surfaces are ASD, hence SD with the reversed orientation. $\square$

Theorem 3.5. *The smooth 4-manifold $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ does not admit any LCF nor SD metrics.*

Proof. Since $\tau = 0$, the argument in the proof of Theorem 3.3 is again valid for this 4-manifold. $\square$

A similar argument also excludes the manifolds $k\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ for any $k > 0$. Since we used it multiple times, it is convenient to sum up the idea of the proof as follows.

[3] Alternatively, the map $(I_3, -I_3)$ is an orientation reversing diffeomorphism of $S^2 \times S^2$. So Hirzebruch signatures mapped onto each other and $\tau = -\tau$.

Theorem 3.6. *Let M be a compact, oriented 4-manifold with signature $\tau = 0$, then we have the following.*

1. A metric g on M is SD iff LCF.

2. If M is simply connected but not diffeomorphic to S^4 then it does not admit any LCF or SD metric.

Theorem 3.7. *The smooth* 4*-manifold $S^1 \times S^1 \times S^1 \times S^1 = T^2 \times T^2$ is LCF hence SD.*

Proof. Since T^4 is a quotient of $\mathbb{R}^2$ by $\mathbb{Z}^2$ isometries, it is flat hence LCF and consequently SD.

The second assertion is alternatively seen as follows: The product of flat metrics on the components is scalar-flat-Kähler (SFK). So by the Theorem 2.1 it is ASD. □

As a by-product of local conformal flatness we also compute $\tau(T^4) = 0$. Bieberbach's theorem states that the only compact manifolds that admit flat metrics are T^n and its finite quotients. Hence these are the only (globally) conformally flat manifolds. Since these are already classified, the phrase "conformally flat" is usually used in place of LCF in the literature. Before checking another possible basic product in dimension four, let us look at its universal cover.

Theorem 3.8. *The smooth* 4*-manifold $S^2 \times \mathbb{R}^2$ is LCF hence SD.*

Proof. Since $S^2 \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^3 - \{0\}$, the inherited standart metric is of constant (positive) sectional curvature. A constant sectional curvature space times $\mathbb{R}$ is LCF. □

Next result indicates that the above manifold can not have isometries that give $S^2 \times T^2$ as the quotient.

Theorem 3.9. *The smooth* 4*-manifold $S^2 \times T^2$ is neither LCF nor SD.*

Proof. Because of the antipodal orientation reversing isometry of the 2-sphere, we have the signature $\tau(S^2 \times T^2) = 0$. Hence this is LCF iff SD. Secondly, since this manifold has Abelian infinite fundamental group $\mathbb{Z}^2$, if it admits a LCF metric then applying Kuiper's second theorem (Theorem 2.5) it should have universal cover $\mathbb{R}^4$ or $\mathbb{R} \times S^3$. Since the universal cover is $S^2 \times \mathbb{R}^2$ this gives a contradiction. In general this argument applies to $S^p \times T^q$ for $p, q \geq 2$. □

This manifold is particularly interesting since even though it is not LCF, the following infimum of all the possible Weyl energies on the manifold,

$$W(M) := \inf \left\{ \int_M |W_g|_g^2 \, \omega_g : g \in \mathcal{M}_M \right\}$$

called the *Weyl invariant* is zero, where W_g is the conformally invariant type $(1,3)$ Weyl tensor, $\mathcal{M}_M$ is the space of smooth metrics. See [Kob] for a proof. Hence the infimum is not attained.

One can easily compute the Euler characteristic of $S^1 \times S^3$ to be $\chi = 0$, since the S^3 component is an oriented closed 3-manifold hence $\chi(S^3) = 0$. Since there is no second homology, the signature $\tau = 0$. In this case it is a candidate of a manifold which may admit LCF metrics. This turns out to be the case as follows.

Theorem 3.10. *The 4-manifold $S^1 \times S^3$ with its standard product metric is LCF and hence SD.*

Proof. The proof is adapted from [JV]. Locally we can think this as $\mathbb{R} \times S^3$, where $\mathbb{R}$ is the flat line with $g_{\mathbb{R}} = dx \otimes dx$ and S^3 has the round metric of curvature $+1$. Rm denoting the Riemann curvature tensor, $\owedge$ is the Kulkarni-Nomizu product which is commutative,

$$\begin{aligned}
\mathrm{Rm}_{\mathbb{R}\times S^3} &= \mathrm{Rm}_{\mathbb{R}} + \mathrm{Rm}_{S^3} \\
&= 0 + \tfrac{1}{2}\, g_{S^3} \owedge g_{S^3} \\
&= \tfrac{1}{2}\,(g_{S^3} + dx^2) \owedge (g_{S^3} - dx^2) \\
&= \tfrac{1}{2}\, g_{\mathbb{R}\times S^3} \owedge (g_{S^3} - dx^2) \\
&= \Psi(\tfrac{1}{2}\,(g_{S^3} - dx^2))
\end{aligned}$$

where by the definition of the product $dx^2 \owedge dx^2 = 0$, and $\Psi : S^2(T^*M) \longrightarrow \mathrm{Ker}\, b$ is defined by $\Psi(h) = h \owedge g_M$. Here, $\mathrm{Ker}\, b$ is the space of curvature-like tensors as in the introduction. The decomposition $\mathrm{Ker}\, b = \mathcal{W} \oplus \Psi(S_0^2(T^*M)) \oplus \Psi(\mathbb{R}g)$ implies that the Weyl tensor vanishes. $\square$

Theorem 3.11. *The 4-manifolds $S^2 \times \Sigma_g$ with their standard product metric is LCF and hence SD for $g \geq 2$.*

Proof. We adapt from [Bes]. Suppose we have the constant sectional curvature $+1$ metric on S^2 and -1 metric on Σ_g. Then we have the following descriptions

$$\mathrm{Rm}_{S^2} = \frac{1}{2}\, g_{S^2} \owedge g_{S^2} \quad \text{and} \quad \mathrm{Rm}_{\Sigma_g} = \frac{-1}{2}\, g_{\Sigma_g} \owedge g_{\Sigma_g}$$

for the Riemann curvature tensors.

$$\begin{aligned}
\mathrm{Rm}_{S^2\times\Sigma_g} &= \mathrm{Rm}_{S^2}+\mathrm{Rm}_{\Sigma_g}\\
&= \tfrac{1}{2}\left(g_{S^2}\owedge g_{S^2}-g_{\Sigma_g}\owedge g_{\Sigma_g}\right)\\
&= \tfrac{1}{2}\left(g_{S^2}+g_{\Sigma_g}\right)\owedge\left(g_{S^2}-g_{\Sigma_g}\right)\\
&= \tfrac{1}{2}\,g_{S^2\times\Sigma_g}\owedge\left(g_{S^2}-g_{\Sigma_g}\right)\\
&= \Psi\big(\tfrac{1}{2}\left(g_{S^2}-g_{\Sigma_g}\right)\big).
\end{aligned}$$

Being in the image of Ψ, the Weyl tensor vanishes. Alternatively, starting with the Kähler metrics, one obtains a scalar-flat-Kähler (SFK) metric on the product. Now apply Theorem 2.1 with both orientations. □

Appendix A.

Using Gauss-Bonnet, and signature formula techniques we can also prove the following.

Theorem A.1. *If a 4-manifold admits a Kähler-Einstein(KE) metric which is also locally conformally flat(LCF) then its Euler characteristic $\chi = 0$.*

Proof. Recall the pointwise identity of Corollary 2.2 for Kähler metrics

$$|W_+|^2 = \frac{s^2}{24}.$$

For Einstein metrics, by definition the trace-free Ricci tensor $\mathring{\mathrm{Ric}} = 0$. Plugging these two identities into the Gauss-Bonnet formula (1) we get

$$8\pi^2\chi(M) = 2\,\|W_+\|^2 + \|W_-\|^2.$$

Locally conformally flatness implies $W_\pm = 0$, hence $\chi = 0$ by above. □

As an application we can prove the following non-existence result.

Theorem A.2. *The product metric on the 4-manifolds $\Sigma_g \times \Sigma_h$, product of surfaces of genus $g, h \geq 2$ is not a LCF nor SD metric.*

Proof. First of all, $\Sigma_g \times \Sigma_h$ admits a Kähler-Einstein metric. One can see this through different ways. One is to use Aubin/Yau theorem, since $c_1 < 0$, a surface of general type, there exists a unique KE metric on this complex surface. Another way is to think in terms of product metrics. If you have the hyperbolic -1 curvature Kähler metrics on both components, the product metric is Kählerian. Besides that, the product of Einstein metrics with common cosmological constant is again Einstein. Combining the two, we obtain a Kähler-Einstein metric on the manifold. If the manifold admits a LCF metric as well, then Theorem A.1 implies that $\chi = 0$, however $\chi = (2-2g)(2-2h)$ which is a contradiction. So the product of two hyperbolic metrics on $\Sigma_g \times \Sigma_h$ is not LCF.

To analyze SD structure we need to find the signature of the manifold. A two dimensional oriented surfaces of genus g always has an orientation reversing diffeomorphism (involution) R_g. One can construct this by using a mirror reflection or reflection through a point after arranging the holes symmetrically. Then (R_g, I) is going to be an orientation reversing diffeomorphism of the 4-manifold. Since the Hirzebruch signature is diffeomorphism invariant and changing the orientation changes its sign, we have $\tau = -\tau$ hence $\tau(\Sigma_g \times \Sigma_h) = 0$. Alternatively one can compute the intersection matrix as $gh(-H_4) \oplus H_2$ and hence the characteristic polynomial $(\lambda^2-1)^{2gh}(\lambda^2-1)$. (Another approach might be exploiting only the parity of the intersection form and use Rokhlin's theorem of divisibility of the signature by 8 in the case of even intersection forms to obtain at least some of the cases.) If the product metric is a SD metric on the manifold then it is LCF by Theorem 3.6, which is already violated. □

Existence of LCF metrics on the product $\Sigma_g \times \Sigma_h$ of surfaces of genus $g \geq 2$ and $h \geq 1$ still remains as an open problem.

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